STUDIA MATHEMATICA 223 (2) (2014)

Thin-shell concentration for convex measures

by

Matthieu Fradelizi, Olivier Guédon and
Alain Pajor (Paris-Est)

Abstract. We prove that for $s < 0$, s-concave measures on $\mathbb{R}^n$ exhibit thin-shell concentration similar to the log-concave case. This leads to a Berry–Esseen type estimate for most of their one-dimensional marginal distributions. We also establish sharp reverse Hölder inequalities for s-concave measures.

1. Introduction. For any subsets $A, B \subset \mathbb{R}^n$, their *Minkowski sum* is defined by

$$A + B = \{a + b : a \in A,\ b \in B\}.$$

Let $s \in [-\infty, 1]$. A measure μ on $\mathbb{R}^n$ is called *s-concave* whenever

$$\mu((1-\lambda)A + \lambda B) \geq ((1-\lambda)\mu(A)^s + \lambda\mu(B)^s)^{1/s}$$

for every $\lambda \in [0,1]$ and any compact subsets $A, B \subset \mathbb{R}^n$ such that $\mu(A)\mu(B) > 0$. When $s = 0$, this inequality should be read as

$$\mu((1-\lambda)A + \lambda B) \geq \mu(A)^{1-\lambda}\mu(B)^{\lambda}$$

and it defines μ as a *log-concave measure*. When $s = -\infty$, the measure is said to be *convex* and the inequality is replaced by

$$\mu((1-\lambda)A + \lambda B) \geq \min(\mu(A), \mu(B)).$$

Notice that the class of s-concave measures on $\mathbb{R}^n$ is decreasing in s so that any s-concave measure is a convex measure. Any s-concave measure with $s \geq 0$ is log-concave, and thin-shell concentration for log-concave measures has been studied in [16, 17, 19, 22, 23]. The purpose of this paper is to prove thin-shell concentration for s-concave measures in the case $s < 0$, which we consider from now on. By *measure*, we always mean probability measure.

The class of s-concave measures was introduced and studied in [10, 11], where a complete characterization was established. An s-concave measure is

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2010 *Mathematics Subject Classification*: Primary 60E15, 60F10, 52A23; Secondary 52A40, 46B09.

Key words and phrases: isotropic, convex measure, concentration inequalities, thin-shell, large-deviation, KLS conjecture.

DOI: 10.4064/sm223-2-2

supported on some convex subset of an affine subspace where it has a density (see Section 2 for more details). When the support of an s-concave measure μ generates the whole space, we say that μ is *full-dimensional*.

A random vector with an s-concave distribution is called *s-concave*. The linear image of an s-concave random vector is also s-concave. We say that a random vector is full-dimensional if its distribution is *full-dimensional*. It is known that any seminorm of an s-concave random vector with $s < 0$ has moments of all order $p \in (0, -1/s)$ (see [10] and [1]). The Euclidean norm of an s-concave random vector X has a finite moment of order 2 if and only if $s > -1/2$. Since we are interested in comparison of moments of the Euclidean norm with the moment of order 2, we will always assume that $-1/2 < s < 0$.

Let $n \geq 1$ be an integer. The Euclidean space $\mathbb{R}^n$ is equipped with its Euclidean norm $|\cdot|_2$ and scalar product $\langle \cdot, \cdot \rangle$. Its unit sphere is denoted by S^{n-1} and its unit ball by B_2^n. We say that a random vector X is *isotropic* if $\mathbb{E}\, X = 0$ and for every $\theta \in S^{n-1}$, $\mathbb{E}\, \langle X, \theta \rangle^2 = 1$. Observe that if X is an s-concave full-dimensional random vector and $s > -1/2$, we can always find an affine transformation A such that AX is isotropic.

Let $p \in \mathbb{R}$ and $X \in \mathbb{R}^n$ be a random vector. Assume that $|X|_2$ has finite moments of order 2 and p with the convention that $(\mathbb{E}\, |X|_2^p)^{1/p} = \exp(\mathbb{E} \ln |X|_2)$ for $p = 0$. We define

$$\alpha_p(X) := \left| \frac{(\mathbb{E}\, |X|_2^p)^{1/p}}{(\mathbb{E}\, |X|_2^2)^{1/2}} - 1 \right|.$$

Our main result is the following

THEOREM 1. *Let $r > 2$. Let $X \in \mathbb{R}^n$ be a full-dimensional $(-1/r)$-concave random vector. If X is isotropic, then for any p such that $|p| \leq c \min(r, n^{1/3})$, we have*

$$\alpha_p(X) \leq \frac{C|p-2|}{r} + \left(\frac{C|p-2|}{n^{1/3}} \right)^{3/5},$$

where C and c are universal constants.

In the general case (when X is not isotropic), let A be an affine transformation such that AX is full-dimensional and isotropic. Then for any $p \in \mathbb{R}$ such that $|p| \leq c \min\big(r, \frac{n^{1/3}}{\|A\|^{2/3} \|A^{-1}\|^{2/3}}\big)$, we have

$$\alpha_p(X) \leq \frac{C|p-2|}{r} + \left(\frac{C|p-2|(\|A\|\, \|A^{-1}\|)^{2/3}}{n^{1/3}} \right)^{3/5},$$

where C and c are universal constants.

We also show (see Remark 15) that for $r > n + \sqrt{n}$, the estimate of $\alpha_p(X)$ in Theorem 1 can be improved and recovers the estimate of the log-concave case from [19].

To present connections between moment inequalities, thin-shell concentration and the Berry–Esseen theorem for one-dimensional marginals, let us introduce some notation.

Let $X \in \mathbb{R}^n$ be an isotropic random vector. Thus $\mathbb{E}\,|X|_2^2 = n$. Define $\varepsilon(X)$ to be the smallest number $\varepsilon > 0$ such that

$$\mathbb{P}\left(\left|\frac{|X|_2}{\sqrt{n}} - 1\right| \geq \varepsilon\right) \leq \varepsilon. \tag{1}$$

If $\varepsilon(X) = o(1)$ with respect to the dimension n, we say that X is *concentrated in a thin shell*. This is the usual jargon of the subject. More rigorously, it suggests that we are considering a sequence (X_n) of random vectors with $X_n \in \mathbb{R}^n$ and that $\varepsilon(X_n) = o(1)$ as n goes to ∞. It was shown in [2] (see also [14, 13]) that if an isotropic random vector X uniformly distributed on a convex body in $\mathbb{R}^n$ is such that $\varepsilon(X) = o(1)$, then almost all one-dimensional marginal distributions of X satisfy a Berry–Esseen theorem. More generally, let $X \in \mathbb{R}^n$ be an isotropic random vector; it was proved in [7] that

$$\sigma_{n-1}\Big(\theta \in S^{n-1} : \sup_{t\in\mathbb{R}} |\mathbb{P}(\langle X, \theta\rangle \leq t) - \Phi(t)| \geq 4\varepsilon(X) + \delta\Big) \leq 4n^{3/8}e^{-cn\delta^4},$$

where σ_{n-1} denotes the rotation invariant probability measure on the unit sphere S^{n-1}, Φ is the standard normal distribution function and $c > 0$ is a universal constant. It is worth noticing that the result from [7] does not assume log-concavity. Assuming only that X is isotropic, we find that if $\varepsilon(X)$ is $o(1)$ then almost all one-dimensional marginal distributions of X are approximately Gaussian. Later in [22, 17] it was proved that indeed $\varepsilon(X) = o(1)$ for all log-concave random vectors, and the best estimate to date [19] is

$$\varepsilon(X) = O(n^{-1/6} \log n).$$

Now let $p > 2$ and assume that X is isotropic and that $|X|_2$ has a finite moment of order p. Then $\varepsilon(X)$ is $o(1)$ if and only if $\alpha_p(X)$ is $o(1)$ (see Remark 4 below). Hence Theorem 1 ensures that if $r \to \infty$ with the dimension n then any isotropic $(-1/r)$-concave random vector exhibits thin-shell concentration and therefore almost all of its one-dimensional marginals satisfy a Berry–Esseen theorem. As a matter of fact, this condition on r is necessary. If r is fixed and does not depend on the dimension n, Proposition 5 gives an example of an isotropic $(-1/r)$-concave random vector $X \in \mathbb{R}^n$ which does not have thin-shell concentraction. Remark 6 also shows the asymptotic sharpness of Theorem 1, since for this example, for a fixed $p > 2$, $\alpha_p(X) \geq C(p-2)/r$ for r and n large enough, where $C > 0$ is a universal constant.

To prove Theorem 1, we need to extend to the case of s-concave measures several tools coming from the study of log-concave measures. This is the purpose of Section 2. Some of them were already established by Bobkov [8], like an analog of Ball's bodies [5] in the s-concave setting. Some others were also noticed previously (see e.g. [8], [1]) but not with the most accurate point of view. These new ingredients are analogous to the results of [12] in the log-concave setting and are at the heart of our proof. As in the approach of [16] or [19], an important ingredient is the log-Sobolev inequality on $\mathrm{SO}(n)$. It follows e.g. from the work of Bakry and Émery [4] and the calculation of the Ricci curvature of $\mathrm{SO}(n)$ (see [21, formula (F6)] for example) that for any Lipschitz function $f : \mathrm{SO}(n) \to \mathbb{R}^+$ (see Sections 3 and 4 for definitions)

$$\mathbb{E}(f(U)\log f(U)) - \mathbb{E}\, f(U)\log(\mathbb{E}\, f(U)) \le \frac{c}{n}\,\mathbb{E}(|\nabla \log f(U)|^2 f(U)), \tag{2}$$

where U is uniformly distributed on $\mathrm{SO}(n)$. This allows one to get reverse Hölder inequalities (see [16, (15)]): for every $f : \mathrm{SO}(n) \to \mathbb{R}$, let L be the log-Lipschitz constant of f (that is, the Lipschitz constant of $\log f$); then for every $q > r > 0$,

$$(\mathbb{E}\,|f(U)|^q)^{1/q} \le \exp\left(\frac{cL^2}{n}(q-r)\right)(\mathbb{E}\,|f(U)|^r)^{1/r}, \tag{3}$$

where U is uniformly distributed on $\mathrm{SO}(n)$.

