Complex rotundities and midpoint local uniform rotundity in symmetric spaces of measurable operators

by

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Abstract. We investigate the relationships between strongly extreme, complex extreme, and complex locally uniformly rotund points of the unit ball of a symmetric function space or a symmetric sequence space E, and of the unit ball of the space $E(\mathcal{M}, \tau)$ of τ-measurable operators associated to a semifinite von Neumann algebra $(\mathcal{M}, \tau)$ or of the unit ball in the unitary matrix space C_E. We prove that strongly extreme, complex extreme, and complex locally uniformly rotund points x of the unit ball of the symmetric space $E(\mathcal{M}, \tau)$ inherit these properties from their singular value function $\mu(x)$ in the unit ball of E with additional necessary requirements on x in the case of complex extreme points. We also obtain the full converse statements for the von Neumann algebra $\mathcal{M}$ with a faithful, normal, σ-finite trace τ as well as for the unitary matrix space C_E. Consequently, corresponding results on the global properties such as midpoint local uniform rotundity, complex rotundity and complex local uniform rotundity follow.

Let E be a symmetric sequence space, and let C_E be the unitary matrix space of compact operators acting on Hilbert space, associated with E. One of the points of interest in the theory of unitary matrix spaces is to investigate what properties of the symmetric sequence space E are inherited by the unitary matrix space C_E [22]. It was shown by Arazy in [2] that E is isometrically embedded in C_E, and that the isometry V can be chosen with respect to any compact operator x in such a way that $V(s(x)) = x$, where $s(x) = \{s_n(x)\}_{n=1}^{\infty}$ is the sequence of singular numbers of x. Therefore many geometrical properties of $x \in C_E$ are also satisfied by $s(x) \in E$. In the same paper Arazy showed that $x \in C_E$ is an extreme (resp. smooth, exposed) point of the unit ball in C_E if and only if $s(x)$ is an extreme (resp. smooth,

exposed) point of the unit ball in E. Lifting the uniform rotundity from E to C_E was considered by Tomczak-Jaegermann in [36].

Later on, more general spaces have been considered, namely the symmetric spaces $E(\mathcal{M}, \tau)$ of τ-measurable operators associated to a symmetric Banach function space E and a semifinite von Neumann algebra $(\mathcal{M}, \tau)$. The space $E(\mathcal{M}, \tau)$ consists of all τ-measurable operators x for which the singular value function $\mu(x)$ belongs to E, and it is equipped with the norm $\|x\|_{E(\mathcal{M}, \tau)} = \|\mu(x)\|_E$. With the development of the theory of symmetric spaces of measurable operators, it was also natural to expect the noncommutative space $E(\mathcal{M}, \tau)$ to reflect the properties of the corresponding symmetric function space E. Until now it has been shown that many geometric properties, like rotundity [5], (local) uniform rotundity [6], (uniform) Kadec–Klee property [15, 16], Banach–Saks property [17] and several others lift from E to $E(\mathcal{M}, \tau)$. In particular, Chilin, Krygin and Sukochev [5] characterized the extreme points of the unit ball in $E(\mathcal{M}, \tau)$ in terms of the singular value function $\mu(x)$ in the unit ball of E. Later on in [6] they showed that local uniform rotundity and uniform rotundity are inherited by $E(\mathcal{M}, \tau)$ from E.

The aim of this paper is to investigate complex extreme points ($\mathbb{C}$-extreme points), complex locally uniformly extreme points ($\mathbb{C}$-LUR points), strongly extreme points (MLUR points) of the unit ball of $E(\mathcal{M}, \tau)$ and the associated properties of complex rotundity ($\mathbb{C}$-rotundity), complex local uniform rotundity ($\mathbb{C}$-LUR), and midpoint local uniform rotundity (MLUR) of $E(\mathcal{M}, \tau)$.

The concepts of $\mathbb{C}$-extreme points and $\mathbb{C}$-rotund spaces have been introduced by Thorp and Whitley in [35] in connection with the strong maximum modulus theorem for vector-valued analytic functions. Its liaison to holomorphic spaces has been further confirmed by Globevnik's work in [21] who investigated complex uniformly rotund spaces and showed among other things that peak points of the ball algebra over a Banach space X are complex extreme points of its unit ball B_X. Along the same lines are for instance the recent results in [1]. The complex geometric properties also found other applications, for instance in studying the (local) geometry of Banach spaces [10]. Moreover, as has recently been observed [23], $\mathbb{C}$-extreme points and $\mathbb{C}$-rotundity of a complex Banach lattice E are equivalent to upper monotone points and strict monotonicity of its real part E_r, respectively. That observation can be very useful in studying complex properties in Banach lattices and we will apply it later in this paper.

In 2000, Wang and Teng [37] defined $\mathbb{C}$-LUR points and $\mathbb{C}$-LUR spaces and obtained criteria for this property in the class of Musielak–Orlicz spaces of vector-valued functions.

The MLUR property in real Banach spaces, its role and relations to other geometric properties are thoroughly discussed in [29].

Observe that complex uniform rotundity in noncommutative spaces has been investigated in [9, 38], where in particular it was shown that the trace class $S_1 = C_{L_1}$ is complex uniformly rotund [9], and that $E(\mathcal{M}, \tau)$ is complex uniformly convexifiable if and only if E has some finite concavity [38].

In Section 1 we provide the terminology and some facts, especially in noncommutative spaces, to be used later.

In Section 2 we prove that if x is order continuous and E is a fully symmetric function space then x is a MLUR point of the unit ball in $E(\mathcal{M}, \tau)$ whenever $\mu(x)$ is a MLUR point of the unit ball in E. Furthermore, under the assumption that E is a symmetric function space and the von Neumann algebra $\mathcal{M}$ has a σ-finite trace we show that if x is a MLUR point then so is $\mu(x)$. Consequently, the MLUR property of a fully symmetric space E lifts to $E(\mathcal{M}, \tau)$, and vice versa, E inherits this property from $E(\mathcal{M}, \tau)$ under the additional assumption that the trace τ on $\mathcal{M}$ is σ-finite.

Section 3 is devoted to $\mathbb{C}$-extreme points and $\mathbb{C}$-rotundity. We provide a characterization of $\mathbb{C}$-extreme points of the unit ball in $E(\mathcal{M}, \tau)$, where E is a symmetric function space, analogous to that of extreme points obtained by Chilin, Krygin and Sukochev in [5, 18]. As a consequence we also find that $\mathbb{C}$-rotundity is inherited by $E(\mathcal{M}, \tau)$ from E, and we have the opposite implication under the same σ-finiteness assumption on the trace τ as for the MLUR property.

