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A product of three projections

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Abstract. Let X and Y be two closed subspaces of a Hilbert space. If we send a point back and forth between them by orthogonal projections, the iterates converge to the projection of the point onto the intersection of X and Y by a theorem of von Neumann.

Any sequence of orthoprojections of a point in a Hilbert space onto a finite family of closed subspaces converges weakly, according to Amemiya and Ando. The problem of norm convergence was open for a long time. Recently Adam Paszkiewicz constructed five subspaces of an infinite-dimensional Hilbert space and a sequence of projections on them which does not converge in norm. We construct three such subspaces, resolving the problem fully. As a corollary we observe that the Lipschitz constant of a certain Whitney-type extension does in general depend on the dimension of the underlying space.

1. Introduction. Let K be a fixed natural number and let $\mathscr{L} = \{L_1, \dots, L_K\}$ be a family of K closed subspaces of a Hilbert space H. Let $z_0 \in H$ and $k_1, k_2, \dots \in \{1, \dots, K\}$ be arbitrary. Consider the sequence of vectors $\{z_n\}$ defined by

$$z_n = P_{k_n} z_{n-1}, \tag{1}$$

where P_k denotes the orthogonal projection of H onto L_k. The sequence $\{z_n\}$ converges weakly by a theorem of Amemiya and Ando [AA]. If each projection appears in the sequence $\{P_{k_n}\}$ infinitely many times, then this limit is equal to the projection of z_0 onto the intersection of all spaces in $\mathscr{L}$.

If $K = 2$ then the sequence $\{z_n\}$ converges even in norm according to a classical result of von Neumann [N].

If $K \geq 3$ then additional assumptions are needed to ensure the norm-convergence. That $\{z_n\}$ converges if H is finite-dimensional was originally proved by Práger [Pr]; this also follows, of course, from [AA].

If H is infinite-dimensional, but the sequence $\{k_n\}$ is periodic, the sequence $\{z_n\}$ converges in norm according to Halperin [Ha]. The result was generalized to quasiperiodic sequences by Sakai [S]. Recall that the sequence

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$\{k_n\}$ is quasiperiodic if there exists $r \in \mathbb{N}$ such that $\{k_m, k_{m+1}, \dots, k_{m+r}\} = \{1, \dots, K\}$ for each $m \in \mathbb{N}$.

The case of H infinite-dimensional, $K \geq 3$ and $\{k_n\}$ arbitrary was open for a long time. In 2012 Paszkiewicz [P1] constructed an ingenious example of five subspaces of an infinite-dimensional Hilbert space and of a sequence $\{z_n\}$ of the form (1) which does not converge in norm. An important input towards the construction comes from Hundal's example ([H], see also [K] and [MR]) of two closed convex subsets of an infinite-dimensional Hilbert space and a sequence of alternating projections onto them which does not converge in norm.

The basic idea of Paszkiewicz was the observation that it is possible to move a unit vector x_1 with an arbitrary precision to another unit vector x_2 orthogonal to x_1 by iterating just three projections. This construction is then used to move the initial vector x_1 to $x_2 \perp x_1$, then to $x_3 \perp \{x_1, x_2\}$ with better and better precision along quarter circles connecting the orthogonal sequence $\{x_1, x_2, \dots\}$. Such an iteration certainly does not converge in norm.

There is a technical difficulty in gluing these "90-degree" steps together in such a way that the next step does not interfere with the preceding ones. In Paszkiewicz's example of five projections this was done by gluing the odd and even steps together. The cases of three or four projections were left open. The goal of this paper is to show that it is possible to glue the Paszkiewicz "90-degree" steps constructions together to obtain three Hilbert space projections with non-convergent iterations. The construction of three projections with this property is not straightforward. In fact, there is a paper [P2] claiming the same result, which is apparently not correct: η_k is chosen on page 6 of [P2] based on M which depends on $m(k,s)$, which in its turn already depends on η_k.

Notation. Let H be a Hilbert space, and $B(H)$ the space of bounded linear operators from H to H. For $M, N \subset H$ we denote by $\bigvee M$ the closed linear hull of M, and by $M \vee N$ the closed linear hull of $M \cup N$. Similarly we use $\vee x$ and $x \vee y$ for $x, y \in H$. If M is a subspace and $N \subset M$ then $M \ominus N$ stands for $M \cap N^{\perp}$. By P_N we denote the orthogonal projection onto the closed linear hull of N.

For $m \in \mathbb{N}$ let $\mathcal{S}_m$ be the free semigroup with generators $g_1, \dots, g_m$ satisfying the relations $g_j^2 = g_j$ $(j = 1, \dots, m)$. If $\varphi = g_{i_r} \cdots g_{i_1} \in \mathcal{S}_m$ (for some $r \in \mathbb{N}$ and $i_j \in \{1, \dots, m\}$ with $i_{j+1} \neq i_j$ for all j) and $A_1, \dots, A_m \in B(H)$ are projections, then we write $\varphi(A_1, \dots, A_m) = A_{i_r} \cdots A_{i_1} \in B(H)$. Denote by $|\varphi| = r$ the "length" of φ.

2. Construction of the example. In this section, let H be an infinite-dimensional Hilbert space. The example is "glued" together from finite-dimensional blocks. In each of these blocks three subspaces and a finite product of projections are constructed so that the product maps a given unit vector u with an arbitrary precision to a unit vector v orthogonal to u.

This idea was already used by Hundal [H] to construct a cone and a half-space in H which intersect at the origin, but the corresponding sequence of alternating nearest point mappings (although weakly convergent to the origin) does not converge pointwise in norm. All of Hundal's blocks are 3-dimensional; here the dimension of the blocks increases exponentially.