Let X be a $(-1/r)$-concave random vector in $\mathbb{R}^n$ with full-dimensional support and distributed according to a measure with a density function $w : \mathbb{R}^n \to \mathbb{R}_+$. For any linear subspace E, denote by P_E the orthogonal projection onto E and for any $x \in E$ denote by

$$\pi_E w(x) = \int_{x+E^\perp} w(y)\,dy$$

the marginal of w on E. Given an integer k between 1 and n, a real number $p \in (-k, r)$, a linear subspace E_0 of $\mathbb{R}^n$ of dimension k, and $\theta_0 \in S(E_0)$, where $S(E_0)$ denotes the unit sphere of E_0, we define the function $h_{k,p} : \mathrm{SO}(n) \to \mathbb{R}_+$ by

$$h_{k,p}(u) := |S^{k-1}| \int_0^\infty t^{p+k-1}\pi_{u(E_0)} w(tu(\theta_0))\,dt \tag{4}$$

for every $u \in \mathrm{SO}(n)$, where $|S^{k-1}|$ denotes the area of the sphere.

Following the approach of [23, 16], we observe that for any $p \in (-k, r)$,

$$\mathbb{E}\,|X|_2^p = \frac{\Gamma((p+n)/2)\Gamma(k/2)}{\Gamma(n/2)\Gamma((p+k)/2)}\,\mathbb{E}\,h_{k,p}(U), \tag{5}$$

where U is uniformly distributed on $\mathrm{SO}(n)$. In view of (5) and the definition of $h_{k,p}$, we notice that it is of importance to work with families of measures which are stable under taking marginals, and it is clear from the definition that for any subspace E, if X is $(-1/r)$-concave, then so is $P_E X$.

In Section 2, we first introduce more notation and recall important facts concerning convex measures. Then we give an example of an isotropic $(-1/r)$-concave random vector $X \in \mathbb{R}^n$ that does not have thin-shell concentration, when r is fixed with respect to the dimension. Finally, we extend to the case of s-concave measures several tools coming from the study of log-concave measures that will be essential in the proof of Theorem 1. Section 3 is devoted to the proof of Theorem 1. Some of the results of these two sections are either classical or variations of known results; their proofs are shifted to the appendix.

2. Preliminary results for s-concave measures. We first recall some properties of s-concave measures and their relation to β-concave functions.

The class of s-concave measures was introduced and studied in [10, 11], where the following complete characterization was established. An s-concave measure μ on $\mathbb{R}^n$ is supported on some convex subset of an affine subspace where it has a density. When this subspace is the whole space, we say that μ is *full-dimensional.* In this case, its density w is β-concave with $\beta = s/(1-ns)$. Recall that a function $f : \mathbb{R}^n \to \mathbb{R}_+$ is called β-*concave* whenever

$$f((1-\lambda)x + \lambda y) \geq \big((1-\lambda)f(x)^\beta + \lambda f(y)^\beta\big)^{1/\beta}$$

for every $\lambda \in [0,1]$ and all $x, y \in \mathbb{R}^n$ such that $f(x)f(y) > 0$, where the right hand side is replaced by $f(x)^{1-\lambda} f(y)^\lambda$ for $\beta = 0$. Note that when $\beta < 0$, which will be the case below, β-concavity means that f^β is convex on its convex support $\{f > 0\}$.

We will use a similar language for probability measure, random vector and function which are related here as distribution, law of a random vector and density of probability. It is important to remember that when $X \in \mathbb{R}^n$ is $(-1/r)$-concave full-dimensional, then the result recalled above states that its distribution has a support that generates $\mathbb{R}^n$ and has a density which is $(-1/(n+r))$-concave.

Recall that for every $x > 0$, $\Gamma(x) = \int_0^\infty u^{x-1} e^{-u}\, du$, and for every $x, y > 0$, $B(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1}\, du = \int_0^\infty u^{x-1}(u+1)^{-(x+y)}\, du$.

The following inequality of Paley–Zygmund type is well known.

LEMMA 2. *Let $2 < p < s$. Let Y be a non-negative random variable with finite s-moment. Then for every $0 \leq t \leq (\mathbb{E}\, Y^p)^{1/p}$ we have*

$$\mathbb{P}(Y \geq t) \geq \left(\frac{\mathbb{E}\, Y^p - t^p}{(\mathbb{E}\, Y^s)^{p/s}}\right)^{s/(s-p)}.$$

Proof. Using the Hölder inequality, we have

$$\mathbb{E}\, Y^p = \mathbb{E}\, Y^p 1_{Y<t} + \mathbb{E}\, Y^p 1_{Y \geq t} \leq t^p + (\mathbb{E}\, Y^s)^{p/s} \mathbb{P}(Y \geq t)^{1-p/s}.$$

Thus

$$\mathbb{P}(Y \ge t) \ge \left(\frac{\mathbb{E}\, Y^p - t^p}{(\mathbb{E}\, Y^s)^{p/s}}\right)^{s/(s-p)}. \ \blacksquare$$

PROPOSITION 3. *Let $2 < p < s$. Let $X \in \mathbb{R}^n$ be an isotropic random vector such that $|X|_2$ has a finite s-moment. Then*

$$\min\left(\frac{\alpha_p(X)}{2}, \left(\frac{p\alpha_p(X)/2}{(\alpha_s(X)+1)^p}\right)^{s/(s-p)}\right) \le \varepsilon(X) \le ((\alpha_p(X)+1)^p - 1)^{1/3}.$$

Proof. Let $\varepsilon > 0$. Applying Lemma 2 to $Y = |X|_2/(\mathbb{E}\,|X|_2^2)^{1/2}$, $t = \varepsilon + 1$ and noticing that $\mathbb{E}\, Y^p = (\alpha_p(X)+1)^p$, $\mathbb{E}\, Y^s = (\alpha_s(X)+1)^s$, we get

$$\mathbb{P}\left(\frac{|X|_2}{(\mathbb{E}\,|X|_2^2)^{1/2}} \ge 1+\varepsilon\right) \ge \left(\frac{(\alpha_p(X)+1)^p - (\varepsilon+1)^p}{(\alpha_s(X)+1)^p}\right)^{s/(s-p)}$$

whenever $0 < \varepsilon \le \alpha_p(X)$. Since $x^p - y^p \ge p(x-y)$ for $p \ge 1$ and $x \ge y \ge 1$, we have

$$\mathbb{P}\left(\frac{|X|_2}{(\mathbb{E}\,|X|_2^2)^{1/2}} \ge 1+\varepsilon\right) \ge \left(\frac{p(\alpha_p(X)-\varepsilon)}{(\alpha_s(X)+1)^p}\right)^{s/(s-p)}.$$

Therefore

$$\mathbb{P}\left(\frac{|X|_2}{(\mathbb{E}\,|X|_2^2)^{1/2}} \ge 1+\varepsilon\right) \ge \left(\frac{p\alpha_p(X)/2}{(\alpha_s(X)+1)^p}\right)^{s/(s-p)}$$

whenever $0 < \varepsilon \le \alpha_p(X)/2$, and the left-hand inequality follows.

Since $|x-1| \le |x^q - 1|$ for $q \ge 1$ and every $x \ge 0$, the Markov inequality gives

$$\mathbb{P}\left(\left|\frac{|X|_2}{(\mathbb{E}\,|X|_2^2)^{1/2}} - 1\right| \ge \varepsilon\right) \le \mathbb{P}\left(\left|\frac{|X|_2^q}{(\mathbb{E}\,|X|_2^2)^{q/2}} - 1\right| \ge \varepsilon\right) \le \frac{\mathbb{E}\left|\frac{|X|_2^q}{(\mathbb{E}\,|X|_2^2)^{q/2}} - 1\right|^2}{\varepsilon^2}.$$

To deduce the right-hand inequality of the statement, take $q = p/2$ and observe that

$$\mathbb{E}\left|\frac{|X|_2^q}{(\mathbb{E}\,|X|_2^2)^{q/2}} - 1\right|^2 = (\alpha_{2q}(X)+1)^{2q} + 1 - 2(\alpha_q(X)+1)^q \le (\alpha_{2q}(X)+1)^{2q} - 1. \ \blacksquare$$

REMARK 4. Let $2 < p < s$. Let $X \in \mathbb{R}^n$ be an isotropic random vector such that $|X|_2$ has a finite s-moment. Proposition 3 shows that $\varepsilon(X)$ is $o(1)$ if and only if $\alpha_p(X)$ is $o(1)$ when $n \to \infty$.

Now we estimate $\varepsilon(X)$ for an example which shows that an isotropic $(-1/r)$-concave random vector $X \in \mathbb{R}^n$ may fail to have thin-shell concentration.

Proposition 5. *Let $r > 2$. There exists a sequence $(X_n)_n$ of isotropic $(-1/r)$-concave random vectors $X_n \in \mathbb{R}^n$ such that*

$$\liminf_{n\to\infty} \varepsilon(X_n) \ge c(r) > 0,$$

where $c(r) > 0$ depends only on r.

Proof. Let $r > 2$ and $2 < p < r$ and let $X_n \in \mathbb{R}^n$ be an isotropic random vector with density

$$f_{n,r}(x) = \frac{c_1}{(1 + c_2|x|_2)^{r+n}},$$

where c_1 and c_2 are normalization factors. From [10, 11], such a random vector is $(-1/r)$-concave. An immediate computation gives

$$\frac{(\mathbb{E}\,|X_n|_2^p)^{1/p}}{(\mathbb{E}\,|X_n|_2^2)^{1/2}} = \left(\frac{B(n+p, r-p)}{B(n,r)}\right)^{1/p} \left(\frac{B(n+2, r-2)}{B(n,r)}\right)^{-1/2}.$$

For fixed r and $2 < p < r$, we have

$$\lim_{n\to\infty} \frac{(\mathbb{E}\,|X_n|_2^p)^{1/p}}{(\mathbb{E}\,|X_n|_2^2)^{1/2}} = \left(\frac{\Gamma(r-p)}{\Gamma(r)}\right)^{1/p} \left(\frac{\Gamma(r-2)}{\Gamma(r)}\right)^{-1/2} \tag{6}$$

and by the strict log-convexity of the Gamma function, we have

$$\lim_{n\to\infty} (\alpha_p(X_n) + 1) = \lim_{n\to\infty} \frac{(\mathbb{E}\,|X_n|_2^p)^{1/p}}{(\mathbb{E}\,|X_n|_2^2)^{1/2}} > 1.$$

As a consequence for any $2 < p < r$, $\lim_{n\to\infty} \alpha_p(X_n) > 0$.

