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Envelope functions and asymptotic structures in Banach spaces

by

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Abstract. We introduce a notion of disjoint envelope functions to study asymptotic structures of Banach spaces. The main result gives a new characterization of asymptotic-ℓ_p spaces in terms of the ℓ_p-behavior of "disjoint-permissible" vectors of constant coefficients. Applying this result to Tirilman spaces we obtain a negative solution to a conjecture of Casazza and Shura. Further investigation of the disjoint envelopes leads to a finite-representability result in the spirit of the Maurey–Pisier theorem.

1. Introduction. Asymptotic structures of infinite-dimensional Banach spaces, introduced in [MMT], reflect the behavior at infinity of finite-dimensional subspaces which repeatedly appear everywhere and far away in the space and are arbitrarily spread out along, for instance, a basis. This approach to infinite-dimensional spaces serves as a bridge between finite-dimensional and infinite-dimensional theories, in view of the outstanding developments in the Banach space theory in the 1990's. For example, asymptotic-ℓ_p spaces were discovered in [MT] in connection with the distortion problem; and the game approach used in [MMT] to define asymptotic structures originated in [G]. For these and many other aspects of asymptotic approaches to infinite-dimensional Banach spaces theory we refer the reader to the exhaustive survey by E. Odell [O].

In its simplest form, the asymptotic structure of a Banach space is defined as follows. Given a Banach space X with a monotone basis, an n-dimensional space E with a monotone basis $\{e_i\}_{i=1}^n$ is an *asymptotic space* for X if there exists a finitely supported normalized vector y_1 (*block*) with support arbitrarily far along the basis $\{x_i\}$, then a normalized block y_2 with support arbitrarily far after the support of y_1, then a normalized block y_3 with support arbitrarily far after the support of y_2, and so on, such that the blocks $y_1, \dots, y_n$ obtained after n steps have behavior as close to the behavior of $\{e_i\}_{i=1}^n$ as we wish. (This means that any linear combination of $\{y_i\}_{i=1}^n$

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has norm in X arbitrarily close to the norm in E of the corresponding linear combination of $\{e_i\}_{i=1}^n$.)

The normalized blocks $y_1, \dots, y_n$ are called *permissible* vectors. The set of all n-dimensional asymptotic spaces of X will be denoted by $\{X\}_n$. The asymptotic structure of X consists of all $E \in \{X\}_n$, for all $n \in \mathbb{N}$.

A Banach space X is called an *asymptotic-ℓ_p space* if there exists a constant $C \geq 1$ such that for all n and $E \in \{X\}_n$, the basis in E is C-equivalent to the unit vector basis of ℓ_p^n (for the precise definition see below). That is, asymptotic-ℓ_p spaces have only one type of asymptotic spaces. In [MMT], it is shown that for $1 < p < \infty$, if for all n and $E \in \{X\}_n$, E is C-isomorphic to ℓ_p^n, then X is asymptotic-ℓ_p. This means that in such situations, there is a natural isomorphism between E and ℓ_p^n, which is the equivalence between respective bases. Up to a constant, for $1 < p < \infty$, asymptotic-ℓ_p spaces have a *unique* asymptotic structure, and this in fact characterizes asymptotic-ℓ_p spaces.

The main result of this paper gives a new characterization of asymptotic-ℓ_p spaces. Our starting point is the following consequence of results in [KOS] which, to build an easier intuition, we state for Banach spaces with a basis:

Suppose that for a Banach space X with an asymptotic unconditional basis there exists a constant $c > 0$ such that for all n and all permissible vectors $\{y_i\}_{i=1}^n$ in X we have $\|\sum_{i=1}^n y_i\| \geq cn$. Then X is an asymptotic-ℓ_1 space.

(This result is not stated in [KOS] as we formulated it here and we will provide a sketch of proof in Corollary 4.3.)

The above result shows that asymptotic-ℓ_1 spaces can be fully characterized by the ℓ_1-behavior of sums with constant coefficients of normalized permissible vectors. A natural question arising in this context is whether this remains true in general.

Question 1.1. *Let $1 < p < \infty$. Suppose that there is a constant $C \geq 1$ so that for all n and for all (normalized) permissible vectors $\{y_i\}_{i=1}^n$ in a Banach space X we have $n^{1/p}/C \leq \|\sum_{i=1}^n y_i\| \leq Cn^{1/p}$. Is X an asymptotic-ℓ_p space?*

It turns out that the answer to this question is negative even for spaces with an unconditional basis (see Section 6). However, in this case, if we extend the assumption to the set of all normalized vectors which have disjoint supports with respect to permissible vectors (called *disjoint-permissible vectors*), then the answer is affirmative. This is our main result (Theorem 4.1). The proof uses a new notion of *disjoint-envelope functions* (which are analogous to envelope functions first introduced and used in [MT]) and relies on a characterization of the unit vector basis of ℓ_p (Proposition 4.2), which is of independent interest.

The notion of disjoint-envelope functions is a convenient tool for studying asymptotic structures of spaces with an unconditional basis (or more generally, with asymptotic unconditional structure). We shall study them at some length here, in particular obtaining a theorem on finite-representability of ℓ_p (Theorem 5.6), in the spirit of the classical Maurey–Pisier theorem. This result is, in a sense, equivalent to a theorem of Milman and Sharir [MS], and can be viewed as a "disjoint-blocks" version of the Maurey–Pisier theorem.

The paper is organized as follows. Section 2 contains basic notation and several preliminary definitions and facts related to asymptotic structures. In Section 3 we introduce the disjoint-envelope functions, and develop some properties of these functions analogous to those of the original ones, which will be used in what follows. The main result, the characterization of asymptotic-ℓ_p spaces, and the characterization of the unit vector basis of ℓ_p mentioned above are given in Section 4.

In Section 5, we return to the study of disjoint-envelope functions in more detail. We introduce a natural notion of *power types* for envelope functions and prove the above mentioned result on finite representability of ℓ_p.

The final Section 6 presents counter-examples to Question 1.1, which turn out to be Tirilman spaces first introduced by Tzafriri [T] and studied in [CS]. As an application of our main results, a negative solution to a conjecture of Casazza and Shura on the structure of Tirilman spaces is obtained: These spaces do not contain any symmetric basic sequences. As a further consequence, unlike Tsirelson's space, Tirilman spaces are shown not to be isomorphic to their modified versions.

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2. Notation and preliminaries. We follow standard Banach space notation which can be found in [LT], and use [MMT] for the notation of asymptotic structures.

A non-zero sequence $\{x_i\}$ is a (Schauder) *basis* for a Banach space X if for all $x \in X$ there exists a unique sequence $\{a_i\}$ of scalars such that $x = \sum_i a_i x_i$. A sequence $\{x_i\}$ is C-*basic* if $\|\sum_{i=1}^n a_i x_i\| \le C \|\sum_{i=1}^m a_i x_i\|$ for all $\{a_i\}$ and integers $n < m$. A basis is *monotone* if it is 1-basic. A basis $\{x_i\}$ is C-*unconditional* if for all $\{a_i\}$ and sequences of signs $\{\theta_i\}$,

$$\Big\|\sum_i \theta_i a_i x_i\Big\| \le C\Big\|\sum_i a_i x_i\Big\|.$$

A basis is *unconditional* if it is C-unconditional for some constant $C \ge 1$. A basis $\{x_i\}$ is C-*equivalent* to a basis $\{y_i\}$, written $\{x_i\} \overset{C}{\sim} \{y_i\}$, if there exist $A, B > 0$ with $AB \le C$ such that for all scalars $\{a_i\}$,

$$\frac{1}{A}\Big\|\sum_i a_i y_i\Big\| \le \Big\|\sum_i a_i x_i\Big\| \le B\Big\|\sum_i a_i y_i\Big\|.$$

A basis $\{x_i\}$ is C-*subsymmetric* if it is C-unconditional and C-equivalent to each of its subsequences. It is C-*symmetric* if it is C-unconditional and C-equivalent to $\{x_{\pi(i)}\}$ for all permutations π of $\mathbb{N}$.

Let X be a Banach space with a basis $\{x_i\}$. The *support* of a vector $x = \sum_i a_i x_i$, denoted by $\operatorname{supp} x$, is the set of all i such that $a_i \ne 0$. A vector is called a *block* if it has finite support. For non-empty subsets I, J of $\mathbb{N}$ we write $I < J$ if $\max I < \min J$. For $n \in \mathbb{N}$ and $x, y \in X$ we write $n < x < y$ if $\{n\} < \operatorname{supp} x < \operatorname{supp} y$. We say x and y are *successive* if $x < y$.

Asymptotic structures of a Banach space X are defined with respect to families $\mathcal{B}(X)$ of subspaces which satisfy the so-called filtration condition: For every $X_1, X_2 \in \mathcal{B}$ there exists $X_3 \in \mathcal{B}$ such that $X_3 \subset X_1 \cap X_2$. The family of finite-codimensional subspaces and the family of *tail subspaces* are two such examples. The tail subspaces are subspaces of the form $X_n = \overline{\operatorname{span}}\,\{x_i\}_{i>n}$, for some $n \in \mathbb{N}$, and where $\{x_i\}$ is a basis (or more generally a minimal system) in X. The following definition of asymptotic structure can be given for an arbitrary family $\mathcal{B}$ of subspaces (see [MMT], where a convenient game terminology is introduced to define asymptotic structures) and the results of the paper can be easily extended to these general settings. However, for the sake of clarity, we will only consider the tail family $\mathcal{B}$ determined by a basis $\{x_i\}$ in X.