Let u and v be two orthonormal vectors. It is very easy to get from u approximately to v be means of finitely many projections onto the lines h_j dissecting the right angle between u and v into small enough angles.

For $\varepsilon > 0$ let $k(\varepsilon)$ be the smallest positive integer k such that $\left(\cos \frac{\pi}{2k}\right)^k > 1 - \varepsilon$. That is, if u and v are two orthonormal vectors, and we project u consecutively onto the lines dividing the right angle between u and v into k angles of size $\pi/(2k)$, then we land at v with error at most ε (see Fig. 1.

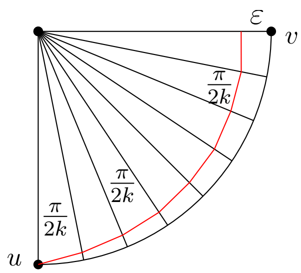

Fig. 1. Approximating v by projections of u

Projecting onto a line can be arbitrarily approximated by iterating projections between two subspaces intersecting at this line. In Hundal's example (see [K]) one of the spaces is always the plane $E = u \vee v$ and the other is a 2-dimensional space V_j intersecting E at h_j. These 2-dimensional planes support a part of the surface of a cone. Paszkiewicz's ingeniously simple idea was to replace the n pieces of 2-dimensional planes V_j by an increasing family of n finite-dimensional spaces $Z_1 \subset \cdots \subset Z_n$. He then replaced the projections onto these spaces by projections onto the largest space $X = Z_n$ and its suitable small variation Y. Lo and behold, instead of projecting onto several spaces, Paszkiewicz is projecting just onto three of them: E, X, and Y. In what follows, we significantly refine this construction in order to be able at the end to glue together the "90-degree" steps to end up with just three subspaces instead of Paszkiewicz's five.

The first statement of the next lemma is taken from [P1]; we supply a slightly different proof.

LEMMA 2.1. *Let $\varepsilon > 0$. Then there exists $\phi_\varepsilon \in \mathcal{S}_{k(\varepsilon)+1}$ with the following properties:*

(i) *If $u \in H$ with $\|u\| = 1$, then there exist $v \perp u$ with $\|v\| = 1$ and subspaces $Z'_1 \subset \cdots \subset Z'_{k(\varepsilon)}$ with $\dim Z'_j = j+1$ for all j such that $v \in Z'_{k(\varepsilon)}$ and*
$$\|\phi_\varepsilon(P_{Z'_1}, \ldots, P_{Z'_{k(\varepsilon)}}, P_{u\vee v})u - v\| < 2\varepsilon.$$

(ii) *If $M, R \subset H$ are finite-dimensional subspaces and $u \in M \cap R^\perp$ with $\|u\| = 1$, then there exist $v \perp M \vee R$ with $\|v\| = 1$ and subspaces $Z'_1 \subset \cdots \subset Z'_{k(\varepsilon)}$ with $\dim Z'_j = j+1$ for all j such that $v \in Z'_{k(\varepsilon)}$, $Z'_{k(\varepsilon)} \perp R$, and*
$$\|\phi_\varepsilon(P_{Z'_1}, \ldots, P_{Z'_{k(\varepsilon)}}, P_{M\vee v})u - v\| < 2\varepsilon.$$

Proof. Write $k := k(\varepsilon)$.

To prove (i), choose orthonormal vectors $z_0, z_1, \ldots, z_{k-1}, v \in H$ orthogonal to u. Let $E = u \vee v$.

Let $\xi = \pi/(2k)$. For $j = 0, \ldots, k$, let $h_j = u\cos j\xi + v \sin j\xi$ be the points on the quarter circle connecting $h_0 = u$ to $h_k = v$. We inductively construct a rapidly decreasing sequence of nonnegative numbers $\alpha_0 > \alpha_1 > \cdots > \alpha_{k-1} > \alpha_k = 0$ in the following way. Choose $\alpha_0 \in (0,1)$ arbitrarily. Let $1 \le j \le k-1$ and suppose that $\alpha_0, \ldots, \alpha_{j-1}$ and subspaces $Z'_1 \subset \cdots \subset Z'_{j-1}$ have already been constructed. Set
$$Z''_j = \bigvee\{h_0 + \alpha_0 z_0, h_1 + \alpha_1 z_1, \ldots, h_{j-1} + \alpha_{j-1} z_{j-1}, h_j\}.$$
Since $E \cap Z''_j = \vee h_j$, we have $(P_{Z''_j} P_E P_{Z''_j})^r x \to P_{h_j} x$ for each $x \in H$ as $r \to \infty$, by [N]. As both spaces are finite-dimensional, there exists $r(j) \in \mathbb{N}$ such that
$$\|(P_{Z''_j} P_E P_{Z''_j})^{r(j)} - P_{h_j}\| < \varepsilon/k.$$
Let $\alpha_j > 0$ be so small that
$$\|(P_{Z'_j} P_E P_{Z'_j})^{r(j)} - P_{h_j}\| < \varepsilon/k, \tag{2}$$
where
$$Z'_j = \bigvee\{h_0 + \alpha_0 z_0, h_1 + \alpha_1 z_1, \ldots, h_{j-1} + \alpha_{j-1} z_{j-1}, h_j + \alpha_j z_j\}.$$
Suppose that $Z'_1 \subset \cdots \subset Z'_{k-1}$ have already been constructed. Set formally $\alpha_k = 0$ and $Z'_k = Z''_k = \bigvee\{h_0 + \alpha_0 z_0, h_1 + \alpha_1 z_1, \ldots, h_{k-1} + \alpha_{k-1} z_{k-1}, h_k\}$. Find $r(k) \in \mathbb{N}$ such that (2) is true also for $j = k$. Then $v = h_k \in Z'_k$. Let
$$\phi_\varepsilon(P_{Z'_1}, \ldots, P_{Z'_k}, P_E) = (P_{Z'_k} P_E P_{Z'_k})^{r(k)} \cdots (P_{Z'_1} P_E P_{Z'_1})^{r(1)}.$$