Now let $2 < p < s < r$. From Proposition 3, we get

$$\liminf_{n\to\infty} \varepsilon(X_n) \ge \lim_{n\to\infty} \min\left(\frac{\alpha_p(X_n)}{2}, \left(\frac{p\alpha_p(X_n)/2}{(\alpha_s(X_n)+1)^p}\right)^{s/(s-p)}\right) > 0. \tag{7}$$

Choose $p = (2+r)/2$ and $s = (p+r)/2$ for which $2 < p < s < r$ and note that the middle term in (7) depends only on r. This concludes the proof. ■

Remark 6. Let $2 < p < r$ and let $r \to \infty$. A calculation applying the Stirling formula in (6) when $r \to \infty$ gives

$$\lim_{r\to\infty} r \lim_{n\to\infty} \alpha_p(X_n) = (p-2)/2.$$

This asymptotic estimate shows that for a fixed $p > 2$ and r and n large enough, then $\alpha_p(X_n) \ge C(p-2)/r$ where $C > 0$ is a universal constant. This proves the sharpness of Theorem 1 under these conditions.

We now prove some inequalities for s-concave measures that will be useful tools in the next section.

Theorem 7.

(i) *Let $f : [0,\infty) \to [0,\infty)$ be a measurable function such that $\|f\|_\infty > 0$. Then*
$$p \mapsto \Big(\int_0^\infty pt^{p-1} f(t)\, dt / \|f\|_\infty \Big)^{1/p}$$
is non-decreasing on its domain of definition.

(ii) *Let $\alpha > 0$ and $f : [0,\infty) \to [0,\infty)$ be $(-1/\alpha)$-concave, continuous and integrable. Define $H_f : [0,\alpha) \to \mathbb{R}_+$ by*
$$H_f(p) = \begin{cases} \dfrac{1}{B(p,\alpha-p)} \displaystyle\int_0^\infty t^{p-1} f(t)\, dt & \text{for } 0 < p < \alpha, \\ f(0) & \text{for } p = 0. \end{cases}$$
Then H_f is log-concave on $[0,\alpha)$.

The proof of (i) may be obtained as in [25, Lemma 2.1] and the proof of (ii) is identical to the well known $(1/n)$-concave case [12]. We postpone the proof of Theorem 7 to the appendix.

We present several consequences of this result such as some reverse Hölder inequalities with sharp constants in the spirit of Borell's [12] and Berwald's [6] inequalities.

Corollary 8. *Let $r > 0$ and μ be a $(-1/r)$-concave measure on $\mathbb{R}^n$. Let $\phi : \mathbb{R}^n \to \mathbb{R}_+ = [0,\infty]$ be such that $\{\phi > 0\}$ is convex and ϕ is concave on $\{\phi > 0\}$. Then the function*
$$p \mapsto \begin{cases} \dfrac{1}{pB(p,r-p)} \displaystyle\int \phi(x)^p\, d\mu(x) & \text{for } 0 < p < r, \\ \mu(\{\phi > 0\}) & \text{for } p = 0, \end{cases}$$
is log-concave on $[0,r)$.

Moreover, if $\mu(\{\phi > 0\}) > 0$ then for any $0 < p \le q < r$,
$$\Big(\int_{\mathbb{R}^n} \phi(x)^q \frac{d\mu(x)}{\mu(\{\phi > 0\})} \Big)^{1/q} \le \frac{(qB(q,r-q))^{1/q}}{(pB(p,r-p))^{1/p}} \Big(\int_{\mathbb{R}^n} \phi(x)^p \frac{d\mu(x)}{\mu(\{\phi > 0\})} \Big)^{1/p}.$$

Proof. By the concavity of ϕ, for all $u, v \ge 0$ and $\lambda \in [0,1]$
$$(1-\lambda)\{\phi > u\} + \lambda\{\phi > v\} \subset \{\phi > (1-\lambda)u + \lambda v\}.$$
By the $(-1/r)$-concavity of μ, the function $f(t) = \mu(\{\phi > t\})$ is $(-1/r)$-concave and it is clearly continuous on $\mathbb{R}_+$. Observe that for any $p > 0$, by Fubini's theorem,
$$\int_{\mathbb{R}^n} \phi(x)^p\, d\mu(x) = \int_0^\infty pt^{p-1} f(t)\, dt.$$

The first part of the result follows from Theorem 7(ii). The "moreover" part follows from log-concavity since $p \mapsto (H_f(p)/f(0))^{1/p}$ is then non-increasing. ∎

The second corollary concerns the function $h_{k,p}$ defined in (4).

COROLLARY 9. *Let $r > 0$ and $u \in \mathrm{SO}(n)$. For any $(-1/(r+n))$-concave function $w : \mathbb{R}^n \to \mathbb{R}_+$ and any subspace E_0 of dimension $k \le n$, the function*

$$p \mapsto \begin{cases} \dfrac{h_{k,p}(u)}{B(p+k, r-p)} & \text{for } p > -k+1, \\ |S^{k-1}|\pi_{u(E_0)}w(0) & \text{for } p = -k+1, \end{cases}$$

is log-concave on $[-k+1, r)$.

Proof. Since w is $(-1/(r+n))$-concave, we note that $t \mapsto \pi_{U(E_0)}w(tu(\theta_0))$ is $(-1/(r+k))$-concave and it is clearly continuous on $\mathbb{R}_+$. Theorem 7 yields the result. ∎

We finish with some geometric properties of a family of bodies introduced by K. Ball [5] in the log-concave case.

COROLLARY 10. *Let $\alpha > 0$. Let $w : \mathbb{R}^n \to \mathbb{R}_+$ be a $(-1/\alpha)$-concave function such that $w(0) > 0$. For $0 < a < \alpha$ let*

$$K_a(w) = \Big\{x \in \mathbb{R}^n : a \int_0^\infty t^{a-1} w(tx)\,dt \ge w(0)\Big\}.$$

Then for any $0 < a \le b < \alpha$,

$$\left(\frac{w(0)}{\|w\|_\infty}\right)^{1/a-1/b} K_a(w) \subset K_b(w) \subset \frac{(bB(b, \alpha-b))^{1/b}}{(aB(a, \alpha-a))^{1/a}} K_a(w).$$

Proof. Notice that the sets K_a are star-shaped with respect to the origin, that is, $\lambda x \in K_a$ for all $x \in K_a$ and $\lambda \in [0,1]$. The radial function of K_a is

$$\rho_{K_a}(x) := \sup\{r : rx \in K_a\} = \left(a \int_0^\infty t^{a-1} \frac{w(tx)}{w(0)}\,dt\right)^{1/a}.$$

For any $x \in \mathbb{R}^n$, let f be the continuous $(-1/\alpha)$-concave function defined on $\mathbb{R}^+$ by $f(t) = w(tx)/w(0)$. By Theorem 7(i), the function

$$a \mapsto \left(\int_0^\infty t^{a-1} \frac{f(t)}{\|f\|_\infty}\,dt\right)^{1/a}$$

is non-decreasing. Hence the left-hand inclusion follows. Moreover, from Theorem 7(ii), the function $H_f : [0, \alpha) \to \mathbb{R}_+$ is log-concave on $[0, \alpha)$ with $H_f(0) = 1$. For $0 < a \le b < \alpha$, we thus have $H_f(b)^{1/b} \le H_f(a)^{1/a}$, implying the right-hand inclusion. ∎

3. Thin-shell concentration for convex measures. The purpose of this section is to prove Theorem 1. We follow the strategy of the log-concave case initiated in [22, 17, 23] and further developed in [16, 19].

The *support function* h_K of a non-empty compact set $K \subset \mathbb{R}^n$ is defined by

$$\forall \theta \in \mathbb{R}^n, \quad h_K(\theta) = \sup_{x \in K} \langle x, \theta \rangle.$$

To any random vector X in $\mathbb{R}^n$ and any $p \geq 1$, we associate its Z_p^+*-body* defined by its support function

$$\forall \theta \in \mathbb{R}^n, \quad h_{Z_p^+(X)}(\theta) = (\mathbb{E}\langle X, \theta \rangle_+^p)^{1/p}.$$

When the distribution of X has a density g, we write $Z_p^+(g) = Z_p^+(X)$. Extending a theorem of Ball [5] for log-concave functions, Bobkov [8, Remark 2.6] (see also [15, Theorem 3.1]) proved that if w is $(-1/(r+n))$-concave on $\mathbb{R}^n$ and $w(0) > 0$, then

$$K_a(w) \text{ is convex and compact for any } 0 < a \leq r + n - 1. \tag{8}$$

In the case of log-concave measures [26, 27, 19, 20], several relations between the Z_p^+-bodies and the convex sets K_a are known. We need their analogue in the setting of s-concave measures for negative s. We start with two technical lemmas. We postpone their proofs to the appendix.

Lemma 11. *Let $x, y \geq 1$. Then*

$$c\frac{x}{x+y} \leq (xB(x,y))^{1/x} \leq C\frac{x}{x+y}, \tag{9}$$

where c, C are universal positive constants. Moreover, for $k, r > 1$, the extension by continuity at 0 of the function

$$p \mapsto \frac{1}{p}\log\frac{B(k+p, r-p)}{B(k,r)}$$

is differentiable on $\left[-\frac{k-1}{2}, \frac{r-1}{2}\right]$ and satisfies

$$0 \leq \frac{d}{dp}\left(\frac{1}{p}\log\frac{B(k+p,r-p)}{B(k,r)}\right) \leq \frac{1}{r-1} + \frac{1}{k-1} \tag{10}$$

for $p \in \left[-\frac{k-1}{2}, \frac{r-1}{2}\right]$.