Let X be a Banach space with a basis $\{x_i\}$. Consider the tail family $\mathcal{B}$ of subspaces X with respect to $\{x_i\}$. An n-dimensional space E with a normalized basis $\{e_i\}_{i=1}^n$ is an *asymptotic space* for X, written $E \in \{X\}_n$ (or $\{e_i\}_{i=1}^n \in \{X\}_n$), if the following holds for every $\varepsilon > 0$: For an arbitrary $X_{m_1} \in \mathcal{B}$ there is a normalized block $y_1 \in X_{m_1}$ such that for an arbitrary $X_{m_2} \in \mathcal{B}$ with $X_{m_2} \subset X_{m_1}$ there is a normalized block $y_2 \in X_{m_2}$, and so on, such that $y_1 < \cdots < y_n$ obtained this way are $(1+\varepsilon)$-equivalent to $\{e_i\}_{i=1}^n$. Formally this means that

$$\forall m_1\ \exists y_1 \in X_{m_1} \dots \forall m_n\ \exists y_n \in X_{m_n} \text{ such that } \{y_i\}_{i=1}^n \overset{1+\varepsilon}{\sim} \{e_i\}_{i=1}^n.$$

Such successive normalized vectors $\{y_i\}_{i=1}^n$ are called ε-*permissible.* Thus, ε-permissible vectors are $(1+\varepsilon)$-representations of the asymptotic space E in X. To avoid repetitions, in the rest of the paper we will use the imprecise term *permissible* to mean ε-permissible for a small $0 < \varepsilon \le 1$. The *asymptotic structure* of X consists of all asymptotic spaces of X, for all n.

A Banach space X has *C-asymptotic unconditional* structure if for all $n \in \mathbb{N}$ and for every asymptotic space $E \in \{X\}_n$ the basis $\{e_i\}_{i=1}^n$ in E is C-unconditional. We say that X has *asymptotic unconditional structure* if it has C-asymptotic unconditional structure for some C. This notion was first introduced and studied in [MS].

A Banach space X is *C-asymptotic-ℓ_p* for $1 \le p \le \infty$ if for all n and $E \in \{X\}_n$ the basis $\{e_i\}$ in E is C-equivalent to the unit vector basis of ℓ_p^n. That is, for some A, B with $AB \le C$,

$$\frac{1}{A}\Big(\sum_{i=1}^n |a_i|^p\Big)^{1/p} \le \Big\|\sum_{i=1}^n a_i e_i\Big\| \le B\Big(\sum_{i=1}^n |a_i|^p\Big)^{1/p}$$

for all scalars (a_i). We call X an *asymptotic-ℓ_p space* if it is C-asymptotic-ℓ_p for some constant $C \ge 1$.

Let c_{00} denote the linear space of finite scalar sequences. If $a = (a_i) \in c_{00}$, then its ℓ_p-norm will be denoted simply by $\|a\|_p$ (the sup norm, corresponding to $p = \infty$, is denoted by $\|a\|_\infty$).

The asymptotic-ℓ_p spaces were first introduced in [MT] in a slightly stronger form. The more general definition given above comes from [MMT].

The notion of envelope functions was introduced in [MT] as well (see also 1.9 in [MMT]). For any sequence of scalars $a = (a_i) \in c_{00}$ the *upper envelope* is the function $r_X(a) = \sup \|\sum_i a_i e_i\|$, where the supremum is taken over all natural bases $\{e_i\}$ of asymptotic spaces $E \in \{X\}_n$ and all n. Similarly, the *lower envelope* is the function $g_X(a) = \inf \|\sum_i a_i e_i\|$, where the infimum is taken over the same set. Clearly, X is an asymptotic-ℓ_p space if and only if both r_X and g_X are equivalent to the norm $\|\cdot\|_p$.

Finally, we will also use the following version of a classical theorem of Krivine's as stated in [M].

KRIVINE'S THEOREM. *Let $r, s \ge 1$, and let X be a Banach space. Suppose that for some $\kappa > 0$ and $K \ge 1$ and for every $n \ge 2$, X contains a normalized K-unconditional sequence $\mathbf{y}^{(n)} = (y_1^{(n)}, \ldots, y_n^{(n)})$ such that $\|\sum_{i\in C} y_i^{(n)}\| \ge \kappa |C|^{1/r}$ (respectively $\|\sum_{i\in C} y_i^{(n)}\| \le \kappa |C|^{1/s}$) for every subset $C \subset \{1, \ldots, n\}$. Then for some $p \le r$ (resp. $p \ge s$) and for every $k \ge 1$ and $\varepsilon > 0$, there is $N(k, \varepsilon)$ such that whenever $n \ge N(k, \varepsilon)$, it is possible to form k successive blocks of $\mathbf{y}^{(n)}$ that are $(1+\varepsilon)$-equivalent to the unit vector basis of ℓ_p^k.*

3. Disjoint-envelope functions. Let X be a Banach space with asymptotic unconditional structure (with a constant $C \ge 1$). Define $\{X\}^{\mathrm{d}}$ to be the set of all normalized sequences of vectors $\{x_i\}$ which have disjoint support with respect to the natural basis of an asymptotic space. That is, for $n \in \mathbb{N}$, $\{x_i\}_{i=1}^n \in \{X\}^{\mathrm{d}}$ if there exist $\{e_j\}_{j=1}^m \in \{X\}_m$ for some $m \geq n$ and a partition $\{A_1, \ldots, A_n\}$ of $\{1, \ldots, m\}$ such that for each $1 \leq i \leq n$, $x_i = \sum_{j \in A_i} \alpha_j e_j$ for some scalars $\alpha = (\alpha_j)$ such that $\|x_i\| = 1$. Obviously, for all $\varepsilon > 0$ and $\{x_i\}_{i=1}^n \in \{X\}^{\mathrm{d}}$, there exists a sequence $\{x_i'\}_{i=1}^n$ of vectors which are disjointly supported on some permissible vectors in X such that $\{x_i'\}_{i=1}^n \overset{1+\varepsilon}{\sim} \{x_i\}_{i=1}^n$. We will call such sequences of vectors *disjoint-permissible vectors.*

First we make a few remarks about the set $\{X\}^{\mathrm{d}}$ (where superscript "d" stands for "disjoint"). Clearly, for all $n \in \mathbb{N}$ and $\{e_i\}_{i=1}^n \in \{X\}_n$ we have $\{e_i\}_{i=1}^n \in \{X\}^{\mathrm{d}}$. That is, $\bigcup_n \{X\}_n \subset \{X\}^{\mathrm{d}}$. If $\{x_i\} \in \{X\}^{\mathrm{d}}$, then $\{x_i\}$ is an unconditional basic sequence (with constant C). It is also clear that if $\{u_j\}$ is a (successive or just disjoint) block basis of some $\{x_i\} \in \{X\}^{\mathrm{d}}$, then $\{u_j\} \in \{X\}^{\mathrm{d}}$ as well. Finally, if $\{x_i\}_{i=1}^n \in \{X\}^{\mathrm{d}}$ then $\{x_{\pi(i)}\}_{i=1}^n \in \{X\}^{\mathrm{d}}$, where π is a permutation of $\{1, \ldots, n\}$. Obviously this property is not shared, in general, by the bases of asymptotic spaces.

We also have the following property of $\{X\}^{\mathrm{d}}$ which is inherited from $\{X\}_n$. If $\{x_i\}_{i=1}^{n_1}$ and $\{y_i\}_{i=1}^{n_2}$ are in $\{X\}^{\mathrm{d}}$, then there exists $\{z_i\}_{i=1}^{n_1+n_2} \in \{X\}^{\mathrm{d}}$ such that $\{z_i\}_{i=1}^{n_1} \overset{1}{\sim} \{x_i\}_{i=1}^{n_1}$ and $\{z_i\}_{i=n_1+1}^{n_1+n_2} \overset{1}{\sim} \{y_i\}_{i=1}^{n_2}$.

Indeed, if $\{x_i\}_{i=1}^{n_1}$ and $\{y_i\}_{i=1}^{n_2}$ are disjoint blocks of the bases $\{e_i\}_{i=1}^{m_1}$ and $\{f_i\}_{i=1}^{m_2}$ of some asymptotic spaces respectively, then we can find an asymptotic space $\{g_i\}_{i=1}^{m_1+m_2}$ such that $\{e_i\}_{i=1}^{m_1} \overset{1}{\sim} \{g_i\}_{i=1}^{m_1}$ and $\{f_i\}_{i=1}^{k} \overset{1}{\sim} \{g_i\}_{i=m_1+1}^{m_1+m_2}$ (cf. [MMT, 1.8.2]). Hence the corresponding disjoint blocks $\{z_i\}_{i=1}^{n_1+n_2}$ of $\{g_i\}_{i=1}^{m_1+m_2}$ have the desired property. When $\{x_i\}_{i=1}^{n_1}$ and $\{y_i\}_{i=1}^{n_2}$ are in $\{X\}^{\mathrm{d}}$, to avoid repetitions, we will simply say that $\{x_i, y_i\} \in \{X\}^{\mathrm{d}}$ without referring to $\{z_i\}$.

We now define the natural analogs of the envelope functions on $\{X\}^{\mathrm{d}}$.

Definition 3.1. Let X be a Banach space with an asymptotic unconditional structure. For $a = (a_i) \in c_{00}$, let $g_X^{\mathrm{d}}(a) = \inf \|\sum_i a_i x_i\|$ and $r_X^{\mathrm{d}}(a) = \sup \|\sum_i a_i x_i\|$, where the inf and the sup are taken over all $\{x_i\} \in \{X\}^{\mathrm{d}}$. We call g_X^{d} and r_X^{d} the *lower* and *upper disjoint-envelope functions* respectively.

It is easy to see that both functions g_X^{d} and r_X^{d} are 1-symmetric and 1-sign unconditional. A function f on c_{00} is 1-*symmetric* if for all $a \in c_{00}$ and permutations π of $\mathbb{N}$, $f(a) = f(a_\pi)$, where $a_\pi = (a_{\pi(i)})$. It is 1-*sign unconditional* if for all sequences of signs (θ_i) and $a \in c_{00}$, $f(\theta_1 a_1, \theta_2 a_2, \ldots) = f(a_1, a_2, \ldots)$. Moreover, while r_X^{d} defines a norm on c_{00}, g_X^{d} satisfies the triangle inequality on disjointly supported vectors (of c_{00}).