In Section 4 we prove analogous results for $\mathbb{C}$-LUR points and the $\mathbb{C}$-LUR property. We show that if x is order continuous and E is a strongly symmetric function space, and if $\mu(x)$ is a $\mathbb{C}$-LUR point of the unit ball in E, then x is a $\mathbb{C}$-LUR point of the unit ball in $E(\mathcal{M}, \tau)$. Therefore if E is order continuous then the $\mathbb{C}$-LUR property lifts from E to $E(\mathcal{M}, \tau)$. The opposite implication holds for a symmetric function space E and a von Neumann algebra $\mathcal{M}$ with a σ-finite trace τ.

Finally in the last section we discuss the definitions of $\mathbb{C}$-MLUR points and $\mathbb{C}$-MLUR spaces that are analogous to those for MLUR points and MLUR spaces in [20] for real Banach spaces. We present several equivalent conditions, and in particular we show that in any complex Banach space these notions are equivalent to $\mathbb{C}$-LUR points and $\mathbb{C}$-LUR spaces, respectively. Therefore $\mathbb{C}$-MLUR and $\mathbb{C}$-LUR are not distinguishable, unlike their real counterparts MLUR and LUR.

1. Preliminaries. As usual, we denote by $\mathbb{C}$, $\mathbb{R}$, and $\mathbb{N}$ the sets of complex, real, and natural numbers, respectively. Given $z \in \mathbb{C}$, we denote by $\overline{z}$, $\operatorname{Re} z$ and $\operatorname{Im} z$ the conjugate of z and the real and imaginary parts of z, respectively.

Let $I = \mathbb{N}$ or $I = [0, \alpha)$, $0 < \alpha \leq \infty$, be equipped with the counting measure or the Lebesgue measure, respectively. We denote by $L^0 = L^0(I)$ the set of all complex-valued measurable functions on I. We say that $(E, \|\cdot\|_E)$ is a *Banach lattice* if $E = E(I) \subset L^0(I)$ is a Banach space with respect to the norm $\|\cdot\|_E$ such that whenever $0 \leq f \leq g$, $f \in L^0(I)$ and $g \in E$, then $f \in E$ and $\|f\|_E \leq \|g\|_E$. When $I = \mathbb{N}$, E is called a *Banach sequence lattice*, and when $I = [0, \alpha)$, it is often called a *Banach function lattice*.

For a Banach lattice E over the field of complex numbers we define its *real part* as
$$E_r = \{f \in E : \operatorname{Im}(f) = 0\},$$
with the norm induced from E. Denote by E^+ the cone of positive elements in E, and by S_{E^+} and B_{E^+} the unit sphere and the unit ball of positive elements of E, respectively. Recall that a Banach lattice E satisfies the *Fatou property* whenever for any $f_n \in E$ and $f \in L^0$, if $f_n \uparrow f$ a.e. and $\sup_n \|f_n\|_E < \infty$ then $f \in E$ and $\|f_n\|_E \uparrow \|f\|_E$. An element $f \in E$ is called *order continuous* if for every $0 \leq f_n \leq |f|$ such that $f_n \downarrow 0$ a.e. we have $\|f_n\|_E \downarrow 0$. We say that E is *order continuous* if every element in E is order continuous.

Given $f \in L^0[0, \alpha)$, $0 < \alpha \leq \infty$, the *distribution function* d_f of the function f is given by
$$d_f(\lambda) = m\{t > 0 : |f(t)| > \lambda\}, \quad \lambda \geq 0,$$
and its *decreasing rearrangement* $\mu_t(f)$ is defined as
$$\mu_t(f) = \inf\{s > 0 : d_f(s) \leq t\}, \quad t \geq 0.$$
We denote by $\mu(f)$ the function $t \mapsto \mu_t(f)$, $t \geq 0$, and set $\mu_\infty(f) = \lim_{t \to \infty} \mu_t(f)$. The symbol E_0 will stand for the space of all elements f of E for which $\mu_\infty(f) = 0$.

We say that $(E, \|\cdot\|_E)$ is a *symmetric (function) space* on $[0, \alpha)$ if E is a Banach function lattice on $[0, \alpha)$ and $\|f\|_E = \|\mu(f)\|_E$ for all $f \in E$. Analogously, when E is a Banach sequence lattice, E is called a *symmetric (sequence) space* whenever $\|\mu(x)\|_E = \|x\|_E$, where $\mu(x)$ is the decreasing rearrangement of the sequence $x \in E$.

We say that f is *submajorized* by g, and write $f \prec g$, whenever
$$\int\limits_0^s \mu_t(f)\,dt \leq \int\limits_0^s \mu_t(g)\,dt \quad \text{for every } s > 0.$$

A symmetric space E is *strongly symmetric* if for $f, g \in E$, if $f \prec g$ then $\|f\|_E \leq \|g\|_E$. If a symmetric space E satisfies the Fatou property or E is order continuous, then E is *fully symmetric*, that is, if $f \in L^0$, $g \in E$ and $f \prec g$ then $f \in E$ and $\|f\|_E \leq \|g\|_E$. For this and all other facts and definitions concerning Banach lattices we refer to monographs [3, 18, 27].

Let H be a complex Hilbert space and $B(H)$ the space of bounded linear operators from H to H. Moreover, we denote by $L(H)$ the space of all linear operators $x : \mathcal{D}(x) \to H$, where $\mathcal{D}(x) \subset H$ is the domain of x. Let $\mathcal{M} \subset B(H)$ be a semifinite von Neumann algebra with a faithful, normal, semifinite trace τ, and let $\mathcal{P}(\mathcal{M})$ be the lattice of all projections in $\mathcal{M}$. If $x \in \mathcal{M}$ then $\|x\|_{\mathcal{M}}$ denotes the operator norm in $B(H)$. A nonzero projection $p \in \mathcal{P}(\mathcal{M})$ is called *minimal* if $q \in \mathcal{P}(\mathcal{M})$ and $q \leq p$ imply that $q = p$ or $q = 0$. The projections p and q are said to be *equivalent* (relative to the von Neumann algebra $\mathcal{M}$), denoted by $p \sim q$, if there exists a partial isometry $v \in \mathcal{M}$ such that $p = v^*v$ and $q = vv^*$. The von Neumann algebra $\mathcal{M}$ is called *nonatomic* if $\mathcal{M}$ contains no minimal projections [34]. A densely defined closed operator x, affiliated with $\mathcal{M}$, is called τ-*measurable* if for each $\epsilon > 0$ there exists $p \in \mathcal{P}(\mathcal{M})$ such that $p(H) \subset \mathcal{D}(x)$ and $\tau(p^\perp) < \epsilon$, where $p^\perp = \mathbf{1} - p$ and $\mathbf{1}$ is the unit element in $\mathcal{M}$. The set $S(\mathcal{M}, \tau)$ of all τ-measurable operators is a $*$-algebra with respect to the strong sum, the strong product, and the adjoint operator. It is well known that in the commutative case the space $S([0, \alpha), m)$ of measurable operators may be identified with