In this paper, we use the notion of *geometric distance* between sets, defined for any compact subsets $K, L \subset \mathbb{R}^n$ containing 0 in their interior by

$$d(K, L) = \inf\{t_2/t_1 : t_1 L \subset K \subset t_2 L,\ t_1, t_2 > 0\}.$$

Let $n \geq 1$, $r \geq 2$ and w be the $(-1/(r+n))$-concave density of a probability measure μ on $\mathbb{R}^n$. Then by Corollary 8 and Lemma 11, for $1 \leq p \leq q \leq r-1$,

$$Z_p^+(w) \subset Z_q^+(w) \subset c\frac{q}{p}\Big(\inf_{\theta \in S^{n-1}} \mu(\{x : \langle x, \theta \rangle > 0\})\Big)^{1/q-1/p} Z_p^+(w).$$

Fix $\theta \in S^{n-1}$ and define $F(t) = \mu(\{x : \langle x, \theta\rangle \le t\})$ for $t \in \mathbb{R}$. Then

$$\int_{\mathbb{R}} tF'(t)\,dt = \int_{\mathbb{R}^n} \langle x, \theta\rangle w(x)\,dx = 0$$

and F is $(-1/r)$-concave. Using Jensen's inequality, we get

$$F(0)^{-1/r} = F\Big(\int_{\mathbb{R}} tF'(t)\,dt\Big)^{-1/r} \le \int_{\mathbb{R}} F(t)^{-1/r} F'(t)\,dt = \left[\frac{F(t)^{1-1/r}}{1-1/r}\right]_{-\infty}^{\infty} = \frac{1}{1-1/r}.$$

Hence $\mu(\{x : \langle x, \theta\rangle > 0\}) \ge (1-1/r)^r \ge 1/4$ for $r \ge 2$. We have recovered here in a simple way a Grünbaum type inequality for convex measures due to Bobkov [8, Theorem 5.2]. We deduce that, for $1 \le p \le q \le r-1$,

$$Z_p^+(w) \subset Z_q^+(w) \subset C\frac{q}{p} Z_p^+(w) \quad \text{and} \quad d(Z_p^+(w), Z_q^+(w)) \le C\frac{q}{p}. \tag{11}$$

Lemma 12. *Let r, m and p be such that m is a positive integer, $r \ge m+1$ and $-m/2 \le p \le r-1$. Let F be a subspace of $\mathbb{R}^n$ of dimension m and let g be a $(-1/(r+m))$-concave density of a probability measure on F such that $\int_F xg(x)\,dx = 0$. Then*

$$d(K_{m+p}(g), Z^+_{\max(m,p)}(g)) \le c,$$

where c is a universal constant.

As in [19], an important ingredient in the proof of the thin-shell concentration inequality is an estimate from above of the log-Lipschitz constant of the map $u \mapsto h_{k,p}(u)$ on $\mathrm{SO}(n)$. Let $\mathcal{M}_n(\mathbb{R})$ be the set of square $n \times n$ matrices. We equip

$$\mathrm{SO}(n) = \{u \in \mathcal{M}_n(\mathbb{R}) : u^t u = \mathrm{Id},\ \det(u) = 1\}$$

with its standard invariant Riemannian metric, which we specify for concreteness on $T_{\mathrm{Id}}\mathrm{SO}(n)$, the tangent space at the identity element $\mathrm{Id} \in \mathrm{SO}(n)$. Since $u^t u = \mathrm{Id}$, this tangent space may be identified with the set of antisymmetric matrices $\{B \in \mathcal{M}_n(\mathbb{R}) : B^t + B = 0\}$. We define the scalar product $\langle B, B\rangle = \frac{1}{2}\operatorname{tr}(B^t B)$ on $T_{\mathrm{Id}}\mathrm{SO}(n)$.

Proposition 13. *Let $n \ge 1$, $r > 10$ and w be the $(-1/(r+n))$-concave density of a probability measure on $\mathbb{R}^n$ such that $\int_{\mathbb{R}^n} xw(x)\,dx = 0$. Let k be an integer such that $k \ge 2$, $2k-1 \le n$ and $2k \le r$. Let p be such that $-k/2 \le p \le r-1$. Denote by $L_{k,p}$ the log-Lipschitz constant of the map $u \mapsto h_{k,p}(u)$ on $\mathrm{SO}(n)$. Then*

$$L_{k,p} \le C \max(k,p)\, d(Z^+_{\max(k,p)}(w), B_2^n),$$

where C is a universal constant.

Proof. For any subspace F of dimension m, the marginal $\pi_F(w)$ is a $(-1/(r+m))$-concave function on F and from (8), to any $a \in [0, r+m-1]$, we associate the convex body $K_a(\pi_F(w))$ in F. Then the proof of Theorem 2.1 in [19, Section 2.2] gives the upper bound:

$$L_{k,p} \le \max_F \{(m+p)\, d(K_{m+p}(\pi_F(w)), B_2(F))\}$$

over all subspaces F of dimension $m = k, k+1, 2k-1$, where $B_2(F)$ is the Euclidean unit ball in F. By the assumptions on k, for these values of m, we have $m \le 2k-1 \le n$ and $r \ge 2k \ge m+1$ and $p \ge -k/2 \ge -m/2$. Hence from Lemma 12, we have

$$d(K_{m+p}(\pi_F(w)), B_2(F)) \le cd(Z^+_{\max(m,p)}(\pi_F(w)), B_2(F)).$$

By definition, if X is a random vector with density w on $\mathbb{R}^n$, the marginal $\pi_F(w)$ is the density of the projection $P_F X$ of X onto F. By identification of the support functions, we see that, for any $\theta \in F$,

$$h^p_{Z^+_p(\pi_F(w))}(\theta) = \mathbb{E}\, \langle P_F X, \theta\rangle^p_+ = \mathbb{E}\, \langle X, \theta\rangle^p_+.$$

This means that $Z^+_p(\pi_F(w)) = P_F(Z^+_p(w))$. Since the distance to the Euclidean ball cannot increase after projections, we conclude that

$$d(K_{m+p}(\pi_F(w)), B_2(F)) \le cd(Z^+_{\max(m,p)}(w), B^n_2).$$

By (11), for $m = k, k+1, 2k-1$, one has

$$d(Z^+_{\max(m,p)}(w), Z^+_{\max(k,p)}(w)) \le c.$$

This finishes the proof. ∎

We define the *q-condition number* of a random vector X to be

$$\rho_q(X) = \frac{\sup_{|\theta|_2=1} (\mathbb{E}\, \langle X, \theta\rangle^q_+)^{1/q}}{\inf_{|\theta|_2=1} (\mathbb{E}\, \langle X, \theta\rangle^q_+)^{1/q}}.$$

Obviously, if w is the density of a full-dimensional random vector X in $\mathbb{R}^n$ then $\rho_q(X) = d(Z^+_q(w), B^n_2)$.

Proposition 14. *With the same assumptions as in Proposition* 13*, if a random vector X with density w is isotropic then*

$$L_{k,p} \le C \max(k,p)^2.$$

More generally if A is such that AX is isotropic then

$$L_{k,p} \le C \max(k,p)^2 \|A\|\, \|A^{-1}\|. \tag{12}$$

Proof. Let $q = \max(k,p)$. Then $1 \le q \le r-1$. Using the triangular inequality we get

$$\rho_q(X) = d(Z^+_q(w), B^n_2) \le d(Z^+_q(w), Z^+_2(w)) d(Z^+_2(w), B^n_2).$$

From (11) we deduce that $d(Z_q^+(w), Z_2^+(w)) \le cq$. For any $\theta \in S^{n-1}$, $\mathbb{E}\langle X,\theta\rangle = 0$, hence $\mathbb{E}\langle X,\theta\rangle_+ = \mathbb{E}\langle -X,\theta\rangle_+$. Using this equality and (11) we deduce that

$$(\mathbb{E}\langle -X,\theta\rangle_+^2)^{1/2} \le c\,\mathbb{E}\langle -X,\theta\rangle_+ = c\,\mathbb{E}\langle X,\theta\rangle_+ \le c(\mathbb{E}\langle X,\theta\rangle_+^2)^{1/2}.$$

Thus

$$\mathbb{E}\langle X,\theta\rangle_+^2 \le \mathbb{E}\langle X,\theta\rangle^2 = \mathbb{E}\langle X,\theta\rangle_+^2 + \mathbb{E}\langle -X,\theta\rangle_+^2 \le C\,\mathbb{E}\langle X,\theta\rangle_+^2.$$

Hence if X is isotropic we deduce that $d(Z_2^+(w), B_2^n) \le c'$. We conclude that

$$\rho_q(X) = d(Z_q^+(w), B_2^n) \le C'q.$$

The first conclusion follows from Proposition 13. In the general case, notice that $Z_q^+(AX) = AZ_q^+(X)$ and $d(AB_2^n, B_2^n) = \|A\|\,\|A^{-1}\|$, thus

$$\rho_q(X) \le \rho_q(AX)\|A\|\,\|A^{-1}\|. \ ■$$

Proof of Theorem 1. Without loss of generality, we can assume $r > 32$. Indeed, if $r \le 32$ then the statement in Theorem 1 is valid for $|p| \le cr$ and it gives only a comparison of $(\mathbb{E}|X|_2^p)^{1/p}$ with $(\mathbb{E}|X|_2^2)^{1/2}$ up to a constant factor. The result is a consequence of Theorem 5.2 in [1].

From now on, we assume that $r > 32$ and $|p| \le r/8$. We start by presenting a complete argument following [16]. This will give a complete proof of a slightly weaker result. In the second part, we just indicate the needed modifications of the argument of [19] to get the complete conclusion.

In this first part, we will prove that for any $p \in [1/\sqrt{n}, \min(cn^{1/8}, r/8)]$,

$$(\mathbb{E}|X|_2^p\,\mathbb{E}|X|_2^{-p})^{1/p} \le 1 + \frac{Cp}{r} + \left(\frac{Cp}{n^{1/3}}\right)^{3/5}. \tag{13}$$

Assuming (13), few elementary steps are needed to prove that for any p such that $|p| \le \min(cn^{1/8}, r/8)$,

$$\left|\frac{(\mathbb{E}|X|_2^p)^{1/p}}{(\mathbb{E}|X|_2^2)^{1/2}} - 1\right| \le \frac{C(1+|p|)}{r} + \left(\frac{C(1+|p|)}{n^{1/3}}\right)^{3/5}, \tag{14}$$

which is already enough to get thin-shell concentration. Indeed, for $p \ge 2$, by the Hölder inequality, we have

$$0 \le \frac{(\mathbb{E}|X|_2^p)^{1/p}}{(\mathbb{E}|X|_2^2)^{1/2}} - 1 \le \frac{(\mathbb{E}|X|_2^p)^{1/p}}{(\mathbb{E}|X|_2^{-p})^{-1/p}} - 1$$

and we conclude by (13). For $p \le -2$, we have $|p| = -p \ge 2$ and from the Hölder inequality and (13),

$$0 \le \frac{(\mathbb{E}|X|_2^2)^{1/2}}{(\mathbb{E}|X|_2^p)^{1/p}} - 1 \le \frac{(\mathbb{E}|X|_2^{|p|})^{1/|p|}}{(\mathbb{E}|X|_2^{-|p|})^{-1/|p|}} - 1 \le \frac{C|p|}{r} + \left(\frac{C|p|}{n^{1/3}}\right)^{3/5}.$$

An elementary computation shows that

$$\left|\frac{(\mathbb{E}\,|X|_2^p)^{1/p}}{(\mathbb{E}\,|X|_2^2)^{1/2}} - 1\right| \le \frac{C|p|}{r} + \left(\frac{C|p|}{n^{1/3}}\right)^{3/5}.$$

For $p \in [-2, 2]$, by the Hölder inequality,

$$0 \le 1 - \frac{(\mathbb{E}\,|X|_2^p)^{1/p}}{(\mathbb{E}\,|X|_2^2)^{1/2}} \le 1 - \frac{(\mathbb{E}\,|X|_2^{-2})^{-1/2}}{(\mathbb{E}\,|X|_2^2)^{1/2}}$$

and we conclude by the previous estimate for $p = -2$. This concludes the proof of (14).