We have

$$\begin{aligned}
&\|\phi_\varepsilon(P_{Z'_1},\dots,P_{Z'_k},P_E)u-v\| \\
&\qquad \le \|(P_{Z'_k}P_EP_{Z'_k})^{r(k)}\cdots((P_{Z'_1}P_EP_{Z'_1})^{r(1)}-P_{h_1})u\| \\
&\qquad\quad + \|(P_{Z'_k}P_EP_{Z'_k})^{r(k)}\cdots((P_{Z'_2}P_EP_{Z'_2})^{r(2)}-P_{h_2})P_{h_1}u\|+\cdots \\
&\qquad\quad + \|((P_{Z'_k}P_EP_{Z'_k})^{r(k)}-P_{h_k})P_{h_{k-1}}\cdots P_{h_1}u\| + \|P_{h_k}\cdots P_{h_1}u-v\| \\
&\qquad \le \frac{\varepsilon}{k}+\cdots+\frac{\varepsilon}{k}+1-\left(\cos\frac{\pi}{2k}\right)^k < 2\varepsilon.
\end{aligned}$$

(ii) Let $M_0 = M\cap u^\perp$. Let $H_0=(R\vee M_0)^\perp$. Then $u\in H_0$.

Clearly, the construction of (i) can be done in H_0, so we can find $v\in (M\vee R)^\perp$ with $\|v\|=1$ and subspaces $Z'_1\subset\cdots\subset Z'_{k(\varepsilon)}\subset H_0\subset R^\perp$ with $\dim Z'_j=j+1$ for all j such that $v\in Z'_{k(\varepsilon)}$ and

$$\|\phi_\varepsilon(P_{Z'_1},\dots,P_{Z'_k},P_{u\vee v})u-v\|<2\varepsilon.$$

All iterations in $\phi_\varepsilon(P_{Z'_1},\dots,P_{Z'_k},P_{u\vee v})u$ belong to $H_0\subset M_0^\perp$, so we may replace $P_{u\vee v}$ by $P_{M\vee v}$ to get

$$\|\phi(P_{Z'_1},\dots,P_{Z'_k},P_{M\vee v})u-v\| = \|\phi(P_{Z'_1},\dots,P_{Z'_k},P_{u\vee v})u-v\|<2\varepsilon. \ \blacksquare$$

The following two corollaries will come in handy when we will be joining the “90-degree” blocks into one single example.

Corollary 2.2. *Let $\varepsilon>0$ and let $\phi_\varepsilon\in\mathcal{S}_{k(\varepsilon)+1}$ be the element constructed in Lemma 2.1. Then there exists $\gamma_\varepsilon\in(0,\min\{1,\varepsilon\})$ (depending only on ε) with the following property: if $M,R\subset H$ are finite-dimensional subspaces, $u\in M\cap R^\perp$ with $\|u\|=1$, and $w\in R^\perp$ with $\|w\|=1$ and $|\langle u,w\rangle|<\gamma_\varepsilon$, then there exist $v\perp M\vee R\vee w$ with $\|v\|=1$ and subspaces $Z_1\subset\cdots\subset Z_{k(\varepsilon)}\subset(R\vee w)^\perp$ with $\dim Z_j=j+1$ for all j such that $v\in Z_{k(\varepsilon)}$ and*

$$\|\phi_\varepsilon(P_{Z_1},\dots,P_{Z_{k(\varepsilon)}},P_{M\vee v})u-v\|<3\varepsilon.$$

Proof. Suppose that $w\in R^\perp$, $\|w\|=1$ and $|\langle u,w\rangle|$ is small enough (how small will be clear from the proof). Let $k=k(\varepsilon)$ and $v,Z'_1,\dots,Z'_k$ be as in Lemma 2.1(ii) with $v,z_1,\dots,z_n\perp w$. We replace the subspaces Z'_j, $j=1,\dots,k$, by the subspaces

$$Z_j=\bigvee_{i=0}^{j}\{h_i+\alpha_i z_i-\cos(i\xi)\langle u,w\rangle w\},$$

which are orthogonal to w. If $|\langle u,w\rangle|$ is small enough, then $\|P_{Z_j}-P_{Z'_j}\|<\varepsilon/|\phi_\varepsilon|$ for all j, hence

$$\|\phi_\varepsilon(P_{Z_1},\dots,P_{Z_k},P_{M\vee v})-\phi_\varepsilon(P_{Z'_1},\dots,P_{Z'_k},P_{M\vee v})\|<\varepsilon$$

and by the triangle inequality

$$\begin{aligned}\|\phi_\varepsilon(P_{Z_1},\dots,P_{Z_k},P_{M\vee v})u-v\| \\ &\le \|\phi_\varepsilon(P_{Z_1},\dots,P_{Z_k},P_{M\vee v})u-\phi_\varepsilon(P_{Z'_1},\dots,P_{Z'_k},P_{M\vee v})u\| \\ &\quad+\|\phi_\varepsilon(P_{Z'_1},\dots,P_{Z'_k},P_{M\vee v})u-v\|<3\varepsilon.\end{aligned}$$