$$\{f \in L^0[0, \alpha) : m([0, \alpha) \setminus A) < \infty \text{ and } f\chi_A \in L_\infty[0, \alpha) \text{ for some } A\},$$

where $\alpha = \tau(\mathbf{1})$. For $x \in S(\mathcal{M}, \tau)$, we denote by $n(x)$ the *null* projection onto $\operatorname{Ker} x$. Then $s(x) = \mathbf{1} - n(x)$ is called the *support* projection, that is, a projection onto $\operatorname{Ker}^\perp x$. Furthermore, if $x = u|x|$ is a polar decomposition of x, then $s(x) = u^*u$ and $s(x^*) = uu^*$. For every subset $X \subset S(\mathcal{M}, \tau)$ we shall denote the set of all self-adjoint (respectively, positive) elements of X by X_h (respectively, X^+). By a *positive operator* x we mean a self-adjoint operator such that $\langle x\xi, \xi \rangle \geq 0$ for all $\xi \in \mathcal{D}(x)$. Hence $S_h(\mathcal{M}, \tau)$ is equipped with the partial order $\geq$, generated by the proper cone $S^+(\mathcal{M}, \tau)$. If $\{x_n\} \subset S_h(\mathcal{M}, \tau)$ is decreasing and $x = \inf_n x_n$ exists, then we write $x_n \downarrow_n x$. For an operator $x \in S(\mathcal{M}, \tau)$, the function $\mu_t(x)$ defined by

$$\mu_t(x) = \inf\{s \geq 0 : \tau(e^{|x|}(s, \infty)) \leq t\}, \quad t \geq 0,$$

where $e^{|x|}(s, \infty)$ is the spectral projection of $|x| = (x^*x)^{1/2}$ corresponding to the interval (s, ∞), is called the *decreasing rearrangement* of x or the *generalized singular value function* of x. The function $\tau(e^{|x|}(s, \infty))$ is called the *spectral distribution* of x. We shall use the notation $\mu(x)$ for the function $\mu_t(x)$, $t \in \mathbb{R}_+$. Observe that in the commutative case, measurable operators are identified with some measurable functions f, and then $\mu(f)$ is the usual decreasing rearrangement of f. Notice that $\mu_t(x) = 0$ for $t \in [\alpha, \infty)$, if $\alpha < \infty$. Put $\mu_\infty(x) = \lim_{t \to \infty} \mu_t(x)$ for $x \in S(\mathcal{M}, \tau)$. Then the set

$$S_0(\mathcal{M}, \tau) = \{x \in S(\mathcal{M}, \tau) : \mu_\infty(x) = 0\}$$

is a $*$-subalgebra in $S(\mathcal{M}, \tau)$.

Let $x, y \in S(\mathcal{M}, \tau)$. We say that x is *submajorized* by y, and write $x \prec y$, whenever $\mu(x) \prec \mu(y)$. It is well known that $\mu(x + y) \prec \mu(x) + \mu(y)$ and $\mu(x) - \mu(y) \prec \mu(x - y)$, for all $x, y \in S(\mathcal{M}, \tau)$.

Recall the following properties of the singular value function. Although the first two properties are certainly well known we sketch their proofs for the sake of completeness.

PROPOSITION 1.1.

(1) *For* $x \in S(\mathcal{M}, \tau)$, $\mu(|x| + \mu_\infty(x)n(x)) = \mu(x)$.

(2) *If* $x \in S(\mathcal{M}, \tau)$ *and* $|x| \geq \mu_\infty(x)s(x)$ *then* $\mu(|x| - \mu_\infty(x)s(x)) = \mu(x) - \mu_\infty(x)$.

(3) [5, Proposition 2.2] *If* $x, y \in S^+(\mathcal{M}, \tau)$, $y \neq 0$ *and* $x \geq \mu_\infty(x)\mathbf{1}$, *then there exists* $s > 0$ *such that* $\mu_s(x) < \mu_s(x + y)$.

(4) [33, Proposition 3] *If* $x, y \in S(\mathcal{M}, \tau)$, $y = y^*$, $x \geq 0$, *then* $\mu_t(x) \leq \mu_t(x + iy)$ *for all* $t > 0$.

(5) [5, Proposition 3.5] *If* $x, y \in S(\mathcal{M}, \tau)$, $y = y^*$, $x \geq \mu_\infty(x)\mathbf{1}$ *and* $\mu(x + iy) = \mu(x)$, *then* $y = 0$.

Proof. (1) For $x \in S(\mathcal{M}, \tau)$, consider the real-valued function $f(t) = t + \mu_\infty(x)\chi_{\{0\}}(t)$, $t \geq 0$. Then, by functional calculus, $f(|x|) = |x| + \mu_\infty(x)n(x) \geq 0$ and for $\lambda > 0$,

$$e^{|x| + \mu_\infty(x)n(x)}(\lambda, \infty) = e^{|x|}(f^{-1}(\lambda, \infty)) = \begin{cases} e^{|x|}(\lambda, \infty) & \text{if } \lambda \geq \mu_\infty(x), \\ e^{|x|}((\lambda, \infty) \cup \{0\}) & \text{if } \lambda < \mu_\infty(x). \end{cases}$$

Since $\mu_\infty(x) = \inf\{\lambda : \tau(e^{|x|}(\lambda, \infty)) < \infty\}$, we have $\tau(e^{|x|}(\lambda, \infty)) = \infty$ for all $\lambda < \mu_\infty(x)$. Hence also $\tau(e^{|x| + \mu_\infty(x)n(x)}(\lambda, \infty)) = \infty$ for $\lambda < \mu_\infty(x)$, and so $\mu_t(|x| + \mu_\infty(x)n(x)) = \mu_t(x)$ for $t > 0$.