Let us start the proof of (13). Let $p \in [1/\sqrt{n}, \min(cn^{1/8}, r/8)]$ and k be an integer greater or equal than 2 such that $p < k \le n$. We will optimize the choice of k at the end of the proof. Recall that by (5),

$$\mathbb{E}\,|X|_2^p = \frac{\Gamma((p+n)/2)\Gamma(k/2)}{\Gamma(n/2)\Gamma((p+k)/2)}\,\mathbb{E}\,h_{k,p}(U),$$

where U is uniformly distributed on $\mathrm{SO}(n)$. Using the fact that the function $\frac{d}{dp}\log\Gamma(p)$ is concave (see for example the proof of Lemma 11 in the appendix), we deduce that

$$\frac{d}{dp}\left(\frac{1}{p}\log\frac{\Gamma((p+n)/2)\Gamma(k/2)}{\Gamma((p+k)/2)\Gamma(n/2)}\right) \le 0. \tag{15}$$

It follows that for any $0 < p < k$,

$$\frac{\Gamma((p+n)/2)\Gamma(k/2)}{\Gamma(n/2)\Gamma((p+k)/2)}\,\frac{\Gamma((-p+n)/2)\Gamma(k/2)}{\Gamma(n/2)\Gamma((-p+k)/2)} \le 1.$$

Then for all $0 < p < r$ and $n \ge k > p$ we have

$$\mathbb{E}\,|X|_2^p\,\mathbb{E}\,|X|_2^{-p} \le \mathbb{E}\,h_{k,p}(U)\,\mathbb{E}\,h_{k,-p}(U). \tag{16}$$

Applying the log-Sobolev inequality (3) to $h_{k,p}$ and $h_{k,-p}$ we get

$$\begin{aligned}\mathbb{E}\,h_{k,p}(U)^2 &\le e^{cL_{k,p}^2/n}\,(\mathbb{E}\,h_{k,p}(U))^2,\\ \mathbb{E}\,h_{k,-p}(U)^2 &\le e^{cL_{k,-p}^2/n}\,(\mathbb{E}\,h_{k,-p}(U))^2.\end{aligned} \tag{17}$$

Since $\mathrm{Var}\,f = \mathbb{E}\,f^2 - (\mathbb{E}\,f)^2$ we deduce that

$$\begin{cases}\mathrm{Var}\,h_{k,p}(U) \le (e^{cL_{k,p}^2/n} - 1)\,(\mathbb{E}\,h_{k,p}(U))^2,\\ \mathrm{Var}\,h_{k,-p}(U) \le (e^{cL_{k,-p}^2/n} - 1)\,(\mathbb{E}\,h_{k,-p}(U))^2.\end{cases} \tag{18}$$

By Corollary 9, we know that $p \mapsto h_{k,p}(u)/B(k+p, r-p)$ is log-concave on $[-k+1, r)$ hence

$$h_{k,p}(u)h_{k,-p}(u) \le \left(\frac{B(k+p, r-p)}{B(k,r)}\,\frac{B(k-p, r+p)}{B(k,r)}\right)h_{k,0}^2(u).$$

Taking the expectation with respect to $\mathrm{SO}(n)$, we get

$$\mathbb{E}\, h_{k,p}(U)h_{k,-p}(U) \le \left(\frac{B(k+p,r-p)}{B(k,r)}\,\frac{B(k-p,r+p)}{B(k,r)}\right)\mathbb{E}\, h_{k,0}^2(U).$$

Since $\mathbb{E}\, h_{k,0}(U) = 1$ we deduce from (17) that

$$\mathbb{E}\, h_{k,0}^2(U) \le e^{cL_{k,0}^2/n}.$$

Assume that $k \le r$. Then by (10), we know that for $p \le (k-1)/2$,

$$\left(\frac{B(k+p,r-p)}{B(k,r)}\,\frac{B(k-p,r+p)}{B(k,r)}\right)^{1/p} \le e^{2p\left(\frac{1}{k-1}+\frac{1}{r-1}\right)} \le e^{4p(1/k+1/r)}$$

since $k, r \ge 2$. Hence

$$\mathbb{E}\, h_{k,p}(U)h_{k,-p}(U) \le e^{cL_{k,0}^2/n+4p^2(1/k+1/r)}. \tag{19}$$

Moreover

$$\begin{aligned}\mathbb{E}\, h_{k,p}(U)h_{k,-p}(U) &= \mathbb{E}\, h_{k,p}(U)\,\mathbb{E}\, h_{k,-p}(U) + \mathrm{Cov}(h_{k,p}(U), h_{k,-p}(U))\\ &\ge \mathbb{E}\, h_{k,p}(U)\,\mathbb{E}\, h_{k,-p}(U) - \sqrt{\mathrm{Var}\, h_{k,p}(U)\,\mathrm{Var}\, h_{k,-p}(U)}\\ &\ge \mathbb{E}\, h_{k,p}(U)\,\mathbb{E}\, h_{k,-p}(U)\Big(1-\sqrt{(e^{cL_{k,p}^2/n}-1)(e^{cL_{k,-p}^2/n}-1)}\Big),\end{aligned} \tag{20}$$

where the last inequality follows from (18). Assume moreover that $2k-1 \le n$ and $2k \le r$. Then for $p \le (k-1)/2$, we can evaluate $L_{k,p}$, $L_{k,-p}$ and $L_{k,0}$ from Proposition 14 since the assumptions are fulfilled. We find that if X is isotropic then $\max(L_{k,p}, L_{k,-p}, L_{k,0}) \le Ck^2$. If $k \le c_0 n^{1/4}$ for a small enough numerical constant c_0, we have

$$\sqrt{(e^{cL_{k,p}^2/n}-1)(e^{cL_{k,-p}^2/n}-1)} \le c'\frac{k^4}{n} \le \frac{1}{10}.$$

Combining this estimate with (20) and (19), we have proved that if k is an integer such that $k \ge 2$, $2k-1 \le n$, $2k \le r$, $k \le c_0 n^{1/4}$ and $2p+1 \le k$ (this set of integers is not empty since $r > 32$ and $p \le r/8$) then

$$\mathbb{E}\, h_{k,p}(U)\,\mathbb{E}\, h_{k,-p}(U) \le \frac{e^{4p^2(1/k+1/r)+ck^4/n}}{1-c'k^4/n} \le e^{4p^2(1/k+1/r)+Ck^4/n}.$$

For $p \le 1$, we also force k to satisfy $k \le C_0 p^{1/4} n^{1/4}$. Hence taking the power $1/p$ in the last expression, we conclude from (16) that

$$(\mathbb{E}\,|X|_2^p\,\mathbb{E}\,|X|_2^{-p})^{1/p} \le e^{4p(1/k+1/r)+Ck^4/(pn)} \le 1 + cp\left(\frac{1}{k}+\frac{1}{r}\right) + c\frac{k^4}{pn},$$

since $p/k, p/r$ and $k^4/(pn)$ are bounded by universal constants.

It remains to optimize the choice of k. Let $p_0 = n^{-1/2}$. In this case we choose $k = 2$ and get

$$(\mathbb{E}\,|X|_2^{p_0}\,\mathbb{E}\,|X|_2^{-p_0})^{1/p_0} \le 1 + C/\sqrt{n}. \tag{21}$$

If $p \geq n^{-1/2}$ we choose k to be an integer such that $\min(r/4,(p^2n)^{1/5}) \leq k \leq 2\min(r/4,(p^2n)^{1/5})$ with the restriction $2p+1 \leq k \leq cn^{1/4}$ and $k \leq cp^{1/4}n^{1/4}$. For any p such that $p_0 \leq p \leq \min(cn^{1/8}, r/8)$, the integer k satisfies $k \geq 2$, $2k-1 \leq n$, $2k \leq r$, $k \leq c_0 n^{1/4}$ and $2p+1 \leq k$ and we get

$$(\mathbb{E}\,|X|_2^p\,\mathbb{E}\,|X|_2^{-p})^{1/p} \leq 1 + \frac{Cp}{r} + \left(\frac{Cp}{n^{1/3}}\right)^{3/5}.$$

This ends the proof of (13).

In the second part, we follow the argument developed in [19] to get a better estimate. We deal now with the case of p being positive or negative and, as already said, we can assume without loss of generality that $r > 34$ and $|p| \leq r/8$. As in [19], our goal is to estimate

$$\frac{d}{dp}\log((\mathbb{E}\,|X|_2^p)^{1/p}) = \frac{d}{dp}\log((\mathbb{E}\,h_{k,p}(U))^{1/p}) + \frac{d}{dp}\left(\frac{1}{p}\log\frac{\Gamma((p+n)/2)\Gamma(k/2)}{\Gamma(n/2)\Gamma((p+k)/2)}\right).$$

Most of the computation of Section 3.2 in [19] can be repeated. All the ingredients needed for the proof have been established and, adapting the argument in [19], we get

$$\frac{d}{dp}\log((\mathbb{E}\,|X|_2^p)^{1/p}) \leq \frac{c}{p^2n}(2L_{k,p}^2 + 3L_{k,0}^2) + \frac{C}{k-1} + \frac{C}{r-1}. \tag{22}$$

For convenience of the reader, we will briefly reproduce the proof of (22) in the appendix.