The exact conditions on $|\langle u,w\rangle|$ depend on $\varepsilon,k,\alpha_1,\dots,\alpha_{k-1}$, where all the parameters are determined by ε. ∎

COROLLARY 2.3. *Let $\varepsilon>0$ and let $k=k(\varepsilon)$. Then $\phi_\varepsilon\in\mathcal{S}_{k+1}$ and $\gamma_\varepsilon>0$ constructed in Corollary 2.2 have the following property: if $R,M\subset H$ are finite-dimensional subspaces, $u\in M\cap R^\perp$, $\|u\|=1$, $u'\in R^\perp$, $\|u-u'\|<\gamma_\varepsilon$ and $u'\perp u'-u$, then there exist $v\perp R\vee M\vee u'$ with $\|v\|=1$ and subspaces $Z_1\subset\cdots\subset Z_k\subset(R\vee(u-u'))^\perp$ with $\dim Z_j=j+1$ for all j such that $v\in Z_k$, $\|\phi_\varepsilon(P_{Z_1},\dots,P_{Z_k},P_{M\vee v})u-v\|<3\varepsilon$ and $u'=P_Xu$, where $X=Z_k\vee u'$.*

Proof. If $u'=u$ then the statement follows from Corollary 2.1. If $u'\neq u$ we set $w=(u'-u)/\|u'-u\|$. Then $\|w\|=1$, and

$$\langle u,w\rangle=\langle u-u',w\rangle=\|u-u'\|<\gamma_\varepsilon.$$

If v and $Z_1,\dots,Z_k\subset(R\vee(u-u'))^\perp$ are constructed as in the proof of Corollary 2.2, then

$$\|\phi_\varepsilon(P_{Z_1},\dots,P_{Z_k},P_{M\vee v})u-v\|<3\varepsilon.$$

Let $X=Z_k\vee u'$. Since $X\perp(u'-u)$, we have $P_Xu=u'$. ∎

Paszkiewicz replaced projections onto an increasing family of n finite-dimensional spaces by projections onto just two spaces: onto the largest space in the family and onto a suitable small variation of it. Again, we modify the proof of his result, so that we can refine it in Lemma 2.5.

LEMMA 2.4. *Let $Z_1\subset\cdots\subset Z_k\subset X\subset H$ be subspaces with $\dim Z_j=j+1$ for $j=1,\dots,k$ and $\dim X=k+2$. Let $\varepsilon,\delta>0$ and $a>0$. Then there exist a subspace $Y\subset H$ and numbers $a<s(k)<s(k-1)<\cdots<s(1)$ such that $X\cap Y=\{0\}$, $\|P_X-P_Y\|<\delta$ and for each $j\in\{1,\dots,k\}$,*

$$\|(P_XP_YP_X)^{s(j)}-P_{Z_j}\|<\varepsilon.$$

Proof. Let $e_0,\dots,e_{k+1}$ be an orthonormal basis in X such that $e_0,e_1\in Z_1$, $e_j\in Z_j\ominus Z_{j-1}$ $(2\le j\le k)$, and $e_{k+1}\in X\ominus Z_k$. Let $w_0,\dots,w_{k+1}$ be orthonormal vectors orthogonal to X. We construct Y as the linear span of the vectors $e_j+\beta_jw_j$, $j\in\{0,\dots,k+1\}$, where $\beta_{k+1}>\cdots>\beta_1=\beta_0>0$ are chosen below.

Note that if Y is constructed in this way, for $m\in\mathbb{N}$ and $j\in\{0,\dots,k+1\}$ we have

$$(P_XP_YP_X)^me_j=\frac{e_j}{(1+\beta_j^2)^m}. \tag{3}$$

Now choose first $\beta_{k+1} > 0$ such that $\|P_{e_{k+1}} - P_{e_{k+1}+\beta_{k+1}w_{k+1}}\| < \delta$. Choose $s(k) > a$ such that $1/(1+\beta_{k+1}^2)^{s(k)} < \varepsilon$.

Inductively choose numbers

$$\beta_k, s(k-1), \beta_{k-1}, s(k-2), \ldots, s(1), \beta_1, \beta_0 = \beta_1$$

such that

$$\begin{aligned}&\beta_{k+1} > \beta_k > \cdots > \beta_1 = \beta_0 > 0,\\ &a < s(k) < s(k-1) < \cdots < s(1),\\ &\frac{1}{(1+\beta_{j+1}^2)^{s(j)}} < \varepsilon \quad \text{and} \quad \left|\frac{1}{(1+\beta_j^2)^{s(j)}} - 1\right| < \varepsilon \qquad \text{for } j = k, \ldots, 1.\end{aligned}$$

If $x = \sum_{i=0}^{k+1} a_i e_i \in X$, then by (3),

$$\begin{aligned}\|(P_X P_Y P_X)^{s(j)} x - P_{Z_j} x\|^2 &= \left\|\sum_{i=0}^{k+1} a_i \frac{e_i}{(1+\beta_i^2)^{s(j)}} - \sum_{i=0}^{j} a_i e_i\right\|^2 \\ &= \sum_{i=0}^{j} a_i^2 \left(1 - \frac{1}{(1+\beta_i^2)^{s(j)}}\right)^2 + \sum_{i=j+1}^{k+1} a_i^2 \frac{1}{(1+\beta_i^2)^{2s(j)}} \\ &\le \varepsilon^2 \sum_{i=0}^{k+1} a_i^2 = \varepsilon^2 \|x\|^2.\end{aligned} \tag{4}$$

For any $z \in H$ we have

$$(P_X P_Y P_X)^{s(j)} z - P_{Z_j} z = (P_X P_Y P_X)^{s(j)} (P_X z) - P_{Z_j}(P_X z),$$

since $Z_j \subset X$. Hence by (4) for $j \in \{1, \ldots, k\}$,

$$\|(P_X P_Y P_X)^{s(j)} - P_{Z_j}\| < \varepsilon.$$

It is easy to see that $\|P_X - P_Y\| = \|P_{e_{k+1}} - P_{e_{k+1}+\beta_{k+1}w_{k+1}}\| < \delta$. ■

The next lemma combines all the technical tools needed for the construction of the example we have developed so far.