Indeed, let $a = (a_i)$ and $b = (b_i)$ be two vectors in c_{00} with disjoint supports and let $\varepsilon > 0$ be arbitrary. Pick $\{x_i\}$ and $\{y_i\}$ in $\{X\}^{\mathrm{d}}$ such that $g_X^{\mathrm{d}}(a) + \varepsilon/2 \geq \|\sum_i a_i x_i\|$ and $g_X^{\mathrm{d}}(b) + \varepsilon/2 \geq \|\sum_i b_i y_i\|$. Then, by the above remark, $\{x_i, y_i\} \in \{X\}^{\mathrm{d}}$ and hence

$$g_X^{\mathrm{d}}(a+b) \le \|a_1x_1 + b_1y_1 + a_2x_2 + b_2y_2 + \cdots\| \\ \le \|a_1x_1+a_2x_2+\cdots\|+\|b_1y_1+b_2y_2+\cdots\| \le g_X^{\mathrm{d}}(a)+g_X^{\mathrm{d}}(a)+\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, it follows that $g_X^{\mathrm{d}}(a+b) \le g_X^{\mathrm{d}}(a) + g_X^{\mathrm{d}}(b)$.

The fact that r_X^{d} is a norm and it is 1-sign unconditional implies that $r_X^{\mathrm{d}}(a) \le r_X^{\mathrm{d}}(b)$ for all $a = (a_i), b = (b_i) \in c_{00}$ with $|a_i| \le |b_i|$. This may not hold for the lower envelopes. However, it is easy to see that $g_X^{\mathrm{d}}(a) \le C g_X^{\mathrm{d}}(b)$, where C is the asymptotic unconditionality constant.

It is convenient to think of both g_X^{d} and r_X^{d} as norms on c_{00} and use the following notation. Let (e_i) be the unit vector basis of c_{00}. For $a = (a_i) \in c_{00}$, occasionally we will write $g_X^{\mathrm{d}}(\sum_i a_ie_i)$ instead of $g_X^{\mathrm{d}}(a)$. Moreover, for any finite number of successive vectors $b^i = (b^i_j) \in c_{00}$ such that $g_X^{\mathrm{d}}(b^i) = 1$ for $i = 1, 2, \ldots$ and for any vector $a = (a_i) \in c_{00}$, we write $g_X^{\mathrm{d}}(\sum_i a_ix_i)$ instead of $g_X^{\mathrm{d}}(\sum_i a_ib^i)$, where $x_i = \sum_j b^i_je_j$ are blocks of the basis (e_i) of c_{00} normalized with respect to g_X^{d}.

The following lemma lists some of the properties of the disjoint envelopes analogous to properties of the original ones as in Lemma 5.3 of [MT].

LEMMA 3.2. *Let X be a Banach space with C-asymptotic unconditional structure. Let $\{x_i\}$ and $\{u_i\}$ be sequences of vectors in c_{00} with disjoint support* (*with respect to the unit vector basis $\{e_i\}$*) *such that $r_X^{\mathrm{d}}(x_i) = 1$ and $g_X^{\mathrm{d}}(u_i) = 1$, for all i.*

(i) r_X^{d} *is submultiplicative*, *that is*, *for all $a = (a_i) \in c_{00}$*,

$$r_X^{\mathrm{d}}\Big(\sum_i a_ix_i\Big) \le r_X^{\mathrm{d}}\Big(\sum_i a_ie_i\Big).$$

(ii) g_X^{d} *is C-supermultiplicative*, *that is*, *for all $a = (a_i) \in c_{00}$*,

$$g_X^{\mathrm{d}}\Big(\sum_i a_ie_i\Big) \le C g_X^{\mathrm{d}}\Big(\sum_i a_iu_i\Big).$$

(iii) *The following inequalities hold for all $a = (a_i) \in c_{00}$*:

$$g_X^{\mathrm{d}}\Big(\sum_i a_iu_i\Big) \le r_X^{\mathrm{d}}\Big(\sum_i a_ie_i\Big), \quad g_X^{\mathrm{d}}\Big(\sum_i a_ie_i\Big) \le C r_X^{\mathrm{d}}\Big(\sum_i a_ix_i\Big).$$

Proof. Since both g_X^{d} and r_X^{d} are 1-symmetric, we may assume that $\{x_i\}$ and $\{u_i\}$ are successive blocks of $\{e_i\}$ in c_{00}.

(i) Let $x_i = \sum_{j=k_i+1}^{k_{i+1}} b_je_j$ for some $1 \le k_1 < k_2 < \cdots$ be a block basis of c_{00} with $r_X^{\mathrm{d}}(x_i) = 1$ for all $i = 1, 2\ldots$, and let $a = (a_1, \ldots, a_l) \in c_{00}$ and $\varepsilon > 0$ be arbitrary. Then there exists $\{v_j\}_{j=1}^{k_l+1} \in \{X\}^{\mathrm{d}}$, which is a disjointly supported sequence of vectors in some asymptotic space E, such that

$$r_X^{\mathrm{d}}\Big(\sum_{i=1}^{l} a_i x_i\Big) - \varepsilon = r_X^{\mathrm{d}}\Big(\sum_{i=1}^{l} a_i\Big(\sum_{j=k_i+1}^{k_{i+1}} b_j e_j\Big)\Big) - \varepsilon \le \Big\|\sum_{i=1}^{l} a_i\Big(\sum_{j=k_i+1}^{k_{i+1}} b_j v_j\Big)\Big\|_E.$$

Set $c_i = \|\sum_{j=k_i+1}^{k_{i+1}} b_j v_j\|_E$; then $c_i \le r_X^{\mathrm{d}}(x_i) = 1$. Let $w_i = c_i^{-1}\sum_{j=k_i+1}^{k_{i+1}} b_j v_j$ for $i = 1,\dots,l$. Then $\{w_i\} \in \{X\}^{\mathrm{d}}$. Thus the latter term above is equal to

$$\Big\|\sum_{i=1}^{l} a_i c_i w_i\Big\|_E \le r_X^{\mathrm{d}}(a_1c_1, a_2c_2, \dots, a_lc_l) \le r_X^{\mathrm{d}}(a_1, a_2, \dots, a_l).$$

The last inequality is due to 1-unconditionality of r_X^{d} and the fact that $c_i \le 1$ for $i = 1,\dots,l$. Since $\varepsilon > 0$ was arbitrary, we obtain

$$r_X^{\mathrm{d}}\Big(\sum_{i=1}^{l} a_i x_i\Big) \le r_X^{\mathrm{d}}\Big(\sum_{i=1}^{l} a_i e_i\Big),$$

as desired.

(ii) The proof proceeds along similar lines (with the inequalities reversed) except that at the end we make use of the fact that $g_X^{\mathrm{d}}(c_1a_1, c_2a_2, \dots) \ge (1/C)g_X^{\mathrm{d}}(a_1, a_2, \dots)$ for $c_i \ge 1$.

(iii) To see the first inequality, let $u_i = \sum_{j=k_i+1}^{k_{i+1}} b_j e_j$ for some $1 \le k_1 < k_2 < \cdots$ be a block basis of c_{00} with $g_X^{\mathrm{d}}(u_i) = 1$ for all $i = 1, 2, \dots$, let $a = (a_1, \dots, a_l)$ be arbitrary scalars and let $\varepsilon > 0$. For each i, pick $\{y_j^i\}_j \in \{X\}^{\mathrm{d}}$, where $\{y_j^i\}_j \in E_i$ for some asymptotic space E_i, such that

$$\Big\|\sum_j b_j y_j^i\Big\|_{E_i} \le g_X^{\mathrm{d}}(u_i) + \varepsilon = 1 + \varepsilon.$$

Then $\{y_j^i\}_{i,j} \in \{X\}^{\mathrm{d}}$, which is a disjointly supported sequence of vectors in some asymptotic space E. Let $c_i = \|\sum_j b_j y_j^i\|_{E_i}$ and $w_i = (1/c_i)\sum_j b_j y_j^i$ for $i = 1, 2, \dots$. Then

$$\begin{aligned} g_X^{\mathrm{d}}\Big(\sum_i a_i u_i\Big) &= g_X^{\mathrm{d}}\Big(\sum_i a_i\Big(\sum_j b_j e_j\Big)\Big) \\ &\le \Big\|\sum_i a_i\Big(\sum_j b_j y_j^i\Big)\Big\|_E = \Big\|\sum_i a_i c_i w_i\Big\|_E, \end{aligned}$$

and since $\{w_i\} \in \{X\}^{\mathrm{d}}$, the latter term above is less than or equal to

$$r_X^{\mathrm{d}}(a_1c_1, \dots, a_lc_l) \le (1+\varepsilon) r_X^{\mathrm{d}}\Big(\sum_i a_i e_i\Big),$$

where the last inequality follows from the unconditionality of r_X^{d}. Finally, since $\varepsilon > 0$ was arbitrary, the desired inequality follows. Again the second inequality is proved similarly with a small difference as in part (ii). ∎

4. A characterization of asymptotic-ℓ_p spaces. The main result of the paper is the following characterization for asymptotic-ℓ_p spaces.

THEOREM 4.1. *Let X be a Banach space with C-asymptotic unconditional structure. Suppose that there exist $1 \le p < \infty$ and a constant $K > 0$ such that for all $n \in \mathbb{N}$ and for all $\{x_i\}_{i=1}^n \in \{X\}^{\mathrm{d}}$, we have*

$$\frac{n^{1/p}}{K} \le \Big\|\sum_{i=1}^n x_i\Big\| \le K n^{1/p}.$$

Then X is $4^{1/p}C^3K^6$-asymptotic-ℓ_p.