Assume that X is isotropic. For any $2|p| \leq k \leq r/2$ (this set of integers is not empty since $r > 32$ and $|p| \leq r/8$), we know by Proposition 14 that $L_{k,p}$ and $L_{k,0}$ are smaller than Ck^2. We get

$$\frac{d}{dp}\log((\mathbb{E}\,|X|_2^p)^{1/p}) \leq C\left(\frac{k^4}{p^2n} + \frac{1}{k} + \frac{1}{r}\right).$$

We have to minimize this expression for k being an integer ≥ 2 in the interval $[2|p|, r/2]$. For $|p| \in [n^{-1/2}, cn^{1/3}]$, we set k to be an integer such that $\min(r/4, 2(p^2n)^{1/5}) \leq k \leq 2\min(r/4, 2(p^2n)^{1/5})$. Therefore k satisfies the restrictions, and for any p such that $|p| \in [n^{-1/2}, cn^{1/3}]$, we get

$$\frac{d}{dp}\log((\mathbb{E}\,|X|_2^p)^{1/p}) \leq C\left(\frac{1}{(p^2n)^{1/5}} + \frac{1}{r}\right). \tag{23}$$

After integration over p, we find that for all $p \in [n^{-1/2}, c\min(r, n^{1/3})]$,

$$\left|\log\frac{(\mathbb{E}\,|X|_2^p)^{1/p}}{(\mathbb{E}\,|X|_2^2)^{1/2}}\right| \leq \frac{C|p-2|}{r} + \frac{C|p^{3/5} - 2^{3/5}|}{n^{1/5}}.$$

Since $|p^{3/5} - 2^{3/5}| \leq |p-2|^{3/5}$ and other terms on the right hand side of the inequality are bounded by a universal constant, we conclude that

$$\left|\frac{(\mathbb{E}\,|X|_2^p)^{1/p}}{(\mathbb{E}\,|X|_2^2)^{1/2}}-1\right| \le \frac{C|p-2|}{r}+\left(\frac{C|p-2|}{n^{1/3}}\right)^{3/5}, \qquad \forall p\in[n^{-1/2},\, c\min(r,n^{1/3})].$$

Since (23) holds only for $|p|\ge n^{-1/2}$, we use (21) to bridge the gap between $-n^{-1/2}$ and $n^{-1/2}$. Indeed, from (21), the previous inequality for $p_0=n^{-1/2}$ and $|p_0-2|=2-p_0\le 2$, we deduce that for $p\in[-p_0,p_0]$,

$$\begin{aligned}(\mathbb{E}\,|X|_2^p)^{1/p} \ge (\mathbb{E}\,|X|_2^{-p_0})^{-1/p_0} &\ge \frac{1}{1+C/\sqrt{n}}(\mathbb{E}\,|X|_2^{p_0})^{1/p_0}\\ &\ge \frac{1-2C/r-(2C/n^{1/3})^{3/5}}{1+C/n^{1/5}}(\mathbb{E}\,|X|_2^2)^{1/2}.\end{aligned}$$

An easy adaptation of the constants leads to the conclusion of Theorem 1 for all $p\in[-n^{-1/2},n^{-1/2}]$.

Integrating (23) again, we get, for $p\in[-c\min(r,n^{1/3}),-n^{-1/2}]$,

$$\frac{(\mathbb{E}\,|X|_2^p)^{1/p}}{(\mathbb{E}\,|X|_2^{-p_0})^{-1/p_0}} \ge 1-\frac{C|p+p_0|}{r}-\left(\frac{C|p+p_0|}{n^{1/3}}\right)^{3/5}.$$

Using $|p+p_0|\le|p-2|$ and the previous comparison of the moment of order $-p_0$ with the moment of order 2 and adjusting the constants proves that for all $p\in[-c\min(r,n^{1/3}),-n^{-1/2}]$,

$$\left|\frac{(\mathbb{E}\,|X|_2^p)^{1/p}}{(\mathbb{E}\,|X|_2^2)^{1/2}}-1\right| \le \frac{C|p-2|}{r}+\left(\frac{C|p-2|}{n^{1/3}}\right)^{3/5}.$$

This concludes the proof of the first part of Theorem 1.

If X is such that AX is isotropic, we know from Proposition 14 that for any integer k such that $2|p|\le k\le r/2$,

$$\max(L_{k,p},L_{k,0})\le Ck^2\|A\|\,\|A^{-1}\|.$$

The proof is identical to the previous one after replacing n by $\frac{n}{\|A\|^2\|A^{-1}\|^2}$. ■

Remark 15. In [19], a preprocessing step consisted in adding a Gaussian isotropic vector to the random vector X in order to start at the very beginning with a better information on the Z_p^+-bodies associated to the measure. In [23, 16], this convolution argument played a role of regularization. It is natural to ask if such a process could be done in the situation of s-concave measure. Adding a Gaussian vector does not help because for $s<0$, the new vector does not belong to any class of s-concave vectors. However, for $r>n$, we can give a similar argument, adding to X a random vector Z uniformly distributed on the Euclidean ball (see also [9]). Since Z is $(1/n)$-concave and X is $(-1/r)$-concave, the new vector $Y=(X+Z)/\sqrt{2}$ will be $(-1/(r-n))$-concave. For any $p\ge 1$, we have (see [19, inequality (4.7)])

$$\alpha_p(X)\le\alpha_{2p}(Y)(2+\alpha_{2p}(Y)),$$

so that it remains to bound $\alpha_{2p}(Y)$. It is easy to see that $(\mathbb{E}\,\langle Y,\theta\rangle_+^q)^{1/q} \geq c\sqrt{q}$ for all $q \geq 2$ and $\theta \in S^{n-1}$. Adapting the proof of Proposition 14, we get $L_{k,p} \leq C\max(k,p)^{3/2}$. As in [19], this improvement leads to the following estimate: if $r-n>2$, then for any p such that $1 \leq p \leq c\min(r-n,\sqrt{n})$,

$$\alpha_{2p}(Y) \leq \frac{C(2p-2)}{r-n} + \left(\frac{C(2p-2)}{\sqrt{n}}\right)^{1/2}.$$

For $r > n+\sqrt{n}$, we recover the same thin-shell concentration as in the log-concave case. It would be interesting to understand in which precise sense s-concave measures are close to log-concave measures for $s \in (-1/n, 1/n)$. Another question is to know what kind of preprocessing argument as in [24] would enable one to recover the small ball estimates from [1].

4. Appendix

Proof of Theorem 7. (i) This result is classical. In the symmetric case, it follows from Lemma 2.1 in [25]. The general case is similar. We provide the proof for completeness. We may assume, without loss of generality, that $\|f\|_\infty = 1$. Denote $I_p(f) = \int_0^\infty t^{p-1} f(t)\,dt$. From the Hölder inequality, the function $p \mapsto \log(I_p(f))$ is convex on its convex support, thus the domain of definition of $I_p(f)$ is an interval. Let $0<p<q$ be fixed such that $I_p(f)<\infty$ and $I_q(f)<\infty$. Let $a=(pI_p(f))^{1/p}$ and $\varphi(t) = t^{p-1}(f(t) - 1_{[0,a]}(t))$. Notice that $\varphi \leq 0$ on $[0,a]$, $\varphi \geq 0$ on $[a,\infty)$ and $\int_0^\infty \varphi(t)\,dt = 0$. Thus

$$I_q(f) - I_q(1_{[0,a]}) = \int_0^\infty t^{q-p}\varphi(t)\,dt = \int_0^\infty (t^{q-p} - a^{q-p})\varphi(t)\,dt \geq 0,$$

since the integrand is non-negative on $\mathbb{R}_+$. We conclude that

$$I_q(f) \geq I_q(1_{[0,a]}) = \frac{a^q}{q} = \frac{1}{q}(pI_p(f))^{q/p}.$$

(ii) Since f is $(-1/\alpha)$-concave, there exists a convex function $\varphi : [0,\infty) \to (0,\infty]$ such that $f = \varphi^{-\alpha}$. Since f is integrable it follows that φ tends to ∞ at ∞. From the convexity of φ, one deduces that $\varphi(t) \geq c(1+t)$ for some constant $c>0$. Thus $f(t) \leq (c+ct)^{-\alpha}$ for every $t \geq 0$. Therefore, $t^{p-1}f$ is integrable for every $p<\alpha$, which means that $H_f(p)<\infty$ for every $0<p<\alpha$. Let $p \in (0,\alpha)$ and $m, M > 0$. Define $g : \mathbb{R}_+ \to \mathbb{R}_+$ by $g(t) = m(1+t/M)^{-\alpha}$. Then

$$\int_0^\infty t^{p-1} g(t)\,dt = mM^p \int_0^\infty v^{p-1}(1+v)^{-\alpha}\,dv = mM^p B(p,\alpha-p).$$

Thus $H_g(p) = mM^p$, which implies that $\log(H_g)$ is affine on $(0,\alpha)$. Take $0<a<b<c<\alpha$. Let $\lambda \in [0,1]$ be such that $b = (1-\lambda)a + \lambda c$. Choose m and M such that $mM^a = H_f(a)$ and $mM^b = H_f(b)$ so that $H_g(a) = H_f(a)$

and $H_g(b) = H_f(b)$. If we prove that

$$\int_0^\infty t^{c-1}(g-f)(t)\,dt \geq 0, \tag{24}$$

that is, $H_g(c) \geq H_f(c)$, then using that $\log(H_g)$ is affine, we will deduce that

$$H_f(b) = H_g(b) = H_g(a)^{1-\lambda} H_g(c)^\lambda \geq H_f(a)^{1-\lambda} H_f(c)^\lambda,$$

and this will prove the log-concavity of H on $(0,\alpha)$. If $f = g$ then (24) is satisfied so that in the following we assume that $h := g - f \not\equiv 0$. Let

$$H_1(t) = \int_t^\infty s^{a-1} h(s)\,ds \quad \text{and} \quad H_2(t) = \int_t^\infty s^{b-a-1} H_1(s)\,ds.$$