(2) Let $x \in S(\mathcal{M}, \tau)$ and $|x| \geq \mu_\infty(x)s(x)$. Consider the function $f(t) = t - \mu_\infty(x)\chi_{(0,\infty)}$. Then $f(|x|) = |x| - \mu_\infty(x)s(x) \geq 0$ and for all $\lambda > 0$,

$$e^{|x| - \mu_\infty(x)s(x)}(\lambda, \infty) = e^{f(|x|)}(\lambda, \infty) = e^{|x|}(f^{-1}(\lambda, \infty)) = e^{|x|}(\lambda + \mu_\infty(x), \infty).$$

Thus $\||x| - \mu_\infty(x)s(x)\| = |x| - \mu_\infty(x)s(x)$, and so

$$\mu_t(|x| - \mu_\infty(x)s(x)) = \inf\{\lambda \geq 0 : \tau(e^{|x| - \mu_\infty(x)s(x)}(\lambda, \infty)) \leq t\}$$
$$= \inf\{\lambda \geq 0 : \tau(e^{|x|}(\lambda + \mu_\infty(x), \infty)) \leq t\}, \quad t > 0.$$

Therefore for all $t > 0$, $\mu_t(|x| - \mu_\infty(x)s(x)) = \mu_t(x) - \mu_\infty(x)$. ∎

Let $(E, \|\cdot\|_E)$ be a symmetric space on $[0, \alpha)$, $\alpha = \tau(\mathbf{1})$. Denote by $E(\mathcal{M}, \tau)$ the set of all $x \in S(\mathcal{M}, \tau)$ such that $\mu(x) \in E$. It has recently been shown [25] in full generality that $\|\mu(x)\|_E$ defines a norm on the symmetric space $E(M, \tau)$ of measurable operators. In addition, if $(E, \|\cdot\|_E)$ is a Banach space then $E(M, \tau)$ equipped with the norm $\|x\|_{E(\mathcal{M}, \tau)} = \|\mu(x)\|_E$ is also a Banach space.

The measure topology in $S(\mathcal{M}, \tau)$ is determined by a fundamental system $\{U(\epsilon, \delta)\}$ of neighborhoods of zero, where

$$U(\epsilon, \delta) = \{x \in S(\mathcal{M}, \tau) : \|xp\|_{\mathcal{M}} \leq \epsilon, \, \tau(p^{\perp}) \leq \delta \text{ for some } p \in \mathcal{P}(\mathcal{M})\}.$$

The notation $x_n \xrightarrow{\tau} x$ will be used to indicate the convergence of the sequence x_n to x in the measure topology generated by the trace τ. It is well known [19, Lemma 3.4] that $x_n \xrightarrow{\tau} x$ if and only if $\mu_t(x_n - x) \to 0$ a.e.

A linear subspace X in $S(\mathcal{M}, \tau)$ provided with a Banach norm $\| \cdot \|_X$ is called a *symmetric space* on $(\mathcal{M}, \tau)$ if it follows from $x \in X$, $y \in S(\mathcal{M}, \tau)$, and $\mu_t(y) \leq \mu_t(x)$ for every $t > 0$ that $y \in X$ and $\|y\|_X \leq \|x\|_X$. A symmetric space X is called *strongly symmetric* if for $x, y \in X$, $y \prec x$ implies $\|y\|_X \leq \|x\|_X$. If from $y \in S(\mathcal{M}, \tau)$, $x \in X$ and $y \prec x$ it follows that $y \in X$ and $\|y\|_X \leq \|x\|_X$ then X is called *fully symmetric*.

An operator $x \in E(\mathcal{M}, \tau)$ is said to be *order continuous* in $E(\mathcal{M}, \tau)$ if for all sequences $\{x_n\} \subset S(\mathcal{M}, \tau)$, whenever $|x| \geq x_n \downarrow 0$ then $\|x_n\|_{E(\mathcal{M}, \tau)} \to 0$. We will denote by $E(\mathcal{M}, \tau)^{\mathrm{oc}}$ the set of all order continuous elements in $E(\mathcal{M}, \tau)$. If $E(\mathcal{M}, \tau)^{\mathrm{oc}} = E(\mathcal{M}, \tau)$ then we say that $E(\mathcal{M}, \tau)$ is an *order continuous space*. It is worth mentioning that $E(\mathcal{M}, \tau)^{\mathrm{oc}}$ is a $*$-ideal of $E(\mathcal{M}, \tau)$ which is norm closed and is contained in $S_0(\mathcal{M}, \tau)$. Furthermore, if x is order continuous then $\|xp_n\|_{E(\mathcal{M}, \tau)} \to 0$ for every sequence $\{p_n\} \subset \mathcal{P}(\mathcal{M})$ satisfying $p_n \downarrow_n 0$. For more details we refer the reader to [18].

It is well known that if E is a strongly [fully] symmetric space, then $E(\mathcal{M}, \tau)$ is a strongly [fully] symmetric space on $(\mathcal{M}, \tau)$. Moreover, if E satisfies the Fatou property (or is order continuous), then so is $E(\mathcal{M}, \tau)$. In particular, if E is a symmetric space on $[0, \alpha)$ which is order continuous or satisfies the Fatou property, then it is fully symmetric, and therefore $E(\mathcal{M}, \tau)$ is fully symmetric. Also note that every order continuous symmetric space on $(\mathcal{M}, \tau)$ is embedded in $S_0(\mathcal{M}, \tau)$ [7]. If $\mathcal{M}$ is a nonatomic von Neumann algebra then for any symmetric space $E(\mathcal{M}, \tau)$ we have $L_1(\mathcal{M}, \tau) \cap \mathcal{M} \subset E(\mathcal{M}, \tau) \subset L_1(\mathcal{M}, \tau) + \mathcal{M}$.

For the theory of operator algebras we refer to [24, 34], for noncommutative Banach function spaces to [30, 18, 13, 12, 11], and for unitary matrix spaces to [22].

Let $E \subset c_0$ be a symmetric sequence space, and C_E be the *unitary matrix space of compact operators* on the Hilbert space H associated with E. Recall that C_E is a subspace of the Banach space of compact operators $K(H) \subset B(H)$ for which the sequence $s(x) = \{s_n(x)\}_{n=1}^{\infty}$ of singular numbers is in E, and it is equipped with the norm $\|x\|_{C_E} = \|s(x)\|_E$. Let G be the set of all real functions $f \in L^1(0, \infty) + L^{\infty}(0, \infty)$ such that

$$\pi(f) = \Big\{ \int_{n-1}^{n} \mu_t(f) \, dt \Big\}_{n=1}^{\infty} \in E,$$

and set $\|f\|_G = \|\pi(f)\|_E$. If E is order continuous then $(G, \|\cdot\|_G)$ is an order continuous symmetric function space on $[0, \infty)$ [7, Proposition 6.1]. It is well known that $S(B(H), \mathrm{tr}) = B(H)$, where tr is the canonical trace on $B(H)$, and the convergence $x_n \xrightarrow{\mathrm{tr}} x$ is equivalent to the norm convergence $\|x - x_n\|_{B(H)} \to 0$, for $x, x_n \in B(H)$. Since $E \subset c_0$, we have $E \neq \ell^\infty$. Then the symmetric space $G(B(H), \mathrm{tr})$ of measurable operators is a proper two-sided $*$-ideal in $B(H)$ and therefore it is contained in $K(H)$. Thus for any $x \in G(B(H), \mathrm{tr})$ the singular value function $\mu(x)$ is of the form $\mu_t(x) = \sum_{n=1}^\infty s_n(x)\chi_{[n-1,n)}(t)$, $t \geq 0$. Therefore the spaces C_E and $G(B(H), \mathrm{tr})$ coincide as sets and they have identical norms.