LEMMA 2.5. *Let $\varepsilon, \delta > 0$. Let $R, M \subset H$ be finite-dimensional subspaces, and let $u \in M \cap R^\perp$ with $\|u\| = 1$ and $u' \in R^\perp$ with $\|u - u'\| < \gamma_\varepsilon$ and $u' \perp u' - u$. Then there exist $v \perp R \vee M \vee u'$ with $\|v\| = 1$, finite-dimensional subspaces $X, Y \subset R^\perp$ with $X \cap Y = \{0\}$ and $\psi \in \mathcal{S}_3$ such that $P_X u = u'$, $\|P_X - P_Y\| < \delta$ and*

$$\|\psi(P_X, P_Y, P_{M\vee v})u - v\| < 4\varepsilon.$$

Moreover, there exists $v' \in Y$ with $\|v'\| = 1$ such that $P_X v' = cv$ for some $c > 0$, $\|v' - v\| < 2\delta$ and $\{u, u'\} \perp \{v, v'\}$.

Proof. Let $v, Z_1, \ldots, Z_k$ and X be as in Corollary 2.3. Let $e_0, \ldots, e_{k+1}$ be an orthonormal basis in X such that $e_0, e_1 \in Z_1$, $e_j \in Z_j \ominus Z_{j-1}$ $(2 \le j \le k)$,

$e_{k+1} \in X \ominus Z_k$. Let $w_0, \dots, w_{k+1}$ be orthonormal vectors orthogonal to $X \vee R \vee M$. As in the proof of the previous lemma, let $Y = \bigvee\{e_i + \beta_i w_i : 0 \le i \le k+1\}$, where $\delta/(k+2) > \beta_{k+1} > \cdots > \beta_2 > \beta_1 = \beta_0 > 0$ are positive numbers which decrease so rapidly that

$$\|P_{e_{k+1}} - P_{e_{k+1}+\beta_{k+1}w_{k+1}}\| < \delta$$

and so that there exist exponents $s(k) < s(k-1) < \cdots < s(1)$ such that

$$\|(P_X P_Y P_X)^{s(j)} - P_{Z_j}\| < \varepsilon/|\phi_\varepsilon|$$

for $j \in \{1, \dots, k\}$. Then $\|P_X - P_Y\| < \delta$. Set

$$\psi(P_X, P_Y, P_{M\vee v}) = \phi_\varepsilon((P_X P_Y P_X)^{s(1)}, \dots, (P_X P_Y P_X)^{s(k)}, P_{M\vee v}).$$

Then

$$\begin{aligned}\|\psi(P_X, P_Y, P_{M\vee v})u - v\| &\\ \le \|\psi(P_X, P_Y, P_{M\vee v})u &- \phi_\varepsilon(P_{Z_1}, \dots, P_{Z_k}, P_{M\vee v})u\| \\ &+ \|\phi_\varepsilon(P_{Z_1}, \dots, P_{Z_k}, P_{M\vee v})u - v\| < 4\varepsilon.\end{aligned}$$

Let $v = \sum_{i=0}^{k+1} \nu_i e_i$. Set

$$v' = \frac{\sum_{i=0}^{k+1} \nu_i(e_i + \beta_i w_i)}{\|\sum_{i=0}^{k+1} \nu_i(e_i + \beta_i w_i)\|}.$$

Then $v' \in Y$, $\|v'\| = 1$ and $P_X v' = cv$, where $c = \|\sum_{i=0}^{k+1} \nu_i(e_i + \beta_i w_i)\|^{-1}$. Since $1 \le \|\sum_{i=0}^{k+1} \nu_i(e_i + \beta_i w_i)\| \le 1 + \delta$, we have $1 \ge c > 1 - \delta$ and $\|v' - P_X v'\| = c\|\sum_{i=0}^{k+1} \nu_i \beta_i e_i\| < \delta$. Thus $1/\|\sum_{i=0}^{k+1} \nu_i(e_i + \beta_i w_i)\| > 1 - \delta$, and

$$\|v' - v\| \le \|v' - P_X v'\| + \|P_X v' - v\| < 2\delta.$$

It is clear from the construction that $\{u, u'\} \perp \{v, v'\}$. ∎

Clearly, $\lim_{s\to\infty} \|P_X P_Y P_X\|^s = 0$. Moreover, as in the previous lemma, we may require that $s(k) = \min\{s(j) : 1 \le j \le k\}$ be arbitrarily large.

Now we are ready to prove our main result: in an infinite-dimensional Hilbert space the iterates of three orthoprojections do not have to converge in norm.

THEOREM 2.6. *Let H be an infinite-dimensional Hilbert space. There exist three orthogonal projections $P_1, P_2, P_3 \in B(H)$, a vector $z_0 \in H$ and a sequence $k_1, k_2, \ldots \in \{1, 2, 3\}$ such that the sequence $\{z_n\}$ of iterates defined by $z_n = P_{k_n} z_{n-1}$ $(n \in \mathbb{N})$ does not converge in norm.*

Proof. For $n \in \mathbb{N}$ let $\varepsilon_n = 1/2^{n+4}$, and let $\gamma_n = \gamma_{\varepsilon_n}$ be defined as in Corollary 2.2.

