

A semigroup analogue of the Fonf–Lin–Wojtaszczyk ergodic characterization of reflexive Banach spaces with a basis

by

DELIO MUGNOLO (Tübingen and Bari)

Abstract. In analogy to a recent result by V. Fonf, M. Lin, and P. Wojtaszczyk, we prove the following characterizations of a Banach space X with a basis.

- (i) X is finite-dimensional if and only if every bounded, uniformly continuous, mean ergodic semigroup on X is uniformly mean ergodic.
- (ii) X is reflexive if and only if every bounded strongly continuous semigroup is mean ergodic if and only if every bounded uniformly continuous semigroup on X is mean ergodic.

1. Introduction. E. R. Lorch proved in the 1930s that if a Banach space X is reflexive, then every power-bounded operator T on X is mean ergodic, i.e., the sequence $(n^{-1} \sum_{k=1}^n T^k x)_{n \in \mathbb{N}}$ converges for all $x \in X$. In a recent article Fonf, Lin, and Wojtaszczyk have proven ([FLW01, Cor. 1]) that the converse is also true if we assume X to have a basis. Indeed, under this assumption, namely that X is a non-reflexive Banach space with a basis, they have been able to construct a power-bounded operator T on X which is not mean ergodic.

Paralleling their technique, we show in Section 3 that an analogous characterization holds if bounded strongly continuous semigroups of operators are considered instead of power-bounded operators. To this purpose, we introduce in Section 2 a semigroup that in turn permits us to prove an ergodic characterization of finite-dimensional Banach spaces that is the semigroup analogue of [FLW01, Cor. 3].

We emphasize that mean ergodicity of a bounded strongly continuous semigroup $(T(t))_{t \geq 0}$ is *not* equivalent to mean ergodicity of the individ-

2000 *Mathematics Subject Classification*: Primary 47A35.

Key words and phrases: Schauder bases, reflexive Banach spaces, ergodic semigroups of operators.

The author is supported by the Istituto Nazionale di Alta Matematica “Francesco Severi”.

ual operators $T(t)$, $t > 0$. If for some $t_0 > 0$ the power-bounded operator $T(t_0)$ is mean ergodic, so is the bounded semigroup $(T(t))_{t \geq 0}$ (see [DS58, Thm. VII.7.1] for $t_0 = 1$). However, there exist examples of bounded, mean ergodic, strongly continuous semigroups $(T(t))_{t \geq 0}$ so that no operator $T := T(t_0)$, $t_0 > 0$, is mean ergodic, cf. [EN00, Expl. V.4.13.3] (see also [Kr85, p. 83]).

2. Introductory results. We begin with a brief reminder of some ergodic theory, and refer to [Kr85] or [EN00, §V.4] for details.

DEFINITION 2.1. Let $\mathcal{T} := (T(t))_{t \geq 0}$ be a strongly continuous semigroup of linear operators on a Banach space X . We denote by

$$C(r)x := \frac{1}{r} \int_0^r T(s)x \, ds, \quad x \in X, r > 0,$$

the Cesàro means of \mathcal{T} . Then the semigroup \mathcal{T} is called *mean ergodic* (*uniformly mean ergodic*, resp.) if $(C(r))_{r > 0}$ converges strongly (with respect to the operator norm, resp.) as $r \rightarrow \infty$.

The following useful characterization of mean ergodicity of a strongly continuous semigroup is due to Nagel ([Na73, Thm. 1.7]). It is the semigroup analogue of the result of Sine stated in Lemma 3.5 below.

LEMMA 2.2. *A bounded strongly continuous semigroup is mean ergodic if and only if its fixed space separates the fixed space of its adjoint.*

REMARK 2.3. We recall that the fixed space of a strongly continuous semigroup equals the null space of its generator (use [EN00, Cor. IV.3.8(i)]). Using the theory of sun dual semigroups, one can also see that the same holds for the adjoint of a strongly continuous semigroup (which in general is *not* strongly continuous). Hence, a bounded strongly continuous semigroup is mean ergodic if and only if the null space of its generator A separates the null space of its adjoint A' .

In particular, a bounded strongly continuous semigroup is mean ergodic if the adjoint of its generator has trivial null space.

For uniformly mean ergodic semigroups the following characterization is well known (cf. [EN00, Thm. V.4.10]).