For the proof, we will require the following characterization of the unit vector basis of ℓ_p, which is of independent interest. The idea of the proof of this proposition is inspired by the proof of Proposition 6.9 in [KOS].

PROPOSITION 4.2. *Let X be a Banach space with a 1-subsymmetric basis $\{x_i\}$. Suppose that there exist $1 \le p < \infty$ and a constant $K \ge 1$ such that for all $n \in \mathbb{N}$ and for all normalized vectors $\{y_i\}_{i=1}^n$ in X with disjoint supports, we have*

$$\frac{n^{1/p}}{K} \le \Big\|\sum_{i=1}^n y_i\Big\| \le K n^{1/p}.$$

Then, for all scalars $(a_i) \in c_{00}$,

$$\frac{1}{2^{1/p}K^3}\Big(\sum_i |a_i|^p\Big)^{1/p} \le \Big\|\sum_i a_i x_i\Big\| \le 2^{1/p}K^3\Big(\sum_i |a_i|^p\Big)^{1/p}.$$

That is, (x_i) is $4^{1/p}K^6$-equivalent to the unit vector basis of ℓ_p.

Proof. First we give the proof of the left hand inequality. Suppose to the contrary that the lower ℓ_p-estimate fails. That is, for some $0 < \varepsilon < 1/(2^{1/p}K^3)$ there exists $a = (a_i)_{i=1}^k$ such that $\|\sum_{i=1}^k a_i x_i\| < \varepsilon$ while $\sum_{i=1}^k |a_i|^p = 1$. By the 1-unconditionality of the basis (x_i) we can assume that all a_i's are positive, and we take the pth root of the sequence (a_i) to rewrite our assumption in the form $\sum_{i=1}^k a_i = 1$ while $\|\sum_{i=1}^k a_i^{1/p} x_i\| < \varepsilon$.

By a slight perturbation if necessary, we assume that a_i's are positive rationals and we write $a_i = n_i/N$ for $1 \le i \le k$, where n_i, N are natural numbers. Put also $N = n_i m_i + k_i$, $0 \le k_i < n_i$, $1 \le i \le k$. Now consider the vector $x = \sum_{i=1}^k a_i^{1/p} \sum_{j=1}^N x_j^i$, where $x_j^i = x_{(i-1)N+j}$ for $1 \le i \le k$ and $1 \le j \le N$. That is, x is of the form $x = (a_1^{1/p}, \dots, a_1^{1/p}, a_2^{1/p}, \dots, a_2^{1/p}, \dots, a_k^{1/p}, \dots, a_k^{1/p})$ with respect to $(x_1, \dots, x_{kN})$, where each block consists of N constant coefficients $a_i^{1/p}$. First, we estimate the norm of x from below.

For each $1 \leq i \leq k$, since $N = k_i(m_i + 1) + (n_i - k_i)m_i$, we may fix a partition

$$\{1, \ldots, N\} = \Big(\bigcup_{\mu=1}^{k_i} A_{\mu,i}\Big) \cup \Big(\bigcup_{v=1}^{n_i-k_i} B_{v,i}\Big),$$

where $|A_{\mu,i}| = m_i + 1$ for each $\mu = 1, \ldots, k_i$ and $|B_{v,i}| = m_i$ for each $v = 1, \ldots, n_i - k_i$. Then

$$\begin{aligned} (1) \qquad \|x\| &= \Big\|\sum_{i=1}^{k} a_i^{1/p} \sum_{j=1}^{N} x_j^i\Big\| = \Big\|\sum_{i=1}^{k} \Big(\frac{n_i}{N}\Big)^{1/p} \sum_{j=1}^{N} x_j^i\Big\| \\ &= \Big\|\sum_{i=1}^{k}\Big(\sum_{\mu=1}^{k_i}\Big(\frac{n_i}{N}\Big)^{1/p} \sum_{j\in A_{\mu,i}} x_j^i + \sum_{v=1}^{n_i-k_i}\Big(\frac{n_i}{N}\Big)^{1/p} \sum_{j\in B_{v,i}} x_j^i\Big)\Big\|. \end{aligned}$$

Now, using the assumption, we obtain lower estimates for the disjoint blocks appearing in (1).

For each $\mu = 1, \ldots, k_i$, since $|A_{\mu,i}| = m_i + 1$, we have

$$\Big\|\Big(\frac{n_i}{N}\Big)^{1/p} \sum_{j\in A_{\mu,i}} x_j^i\Big\| \geq \frac{(m_i+1)^{1/p}}{K}\Big(\frac{n_i}{N}\Big)^{1/p} = \frac{(n_i m_i + n_i)^{1/p}}{N^{1/p}K} \geq \frac{1}{K}.$$

For each $v = 1, \ldots, n_i - k_i$, since $|B_{v,i}| = m_i$, we have

$$\Big\|\Big(\frac{n_i}{N}\Big)^{1/p} \sum_{j\in B_{v,i}} x_j^i\Big\| \geq \frac{n_i^{1/p}}{N^{1/p}}\,\frac{m_i^{1/p}}{K} \geq \frac{1}{2^{1/p}K}.$$

Let

$$u_\mu^i = \frac{\sum_{j\in A_{\mu,i}} x_j^i}{\|\sum_{j\in A_{\mu,i}} x_j^i\|}, \qquad w_v^i = \frac{\sum_{j\in B_{v,i}} x_j^i}{\|\sum_{j\in B_{v,i}} x_j^i\|}.$$

By the 1-unconditionality of the basis and by the above estimates for the blocks u_μ^i and w_v^i appearing in (1), the expression (1) is greater than or equal to

$$(2) \qquad \frac{1}{2^{1/p}K}\Big\|\sum_{i=1}^{k}\Big(\sum_{\mu=1}^{k_i} u_\mu^i + \sum_{v=1}^{n_i-k_i} w_v^i\Big)\Big\|.$$

The blocks u_μ^i and w_v^i have disjoint supports (in fact, note that the partition can be chosen so that they become successive) and normalized, therefore by the assumption, (2) is greater than or equal to

$$\frac{1}{2^{1/p}K^2}\Big(\sum_{i=1}^{k} n_i\Big)^{1/p} = \frac{N^{1/p}}{2^{1/p}K^2}.$$

Here we have used $1 = \sum_{i=1}^{k} a_i = \sum_{i=1}^{k} n_i/N$. Thus we have obtained

$$\|x\| \geq \frac{N^{1/p}}{2^{1/p}K^2}. \tag{3}$$

On the other hand, letting $y_j = \sum_{i=1}^{k} a_i^{1/p} x_j^i$ for $1 \leq j \leq N$, by subsymmetry of the basis $\{x_i\}$, we have $\|y_j\| < \varepsilon$. Since $\{y_j\}$ have disjoint supports, it follows from the assumption that

$$\|x\| = \Big\|\sum_{i=1}^{k} a_i^{1/p} \sum_{j=1}^{N} x_j^i\Big\| = \Big\|\sum_{j=1}^{N} y_j\Big\| < \varepsilon K N^{1/p}. \tag{4}$$

From (3) and (4) it follows that

$$\varepsilon \geq \frac{1}{2^{1/p}K^3},$$

which is a contradiction.

The proof of the upper ℓ_p-estimate is similar. Suppose to the contrary that for some $M > 2^{1/p}K^3$ there exists a positive scalar sequence $(a_i^{1/p})_{i=1}^{k}$ such that $\|\sum_{i=1}^{k} a_i^{1/p} x_i\| > M$ while $\sum_{i=1}^{k} a_i = 1$. With the same setup as in the first part of the proof, we estimate the norm of the vector x in (1) from above. Thus,

$$\Big\|\sum_{i=1}^{k} a_i^{1/p} \sum_{j=1}^{N} x_j^i\Big\| \leq K^2 2^{1/p} N^{1/p}.$$

On the other hand, as in (4), using the assumption again we have

$$\Big\|\sum_{i=1}^{k} a_i^{1/p} \sum_{j=1}^{N} x_j^i\Big\| \geq \frac{MN^{1/p}}{K}.$$

From these two estimates we conclude that $M \leq 2^{1/p}K^3$, a contradiction. The proof is now complete. ∎

Let us remark that in the assumption of the above proposition disjointly supported vectors cannot be replaced with successive blocks (see Section 6, Theorem 6.1).

Moreover, note that the above proof also works in the "space" $(c_{00}, g_X^{\mathrm{d}})$, if the assumptions are satisfied. That is, if for all disjointly supported vectors $\{u_i\}$ in c_{00} with $g_X^{\mathrm{d}}(u_i) = 1$ we have $g_X^{\mathrm{d}}(\sum_{i=1}^{n} u_i) \overset{K^2}{\sim} n^{1/p}$, then

$$(2^{1/p}CK^3)^{-1}\|a\|_p \leq g_X^{\mathrm{d}}(a) \leq 2^{1/p}CK^3\|a\|_p,$$

where C is the asymptotic unconditionality constant of X.

Proof of Theorem 4.1. Since for all $n \in \mathbb{N}$ and $\{e_i\}_{i=1}^n \in \{X\}_n$, we have $g_X^{\mathrm{d}}(a) \le \|\sum_{i=1}^n a_i e_i\| \le r_X^{\mathrm{d}}(a)$, it is clearly sufficient to show that

$$g_X^{\mathrm{d}}(a) \ge \frac{1}{2^{1/p}C^2K^3}\|a\|_p, \qquad r_X^{\mathrm{d}}(a) \le 2^{1/p}CK^3\|a\|_p, \tag{5}$$

for all $a \in c_{00}$. The assumption of the theorem already implies that

$$\frac{n^{1/p}}{K} \le g_X^{\mathrm{d}}\Big(\sum_{i=1}^n e_i\Big) \le r_X^{\mathrm{d}}\Big(\sum_{i=1}^n e_i\Big) \le Kn^{1/p},$$

where $\{e_i\}$ is the unit vector basis of c_{00}.