Let $u_1 \in H$ with $\|u_1\| = 1$. Set formally $Y_0 = \vee\{u_1\}$ and $X_0 = \{0\}$. Let u_1' be any vector satisfying $\|u_1' - u_1\| < \gamma_1$ and $u_1' \perp u_1' - u_1$. Using

Lemma 2.5 (for $R = \{0\}$ and $M = \vee u_1$), we find $X_1, Y_1 \subset H$, $v_1 \in X_1$ and $\psi_1 \in \mathcal{S}_3$ such that $\|v_1\| = 1$, $v_1 \perp u_1$ and

$$\|\psi_1(P_{X_1}, P_{Y_1}, P_{u_1 \vee v_1})u_1 - v_1\| < 4\varepsilon_1.$$

Let $t_1 \in \mathbb{N}$ satisfy $\|(P_{X_1}P_{Y_1}P_{X_1})^{t_1}\| \le \varepsilon_1$.

Let $v_1' \in Y_1$ satisfy $\|v_1'\| = 1$, $\|v_1' - P_{X_1}v_1'\| \le \|v_1' - v_1\| < \gamma_2$, where $P_{X_1}v_1'$ is a multiple of v_1 and $\{u_1, u_1'\} \perp \{v_1, v_1'\}$.

Set $u_2 = v_1'$, $u_2' = P_{X_1}u_2$ and continue the construction using Lemma 2.5.

If $n \ge 2$ and $X_1, \ldots, X_{n-1}, Y_1, \ldots, Y_{n-1} \subset H$, $u_1, \ldots, u_{n-1}$, $v_1, \ldots, v_{n-1}$, $u_1', \ldots, u_{n-1}'$ and $v_1', \ldots, v_{n-1}'$ have already been constructed, then set $u_n := v_{n-1}'$, $u_n' := P_{X_{n-1}}u_n$ (which is a multiple of v_{n-1}), $M_n = Y_{n-1}$ and $R_n = \bigvee_{j=0}^{n-1}(X_j \vee Y_j) \ominus \vee\{u_n, u_n'\}$. Construct $X_n, Y_n \subset R_n^\perp$, $\psi_n \in \mathcal{S}_3$ and $v_n, v_n' \perp R_n \vee \{u_n, u_n'\}$ as in Lemma 2.5 such that $\|v_n' - P_{X_n}v_n'\| \le \|v_n - v_n'\| < \gamma_{n+1}$ and

$$\|\psi_n(P_{X_n}, P_{Y_n}, P_{Y_{n-1}\vee v_n})u_n - v_n\| < 4\varepsilon_n.$$

Moreover, we require that $\|P_{X_n} - P_{Y_n}\| < \varepsilon_n/|\phi_{\varepsilon_{n-1}}|$ and that any two consecutive occurrences of $P_{Y_{n-1}\vee v_n}$ in $\psi_n(P_{X_n}, P_{Y_n}, P_{Y_{n-1}\vee v_n})$ are separated by $(P_{X_n}P_{Y_n}P_{X_n})^s$ with s so large that $\varepsilon_{n-2}^{s/t_{n-2}} < \varepsilon_n/|\phi_n|$. This is possible according to the remark after the proof of Lemma 2.5; if $n=2$ then this condition is not relevant. Let $t_n \in \mathbb{N}$ satisfy $\|(P_{X_n}P_{Y_n}P_{X_n})^{t_n}\| < \varepsilon_n$. We now continue the construction.

Let $L_n = X_n \vee Y_n \vee u_n$ and $\tilde{L}_n = L_n \ominus \{u_n, u_n', v_n, v_n'\}$. By the construction, $\tilde{L}_n \perp \bigvee_{j=1}^{n-1} L_j$, and if $|n-j| \ge 2$, then $L_n \perp L_j$.

Let further $\tilde{X}_n = \tilde{L}_n \cap X_n = X_n \ominus \vee\{u_n', v_n\}$.

By the construction,

$$\|\psi_n(P_{X_n}, P_{Y_n}, P_{Y_{n-1}\vee v_n})u_n - v_n\| < 4\varepsilon_n.$$

Set $\widehat{X}_n = X_n \vee X_{n-1} \vee X_{n-2} \vee \cdots$, $\widehat{Y}_n = Y_n \vee Y_{n-2} \vee Y_{n-4} \vee \cdots$ and $\widehat{E}_n = v_n \vee Y_{n-1} \vee Y_{n-3} \vee \cdots$.

For each $x \in X_n$ we have $P_{Y_n}x = P_{\widehat{Y}_n}x$ and $P_{Y_{n-1}\vee v_n}x = P_{\widehat{E}_n}x$. Since in the product $\psi_n(P_{X_n}, P_{Y_n}, P_{Y_{n-1}\vee v_n})$, both P_{Y_n} and $P_{Y_{n-1}\vee v_n}$ always follow P_{X_n}, we can replace P_{Y_n} by $P_{\widehat{Y}_n}$, and $P_{Y_{n-1}\vee v_n}$ by $P_{\widehat{E}_n}$ without any change. So we have

$$\|\psi_n(P_{X_n}, P_{\widehat{Y}_n}, P_{\widehat{E}_n})u_n - v_n\| < 4\varepsilon_n. \tag{5}$$

Note that for $n = 1$ we have $\widehat{X}_1 = X_1$ and so we may replace P_{X_1} by $P_{\widehat{X}_1}$ in (5).