LEMMA 2.4. *A bounded strongly continuous semigroup is uniformly mean ergodic if and only if either its generator is invertible with bounded inverse, or the origin is a pole for its resolvent operator.*

For a thorough treatment of the theory of Schauder bases and Schauder decompositions on infinite-dimensional spaces we refer the reader to [Si70, §I.14] or [GD92, Chapt. 1]. We only recall that the notions of basis and

Schauder basis are equivalent on Banach spaces (see [Si70, p. 170]) and that a Schauder basis is always associated with a Schauder decomposition, but the converse implication fails to hold.

Let us moreover mention the following general result: If X is a Banach space with a Schauder decomposition $X = \sum_{k \in \mathbb{N}} X_k$, then the norm of X can, without loss of generality, be assumed to make the operators Q_h and P_h , $h \in \mathbb{N}$, defined by

$$(2.1) \quad Q_h x := x_h, \quad P_h x := \sum_{k=1}^h x_k, \quad x := \sum_{k \in \mathbb{N}} x_k \in X,$$

contractive. Observe that, due to the uniqueness of the Schauder expansion of $x \in X$, $x = 0$ if and only if $P_n x = 0$ for all $n \in \mathbb{N}$.

With this notation we now prove the following.

LEMMA 2.5. *Let X be a Banach space with a Schauder decomposition. Then*

$$M(t)x := \sum_{h \in \mathbb{N}} e^{-t/h} Q_h x, \quad x \in X, t \geq 0,$$

defines a bounded uniformly continuous semigroup $\mathcal{M} := (M(t))_{t \geq 0}$ on X .

For the proof (and later on) we will repeatedly use the auxiliary family $(b_{n,t})_{n \in \mathbb{N}, t \geq 0}$, of sequences of positive numbers defined by

$$(2.2) \quad b_{1,t} := e^{-t}, \quad b_{n,t} := e^{-t/n} - e^{-t/(n-1)} \quad \text{for } n = 2, 3, \dots, t \geq 0.$$

Observe that

$$(2.3) \quad \begin{aligned} \sum_{h=m+1}^n b_{h,t} &= e^{-t/n} - e^{-t/m}, & \sum_{h=1}^n b_{h,t} &= e^{-t/n}, \\ \sum_{h=m+1}^{\infty} b_{h,t} &= 1 - e^{-t/m} \end{aligned}$$

for all $m, n \in \mathbb{N}$, $m < n$, $t \geq 0$.

Proof of Lemma 2.5. We first show that \mathcal{M} is well defined on X . Let $x = \sum_{k \in \mathbb{N}} x_k \in X$. Then

$$\begin{aligned} \sum_{k=m+1}^n e^{-t/k} x_k &\stackrel{(2.3)}{=} \sum_{k=m+1}^n \left(\sum_{h=m+1}^k b_{h,t} + e^{-t/m} \right) x_k \\ &= e^{-t/m} \sum_{k=m+1}^n x_k + \sum_{j=m+1}^n b_{j,t} \left(\sum_{h=j}^n x_h \right), \end{aligned}$$

for all $m, n \in \mathbb{N}$, $m < n$, and the convergence of $\sum_{k \in \mathbb{N}} x_k$ implies that $(\sum_{k=1}^n e^{-t/k} x_k)_{n \in \mathbb{N}}$ is a Cauchy sequence. It is also clear that \mathcal{M} is a semi-

group, and that its generator A , defined by

$$Ax := - \sum_{h \in \mathbb{N}} \frac{1}{h} Q_h x, \quad x \in X,$$

is bounded, hence \mathcal{M} is uniformly continuous. Also, \mathcal{M} is bounded since for $t \geq 0$ and $m \in \mathbb{N}$

$$\begin{aligned} \sum_{k=1}^m (1 - e^{-t/k}) x_k &\stackrel{(2.3)}{=} \sum_{k=1}^m \sum_{h=k+1}^{\infty} b_{h,t} x_k \\ &= \sum_{h=2}^m b_{h,t} \left(\sum_{k=1}^{h-1} x_k \right) + \sum_{h=m+1}^{\infty} b_{h,t} \left(\sum_{k=1}^m x_k \right) \\ &= \sum_{h=2}^m b_{h,t} P_{h-1} x + \sum_{h=m+1}^{\infty} b_{h,t} P_m x. \end{aligned}$$

Recall that the operators P_h , $h \in \mathbb{N}$, are contractive, and hence

$$\begin{aligned} \left\| \sum_{k=1}^m (1 - e^{-t/k}) x_k \right\| &\leq \sum_{h=2}^m b_{h,t} \|x\| + \sum_{h=m+1}^{\infty} b_{h,t} \|x\| \\ &= \sum_{h=2}^{\infty} b_{h,t} \|x\| \stackrel{(2.3)}{=} (1 - e^{-t}) \|x\|. \end{aligned}$$