Given any linear operator $x : H \to H$ and a subspace $H_0 \subset H$, we denote by $x|_{H_0}$ the restriction of x to H_0. For a von Neumann algebra $\mathcal{M} \subset B(H)$ and a projection $p \in \mathcal{P}(\mathcal{M})$ define

$$\mathcal{M}_p = \{x_p : x \in \mathcal{M}\},$$

where $x_p = px|_{p(H)}$. It is well known that $\mathcal{M}_p$ is a von Neumann algebra contained in $B(p(H))$, with the unit element p. We define $\tau_p : \mathcal{M}_p^+ \to [0, \infty]$ by setting

$$\tau_p(x_p) = \tau(pxp),$$

where $x \in \mathcal{M}^+$. Thus for any $y \in \mathcal{M}_p^+$, $\tau_p(y) = \tau_p(x_p) = \tau(pxp)$, where $y = px|_{p(H)} = x_p$ for some $x \in \mathcal{M}^+$. It is well known that τ_p is a semifinite, normal, faithful trace on $\mathcal{M}_p^+$, and it is finite if and only if $\tau(p) < \infty$. Furthermore, if $\mathcal{M}$ is nonatomic, then $\mathcal{M}_p$ is also nonatomic. Let

$$pS(\mathcal{M}, \tau)p = \{pxp : x \in S(\mathcal{M}, \tau)\}.$$

Then $pS(\mathcal{M}, \tau)p$ is a $*$-subalgebra of $S(\mathcal{M}, \tau)$ with the unit element p. It is also well known [18, Chapter 3, Section 7] that if $x \in S(\mathcal{M}, \tau)$ then $x_p \in S(\mathcal{M}_p, \tau_p)$, and the map $\Phi_p : x \mapsto x_p$, $x \in pS(\mathcal{M}, \tau)p$, is a unital $*$-isomorphism from $pS(\mathcal{M}, \tau)p$ onto $S(\mathcal{M}_p, \tau_p)$. Moreover for any $x \in S(\mathcal{M}, \tau)$ we have $e^{|x_p|}(s, \infty) = e^{|pxp|}(s, \infty)$, $s \geq 0$, and consequently $\mu^{\tau_p}(x_p) = \mu(pxp)$, where $\mu^{\tau_p}(x_p)$ is the singular value function of x_p computed with respect to the reduced von Neumann algebra $\mathcal{M}_p$ and the trace τ_p.

The following two results will be very useful to show that certain properties of x in $E(\mathcal{M}, \tau)$ are inherited by $\mu(x)$.

PROPOSITION 1.2 ([18, 13]). *Suppose that $\mathcal{M}$ is a nonatomic von Neumann algebra with a faithful, normal, σ-finite trace τ. Let $x \in L_1(\mathcal{M}, \tau) + \mathcal{M}$ and $x \in S_0^+(\mathcal{M}, \tau)$. Then there exist a nonatomic commutative von Neumann subalgebra $\mathcal{N} \subset \mathcal{M} \subset B(H)$ and a $*$-isomorphism V from the $*$-algebra $S([0, \tau(\mathbf{1})), m)$ into the $*$-algebra $S(\mathcal{N}, \tau)$ such that*

$$V(\mu(x)) = x \quad and \quad \mu(V(f)) = \mu(f) \quad for\ all\ f \in S([0, \tau(\mathbf{1})), m).$$

Consequently, the Banach function lattice $E = E[0, \tau(\mathbf{1}))$ is isometrically embedded into $E(\mathcal{M}, \tau)$.

PROPOSITION 1.3. *Suppose that $\mathcal{M}$ is a nonatomic von Neumann algebra with a faithful, normal, σ-finite trace τ. Let $x \in S_0^+(\mathcal{M}, \tau)$ and $p \in \mathcal{P}(\mathcal{M})$ be such that $\tau(p) = \infty$ and $xp = px = x$. Then there exists a $*$-isomorphism W from the $*$-algebra $S([0, \infty), m)$ into the $*$-algebra $pS(\mathcal{M}, \tau)p$ such that*

$$W(\mu(x)) = x \quad and \quad \mu(W(f)) = \mu(f) \quad for\ all\ f \in S([0, \infty), m).$$

Proof. Let $x \in S_0^+(\mathcal{M}, \tau)$ and $xp = px = x$. Then $x_p \in S_0^+(\mathcal{M}_p, \tau_p)$. In fact by the remarks before, $x_p \in S^+(\mathcal{M}_p, \tau_p)$ and $\mu^{\tau_p}(x_p) = \mu(pxp) = \mu(x)$. Hence $\mu_\infty^{\tau_p}(x_p) = \mu_\infty(x) = 0$. Applying now Proposition 1.2 to the element $x_p \in S_0^+(\mathcal{M}_p, \tau_p)$ and in view of $\tau_p(\mathbf{1}_p) = \tau(p) = \infty$, there exists a $*$-isomorphism V from $S([0, \infty), m)$ into $S(\mathcal{M}_p, \tau)$ such that $V(\mu^{\tau_p}(x_p)) = x_p$ and $\mu^{\tau_p}(V(f)) = \mu(f)$ for all $f \in S([0, \infty), m)$. The function $\Psi_p(y_p) = \Phi_p^{-1}(y_p) = pyp$ for $y_p \in S(\mathcal{M}_p, \tau_p)$ is a $*$-isomorphism from $S(\mathcal{M}_p, \tau_p)$ onto $pS(\mathcal{M}, \tau)p$.

Let $W = \Psi_p \circ V$; then W is a $*$-isomorphism from $S([0, \infty), m)$ into $pS(\mathcal{M}, \tau)p$. Since $pxp = x$, $\mu(x) = \mu^{\tau_p}(x_p)$, we have

$$W(\mu(x)) = \Psi_p(V(\mu(x))) = \Psi_p(V(\mu^{\tau_p}(x_p))) = \Psi_p(x_p) = pxp = x.$$

Note that for any $y \in S(\mathcal{M}_p, \tau_p)$, $py = y$ and $y = yp|_{p(H)} = (yp)_p$. Finally, since $V(f) \in S(\mathcal{M}_p, \tau_p)$,

$$\begin{aligned}
\mu(W(f)) &= \mu(\Psi_p(V(f))) = \mu(\Psi_p((V(f)p)_p)) = \mu(pV(f)p) \\
&= \mu(p|V(f)|p) = \mu(p|V(f)|pp) = \mu^{\tau_p}((|V(f)|p)_p) \\
&= \mu^{\tau_p}(|V(f)|) = \mu^{\tau_p}(V(f)) = \mu(f),
\end{aligned}$$

which finishes the proof. ∎

Let $(X, \|\cdot\|)$ be a Banach space over the field of complex numbers. We denote by B_X and S_X the unit ball and the unit sphere in X, respectively. We now define several geometric properties of X which will be studied in this paper.