Let $n \ge 2$. Note that in ψ_n two consecutive positions of $P_{\widehat{E}_n}$ are separated by $(P_{X_n}P_{\widehat{Y}_n}P_{X_n})^s$ where s satisfies $\varepsilon_{n-2}^{s/t_{n-2}} < \varepsilon_n/|\phi_n|$. For $x \in X_n$ we have $P_{\widehat{E}_n}x = P_{Y_{n-1}\vee v_n}x$ and $P_{\widehat{X}_n}P_{\widehat{E}_n}x = P_{X_n}P_{\widehat{E}_n}x + x' + x''$ for some $x' \in \tilde{X}_{n-1}$ and $x'' \in \vee u_{n-1}'$. Furthermore, $P_{\widehat{Y}_n}x' = 0$. Moreover, for each $y \in L_n$ we have $P_{X_n}y = P_{\hat{X}_n}y$ and $P_{Y_n}y = P_{\hat{Y}_n}y$. Hence

$$\begin{aligned}\|(P_{\widehat{X}_n}P_{\widehat{Y}_n}P_{\widehat{X}_n})^s P_{\widehat{E}_n}x - (P_{X_n}P_{\widehat{Y}_n}P_{X_n})^s P_{\widehat{E}_n}x\| \\ \le \|(P_{\widehat{X}_n}P_{\widehat{Y}_n}P_{\widehat{X}_n})^s x''\| = \|(P_{X_{n-2}}P_{Y_{n-2}}P_{X_{n-2}})^s x''\| < \varepsilon_{n-2}^{s/t_{n-2}} < \varepsilon_n/|\phi_n|.\end{aligned}$$

So

$$\begin{aligned}&\|\psi_n(P_{\widehat{X}_n}, P_{\widehat{Y}_n}, P_{\widehat{E}_n})u_n - v_n\| \\ &\qquad \le \|\psi_n(P_{\widehat{X}_n}, P_{\widehat{Y}_n}, P_{\widehat{E}_n})u_n - \psi_n(P_{X_n}, P_{\widehat{Y}_n}, P_{\widehat{E}_n})u_n\| \\ &\qquad\quad + \|\psi_n(P_{X_n}, P_{\widehat{Y}_n}, P_{\widehat{E}_n})u_n - v_n\| < 5\varepsilon_n.\end{aligned}$$

Let $X = \bigvee_{j=1}^\infty X_j$, $Y_{\mathrm{odd}} = \bigvee_{j=0}^\infty Y_{2j+1}$ and $Y_{\mathrm{even}} = \bigvee_{j=0}^\infty Y_{2j}$. We show that $P_1 = P_X$, $P_2 = P_{Y_{\mathrm{even}}}$, and $P_3 = P_{Y_{\mathrm{odd}}}$ have the desired properties.

Suppose that n is even. All iterations in $\psi_n(P_{\widehat{X}_n}, P_{\widehat{Y}_n}, P_{\widehat{E}_n})u_n$ belong to $\bigvee_{j=1}^n L_j$, so we may replace $P_{\widehat{X}_n}$ by P_X without any change. Thus

$$\|\psi_n(P_X, P_{\widehat{Y}_n}, P_{\widehat{E}_n})u_n - v_n\| < 5\varepsilon_n.$$

Similarly, we may replace $P_{\widehat{Y}_n}$ by $P_{Y_{\mathrm{even}}}$. Thus

$$\|\psi_n(P_X, P_{Y_{\mathrm{even}}}, P_{\widehat{E}_n})u_n - v_n\| < 5\varepsilon_n.$$

Let $\widetilde{E} = \widehat{E}_n \vee X_{n+1} \vee Y_{n+3} \vee Y_{n+5} \vee \cdots$. Then we have $\|P_{\widetilde{E}} - P_{Y_{\mathrm{odd}}}\| = \|P_{X_{n+1}} - P_{Y_{n+1}}\| < \varepsilon_{n+1}/|\varphi_{\varepsilon_n}|$ and

$$\|\psi_n(P_X, P_{Y_{\mathrm{even}}}, P_{\widetilde{E}})u_n - v_n\| < 5\varepsilon_n.$$

So

$$\begin{aligned}&\|\psi_n(P_X, P_{Y_{\mathrm{even}}}, P_{Y_{\mathrm{odd}}})u_n - v_n\| \\ &\qquad \le \|\psi_n(P_X, P_{Y_{\mathrm{even}}}, P_{\widetilde{E}})u_n - v_n\| + \|P_{X_{n+1}} - P_{Y_{n+1}}\| \cdot |\phi_{\varepsilon_n}| < 6\varepsilon_n.\end{aligned}$$

Similarly, for odd n we have

$$\|\psi_n(P_X, P_{Y_{\mathrm{odd}}}, P_{Y_{\mathrm{even}}})u_n - v_n\| < 6\varepsilon_n.$$