Let $\{u_i\}$ and $\{w_i\}$ be arbitrary vectors with disjoint supports in c_{00} such that $g_X^{\mathrm{d}}(u_i) = 1$ and $r_X^{\mathrm{d}}(w_i) = 1$ for all $i = 1, 2, \ldots$. From Lemma 3.2 and the above inequalities, it follows that

$$\frac{n^{1/p}}{CK} \le \frac{1}{C}\, g_X^{\mathrm{d}}\Big(\sum_{i=1}^n e_i\Big) \le g_X^{\mathrm{d}}\Big(\sum_{i=1}^n u_i\Big) \le r_X^{\mathrm{d}}\Big(\sum_{i=1}^n e_i\Big) \le Kn^{1/p}$$

and

$$\frac{n^{1/p}}{CK} \le \frac{1}{C}\, g_X^{\mathrm{d}}\Big(\sum_{i=1}^n e_i\Big) \le r_X^{\mathrm{d}}\Big(\sum_{i=1}^n w_i\Big) \le r_X^{\mathrm{d}}\Big(\sum_{i=1}^n e_i\Big) \le Kn^{1/p}.$$

That is, $g_X^{\mathrm{d}}(\sum_{i=1}^n u_i) \overset{CK^2}{\sim} n^{1/p}$ and $r_X^{\mathrm{d}}(\sum_{i=1}^n w_i) \overset{CK^2}{\sim} n^{1/p}$ for all normalized vectors (u_i) and (w_i) with disjoint supports in $(c_{00}, g_X^{\mathrm{d}})$ and $(c_{00}, r_X^{\mathrm{d}})$ respectively. Thus, using the fact that r_X^{d} is 1-unconditional and g_X^{d} is C-unconditional, the inequalities in (5) follow from the proof of Proposition 4.2 (see the remark preceding the proof). Hence, X is $4^{1/p}C^3K^6$-asymptotic-ℓ_p. ∎

As pointed out in the introduction, for $p = 1$ this result can be improved.

COROLLARY 4.3. *Suppose that for a Banach space X with asymptotic unconditional structure there exists a constant $K > 0$ such that for all n and $\{e_i\}_{i=1}^n \in \{X\}_n$ we have*

$$\Big\|\sum_{i=1}^n e_i\Big\| \ge n/K.$$

Then X is an asymptotic-ℓ_1 space.

Proof (sketch). It is sufficient to show that the lower (original) envelope function satisfies $g_X(a) \ge c\|a\|_1$ for some constant c. (The upper estimate trivially follows from the triangle inequality.)

The proof runs along the same lines as the first part of the proof of Proposition 4.2, when the argument is applied to the "space" (c_{00}, g_X), so we only indicate the few differences.

With the same setup as in the first part of the proof of Proposition 4.2, assume that the above estimate fails and consider the vector x in (1). Then the estimate $g_X(x) \geq N/2CK^2$ in (3) holds because, as we remarked there, the blocks appearing in (2) can be chosen to be successive and we have the asymptotic unconditionality assumption (with constant C). On the other hand, the upper estimate $g_X(x) < \varepsilon N$ in (4) simply follows from the triangle inequality for g_X on vectors with disjoint supports. Thus we arrive at a contradiction for small enough $\varepsilon > 0$. ∎

5. ℓ_p-estimates and finite representability of envelopes. The most interesting fact about envelope functions is that they are always close to some ℓ_p-norm. The following result for the (original) envelope functions is stated in [MMT].

PROPOSITION 5.1. *There exist $1 \leq p, q \leq \infty$ and $C, c > 0$ and for every $\varepsilon > 0$ there exist $C_\varepsilon, c_\varepsilon > 0$ such that for all $a \in c_{00}$ we have*

$$c_\varepsilon \|a\|_{q+\varepsilon} \leq g_X(a) \leq C\|a\|_q, \qquad c\|a\|_p \leq r_X(a) \leq C_\varepsilon \|a\|_{p-\varepsilon}.$$

The proof of the r_X case is sketched in [MMT], it follows from standard arguments using the submultiplicativity of the function and an application of Krivine's theorem. Below we prove an analogous result for the disjoint-envelope functions. However, since the lower disjoint envelope g_X^{d} is not necessarily a norm, to be able to use Krivine's theorem, one needs to check that the theorem holds in a more general setting, namely for functions which satisfy the triangle inequality for vectors with disjoint supports. To avoid this cumbersome work, we postpone the proof of the g_X^{d} case to the end of this section, where we give a different and self-contained proof. We also observe that in our case the corresponding constants $C, c > 0$ of the above inequalities can be taken to be 1.

PROPOSITION 5.2. *Let X be a Banach space with asymptotic unconditional structure. Then there exist $1 \leq p, q \leq \infty$ such that for all $\varepsilon > 0$ there exist $C_\varepsilon, c_\varepsilon > 0$ such that for all $a \in c_{00}$ we have*

$$c_\varepsilon \|a\|_{q+\varepsilon} \leq g_X^{\mathrm{d}}(a) \leq \|a\|_q, \qquad \|a\|_p \leq r_X^{\mathrm{d}}(a) \leq C_\varepsilon \|a\|_{p-\varepsilon}.$$

Here it is understood that if $q = \infty$ (resp. $p = 1$), then g_X^{d} is equivalent to $\|\cdot\|_\infty$ (resp. r_X^{d} is equivalent to $\|\cdot\|_1$). If $p = \infty$, then for all $r < \infty$ there exists $C_r < \infty$ such that $r_X^{\mathrm{d}}(a) \leq C_r \|a\|_r$.

Proof (the r_X^{d} case). The proof of this case is identical for disjoint and original envelopes. For the reader's convenience we give the details.

For $n, m \in \mathbb{N}$, the submultiplicativity of r_X^{d} implies that

$$r_X^{\mathrm{d}}\Big(\sum_{i=1}^{nm} e_i\Big) \leq r_X^{\mathrm{d}}\Big(\sum_{i=1}^{n} e_i\Big) r_X^{\mathrm{d}}\Big(\sum_{i=1}^{m} e_i\Big).$$

Hence, by induction, we get $r_X^{\mathrm{d}}(\sum_{i=1}^{n^k} e_i) \le r_X^{\mathrm{d}}(\sum_{i=1}^{n} e_i)^k$ for all $n, k \in \mathbb{N}$. Let

$$1/p = \inf \ln r_X^{\mathrm{d}}\Big(\sum_{i=1}^{n} e_i\Big)/\ln n.$$

Then, clearly, $r_X^{\mathrm{d}}(\sum_{i=1}^{n} e_i) \ge n^{1/p}$ for all $n \in \mathbb{N}$. Moreover, for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that for all $n \in \mathbb{N}$, we have $r_X^{\mathrm{d}}(\sum_{i=1}^{n} e_i) \le C_\varepsilon n^{1/p-\varepsilon}$.

Now consider the space $(c_{00}, r_X^{\mathrm{d}})$. The unit vector basis $\{e_i\}$ is symmetric and since $r_X^{\mathrm{d}}(\sum_{i=1}^{n} e_i) \ge n^{1/p}$ for all n, it follows from Krivine's theorem that there exists $r \le p$ such that ℓ_r is block finitely representable in the space $(c_{00}, r_X^{\mathrm{d}})$. That is, for all $\delta > 0$ and $n \in \mathbb{N}$, there exists a sequence of successive blocks $\{x_i\}_{i=1}^n$ in $(c_{00}, r_X^{\mathrm{d}})$ such that $\{x_i\}_{i=1}^n \overset{1+\delta}{\sim} \ell_r^n$. Moreover, by the submultiplicativity of r_X^{d}, for all n, we have

$$\frac{n^{1/r}}{1+\delta} \le r_X^{\mathrm{d}}\Big(\sum_{i=1}^{n} x_i\Big) \le r_X^{\mathrm{d}}\Big(\sum_{i=1}^{n} e_i\Big) \le C_\varepsilon n^{1/p-\varepsilon}.$$

Since this is true for all ε and n, it follows that $r \ge p - \varepsilon$ for all $\varepsilon > 0$, and thus $r = p$.

Finally, by the submultiplicativity of r_X^{d} and the fact that $\delta > 0$ can be chosen arbitrarily, it follows that $r_X^{\mathrm{d}}(a) \ge \|a\|_p$ for all $a \in c_{00}$.

To prove the upper $\ell_{p-\varepsilon}$ estimate for r_X^{d} we make use of an auxiliary norm $\sigma_{p-\varepsilon}$. For $1 \le s \le \infty$, the unit ball of the norm σ_s is the convex hull of all vectors $\alpha = (\sum_i |\alpha_i|)^{-1/s}(\alpha_i)_{i=1}^n$, where $\alpha_i = \pm 1$ or 0. (The norm σ_s is equivalent to the norm of the Lorentz sequence space $d(w,1)$, where the weight $w = (w_i)$ satisfies $\sum_{i=1}^n w_i = n^{1/s}$.) A direct estimate shows that for all $s' < s$ there exists $C_{s'} < \infty$ independent of n so that $\sigma_s(a) \le C_{s'}\|a\|_{s'}$ for all $a \in c_{00}$. Now fix $\varepsilon > \delta > 0$. As remarked earlier, there exists $C_\delta > 0$ such that $r_X^{\mathrm{d}}(\sum_{i=1}^n e_i) \le C_\delta n^{1/p-\delta}$. Put $p - \delta = s$; hence $r_X^{\mathrm{d}}(a) \le C_\delta \sigma_s(a) \le C_\delta C_{p-\varepsilon}\|a\|_{p-\varepsilon}$ for all $a \in c_{00}$. Then for all $a \in c_{00}$ we have $r_X^{\mathrm{d}}(a) \le C_\varepsilon \|a\|_{p-\varepsilon}$, where $C_\varepsilon = C_\delta C_{p-\varepsilon}$. ∎

Using the C-supermultiplicativity of g_X^{d}, as in the first part of the above proof, we easily obtain the following:

There exists $1 \le q \le \infty$ such that for all $\varepsilon > 0$ there exists a constant c_ε such that for all n,

$$c_\varepsilon n^{1/q+\varepsilon} \le g_X^{\mathrm{d}}\Big(\sum_{i=1}^{n} e_i\Big) \le C n^{1/q}, \tag{6}$$

where C is the asymptotic unconditionality constant.

