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Unitary equivalence of operators and dilations

by

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Abstract. Given two contractions T and T' such that $T'-T$ is an operator of finite rank, we prove, under some conditions, the unitary equivalence of the unitary parts of the minimal isometric dilations (respectively minimal co-isometric extensions) of T and T'.

1. Introduction. Let $\mathcal{H}$ be a separable Hilbert space. $\mathcal{L}(\mathcal{H})$ will denote the algebra of all bounded operators on $\mathcal{H}$. An operator $T \in \mathcal{L}(\mathcal{H})$ is a *contraction* if $\|T\| \le 1$ or equivalently $I - T^*T$ is positive. Note also that any contraction T has a canonical decomposition $T = T_1 \oplus T_2$ into a completely nonunitary contraction T_1 and a unitary operator T_2. We say that T is *absolutely continuous* if the unitary part T_2 is absolutely continuous, which means that the spectral measure of T_2 is absolutely continuous with respect to the Lebesgue measure.

The *defect operators* are the positive operators defined by $D_T = (I - T^*T)^{1/2}$, $D_{T^*} = (I - TT^*)^{1/2}$; the *defect spaces* are $\mathcal{D}_T = \overline{D_T\mathcal{H}}$ and $\mathcal{D}_{T^*} = \overline{D_{T^*}\mathcal{H}}$.

2. Main result. Recall the following result which is due to Carey ([2]).

LEMMA 2.1. *Let W_1, W_2 be isometries on a Hilbert space such that $W_1 - W_2$ has finite rank. If $W_i = W_i^u \oplus W_i^p$ is the Wold decomposition of W_i $(i = 1, 2)$, then the absolutely continuous parts of W_1^u and W_2^u are unitarily equivalent.*

We shall use this result to prove the following one.

PROPOSITION 2.2. *Let $T_1, T_2 \in \mathcal{L}(\mathcal{H})$ be contractions such that $T_2 - T_1$ is a finite rank operator and the dimension of $\bigvee_{k\ge 0}(D_{T_i}^2)^k(T_1^*T_1 - T_2^*T_2)\mathcal{H}$ is finite for $i = 1$ or $i = 2$. If W_{T_i} is the minimal isometric dilation of T_i for $i = 1, 2$, then the absolutely continuous parts of $W_{T_1}^u$ and $W_{T_2}^u$ are unitarily equivalent.*

To prove the last proposition, we need the following lemma:

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Lemma 2.3. *Let A, B be positive operators on a Hilbert space $\mathcal{H}$ such that $A - B = R$ is a finite rank operator. If the dimension of $\bigvee_{k\geq 0} B^k R\mathcal{H} = \mathcal{M}$ is finite, then $\sqrt{A} - \sqrt{B}$ is a finite rank operator.*

Proof. Consider a family of polynomials q_n that is convergent to the square root function uniformly on $\sigma(A) \cup \sigma(B)$. We then have $\sqrt{A} = \lim_n q_n(A)$ and $\sqrt{B} = \lim_n q_n(B)$. Let $q_n = \sum_{m=0}^{M_n} a_{n,m} x^m$. Then

$$\sqrt{A} - \sqrt{B} = \lim_n (q_n(A) - q_n(B)) = \lim_n \sum_{m=0}^{M_n} a_{n,m}(A^m - B^m).$$

It is quite clear that we have by induction

$$A^m - B^m = RA^{m-1} + BRA^{m-2} + \cdots + B^{m-2}RA + B^{m-1}R.$$

Under the assumptions made on A and B, the range $A^m - B^m$ has a finite dimension since it is included in $\mathcal{M}$. This ends the proof. ∎

Remark. It is an interesting question whether the statement of the lemma given above remains true without the condition on $\mathcal{M}$. For example, consider $A = \sum_{n\geq 0} \alpha_n^2 e_n \otimes e_n$ and $B = A + f \otimes f$, which is a rank one perturbation of A. Assume that $f = \sum_{n\geq 0} \beta_n e_n$. It is clear that A and B are positive operators. Could we choose the α_n and β_n such that $\sqrt{A} - \sqrt{B}$ is not a finite rank operator?