We say that $x \in S_X$ is a *strongly extreme point* of the unit ball B_X, or *MLUR point* of B_X [29], if for any $\{y_n\}, \{z_n\} \subset B_X$, $\|2x - y_n - z_n\| \to 0$ implies that $\|y_n - z_n\| \to 0$. Equivalently, $x \in S_X$ is a strongly extreme point if for any $\{y_n\} \subset X$, $\|x \pm y_n\| \to 1$ implies $\|y_n\| \to 0$. A Banach space X is called *midpoint locally uniformly rotund (MLUR)* if every element of S_X is a strongly extreme point.

A point x of S_X is said to be a *complex extreme point* ($\mathbb{C}$-*extreme point*) of the unit ball B_X [35] if for every $\lambda \in \mathbb{C}$ with $|\lambda| \leq 1$ and y in X, whenever $x + \lambda y \in B_X$ then $y = 0$. Equivalently, x is a complex extreme point for B_X if whenever $x \pm y, x \pm iy \in B_X$, $y \in B_X$, then $y = 0$. The

space X is said to be *complex rotund* ($\mathbb{C}$-*rotund*) if every element of S_X is a $\mathbb{C}$-extreme point. We denote by $\mathbb{C}$-ext(B_X) the set of all complex extreme points of B_X.

A point $x \in S_X$ is a *point of complex local uniform rotundity* ($\mathbb{C}$-*LUR point*) [37] if for every $\epsilon > 0$ there exists $\delta(x, \epsilon) > 0$ such that

$$\sup_{\lambda = \pm 1, \pm i} \|x + \lambda y\| \geq 1 + \delta(x, \epsilon)$$

for every $y \in X$ satisfying $\|y\| \geq \epsilon$. Equivalently, x is a $\mathbb{C}$-LUR point if from $\|x + \lambda y_n\| \to 1$, $\{y_n\} \subset X$, $\lambda = \pm 1, \pm i$ it follows that $\|y_n\| \to 0$. If every point of S_X is a $\mathbb{C}$-LUR point, then X is called a *complex locally uniformly rotund* ($\mathbb{C}$-*LUR*) *space*.

It is clear that real geometric properties such as uniform rotundity, local uniform rotundity and rotundity imply their complex analogues, that is, complex uniform rotundity, complex local uniform rotundity and complex rotundity, respectively.

There also exist the notions of complex strongly extreme points ($\mathbb{C}$-MLUR points) and $\mathbb{C}$-MLUR spaces. They are defined analogously to MLUR points and MLUR spaces following the idea in [20]. However, as we will show in the last section, the notions of $\mathbb{C}$-LUR points and $\mathbb{C}$-MLUR points coincide in any complex Banach space, and consequently the $\mathbb{C}$-LUR and $\mathbb{C}$-MLUR properties are equivalent. We wish to point out that there exist $\mathbb{C}$-LUR spaces that are not complex uniformly rotund, and complex rotund spaces that are not $\mathbb{C}$-LUR. We will briefly discuss some examples in the last section.

It is well known that monotone properties of Banach lattices are closely related to their complex rotundity properties [23, 28]. The interplay between those properties is an important factor in investigating complex properties of Banach lattices. Let us recall some monotonicity notions employed further in this paper.

A point f in a Banach lattice $(E, \|\cdot\|_E)$ is called *upper monotone*, or *UM*, if for any $g \in E$ the condition $g \geq f$ and $g \neq f$ implies that $\|g\|_E > \|f\|_E$.

An element $0 \leq f \in S_E$ is called *upper locally uniformly monotone*, or *ULUM*, if for any sequence $\{f_n\}$ such that $f \leq f_n$, if $\|f_n\|_E \to 1$ then $\|f_n - f\|_E \to 0$.

The following results relating monotonicity and complex rotundity properties for the Banach lattice E appear very useful in our investigations.

Theorem 1.4 ([23]).

(1) *An element f of a complex Banach lattice E is a $\mathbb{C}$-extreme point if and only if $|f|$ is an UM point in its real part E_r.*

(2) *If f is a $\mathbb{C}$-LUR point of a complex Banach lattice E, then $|f|$ is an ULUM point in its real part E_r.*

We finish this section with an elementary result on Banach spaces.

LEMMA 1.5. *Let $(X, \|\cdot\|)$ be a normed space. If $\{y_n\} \subset X$, $\|x\| = 1$ and $\overline{\lim}_n \|x \pm y_n\| \leq 1$, then $\lim_n \|x \pm y_n\| = 1$.*

Proof. Since $\|x\| = 1$ and $\overline{\lim}_n \|x \pm y_n\| \leq 1$, from the inequality

$$2 = 2\|x\| \leq \|x + y_n\| + \|x - y_n\|$$

it follows that $\overline{\lim}_n \|x \pm y_n\| = 1$. Note that $1 - \|x - y_n\|/2 \leq \|x + y_n\|/2$ and therefore

$$1 - \overline{\lim_n} \|x - y_n\|/2 \leq \underline{\lim_n}(1 - \|x - y_n\|/2) \leq \underline{\lim_n} \|x + y_n\|/2.$$

Since $\overline{\lim}_n \|x \pm y_n\| = 1$, we get $1 \leq \underline{\lim}_n \|x + y_n\| \leq \overline{\lim}_n \|x + y_n\| = 1$, and so $\lim_n \|x + y_n\| = 1$. Similarly one can show that $\lim_n \|x - y_n\| = 1$. ∎

Convention. Throughout the paper, E will stand for a symmetric function space on $[0, \alpha)$, where $\alpha = \tau(\mathbf{1})$. Also the semifinite von Neumann algebra $\mathcal{M}$ is always fixed and has a faithful, normal, semifinite trace τ.

2. Strongly extreme points and midpoint local uniform rotundity. In this section we will discuss strongly extreme points and midpoint local uniform rotundity of $E(\mathcal{M}, \tau)$. We will first show that if $\mu(x)$ is an order continuous, strongly extreme point of the unit ball in E, then x is a strongly extreme point of the unit ball in $E(\mathcal{M}, \tau)$. We shall need the following two lemmas. To ensure that E is an interpolation space between L^1 and L^∞ we need to assume that E is a fully symmetric space [3, 27].