It follows that

$$\|M(t)x - x\| = \left\| \sum_{k \in \mathbb{N}} (1 - e^{-t/k}) x_k \right\| \leq (1 - e^{-t}) \|x\| \leq \|x\|,$$

and therefore $\|M(t) - I\| \leq 1$, hence $\|M(t)\| \leq 2$ for all $t \geq 0$. ■

THEOREM 2.6. *Let X be a Banach space with a basis. Then X is finite-dimensional if and only if every bounded, mean ergodic, uniformly continuous semigroup on X is uniformly mean ergodic.*

Proof. The necessity follows by [EN00, Cor. I.2.11 and Thm. V.4.10].

Assume now X to be infinite-dimensional with a basis, thus in particular with a Schauder decomposition. Hence, we can define, as in Lemma 2.5, the bounded uniformly continuous semigroup \mathcal{M} generated by A . We prove that \mathcal{M} is mean ergodic, but not *uniformly* mean ergodic.

Let $x' \in \ker(A')$. Then

$$0 = \langle Ax, x' \rangle = - \sum_{k \in \mathbb{N}} \frac{\langle Q_k x, x' \rangle}{k}$$

for all $x \in X$. Taking into account the uniqueness of the expansion of elements of X we obtain $x' = 0$. By Remark 2.3 this implies that \mathcal{M} is mean ergodic.

The generator A is injective, and its inverse is given by

$$(2.4) \quad A^{-1}x = - \sum_{h \in \mathbb{N}} h Q_h x$$

for those $x \in X$ such that the right-hand side of (2.4) converges. Hence A^{-1} is not bounded. Moreover, 0 is an accumulation point for the set of eigenvalues of A . The claim now follows by Lemma 2.4. ■

3. Ergodic characterizations of reflexivity. The technique involved in the proof of Theorem 3.4 below relies on a general result on the geometry of Banach spaces. Answering a question raised by Singer ([Si62, Problem P2]), Zippin was able to prove ([Zi68, Thm. 1]) that on every non-reflexive Banach space with a basis there exists a non-shrinking basis (we refer to [Zi68] for details). Using this property, Fonf, Lin, and Wojtaszczyk have obtained ([FLW01]) a result that can be summarized as follows.

LEMMA 3.1. *Let X be a non-reflexive Banach space with a basis. Then there exists a Schauder decomposition $X = \sum_{k \in \mathbb{N}} X_k$, an (equivalent) norm $\|\cdot\|$ on X , a functional $\mathbf{f} \in X'$, and a sequence $(\mathbf{e}_k)_{k \in \mathbb{N}} \in \prod_{k \in \mathbb{N}} X_k$ such that $\|\mathbf{e}_k\| \leq 1$ and $\mathbf{f}(\mathbf{e}_k) = 1$ for all $k \in \mathbb{N}$, and the operators P_h and Q_h , $h \in \mathbb{N}$, defined as in (2.1) are contractive with respect to $\|\cdot\|$.*

REMARK 3.2. The proof ([FLW01, p. 149]) shows in particular that the sequence $(\mathbf{e}_k)_{k \in \mathbb{N}}$ is in general *not* a basis; in fact, the space X_1 need not be one-dimensional and accordingly the Schauder decomposition $X = \sum_{k \in \mathbb{N}} X_k$ need not be associated with a basis.

In the following we use the notations of Lemmas 2.4 and 3.1.

LEMMA 3.3. *Let X be a non-reflexive Banach space with a basis. Define a family of operators on X by*

$$N_t x := \sum_{h \in \mathbb{N}} \mathbf{f}(P_h x) b_{h+1,t} \mathbf{e}_{h+1}, \quad x \in X, t \geq 0,$$

with $(b_{n,t})_{n \in \mathbb{N}}, t \geq 0$, as in (2.2). Then

$$T(t)x := M(t)x + N_t x, \quad x \in X, t \geq 0,$$

defines a bounded uniformly continuous semigroup $\mathcal{T} := (T(t))_{t \geq 0}$ on X .