Proof of Proposition 2.2. Set $\mathcal{D} = \mathcal{D}_{T_1} \vee \mathcal{D}_{T_2}$ and $\mathcal{K} = \mathcal{H} \oplus \mathcal{D} \oplus \mathcal{D} \oplus \cdots$. Consider on $\mathcal{K}$ the two isometries

$$\widetilde{W_i} = \begin{pmatrix} T_i & 0 & 0 & \cdots \\ D_{T_i} & 0 & 0 & \cdots \\ 0 & I & 0 & \cdots \\ 0 & 0 & I & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} T_i & 0 \\ D_{T_i} & S^\alpha \end{pmatrix}, \qquad i = 1, 2.$$

where α is the dimension of $\mathcal{D}$. It is easy to see, by the construction of the isometric dilation given in [3], that $\widetilde{W_i}$ is the direct sum of the minimal isometric dilation T_i and an (eventual) shift. Then

$$\widetilde{W_1} - \widetilde{W_2} = \begin{pmatrix} T_1 - T_2 & 0 \\ D_{T_1} - D_{T_2} & 0 \end{pmatrix}.$$

By using Lemma 2.3 with $D_{T_1}^2$ and $D_{T_2}^2$ instead of A and B, we see that $\widetilde{W_1} - \widetilde{W_2}$ is a finite rank operator and Carey's result gives the announced statement. ∎

Corollary 2.4. *Let $T_1, T_2 \in \mathcal{L}(\mathcal{H})$ be contractions such that $T_2 - T_1$ is a finite rank operator. If W_{T_i} is the minimal isometric dilation of T_i for*

$i = 1, 2$, *then the absolutely continuous parts of* $W^u_{T_1}$ *and* $W^u_{T_2}$ *are unitarily equivalent when one of the following statements is satisfied*:

(i) $T_1|\mathcal{D}_{T_1} = 0$,
(ii) $(D^2_{T_1} - D^2_{T_2})\mathcal{H}$ *is invariant under* $D^2_{T_1}$ *or* $D^2_{T_2}$,
(iii) D_{T_1} *or* D_{T_2} *is a finite rank operator.*

Proof. (i) Let $T_1|\mathcal{D}_{T_1} = 0$. In this special case $D_{T_1} = P_{\mathcal{D}_{T_1}}$ is the orthogonal projection on $\mathcal{D}_{T_1}$, and this obviously gives the assumption of Proposition 2.2.

(ii) If $(D^2_{T_1} - D^2_{T_2})\mathcal{H}$ is invariant under $D^2_{T_1}$ or $D^2_{T_2}$, then

$$\bigvee_{k\geq 0} (D^2_{T_i})^k (T_1^*T_1 - T_2^*T_2)\mathcal{H} = \bigvee_{k\geq 0} (D^2_{T_i})^k (D^2_{T_1} - D^2_{T_2})\mathcal{H}$$

has finite dimension for $i = 1$ or $i = 2$.

(iii) If D_{T_1} or D_{T_2} is a finite rank operator, then both are, since $D^2_{T_1} - D^2_{T_2}$ has finite rank.

Note that with (iii) we retrieve the result obtained in [1]. ∎

Of course there is a dual version of Proposition 2.2. This is due to the fact that the adjoint of the minimal isometric dilation of T^* is actually the minimal co-isometric extension of T. The corresponding result can be formulated as follows.

PROPOSITION 2.5. *Let* $T_1, T_2 \in \mathcal{L}(\mathcal{H})$ *be contractions such that* $T_2 - T_1$ *is a finite rank operator and the dimension of* $\bigvee_{k\geq 0}(D^2_{T_i^*})^k(T_1T_1^* - T_2T_2^*)\mathcal{H}$ *is finite for* $i = 1$ *or* $i = 2$. *If* B_{T_i} *is the minimal co-isometric extension of* T_i *for* $i = 1, 2$, *then the absolutely continuous parts of* $B^u_{T_1}$ *and* $B^u_{T_2}$ *are unitarily equivalent.*

References

[1] C. Benhida and D. Timotin, *Finite rank perturbations of contractions*, Integral Equations Operator Theory 36 (2000), 253–268.

[2] R. W. Carey, *Trace class perturbations of isometries and unitary dilations*, Proc. Amer. Math. Soc. 45 (1974), 229–234.

[3] B. Sz.-Nagy and C. Foiaş, *Harmonic Analysis of Operators on Hilbert Space*, North-Holland, Amsterdam, 1970.

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