LEMMA 2.1. *Suppose that E is fully symmetric and $\mu(x)$ is a MLUR point of B_E. Then for any $\{y_n\}, \{z_n\} \subset B_{E(\mathcal{M}, \tau)}$, if $\|2x - y_n - z_n\| \to 0$ then $\|\mu(x) - \mu(y_n)\| \to 0$ and $\|\mu(x) - \mu(z_n)\| \to 0$.*

Proof. Suppose that $\|2x - y_n - z_n\|_{E(\mathcal{M}, \tau)} \to 0$, $\{y_n\}, \{z_n\} \subset B_{E(\mathcal{M}, \tau)}$. Note that $\mu(2x) \prec \mu(2x - y_n - z_n) + \mu(y_n) + \mu(z_n)$. By [3, Chapter 3, Theorems 2.10 and 2.12] and [27, Chapter 2, Theorem 4.3], for any $n \in \mathbb{N}$, there exists a substochastic linear operator $T_n : E \to E$ such that

$$T_n \mu(2x - y_n - z_n) + T_n \mu(y_n) + T_n \mu(z_n) = \mu(2x).$$

Therefore

$$\|\mu(2x) - T_n \mu(y_n) - T_n \mu(z_n)\|_E = \|T_n \mu(2x - y_n - z_n)\|_E \leq \|2x - y_n - z_n\|_{E(\mathcal{M}, \tau)},$$

which implies that $\|2\mu(x) - T_n\mu(y_n) - T_n\mu(z_n)\|_E \to 0$. Applying the fact that $\mu(x)$ is a MLUR point of B_E, we get

$$\|\mu(x) - T_n\mu(y_n)\|_E \to 0 \quad \text{and} \quad \|\mu(x) - T_n\mu(z_n)\|_E \to 0.$$

It remains to show that $\|T_n\mu(y_n) - \mu(y_n)\|_E \to 0$ and $\|T_n\mu(z_n) - \mu(z_n)\|_E \to 0$. Define $f_n = T_n\mu(y_n)/2 - \mu(y_n)/2$. For every $n \in \mathbb{N}$,

$$\begin{aligned}
\|\mu(x) - f_n\|_E &= \|\mu(x) - T_n\mu(y_n)/2 + \mu(y_n)/2\|_E \\
&\leq \|\mu(x) - T_n\mu(y_n)\|_E + \|T_n\mu(y_n)\|_E/2 + \|\mu(y_n)\|_E/2 \\
&\leq \|\mu(x) - T_n\mu(y_n)\|_E + 1,
\end{aligned}$$

and in view of $\mu(x) - \frac{1}{2}\mu(y) \prec \mu\left(x - \frac{1}{2}y\right)$,

$$\begin{aligned}
\|\mu(x) + f_n\|_E &= \|\mu(x) + T_n\mu(y_n)/2 - \mu(y_n)/2\|_E \\
&\leq \|\mu(x) - \mu(y_n/2)\|_E + \|T_n\mu(y_n)\|_E/2 \\
&\leq \|x - y_n/2\|_{E(\mathcal{M},\tau)} + \|y_n\|_{E(\mathcal{M},\tau)}/2 \\
&\leq \|x - y_n/2 - z_n/2\|_{E(\mathcal{M},\tau)} + \|y_n\|_{E(\mathcal{M},\tau)}/2 + \|z_n\|_{E(\mathcal{M},\tau)}/2 \\
&\leq \|x - y_n/2 - z_n/2\|_{E(\mathcal{M},\tau)} + 1.
\end{aligned}$$

Consequently,

$$\varlimsup_n \|\mu(x) \pm f_n\|_E \leq 1.$$

Now by Lemma 1.5 we have $\lim_n \|\mu(x) \pm f_n\|_E = 1$. As $\mu(x)$ is a MLUR point, we deduce that $\|f_n\|_E = \frac{1}{2}\|T_n\mu(y_n) - \mu(y_n)\|_E \to 0$. Similarly, one can show that $\|T_n\mu(z_n) - \mu(z_n)\|_E \to 0$. ∎

Let $\mathcal{N} = \{N_f : L^2(0,1) \to L^2(0,1) : f \in L^\infty(0,1)\}$ be a commutative von Neumann algebra with the trace $\eta(N_f) = \int_0^1 f\,dm$, where m is the Lebesgue measure on $[0,1]$. Let $\mathcal{A} = \mathcal{N}\,\overline{\otimes}\,\mathcal{M}$ be the tensor product of the von Neumann algebras $\mathcal{N}$ and $\mathcal{M}$, and $\kappa = \eta \otimes \tau$ be the tensor product of the traces η and τ, that is, $\kappa(N_f \otimes x) = \eta(N_f)\tau(x)$, [24, 34]. It is well known that $\mathcal{A}$ has no atoms.

Let $\mathbb{1}$ be the identity operator in $L^2(0,1)$ and set $\mathbb{C}\mathbb{1} = \{\lambda\mathbb{1} : \lambda \in \mathbb{C}\}$. Let $x \in S(\mathcal{M},\tau)$ and consider a linear subspace D in $L^2(0,1)\otimes H$ generated by the vectors of the form $\zeta \otimes \xi$, where $\zeta \in L^2(0,1)$ and $\xi \in \mathcal{D}(x) \subset H$. For every $\xi = \sum_{i=1}^n \zeta_i \otimes \xi_i \in D$ define $(\mathbb{1} \otimes x)(\xi) = \sum_{i=1}^n \zeta_i \otimes x\xi_i$. The linear operator $\mathbb{1} \otimes x : D \to L^2(0,1) \otimes H$ with domain D is preclosed and its closure $\mathbb{1}\,\overline{\otimes}\,x$ is contained in $S(\mathbb{C}\mathbb{1} \otimes \mathcal{M}, \kappa)$ [18, 31, 32].

The map $\pi : x \mapsto \mathbb{1} \otimes x$, $x \in \mathcal{M}$, is a unital trace-preserving $*$-isomorphism from $\mathcal{M}$ onto the von Neumann subalgebra $\mathbb{C}\mathbb{1} \otimes \mathcal{M}$. Consequently, π extends uniquely to a $*$-isomorphism $\tilde{\pi}$ from $S(\mathcal{M},\tau)$ onto $S(\mathbb{C}\mathbb{1} \otimes \mathcal{M}, \kappa)$. In fact one can show that $\tilde{\pi}(x) = \mathbb{1}\,\overline{\otimes}\,x$.