DEFINITION 5.3. Let p be as in Proposition 5.2 and let q be as in (6). We say that the lower disjoint envelope g_X^{d} has *power type* q and the uppert disjoin envelope r_X^{d} has *power type* p.

Define the power types of the original envelope functions similarly. The functions r_X and g_X have power types p and q respectively if $1 \le p, q \le \infty$ are as in Proposition 5.1.

The following example shows that the power types of the original and the disjoint-envelope functions can be very different.

EXAMPLE 5.4. *There exists a Banach space X with an unconditional basis such that for every block subspace Y of X the power type of g_Y is* 1 *but g_Y^{d} is equivalent to $\|\cdot\|_\infty$.*

Proof. The Schlumprecht space S has this property. Recall that $S = c_{00}$ with the norm defined as follows ([S]): For $a \in c_{00}$, put

$$\|a\| = \max\bigg\{\|a\|_\infty, \sup_{l\ge 2} \frac{1}{\log_2(l+1)} \sum_{i=1}^{l} \|E_i(a)\|\bigg\},$$

where the inner sup runs through all subsets E_i of $\mathbb{N}$ such that $\max E_i < \min E_{i+1}$. Here $\|a\|_\infty = \sup_i |a_i|$ and $E_i(a) = \sum_{j\in E_i} a_j e_j$ for $a = \sum_i a_i e_i \in c_{00}$. The unit vector basis $\{e_i\}$ is 1-subsymmetric and 1-unconditional.

From the definition of the norm, $\|\sum_{i=1}^n x_i\| \ge n/\log_2(n+1)$ for all successive normalized blocks $\{x_i\}_{i=1}^n$ in S. This implies that for every block subspace Y of S the power type of g_Y is 1.

On the other hand, it is shown in [KL] by a delicate calculation that c_0 is disjointly finitely representable in every subspace of S. That is, for all $n \in \mathbb{N}$ and $\varepsilon > 0$, there exists a sequence $\{x_i\}_{i=1}^n$ of vectors in Y with disjoint supports such that $\{x_i\}_{i=1}^n \overset{1+\varepsilon}{\sim} \ell_\infty^n$. Moreover, it can be deduced from the proof in [KL] that one can find disjoint permissible vectors $\{x_i\}_{i=1}^n$ such that $\{x_i\}_{i=1}^n \overset{1+\varepsilon}{\sim} \ell_\infty^n$ in *every* block subspace Y of S. Thus, for every block subspace Y of S the envelope g_Y is close to the ℓ_1-norm and g_Y^{d} is equivalent to $\|\cdot\|_\infty$.

Also it is easy to verify that for the dual space S^*, the envelope r_{S^*} has power type ∞ and $r_{S^*}^{\mathrm{d}}$ is equivalent to the ℓ_1-norm. ∎

Next we consider a finite representability problem for the envelope functions. We start with the following observation.

PROPOSITION 5.5. *Let X be a Banach space with asymptotic unconditional structure. Then*

(i) g_X *(resp. g_X^{d}) is equivalent to $\|\cdot\|_\infty$ if and only if $\ell_\infty^n \in \{X\}_n$ (resp. $\ell_\infty^n \in \{X\}^{\mathrm{d}}$) for all n.*

(ii) r_X (*resp.* r_X^{d}) *is equivalent to* $\|\cdot\|_1$ *if and only if* $\ell_1^n \in \{X\}_n$ (*resp.* $\ell_1^n \in \{X\}^{\mathrm{d}}$) *for all* n.

Proof. The proof of part (i) is trivial. To prove the second part for r_X, assume for simplicity that r_X is 1-equivalent to $\|\cdot\|_1$ (for the general case the constants involved should be modified appropriately). Fix $n \in \mathbb{N}$ and pick an asymptotic space $E \in \{X\}_n$ with the natural basis $\{e_i\}_{i=1}^n$ such that $\|\sum_{i=1}^n e_i\| \geq (1/2)r_X(1,\ldots,1) \geq n/2$. Pick $x^* \in E^*$ with $\|x^*\| = 1$ and $x^*(\sum_{i=1}^n e_i) = \|\sum_{i=1}^n e_i\|$. Consider the set $I = \{i : |x^*(e_i)| \geq 1/4\}$. Since $|x^*(e_i)| \leq 1$ for all i, a standard argument shows that the cardinality k of I satisfies $k = |I| \geq n/3$. For an arbitrary scalar sequence $a = (a_i)$, let $\varepsilon_i = \operatorname{sgn} a_i x^*(e_i)$ for $i \in I$. Then

$$\Big\|\sum_{i\in I}\varepsilon_i a_i e_i\Big\| \geq x^*\Big(\sum_{i\in I}\varepsilon_i a_i e_i\Big) = \sum_{i\in I}|a_i|\,|x^*(e_i)| \geq (1/4)\sum_{i\in I}|a_i|.$$

This shows that $\{e_i\}_{i\in I}$ is $4C$-equivalent to the unit vector basis in an ℓ_1^k, by the unconditionality of the basis (with constant C). Since a block basis of the basis in an asymptotic space spans an asymptotic space, we reduce the constant to $1+\varepsilon$, by a well known blocking argument of James (cf. e.g. Proposition 2 of [OS]). The proof of the r_X^{d} case is similar. ∎

A natural question we consider here is whether for every Banach space X with asymptotic unconditional structure, $\ell_q^n, \ell_p^n \in \{X\}_n$ ($\ell_q^n, \ell_p^n \in \{X\}^{\mathrm{d}}$) for all $n \in \mathbb{N}$, where q and p are the power types of g_X and r_X (g_X^{d} and r_X^{d}) respectively.

Quite remarkably, the disjoint-envelopes case has an affirmative answer. Namely, we prove the following theorem.

THEOREM 5.6. *Let X be a Banach space with asymptotic unconditional structure. Let $1 \leq p \leq q \leq \infty$ be the power types of r_X^{d} and g_X^{d} respectively. Then $\ell_p^n, \ell_q^n \in \{X\}^{\mathrm{d}}$ for all $n \in \mathbb{N}$.*

This theorem can be viewed as a "disjoint-block" version of the classical Maurey–Pisier Theorem ([MP]). Such a "disjoint-block" version was already proved by Milman and Sharir [MS] in a different formulation. They have defined the notion of "asymptotic block type and cotype" and showed, analogously to the Maurey–Pisier Theorem, that if q is the infimum of asymptotic block cotype and p is the supremum of asymptotic block type of the space X with an asymptotic unconditional structure, then ℓ_q and ℓ_p are "disjointly" block finitely representable in X.

Although they make use of different notions, Theorem 5.6 is equivalent to Milman–Sharir's result. However, our proof here, which is based on a recent presentation of the proof of the Maurey–Pisier Theorem given by Maurey [M], is shorter than that in [MS].

Proof of Theorem 5.6. For simplicity we assume that the asymptotic unconditionality constant is $C = 1$ (in the general case the estimates in the proof should be multiplied by C).

The g_X^{d} case. Let q be the power type of g_X^{d}. If $q = 1$, then since $g_X^{\mathrm{d}} \le g_X$, the power type of g_X is also equal to 1. Thus it follows immediately from Krivine's theorem that $\ell_1^n \in \{X\}_n$ for all n.

Now suppose that $q > 1$. Let $1 < s < q$ and for all $n \in \mathbb{N}$, let $\phi(n)$ be the smallest real number for which

$$\sum_{i=1}^n |a_i|^s \le \phi(n)^s \Big\| \sum_{i=1}^n a_i x_i \Big\|^s$$

for all $\{x_i\}_{i=1}^n \in \{X\}^{\mathrm{d}}$ and scalars $\{a_i\}$.

Since the power type of g_X^{d} is q and $s < q$, it follows that ϕ is not bounded as a function of n, and it is easy to see that it is increasing.

We refer to the following argument as the "exhaustion" argument.

Fix $0 < \varepsilon < 1/2$ and pick $\{x_i\}_{i=1}^n \in \{X\}^{\mathrm{d}}$ and scalars $\{a_i\}$ such that $\sum_{i=1}^n |a_i|^s = 1$ and

$$1 > (1-\varepsilon)\phi(n)^s \Big\| \sum_{i=1}^n a_i x_i \Big\|^s. \tag{7}$$

Let $(B_\alpha)_{\alpha \in I}$ be a maximal family of mutually disjoint subsets of $\{1, \ldots, n\}$, possibly empty, such that

$$\sum_{i \in B_\alpha} |a_i|^s \le \varepsilon \Big\| \sum_{i \in B_\alpha} a_i x_i \Big\|^s. \tag{8}$$

Let $B = \bigcup_{\alpha \in I} B_\alpha$ and $m = |I|$ (note that $m < n$ because $|B_\alpha| > 1$). Then

$$\begin{aligned}\sum_{i \in B} |a_i|^s &= \sum_{\alpha \in I} \sum_{i \in B_\alpha} |a_i|^s \le \sum_{\alpha \in I} \varepsilon \Big\| \sum_{i \in B_\alpha} a_i x_i \Big\|^s \\ &\le \varepsilon \phi(m)^s \Big\| \sum_{\alpha \in I} \sum_{i \in B_\alpha} a_i x_i \Big\|^s \le \varepsilon \phi(n)^s \Big\| \sum_{i=1}^n a_i x_i \Big\|^s;\end{aligned} \tag{9}$$

here the second inequality uses the definition of $\phi(m)$ applied to vectors $\{u_\alpha\}_{\alpha=1}^m \in \{X\}^{\mathrm{d}}$, where $u_\alpha = \sum_{i \in B_\alpha} a_i x_i / \|\sum_{i \in B_\alpha} a_i x_i\|$ for all $\alpha \in I$, and the last inequality uses the unconditionality of $\{x_i\}$ and the fact that $\phi(m) \le \phi(n)$.