Since $h(t) = O(t^{-\alpha})$ at infinity, we deduce that $H_1(t) = O(t^{a-\alpha})$ and $H_2(t) = O(t^{b-\alpha})$. We have $\int_0^\infty t^{a-1}h(t)\,dt = 0$, thus $H_1(\infty) = H_1(0) = 0$. Obviously $H_2(\infty) = 0$. We also observe that

$$\begin{aligned} 0 = \int_0^\infty t^{b-1} h(t)\,dt &= \int_0^\infty t^{b-a} t^{a-1} h(t)\,dt = -\int_0^\infty t^{b-a} H_1'(t)\,dt \\ &= [t^{b-a} H_1(t)]_0^\infty + (b-a)\int_0^\infty t^{b-a-1} H_1(t)\,dt = (b-a)H_2(0), \end{aligned}$$

whence $H_2(\infty) = H_2(0) = 0$. Since $\int_0^\infty t^{b-a-1} H_1(t)\,dt = 0$ and $H_1 \not\equiv 0$, the function H_1 has at least one change of sign. Moreover, using that $H_1(0) = H_1(\infty) = 0$, we deduce that H_1' and therefore h has at least two sign changes. Since $h = g - f$ has the same sign as $f^{-\alpha} - g^{-\alpha}$ which is convex, it cannot have more than two sign changes. Thus it has exactly two sign changes at some $0 < t_1 < t_2$. Moreover, from the convexity of $f^{-\alpha} - g^{-\alpha}$, h has to be negative on (t_1, t_2) and positive on $(0, t_1)$ and (t_2, ∞). From an easy study of the function H_2, we deduce that $H_2 \geq 0$. Therefore, using $H_1(0) = H_1(\infty) = H_2(0) = H_2(\infty) = 0$, we get

$$\begin{aligned} \int_0^\infty t^{c-1} h(t)\,dt &= \int_0^\infty t^{c-a} t^{a-1} h(t)\,dt = -\int_0^\infty t^{c-a} H_1'(t)\,dt \\ &= [-t^{c-a} H_1(t)]_0^\infty + (c-a)\int_0^\infty t^{c-a-1} H_1(t)\,dt \\ &= (c-a)\int_0^\infty t^{c-b} t^{b-a-1} H_1(t)\,dt \\ &= (c-a)[-t^{c-b} H_2(t)]_0^\infty + (c-a)(c-b)\int_0^\infty t^{c-b-1} H_2(t)\,dt \\ &= (c-a)(c-b)\int_0^\infty t^{c-b-1} H_2(t)\,dt \geq 0. \end{aligned}$$

This proves (24) and establishes the log-concavity of H_f on $(0,\alpha)$. To get it on $[0,\alpha)$, it is enough to prove that H_f is continuous at 0. This follows from the observation that

$$B(p,\alpha-p) \underset{p\to 0}{\sim} \Gamma(p) \underset{p\to 0}{\sim} \frac{1}{p}, \quad \text{thus} \quad H_f(p) \underset{p\to 0}{\sim} p\int_0^\infty t^{p-1} f(t)\,dt.$$

And it is classical that, for a continuous function f, the right-hand side term tends to $f(0)$ when $p\to 0$. ■

Proof of Lemma 11. Estimates (9) follow easily from the classical bounds for the Gamma function (see [3]), valid for $x\ge 1$:

$$\sqrt{2\pi}\, x^{x-1/2}e^{-x} \le \Gamma(x) \le \sqrt{2\pi}\, x^{x-1/2}e^{-x+1/12}.$$

For (10), we write

$$\frac{B(k+p,r-p)}{B(k,r)} = \frac{\Gamma(k+p)\Gamma(r-p)}{\Gamma(k)\Gamma(r)}.$$

Denote $G(p)=\log\Gamma(p)$ for $p>0$. We know that $G''(p)=\sum_{i\ge 0} 1/(p+i)^2$, hence G'' is non-increasing and $0\le G''(p)\le 1/(p-1)$ for $p>1$. Denote

$$F_k(p) = \frac{G(k+p)-G(k)}{p} \qquad \text{for } k>0 \text{ and } p>-k.$$

We have $F_k(p)=\int_0^1 G'(k+up)\,du$. Using that G'' is non-increasing, we deduce that for $k>1$ and $p\ge -(k-1)/2$,

$$\begin{aligned} F_k'(p) &= \int_0^1 G''(k+up)u\,du \le G''\Big(\frac{k+1}{2}\Big)\int_0^1 u\,du \\ &= \frac{1}{2}G''\Big(\frac{k+1}{2}\Big) \le \frac{1}{k-1} \end{aligned}$$

and $F_k'(p)\ge 0$. Therefore, if $k>1$, $r>1$ and $-\frac{k-1}{2}\le p\le \frac{r-1}{2}$ then

$$\begin{aligned} 0 \le \frac{d}{dp}\Big(\frac{1}{p}\log\frac{B(k+p,r-p)}{B(k,r)}\Big) &= \frac{d}{dp}(F_k(p)-F_r(-p)) \\ &= F_k'(p)+F_r'(-p) \le \frac{1}{k-1}+\frac{1}{r-1}. \ ■ \end{aligned}$$

Proof of Lemma 12. We present here a similar proof to one in the appendix of [19]. Applying Corollary 10 to $w=g$, $n=m$, $\alpha=r+m$, we deduce that, for $m/2\le a\le b\le r+m-1$,

$$\Big(\frac{g(0)}{\|g\|_\infty}\Big)^{1/a-1/b} K_a(g) \subset K_b(g) \subset \frac{(bB(b,r+m-b))^{1/b}}{(aB(a,r+m-a))^{1/a}} K_a(g).$$

From Lemma 11, we have

$$\frac{(bB(b,r+m-b))^{1/b}}{(aB(a,r+m-a))^{1/a}} \le c\frac{b}{a}$$

Moreover since $\int xg(x)\,dx = 0$, from Lemma 7.2 of [1], one has

$$\frac{g(0)}{\|g\|_\infty} \ge \left(\frac{r-1}{r+m-1}\right)^{r+m} \ge e^{-2m}.$$

Since $1/a - 1/b \le 1/a \le 2/m$, we deduce that $\left(\frac{g(0)}{\|g\|_\infty}\right)^{1/a-1/b} \ge e^{-4}$. We conclude that for $m/2 \le a \le b \le r+m-1$,

$$e^{-4}K_a(g) \subset K_b(g) \subset c\frac{b}{a}K_a(g). \tag{25}$$

By integration in polar coordinates, it is well known [26] (see also [20]) that we have the following relation between the Z_q^+-bodies associated with g and the Z_q^+-bodies associated with one of the convex bodies $K_a(g)$: for any $0 < q < r$,

$$Z_q^+(g) = g(0)^{1/q} Z_q^+(K_{m+q}(g)), \tag{26}$$

where for any body K, $Z_q^+(K)$ denotes the convex body whose support function is defined by

$$\forall \theta \in \mathbb{R}^m, \qquad h_{Z_q^+(K)}(\theta) = \Big(\int_K \langle x,\theta\rangle_+^q\,dx\Big)^{1/q}.$$

Let $\theta \in \mathbb{R}^m$ and K be a convex body containing 0. From Berwald's inequalities [6] applied to $K \cap \{\langle x,\theta\rangle \ge 0\}$ and the function $x \mapsto \langle x,\theta\rangle_+$ which is concave on $K \cap \{\langle x,\theta\rangle \ge 0\}$, the function

$$p \mapsto \left(\frac{\int_K \langle x,\theta\rangle_+^p\,dx}{mB(p+1,m)\operatorname{Vol}(K\cap\{\langle x,\theta\rangle \ge 0\})}\right)^{1/p}$$

is decreasing. Observe that $\lim_{p\to\infty}(\int_K \langle x,\theta\rangle_+^p\,dx)^{1/p} = h_K(\theta)$ for all $\theta \in \mathbb{R}^m$, and

$$(mB(p+1,m))^{1/p} = \Big(m\int_0^1 u^p(1-u)^{m-1}\,du\Big)^{1/p} \xrightarrow[p\to\infty]{} 1.$$

We deduce that

$$\left(\frac{\int_K \langle x,\theta\rangle_+^q dx}{mB(q+1,m)\operatorname{Vol}(K\cap\{\langle x,\theta\rangle \ge 0\})}\right)^{1/q} \ge h_K(\theta).$$

Note also that $\int_K \langle x,\theta\rangle_+^q\,dx \le h_K(\theta)^q \operatorname{Vol}(K\cap\{\langle x,\theta\rangle \ge 0\})$ and that $mB(q+1,m) = qB(q,m+1)$. Therefore

$$h_K(\theta) \ge \frac{h_{Z_q^+(K)}(\theta)}{\operatorname{Vol}(K\cap\{\langle x,\theta\rangle \ge 0\})^{1/q}} \ge (qB(q,m+1))^{1/q} h_K(\theta). \tag{27}$$

Now we establish that for $q = \max(p, m)$,

$$d(K_{m+q}(g), Z_q^+(g)) \le c. \tag{28}$$

By Lemma 11, for any $q \ge m \ge 1$, $(qB(q, m+1))^{1/q} \ge cq/(m+q+1) \ge c/3$ and we deduce from (27) that for every $\theta \in \mathbb{R}^n$,

$$h_{K_{m+q}(g)}(\theta) \ge \frac{h_{Z_q^+(K_{m+q}(g))}(\theta)}{\mathrm{Vol}(K_{m+q}(g) \cap \{\langle x, \theta\rangle \ge 0\})^{1/q}} \ge \frac{c}{3} h_{K_{m+q}(g)}(\theta),$$

where c is a universal constant. Together with (26), we conclude that

$$\begin{aligned} d(K_{m+q}(g), Z_q^+(g)) &= d(K_{m+q}(g), Z_q^+(K_{m+q}(g))) \\ &\le c \frac{\sup_{\theta\in\mathbb{R}^n} \mathrm{Vol}(K_{m+q}(g) \cap \{\langle x, \theta\rangle \ge 0\})^{1/q}}{\inf_{\theta\in\mathbb{R}^n} \mathrm{Vol}(K_{m+q}(g) \cap \{\langle x, \theta\rangle \ge 0\})^{1/q}} \end{aligned} \tag{29}$$

for a universal constant c. Applying (25) for $a = m + 1$ and $b = m + q$, we get

$$e^{-4} K_{m+1}(g) \subset K_{m+q}(g) \subset c\frac{m+q}{m+1} K_{m+1}(g).$$

Since $q \ge m$ and $\left(\frac{m+q}{m+1}\right)^{m/q} \le e$, from (29) we get

$$d(K_{m+q}(g), Z_q^+(g)) \le C\ \frac{\sup_{\theta\in\mathbb{R}^n} \mathrm{Vol}(K_{m+1}(g) \cap \{\langle x, \theta\rangle \ge 0\})^{1/q}}{\inf_{\theta\in\mathbb{R}^n} \mathrm{Vol}(K_{m+1}(g) \cap \{\langle x, \theta\rangle \ge 0\})^{1/q}}$$

for a universal constant C. Since g has its barycenter at the origin, so does $K_{m+1}(g)$, and we deduce from a classical result of Grünbaum [18] that

$$\frac{\sup_{\theta\in\mathbb{R}^n} \mathrm{Vol}(K_{m+1}(g) \cap \{\langle x, \theta\rangle \ge 0\})^{1/q}}{\inf_{\theta\in\mathbb{R}^n} \mathrm{Vol}(K_{m+1}(g) \cap \{\langle x, \theta\rangle \ge 0\})^{1/q}} \le (e-1)^{1/q} \le e - 1.$$

Thus (28) is proved.