Proof. Since $M(0) = I$ and $N_0 = 0$, we have $T(0) = I$. Let now $x_k \in X_k$, $k \in \mathbb{N}$, $t, s \geq 0$. To show that \mathcal{T} satisfies the semigroup law, it suffices to check that $T(t+s)x_k = T(t)T(s)x_k$, or rather, since $(M(t))_{t \geq 0}$ is a semigroup by Lemma 2.5, that

$$M(t)N_s x_k + N_t M(s)x_k + N_t N_s x_k = N_{t+s} x_k.$$

Observe that

$$N_t x_k = \mathbf{f}(x_k) \sum_{h=k+1}^{\infty} b_{h,t} \mathbf{e}_h, \quad \text{and in particular} \quad N_t \mathbf{e}_k = \sum_{h=k+1}^{\infty} b_{h,t} \mathbf{e}_h.$$

Hence, we obtain

$$\begin{aligned} & M(t)N_s x_k + N_t M(s)x_k + N_t N_s x_k \\ &= \sum_{j=k+1}^{\infty} \mathbf{f}(x_k) b_{j,s} e^{-t/j} \mathbf{e}_j + e^{-s/k} \sum_{h=k+1}^{\infty} \mathbf{f}(x_k) b_{h,t} \mathbf{e}_h \\ &+ \sum_{j=k+1}^{\infty} \mathbf{f}(x_k) b_{j,s} \left(\sum_{h=j+1}^{\infty} b_{h,t} \mathbf{e}_h \right) \\ &= \mathbf{f}(x_k) \left(\sum_{j=k+1}^{\infty} (e^{-t/j} b_{j,s} + e^{-s/k} b_{j,t}) \mathbf{e}_j + \sum_{l=k+2}^{\infty} b_{l,t} \left(\sum_{j=k+1}^{l-1} b_{j,s} \right) \mathbf{e}_l \right) \\ &\stackrel{(2.3)}{=} \mathbf{f}(x_k) \left(\sum_{j=k+1}^{\infty} (e^{-t/j} b_{j,s} + e^{-s/k} b_{j,t}) \mathbf{e}_j + \sum_{j=k+1}^{\infty} b_{j+1,t} (e^{-s/j} - e^{-s/k}) \mathbf{e}_{j+1} \right) \\ &= \mathbf{f}(x_k) \sum_{j=k+1}^{\infty} (e^{-(t+s)/j} - e^{-(t+s)/(j-1)}) \mathbf{e}_j = \mathbf{f}(x_k) \sum_{j=k+1}^{\infty} b_{j,t+s} \mathbf{e}_j = N_{t+s} x_k. \end{aligned}$$

Observe now that N_t/t converges strongly as $t \rightarrow 0^+$, and we obtain

$$\dot{N}x := \lim_{t \rightarrow 0^+} \frac{N_t x}{t} = \sum_{h \in \mathbb{N}} \frac{\mathbf{f}(P_h x)}{h^2 + h} \mathbf{e}_{h+1}, \quad x \in X.$$

The semigroup \mathcal{T} is then generated by the operator B given by

$$Bx := Ax + \dot{N}x = - \sum_{h \in \mathbb{N}} \left(\frac{1}{h} Q_h x - \frac{\mathbf{f}(P_h x)}{h^2 + h} \mathbf{e}_{h+1} \right), \quad x \in X.$$

Since $\|\dot{N}\| \leq \|\mathbf{f}\|$, and hence also B is bounded, it follows that \mathcal{T} is uniformly continuous.

Finally, observe that

$$\begin{aligned} \|N_t x\| &= \left\| \sum_{j \in \mathbb{N}} \mathbf{f}(P_j x) b_{j+1,t} \mathbf{e}_{j+1} \right\| \\ &\leq \|\mathbf{f}\| \sum_{j \in \mathbb{N}} b_{j+1,t} \|x\| \stackrel{(2.3)}{\leq} \|\mathbf{f}\| \|x\| \quad \text{for all } x \in X. \end{aligned}$$

Here, we have used the fact that the operators P_h , $h \in \mathbb{N}$, are contractive by Lemma 3.1. Since the semigroup $(M(t))_{t \geq 0}$ is bounded, it also follows that $\|T(t)\| \leq \|M(t)\| + \|N_t\| \leq 2 + \|\mathbf{f}\|$ for all $t \geq 0$. ■

THEOREM 3.4. *If X is a Banach space with a basis, then the following are equivalent:*

- (a) X is reflexive.
- (b) Every bounded strongly continuous semigroup on X is mean ergodic.
- (c) Every bounded uniformly continuous semigroup on X is mean ergodic.

Proof. (a) \Rightarrow (b). This is a well known result due to Lorch (see [EN00, Expl. V.4.7]).

(b) \Rightarrow (c). Obvious.