Let A denote the complement of B and for every $j \ge 0$ let

$$A_j = \{i \in A : 2^{-j-1} < |a_i| \le 2^{-j}\}.$$

Then from (7) and (9) it follows that

$$\sum_{i \in A} |a_i|^s > (1-2\varepsilon)\phi(n)^s \Big\| \sum_{i=1}^n a_i x_i \Big\|^s. \tag{10}$$

Let j_1 be the smallest $j \geq 0$ such that A_j is non-empty, and let $k = |A_{j_0}|$ be the cardinality of the largest set A_{j_0} among all A_j's. Then by (10),

$$k \sum_{j=j_1}^{\infty} 2^{-js} \geq \sum_{j=j_1}^{\infty} 2^{-js} |A_j| \geq \sum_{i \in A} |a_i|^s$$
$$> (1-2\varepsilon)\phi(n)^s \Big\| \sum_{i=1}^{n} a_i x_i \Big\|^s \geq (1-2\varepsilon)\phi(n)^s 2^{-j_1 s - s}.$$

This shows that k is large when $\phi(n)$ is large, i.e., since $\phi(n)$ increases to infinity with n, so does k.

Now by maximality of B,

$$\sum_{i \in C} |a_i|^s > \varepsilon \Big\| \sum_{i \in C} a_i x_i \Big\|^s$$

for every non-empty subset $C \subset A_{j_0}$. Since $2^{-j_0-1} < |a_i| \leq 2^{-j_0}$ for every $i \in A_{j_0}$, it follows that

$$\Big\| \sum_{i \in C} x_i \Big\| \leq 2(1/\varepsilon)^{1/s} |C|^{1/s} \leq (2/\varepsilon)|C|^{1/s}$$

for all $C \subset A_{j_0}$.

Therefore we have shown that there exists a constant $\kappa = 2/\varepsilon$ such that for all $k \in \mathbb{N}$ there exists $\{x_i\}_{i=1}^k \in \{X\}^{\mathrm{d}}$ such that $\|\sum_{i \in C} x_i\| \leq \kappa |C|^{1/s}$ for all $C \subset \{1, \ldots, k\}$ and $s < q$.

Now by Krivine's theorem, there is $q' \geq s$ such that $\ell_{q'}^n \in \{X\}^{\mathrm{d}}$ for all n. But since $s < q$ was arbitrary and q is the power type of g_X^{d}, it follows that $q' = q$, hence the proof of this case is complete.

The r_X^{d} case. The proof of this case is similar but there are slight differences.

Let p be the power type of r_X^{d}. If $p = \infty$, then since $r_X \leq r_X^{\mathrm{d}}$, the power type of r_X is also equal to infinity. Again it follows immediately from Krivine's theorem that $\ell_\infty^n \in \{X\}_n$ for all n.

Now suppose that $p < \infty$, and fix $p < r$. For each $n \geq 1$, let $\psi(n)$ be the smallest constant such that

$$\Big\| \sum_{i=1}^{n} a_i x_i \Big\|^r \leq \psi(n)^r \sum_{i=1}^{n} |a_i|^r$$

for all $\{x_i\}_{i=1}^n \in \{X\}^{\mathrm{d}}$ and scalars $\{a_i\}$. Since the power type of r_X^{d} is p and $p < r$, it follows that $\psi(n)$ increases to infinity.

Fix $0 < \varepsilon < 1/2$ and pick $\{x_i\}_{i=1}^n \in \{X\}^{\mathrm{d}}$ and scalars $\{a_i\}$ such that $\sum_{i=1}^n |a_i|^r = 1$ and

$$\Big\| \sum_{i=1}^{n} a_i x_i \Big\|^r > (1-\varepsilon)\psi(n)^r. \tag{11}$$

Let $(B_\alpha)_{\alpha\in I}$ be a maximal family of mutually disjoint subsets of $\{1,\ldots,n\}$ such that

$$\Big\|\sum_{i=1}^n a_i x_i\Big\|^r \le \varepsilon \sum_{i\in B_\alpha} |a_i|^r. \tag{12}$$

Let $B=\bigcup_{\alpha\in I} B_\alpha$ and $m=|I|$. Then

$$\begin{aligned}\Big\|\sum_{i\in B} a_i x_i\Big\|^r &= \Big\|\sum_{\alpha\in I}\sum_{i\in B_\alpha} a_i x_i\Big\|^r \le \psi(m)^r \sum_{\alpha\in I}\Big\|\sum_{i\in B_\alpha} a_i x_i\Big\|^r \\ &\le \varepsilon\psi(m)^r \sum_{\alpha\in I}\sum_{i\in B_\alpha} |a_i|^r \le \varepsilon\psi(n)^r.\end{aligned} \tag{13}$$

Let A denote the complement of B and for every $j\ge 0$ let

$$A_j=\{i\in A: 2^{-j-1}<|a_i|\le 2^{-j}\}.$$

Then $A=\bigcup_{j=0}^\infty A_j$ because $\sum_{i=1}^n |a_i|^r=1$. Let $k=\max_{j\ge0}|A_j|$. Then

$$\Big\|\sum_{i\in A} a_i x_i\Big\| = \Big\|\sum_{j=0}^\infty\sum_{i\in A_j} a_i x_i\Big\| \le \sum_{j=0}^\infty\Big\|\sum_{i\in A_j} a_i x_i\Big\| \le k\sum_{j=0}^\infty 2^{-j}=2k. \tag{14}$$

Hence, using (11), (13) and (14), we obtain

$$(1-\varepsilon)^{1/r}\psi(n) < \Big\|\sum_{i=1}^n a_i x_i\Big\| \le \Big\|\sum_{i\in B} a_i x_i\Big\| + \Big\|\sum_{i\in A} a_i x_i\Big\| \le \varepsilon^{1/r}\psi(n)+2k,$$

which shows that k is large whenever $\psi(n)$ is. Let j_0 be such that $|A_{j_0}|=k$. By maximality of B we deduce that for every non-empty subset C of A_{j_0},

$$\Big\|\sum_{i\in C} a_i x_i\Big\|^r > \varepsilon\sum_{i\in C}|a_i|^r \ge \varepsilon 2^{-(j_0+1)r}|C|.$$

It follows that

$$\Big\|\sum_{i\in C} x_i\Big\| \ge (1/2)\varepsilon^{1/r}|C|^{1/r}.$$

Since we can find such vectors $\{x_i\}_{i=1}^k\in\{X\}^{\mathrm{d}}$ for all $k\in\mathbb{N}$, the result again follows from Krivine's theorem. That is, $\ell_p^n\in\{X\}^{\mathrm{d}}$ for all $n\in\mathbb{N}$. ■

We now give the proof of the remaining part of Proposition 5.2, as was promised before.

Proof of Proposition 5.2 (the g_X^{d} case). As already remarked in (6), there exists $1\le q\le\infty$ such that for all $\varepsilon>0$ there exists a constant $c_\varepsilon>0$ such that for all n, we have

$$c_\varepsilon n^{1/q+\varepsilon} \le g_X^{\mathrm{d}}\Big(\sum_{i=1}^n e_i\Big) \le Cn^{1/q}.$$

We first show the lower estimate, that is, for all $\varepsilon > 0$ there exists c'_ε such that $g^{\mathrm{d}}_X(a) \geq c'_\varepsilon \|a\|_{q+\varepsilon}$.

For every $\varepsilon > 0$ and $n \in \mathbb{N}$, let $\phi_\varepsilon(n)$ be the smallest constant such that

$$\|a\|_{q+\varepsilon} \leq \phi_\varepsilon(n) g^{\mathrm{d}}_X\Big(\sum_{i=1}^n a_i x_i\Big)$$

for all vectors $\{x_i\}_{i=1}^n$ with disjoint supports (in c_{00}) such that $g^{\mathrm{d}}_X(x_i) = 1$ for all i, and scalars $a \in c_{00}$.

If $\sup_n \phi_\varepsilon(n) < \infty$ for every $\varepsilon > 0$, then there is nothing to prove.

Suppose that $\sup_n \phi_{\varepsilon_0}(n) = \infty$ for some $\varepsilon_0 > 0$. Then it follows from the exhaustion argument as in the proof of Theorem 5.6 (the g^{d}_X case) that there exists a constant $\kappa > 0$ such that for all n, there exist vectors $\{x_i\}_{i=1}^n$ with disjoint supports such that $g^{\mathrm{d}}_X(x_i) = 1$ for all i, and

$$g^{\mathrm{d}}_X\Big(\sum_{i=1}^n x_i\Big) \leq \kappa n^{1/q+\varepsilon_0}.$$

Now fix $\varepsilon_1 < \varepsilon_0$. Then there exists c_{ε_1} such that for all n,

$$c_{\varepsilon_1} n^{1/q+\varepsilon_1} \leq g^{\mathrm{d}}_X\Big(\sum_{i=1}^n e_i\Big) \leq C g^{\mathrm{d}}_X\Big(\sum_{i=1}^n x_i\Big) \leq C\kappa n^{1/q+\varepsilon_0}.$$

When n is large enough, this is a contradiction. Therefore, for every $\varepsilon > 0$, there exists $1/c'_\varepsilon = \sup_n \phi_\varepsilon(n) < \infty$ such that $g^{\mathrm{d}}_X(a) \geq c'_\varepsilon \|a\|_{q+\varepsilon}$, as desired.