It is now enough to establish that $d(K_{m+q}, K_{m+p}) \le c$, where $q = \max(m, p)$. For $q = p$, this is obvious, so we may assume that $q = m \ge p$. Then $m/2 \le m + p \le m + q = 2m$ and using (25) for $a = m + p \le b = 2m$, we deduce that

$$d(K_{m+p}(g), K_{2m}(g)) \le ce^4 \frac{2m}{m+p} \le 4ce^4. \ \blacksquare$$

Proof of inequality (22). Our goal is to estimate

$$\begin{aligned} &\frac{d}{dp} \log((\mathbb{E}\,|X|_2^p)^{1/p}) \\ &\quad = \frac{d}{dp} \log\big((\mathbb{E}\, h_{k,p}(U))^{1/p}\big) + \frac{d}{dp}\left(\frac{1}{p} \log \frac{\Gamma((p+n)/2)\Gamma(k/2)}{\Gamma(n/2)\Gamma((p+k)/2)}\right). \end{aligned}$$

As already mentioned in (15), by concavity of $p \mapsto \frac{d}{dp} \log \Gamma(p)$, we have

$$\frac{d}{dp}\left(\frac{1}{p} \log \frac{\Gamma((p+n)/2)\Gamma(k/2)}{\Gamma(n/2)\Gamma((p+k)/2)}\right) \le 0.$$

We use the following notation. Let (Ω,μ) be a measurable space. For any measurable function $f:\Omega\to\mathbb{R}^+$, we set

$$\mathbb{E}_\mu(f)=\int f\,d\mu \quad\text{and}\quad \mathrm{Ent}_\mu(f)=\mathbb{E}_\mu(f\log f)-\mathbb{E}_\mu(f)\log\mathbb{E}_\mu(f).$$

Let w be the density of the distribution of X on $\mathbb{R}^n$. Since X is $(-1/r)$-concave, w is $(-1/(r+n))$-concave on $\mathbb{R}^n$. To any fixed $u\in\mathrm{SO}(n)$, we associate the measure μ_u on $\mathbb{R}^+$ with density

$$t\mapsto |S^{k-1}|t^{k-1}\pi_{u(E_0)}w(tu(\theta_0))$$

so that

$$h_{k,p}(u)=|S^{k-1}|\int_0^\infty t^{p+k-1}\pi_{u(E_0)}w(tu(\theta_0))\,dt=\mathbb{E}_{\mu_u}(t^p).$$

Define also $\mu_{k,p}$ to be the measure on $\mathbb{R}^+$ with density

$$t\mapsto |S^{k-1}|t^{k-1}\,\mathbb{E}\,\pi_{U(E_0)}w(tU(\theta_0)).$$

Then $\mathbb{E}\,h_{k,p}(U)=\mathbb{E}_U\,\mathbb{E}_{\mu_U}(t^p)=\mathbb{E}_{\mu_{k,p}}(t^p)$. Since w is a density of probability, $\mu_{k,p}$ is a probability measure on $\mathbb{R}^+$. A classical fact, verified by direct computation, is that

$$\frac{d}{dp}\log\big((\mathbb{E}_\mu(f^p))^{1/p}\big)=\frac{1}{p^2}\,\frac{\mathrm{Ent}_\mu(f^p)}{\mathbb{E}_\mu(f^p)}.$$

Therefore

$$\begin{aligned}\text{(30)}\qquad \frac{d}{dp}\log\big((\mathbb{E}\,h_{k,p}(U))^{1/p}\big)&=\frac{d}{dp}\log\big((\mathbb{E}_{\mu_{k,p}}(t^p))^{1/p}\big)\\ &=\frac{1}{p^2}\,\frac{\mathrm{Ent}_{\mu_{k,p}}(t^p)}{\mathbb{E}_{\mu_{k,p}}(t^p)}=\frac{1}{p^2}\,\frac{\mathrm{Ent}_{\mu_{k,p}}(t^p)}{\mathbb{E}\,h_{k,p}(U)}.\end{aligned}$$

The numerator can be decomposed into two terms:

$$\mathrm{Ent}_{\mu_{k,p}}(t^p)=\mathbb{E}_U\,\mathrm{Ent}_{\mu_U}(t^p)+\mathrm{Ent}_U\,\mathbb{E}_{\mu_U}(t^p)=\mathbb{E}_U\,\mathrm{Ent}_{\mu_U}(t^p)+\mathrm{Ent}_U\,h_{k,p}(U).$$

To control the second term, we use the log-Sobolev inequality (2):

$$\text{(31)}\qquad \frac{1}{p^2}\,\frac{\mathrm{Ent}_U\,h_{k,p}(U)}{\mathbb{E}\,h_{k,p}(U)}\le\frac{c}{p^2n}\,\frac{\mathbb{E}(|\nabla\log h_{k,p}|^2(U)h_{k,p}(U))}{\mathbb{E}\,h_{k,p}(U)}\le\frac{cL_{k,p}^2}{p^2n}.$$

To control the first term, we start by observing that for a fixed $u\in\mathrm{SO}(n)$,

$$\begin{aligned}\frac{1}{p^2}\,\frac{\mathrm{Ent}_{\mu_u}(t^p)}{\mathbb{E}_{\mu_u}(t^p)}&=\frac{d}{dp}\log\big((\mathbb{E}_{\mu_u}(f^p))^{1/p}\big)=\frac{d}{dp}\bigg(\frac{1}{p}\log h_{k,p}(u)\bigg)\\ &=\frac{d}{dp}\,\frac{1}{p}\bigg(\log\frac{h_{k,p}(u)}{B(p+k,r-p)}-\log\frac{h_{k,0}(u)}{B(k,r)}\\ &\qquad+\log\frac{B(p+k,r-p)}{B(k,r)}+\log h_{k,0}(u)\bigg).\end{aligned}$$

By Corollary 9, the map $p \mapsto \frac{h_{k,p}(u)}{B(p+k,r-p)}$ is log-concave on $(-k+1,r)$. This implies that

$$\frac{d}{dp}\,\frac{1}{p}\biggl(\log\frac{h_{k,p}(u)}{B(p+k,r-p)}-\log\frac{h_{k,0}(u)}{B(k,r)}\biggr)\le 0.$$

We know from Lemma 11 that, for all $p\in\bigl[-\frac{k-1}{2},\frac{r-1}{2}\bigr]$,

$$\frac{d}{dp}\biggl(\frac{1}{p}\log\frac{B(k+p,r-p)}{B(k,r)}\biggr)\le C\biggl(\frac{1}{k-1}+\frac{1}{r-1}\biggr).$$

Therefore, for any fixed $u\in \mathrm{SO}(n)$,

$$\frac{1}{p^2}\operatorname{Ent}_{\mu_u}(t^p)\le C h_{k,p}(u)\biggl(\frac{1}{k-1}+\frac{1}{r-1}\biggr)-\frac{1}{p^2}h_{k,p}(u)\log h_{k,0}(u).$$

Integrating over $u\in\mathrm{SO}(n)$, we deduce that

$$\begin{aligned}\text{(32)}\qquad \frac{1}{p^2}\,\frac{\mathbb{E}\operatorname{Ent}_{\mu_U}(t^p)}{\mathbb{E}\,h_{k,p}(U)}&\le C\biggl(\frac{1}{k-1}+\frac{1}{r-1}\biggr)\\&\quad+\frac{1}{p^2}\,\frac{\mathbb{E}\,h_{k,p}(U)\log(h_{k,0}(U)^{-1})}{\mathbb{E}\,h_{k,p}(U)}.\end{aligned}$$

From the Jensen and Hölder inequalities,

$$\begin{aligned}\frac{\mathbb{E}(h_{k,p}(U)\log h_{k,0}(U)^{-1})}{\mathbb{E}\,h_{k,p}(U)}&\le\log\biggl(\frac{\mathbb{E}(h_{k,p}(U)h_{k,0}(U)^{-1})}{\mathbb{E}\,h_{k,p}(U)}\biggr)\\&\le\log\biggl(\frac{(\mathbb{E}\,h_{k,p}(U)^2)^{1/2}}{\mathbb{E}\,h_{k,p}(U)}\biggr)\\&\quad+\log\bigl((\mathbb{E}(h_{k,0}(U)^{-2}))^{1/2}\bigr).\end{aligned}$$

From (3), the first term is upper bounded by $(c/n)L_{k,p}^2$. For the second term, we first use (3) with $f=h_{k,0}^{-1}$, $q=2$ and $r=0$, then we use (3) again with $f=h_{k,0}$, $q=1$ and $r=0$. Since $\mathbb{E}\,h_{k,0}(U)=\mathbb{E}_{\mu_{k,0}}(1)=1$, we deduce that this term is bounded by $(3c/n)L_{k,0}^2$. Combining this last inequality with (32), (31) and (30), we conclude that

$$\frac{d}{dp}\log((\mathbb{E}\,|X|_2^p)^{1/p})\le\frac{c}{p^2n}(2L_{k,p}^2+3L_{k,0}^2)+\frac{C}{k-1}+\frac{C}{r-1}.\ \blacksquare$$

Acknowledgements. We deeply and warmly thank the referee for his constructive comments on the first version of the paper. It forced us to clarify several main points and we hope that it improved the presentation of the paper.

The research of the three authors was partially supported by the ANR project GeMeCoD, ANR 2011 BS01 007 01.

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Matthieu Fradelizi, Olivier Guédon, Alain Pajor
Université Paris-Est
Laboratoire d'Analyse et Mathématiques Appliquées (UMR 8050)
UPEMLV
F-77454 Marne-la-Vallée Cedex 2, France
E-mail: matthieu.fradelizi@u-pem.fr
olivier.guedon@u-pem.fr
alain.pajor@u-pem.fr

Received July 18, 2013
Revised version September 9, 2014 (7820)