Write $A_n = \psi_n(P_X, P_{Y_{\mathrm{even}}}, P_{Y_{\mathrm{odd}}})$ if n is even and $A_n = \psi_n(P_X, P_{Y_{\mathrm{odd}}}, P_{Y_{\mathrm{even}}})$ if n is odd. So $\|A_n u_n - v_n\| < 6\varepsilon_n$ and $\|A_n\| \le 1$ for all n. We have

$$\begin{aligned}\|A_n A_{n-1} \cdots A_1 u_1 - v_n\| &\le \|A_n \cdots A_2(A_1u_1 - v_1)\| \\ &\qquad + \|A_n \cdots A_2(v_1 - u_2)\| + \|A_n \cdots A_2 u_2 - v_n\| \\ &\le 6\varepsilon_1 + \gamma_2 + \|A_n \cdots A_2 u_2 - v_n\| \\ &\le 7\varepsilon_1 + \|A_n \cdots A_2 u_2 - v_n\|\end{aligned}$$

and by induction,

$$\|A_n A_{n-1} \cdots A_1 u_1 - v_n\| \le 7\varepsilon_1 + 7\varepsilon_2 + \cdots + 7\varepsilon_n < 14\varepsilon_1 < 1/2.$$

Since $\{v_n\}$ is an orthonormal sequence, the limit $\lim_{n\to\infty} A_n \cdots A_1 u_1$ does not exist. ■

3. Dimension dependent constant in an extension theorem. Let $\mathscr{L}$ be a family of K closed subspaces of finite dimension or codimension of a Hilbert space H. Let $\{z_n\}$ be a sequence of vectors defined as in (1). It follows from [Pr] that the sequence converges in norm. In [KKM] the following estimate of the rate of convergence was given, sometimes called "condition (K)" (see, e.g., [DR1] and [DR2]).

THEOREM 3.1. *Let $\mathscr{L}$ be a finite family of closed subspaces of ℓ_2 of finite dimension or codimension. Let $\{z_i\}$ be a sequence of projections on the spaces in $\mathscr{L}$ as defined in* (1). *Then for all $j \le k$,*

$$|z_j - z_k|^2 \le c(K,d)(|z_j|^2 - |z_k|^2),$$

where the constant $c(K,d) > 0$ depends on the number K of subspaces and their maximal dimension or codimension d (*for each subspace we choose the one which is finite*) *only. Consequently, the sequence $\{z_i\}$ converges in norm.*

The main tool in [KKM] for proving the above estimate is a Whitney-type extension theorem involving derivatives. Given two points a and b in $\mathbb{R}^d$ with $|b-a| = 1$, there is a differentiable function Φ such that $\Phi(b)-\Phi(a) = 1$, and on K given affine spaces, the derivative of Φ is parallel to these spaces. Moreover, the Lipschitz constant of Φ' depends on K and d only.

THEOREM 3.2. *Let $L_1, \dots, L_K$ be subspaces of $\mathbb{R}^d$ and $\tilde{L}_i$ their affine translates. Let $a, b \in \mathbb{R}^d$ be two points with $|b-a| = 1$. Then there exists a differentiable function $\Phi : \mathbb{R}^d \to \mathbb{R}$ such that*

(i) $\Phi(b) - \Phi(a) = 1$;
(ii) $\Phi'(\tilde{L}_i) \subset L_i$ *for* $i = 1, \dots, K$;
(iii) *the mapping $\Phi' : \mathbb{R}^d \to \mathbb{R}^d$ is Lipschitz with a constant c depending on K and d only.*

The question whether it is possible to choose c independently of the dimension d was left open in [KKM]. According to [KR], if $K = 2$ this is indeed the case.

In view of Theorem 2.6, for $K \ge 3$ the Lipschitz constant c of Φ' *does depend* on the dimension d. If c depended on K only, then according to Theorem 2.8 of [KKM] the rate of convergence as in Theorem 3.1 and hence convergence in norm of $\{z_n\}$ would be available for any K closed subspaces of *any* Hilbert space H. Theorem 2.6 proves that in an infinite-dimensional Hilbert space H this is not always the case.

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References

[AA] I. Amemiya and T. Ando, *Convergence of random products of contractions in Hilbert space*, Acta Sci. Math. (Szeged) 26 (1965), 239–244.

[DR1] J. M. Dye and S. Reich, *On the unrestricted iteration of projections in Hilbert space*, J. Math. Anal. Appl. 156 (1991), 101–119.

[DR2] J. Dye and S. Reich, *Random products of nonexpansive mappings*, in: Pitman Res. Notes Math. Ser. 244, Longman, Harlow, 1992, 106–118.

[Ha] I. Halperin, *The product of projection operators*, Acta Sci. Math. (Szeged) 23 (1962), 96–99.

[H] H. S. Hundal, *An alternating projection that does not converge in norm*, Nonlinear Anal. 57 (2004), 35–61.

[KKM] B. Kirchheim, E. Kopecká and S. Müller, *Monotone curves*, Math. Ann. 351 (2011), 81–94.

[K] E. Kopecká, *Spokes, mirrors and alternating projections*, Nonlinear Anal. 68 (2008), 1759–1764.

[KR] E. Kopecká and S. Reich, *Alternating projections and orthogonal decompositions*, J. Nonlinear Convex Anal. 12 (2011), 155–159.

[MR] E. Matoušková and S. Reich, *The Hundal example revisited*, J. Nonlinear Convex Anal. 4 (2003), 411–427.

[N] J. von Neumann, *On rings of operators. Reduction theory*, Ann. of Math. 50 (1949), 401–485.

[P1] A. Paszkiewicz, *The Amemiya–Ando conjecture falls*, arXiv:1203.3354.

[P2] A. Paszkiewicz, *In the Amemiya–Ando problem 3 is enough*, arXiv:1208.0905.

[Pr] M. Práger, *Über ein Konvergenzprinzip im Hilbertschen Raum*, Czechoslovak Math. J. 10 (1960), 271–282 (in Russian).

[S] M. Sakai, *Strong convergence of infinite products of orthogonal projections in Hilbert space*, Appl. Anal. 59 (1995), 109–120.

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