For the upper estimate, note that by Theorem 5.6, $\ell_q^n \in \{X\}^{\mathrm{d}}$ for all $n \in \mathbb{N}$. This immediately implies that $g^{\mathrm{d}}_X(a) \leq \|a\|_q$ for all $a \in c_{00}$. The proof is now complete. ■

We end this section with a few remarks concerning the finite representability problem for the (original) envelope functions.

First, observe that the answer to this problem is negative in general. For instance, if (e_i) is the summing basis for $X = c_0$, then r_X is equivalent to $\|\cdot\|_1$, where the asymptotic structure is with respect to the summing basis (e_i), but $\ell_1^n \notin \{X\}_n$ for large n. Moreover, a (non-reflexive) Banach space X constructed in [KOS, Example 6.4] has the property that for all n, there exists $\{e_i\}_{i=1}^n \in \{X\}_n$ such that $\|\sum_{i=1}^n e_i\| = 1$, in particular, $g_X \sim \|\cdot\|_\infty$, and yet c_0 is not block finitely representable in X, in particular, $\ell_\infty^n \notin \{X\}_n$ for large n.

In these examples, the asymptotic structures are (necessarily) not unconditional (by Proposition 5.5).

It is likely that there are also examples of Banach spaces with asymptotic unconditional structure with power types of the envelopes satisfying $1 < p, q < \infty$ and yet $\ell_p^n, \ell_q^n \notin \{X\}_n$ for large n. However we do not know how to construct such examples.

Finally, we do not know if reflexivity plays a role in this problem. It is open, for instance, if there exists a reflexive space X for which $g_X \sim \|\cdot\|_\infty$ and yet $\ell_\infty^n \notin \{X\}_n$ for large n. This was raised in [KOS, Problem 6.5].

6. Tirilman spaces. To complement the main result of the paper, we show that the characterization of asymptotic-ℓ_p spaces given in Theorem 4.1 cannot be strengthened further, as stated in Question 1.1. Namely, we show that for all $1 < p < \infty$, there is a Tirilman space X with the property that for all n and permissible vectors $\{x_i\}_{i=1}^n$ in X, we have $\|\sum_{i=1}^n x_i\| \overset{K}{\sim} n^{1/p}$ for some constant independent of n, and yet X is not an asymptotic-ℓ_p space.

Additionally, as a consequence of Proposition 4.2, we also obtain a solution to a conjecture of Casazza and Shura on the structure of Tirilman spaces.

The Tirilman spaces are introduced and studied by Casazza and Shura [CS]. Their definitions depend on a slight modification of the original spaces constructed by L. Tzafriri [T] (the name "Tirilman" comes from the Romanian surname of L. Tzafriri).

We now recall the definition and a few properties of these spaces, which we shall use subsequently.

Let $1 < p < \infty$. Fix $0 < \gamma < 1$. For all $a = (a_i) \in c_{00}$, let

$$\|a\| = \max\left\{\|a\|_\infty, \gamma \sup \frac{\sum_{j=1}^k \|E_j a\|}{k^{1/q}}\right\},$$

where the inner supremum is taken over all finite successive sets of natural numbers $1 \le E_1 < \cdots < E_k$ and all k, and $1/p + 1/q = 1$.

The Banach space $(c_{00}, \|\cdot\|)$, which is defined with the parameters p and γ, is called a *Tirilman space* and denoted by $\mathrm{Ti}(p,\gamma)$.

It is immediate from the definition that the unit vectors $\{e_i\}_{i=1}^\infty$ form a normalized 1-subsymmetric basis for $\mathrm{Ti}(p,\gamma)$.

Some of the known properties of these spaces, which we shall use, are listed in the following theorem. For the proofs, see Lemma X.d.4 and Theorem X.d.6 of [CS] (note that in [CS] the proofs are given for $p = 2$ only, appropriate modifications are necessary for the general case).

THEOREM 6.1. *Let $1 < p < \infty$. There exists $0 < \gamma < 1$ such that the following hold for* $\mathrm{Ti}(p,\gamma)$.

(1) *For any normalized successive blocks $\{x_j\}_{j=1}^n$ of the basis $\{e_i\}_i$, we have*

$$\gamma n^{1/p} \le \Big\|\sum_{j=1}^n x_j\Big\| \le 3^{1/q} n^{1/p}.$$

(2) $\mathrm{Ti}(p,\gamma)$ *does not contain isomorphs of any* ℓ_p $(1 \le p < \infty)$ *or of* c_0. *In particular*, $\mathrm{Ti}(p,\gamma)$ *is a reflexive space.*

EXAMPLE 6.2. *Let* $1 < p < \infty$. *Then there exists* $0 < \gamma < 1$ *such that the Tirilman space* $X = \mathrm{Ti}(p,\gamma)$ *has the property that for all* n *and all* $\{e_i\}_{i=1}^n \in \{X\}_n$, *we have* $\|\sum_{i=1}^n e_i\| \overset{K}{\sim} n^{1/p}$, *where* K *depends on* γ *and* p *only*, *and yet* X *is not an asymptotic-*ℓ_p *space.*

Proof. By Theorem 6.1, there exists $0 < \gamma < 1$ such that the Tirilman space $X = \mathrm{Ti}(p,\gamma)$ has the property that for all n and successive blocks $\{x_j\}_{j=1}^n$ of the basis, we have $\gamma n^{1/p} \le \|\sum_{j=1}^n x_j\| \le 3^{1/q} n^{1/p}$, and (2) holds. In particular, the same estimates hold for all $\{e_i\}_{i=1}^n \in \{X\}_n$, for all n. On the other hand, since the basis $\{e_i\}$ is subsymmetric, if X were asymptotic-ℓ_p, this would imply that the basis $\{e_i\}$ is equivalent to the unit vector basis of ℓ_p. However, this contradicts part (2) of Theorem 6.1. ∎

Moreover, Casazza and Shura conjecture that $\mathrm{Ti}(2,\gamma)$, where $0 < \gamma < 10^{-6}$, has a symmetric basis ([CS, Conjecture X.d.9]). (As shown in [CS], for $0 < \gamma < 10^{-6}$ the conclusion of Theorem 6.1 holds.) However, this is not the case, as the next theorem shows.

THEOREM 6.3. *Let* $1 < p < \infty$ *and let* $0 < \gamma < 1$ *be as in Theorem* 6.1. *Then* $\mathrm{Ti}(p,\gamma)$ *contains no symmetric basic sequences.*

Proof. Suppose to the contrary that there is a symmetric basic sequence $\{x_i\}_{i=1}^\infty$ in $\mathrm{Ti}(p,\gamma)$. By Theorem 6.1, $\mathrm{Ti}(p,\gamma)$ is reflexive, thus $\{x_i\}$ is weakly null and by a sliding hump argument there exists a subsequence which is equivalent to a block basis of the unit vector basis $\{e_i\}$ of $\mathrm{Ti}(p,\gamma)$ (cf. Proposition 1.a.12 of [LT]). Since the sequence $\{x_i\}$ is symmetric, it is equivalent to all of its subsequences, in particular, $\{x_i\}$ itself is equivalent to a block basis of $\{e_i\}$. Now it follows from the first part of Theorem 6.1 that for all n and all normalized successive blocks $\{u_i\}_{i=1}^n$ of $\{x_i\}$, we have

$$\gamma n^{1/p} \le \Big\|\sum_{i=1}^n u_i\Big\| \le 3^{1/q} n^{1/p}.$$

By symmetry of $\{x_i\}$, the same estimates hold for all normalized vectors $\{u_i\}_{i=1}^n$ with disjoint supports with respect to $\{x_i\}$. Thus by Proposition 4.2, $\{x_i\}$ must be equivalent to the unit vector basis of ℓ_p, which contradicts the second part of Theorem 6.1. ∎

The definition of $\mathrm{Ti}(p,\gamma)$ in [CS] was modelled on spaces constructed by Tzafriri in [T]. This definition was fully analogous to that of the Tirilman spaces, except that in the implicit equation of the norm the inner supremum is taken over all *disjoint* subsets E_j of the natural numbers (rather than successive ones) [T]. In this case, as is easily seen, the unit vectors form a symmetric basis for the space. In the literature on Tsirelson-like spaces, the Tzafriri spaces are the *modified* Tirilman spaces (cf. [CS]).

It is well known, for instance, that the modified Tsirelson space is canonically isomorphic to the Tsirelson space, that is, the unit vector bases are equivalent (cf. [CS]).

A natural question then was raised in [CS] (see X.D. Notes and Remarks 3) whether the same holds for the Tirilman spaces. It follows immediately from Theorem 6.3 that the answer is negative. In fact, Theorem 6.3 has the following consequence.

COROLLARY 6.4. *Let* $1 < p < \infty$ *and let* $0 < \gamma < 1$ *be as in Theorem* 6.1. *Then the Tzafriri space with these parameters* p *and* γ *does not imbed into* $\mathrm{Ti}(p, \gamma)$.

FINAL REMARKS. Recently, M. Junge, D. Kutzarova and E. Odell [JKO] proved that a Banach space X satisfying the assumptions of Question 1.1 contains an asymptotic-ℓ_p subspace. In particular, any Tirilman space $\mathrm{Ti}(p, \gamma)$ has an asymptotic-ℓ_p subspace. They have also shown that c_0 is disjointly finitely representable in $\mathrm{Ti}(p, \gamma)$. The latter result gives another proof for Corollary 6.4, because c_0 cannot be finitely representable in the modified version of these spaces.

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