(c) \Rightarrow (a). In Lemma 3.3 we have constructed a semigroup \mathcal{T} on a non-reflexive Banach space with a basis. We prove that \mathcal{T} is not mean ergodic by showing that its generator B has trivial null space, and that $\mathbf{f} \neq 0$ is in the null space of B' .

Let $x \in \ker(B)$. Then $-Ax = \dot{N}x$ and hence

$$\sum_{k \in \mathbb{N}} \frac{Q_k x}{k} = \sum_{k \in \mathbb{N}} \frac{\mathbf{f}(P_k x)}{k^2 + k} \mathbf{e}_{k+1}.$$

By the uniqueness of the Schauder expansion of elements of X we deduce that x solves the system

$$\begin{cases} Q_1 x = 0, \\ Q_k x = \frac{\mathbf{f}(P_{k-1} x)}{k-1} \mathbf{e}_k, \quad k = 2, 3, \dots \end{cases}$$

It suffices to show by induction that $P_n x = 0$ for all $n \in \mathbb{N}$, with $P_1 x = x_1 = 0$. Let $P_{n-1} x = 0$. It follows that $Q_n x = 0$, hence $P_n x = P_{n-1} x + Q_n x = 0$, and we conclude that $\ker(B) = \{0\}$.

Let now $x_k \in X_k$, $k \in \mathbb{N}$. Then

$$Bx_k = -\frac{1}{k} x_k + \mathbf{f}(x_k) \sum_{h=k}^{\infty} \frac{1}{h^2 + h} \mathbf{e}_{h+1},$$

and we obtain

$$\langle x_k, B' \mathbf{f} \rangle = \langle Bx_k, \mathbf{f} \rangle = \mathbf{f}(x_k) \left(-\frac{1}{k} + \sum_{h=k}^{\infty} \frac{1}{h^2 + h} \right) = 0.$$

It follows by linearity that $\mathbf{f} \in \ker(B')$, and by Lemma 2.2 the claim holds. ■

The following alternative proof of Theorem 3.4 is due R. Nagel, and seems to be much shorter, but is based on the original result of Fonf, Lin, and Wojtaszczyk.

Recall the following, due to R. Sine ([Se70]).

LEMMA 3.5. *A power-bounded operator is mean ergodic if and only if its fixed space separates the fixed space of its adjoint.*

Acknowledgments. The topic of this paper was suggested to me by Prof. Jerry Goldstein. I wish to thank him and Prof. Gisèle Goldstein for their kind hospitality, for many inspiring discussions, and for their constant support during my visit at the University of Memphis in Spring 2002.

References

- [DS58] N. Dunford and J. T. Schwartz, *Linear Operators. I. General Theory*, Pure Appl. Math. 7, Interscience, 1958.
- [EN00] K.-J. Engel and R. Nagel, *One-Parameter Semigroups for Linear Evolution Equations*, Grad. Texts in Math. 194, Springer, 2000.
- [FLW01] V. P. Fonf, M. Lin, and P. Wojtaszczyk, *Ergodic characterizations of reflexivity of Banach spaces*, J. Funct. Anal. 187 (2001), 146–162.
- [GD92] S. Guerre-Delabrière, *Classical Sequences in Banach Spaces*, Monogr. Textbooks Pure Appl. Math. 166, Marcel Dekker, 1992.
- [Kr85] U. Krengel, *Ergodic Theorems*, de Gruyter, 1985.
- [Na73] R. J. Nagel, *Mittelergodische Halbgruppen linearer Operatoren*, Ann. Inst. Fourier (Grenoble) 23 (1973), no. 4, 75–87.
- [Se70] R. Sine, *A mean ergodic theorem*, Proc. Amer. Math. Soc. 24 (1970), 438–439.
- [Si62] I. Singer, *Basic sequences and reflexivity of Banach spaces*, Studia Math. 21 (1962), 351–369.
- [Si70] —, *Bases in Banach Spaces. I*, Grundlehren Math. Wiss. 154, Springer, 1970.
- [Zi68] M. Zippin, *A remark on bases and reflexivity in Banach spaces*, Israel J. Math. 6 (1968), 74–79.

Arbeitsbereich Funktionalanalysis
 Mathematisches Institut
 der Universität Tübingen
 Auf der Morgenstelle 10
 D-72076 Tübingen, Germany
 E-mail: demu@fa.uni-tuebingen.de

Dipartimento di Matematica
 Università degli Studi di Bari
 Via E. Orabona 4
 I-70125 Bari, Italy

Received July 22, 2002
Revised version March 24, 2004

(4995)