Moreover $\tilde{\pi}$ preserves the singular value function in the sense that $\tilde{\mu}(\tilde{\pi}(x)) = \mu(x)$, where $\tilde{\mu}(\tilde{\pi}(x))$ is the singular value function of $\tilde{\pi}(x)$ computed with respect to the von Neumann algebra $\mathbb{C}\mathbb{1} \otimes \mathcal{M}$ and the trace κ.

Thus
$$E(\mathbb{C}\mathbb{1} \otimes \mathcal{M}, \kappa) = \{y \in S(\mathbb{C}\mathbb{1} \otimes \mathcal{M}, \kappa) : \tilde{\mu}(y) \in E\}$$
$$= \{\tilde{\pi}(x) : x \in S(\mathcal{M}, \tau) \text{ and } \mu(x) \in E\},$$

where
$$\|\tilde{\pi}(x)\|_{E(\mathbb{C}\mathbb{1} \otimes \mathcal{M}, \kappa)} = \|\tilde{\mu}(\tilde{\pi}(x))\|_E = \|\mu(x)\|_E = \|x\|_{E(\mathcal{M}, \tau)}.$$

Hence $\tilde{\pi}$ is a $*$-isomorphism which is also an isometry from $E(\mathcal{M}, \tau)$ onto $E(\mathbb{C}\mathbb{1} \otimes \mathcal{M}, \kappa)$. We refer the reader to [18] for the details.

With this preparation we can prove the next result for any von Neumann algebra. For nonatomic von Neumann algebras it was shown implicitly in the proof of Theorem 2.1 in [6].

LEMMA 2.2. *For any* $x, x_n \in E_0(\mathcal{M}, \tau)$, $n \in \mathbb{N}$, *we have* $x_n \xrightarrow{\tau} x$ *whenever* $\|\mu(x) - \mu(x_n)\|_E \to 0$ *and* $\|\mu(x) - \mu((x + x_n)/2)\|_E \to 0$.

Proof. Suppose that $x_n \in E_0(\mathcal{M}, \tau)$, $n \in \mathbb{N}$, $\|\mu(x) - \mu(x_n)\|_E \to 0$ and $\|\mu(x) - \mu((x + x_n)/2)\|_E \to 0$. In view of the assumptions and the remarks preceding the lemma,
$$\|\tilde{\mu}(\mathbb{1} \overline{\otimes} x) - \tilde{\mu}(\mathbb{1} \overline{\otimes} x_n)\|_E \to 0$$
and
$$\|\tilde{\mu}(\mathbb{1} \overline{\otimes} x) - \tilde{\mu}((\mathbb{1} \overline{\otimes} x + \mathbb{1} \overline{\otimes} x_n)/2)\|_E = \|\tilde{\mu}(\mathbb{1} \overline{\otimes} x) - \tilde{\mu}(\mathbb{1} \overline{\otimes} ((x + x_n)/2))\|_E \to 0.$$
Since $\mathbb{C}\mathbb{1} \otimes \mathcal{M}$ is nonatomic it follows that $\mathbb{1} \overline{\otimes} x - \mathbb{1} \overline{\otimes} x_n \xrightarrow{\kappa} 0$. The latter is equivalent to the fact that for a.e. $t > 0$, $\tilde{\mu}_t(\mathbb{1} \overline{\otimes} x - \mathbb{1} \overline{\otimes} x_n) \to 0$. Thus for a.e. $t > 0$,
$$\mu_t(x - x_n) = \tilde{\mu}_t(\mathbb{1} \overline{\otimes} (x - x_n)) = \tilde{\mu}_t(\mathbb{1} \overline{\otimes} x - \mathbb{1} \overline{\otimes} x_n) \to 0$$
and so $x - x_n \xrightarrow{\tau} 0$. ∎

Before we state the first main theorem of this section we need a few facts about order continuous elements of $E(\mathcal{M}, \tau)$. The next proposition relates the order continuity of an operator x to the order continuity of its singular value function.

PROPOSITION 2.3. *An operator* $x \in E(\mathcal{M}, \tau)$ *is order continuous if and only if* $\mu(x)$ *is order continuous in* E. *Therefore* $E(\mathcal{M}, \tau)^{\mathrm{oc}} = E^{\mathrm{oc}}(\mathcal{M}, \tau)$.

Proof. Suppose that $\mu(x)$ is order continuous in E and $0 \downarrow_n x_n \le |x|$. Then $\mu_\infty(x) = 0$ and by [14, Lemma 3.5], $\mu_t(x_n) \downarrow_n 0$ for all $t > 0$. Therefore $\|x_n\|_{E(\mathcal{M}, \tau)} = \|\mu(x_n)\|_E \to 0$ and x is an order continuous element of $E(\mathcal{M}, \tau)$.

Let now x be an order continuous element of $E(\mathcal{M}, \tau)$. Hence $\mu_\infty(x) = 0$ and for $p = s(x) \vee s(x^*)$ the trace τ_p on $\mathcal{M}_p$ is σ-finite. It is easy to show that x_p is an order continuous element of $E(\mathcal{M}_p, \tau_p)$. Indeed, let $\{y_n\} \subset E(\mathcal{M}_p, \tau_p)$ be such that $0 \downarrow_n y_n \le |x_p| = |x|_p$. Then for all $n \in \mathbb{N}$, $y_n = (x_n)_p$

for some positive $x_n \in E(\mathcal{M}, \tau)$. Moreover, from $0 \downarrow_n (x_n)_p \le |x|_p$ it follows that $0 \downarrow_n px_np \le p|x|p \le |x|$. As x is order continuous, $\|y_n\|_{E(\mathcal{M}_p, \tau_p)} = \|px_np\|_{E(\mathcal{M}, \tau)} \to 0$, proving that x_p is order continuous. Moreover $\mu^{\tau_p}(x_p) = \mu(pxp) = \mu(x)$, where μ^{τ_p} is the singular value function computed with respect to the trace τ_p and the von Neumann algebra $\mathcal{M}_p$. Therefore we can assume that the trace τ is σ-finite. Consider a $*$-isomorphism $\tilde{\pi}$ from $E(\mathcal{M}, \tau)$ onto $E(\mathbb{C}1 \otimes \mathcal{M}, \kappa)$, where $\mathbb{C}1 \otimes \mathcal{M}$ is a nonatomic von Neumann algebra. It is not difficult to see that x is order continuous in $E(\mathcal{M}, \tau)$ if and only if $\tilde{\pi}(x)$ is order continuous in $E(\mathbb{C}1 \otimes \mathcal{M}, \kappa)$. Since for the singular value function $\tilde{\mu}(\tilde{\pi}(x))$ of $\tilde{\pi}(x)$ computed with respect to the von Neumann algebra $\mathbb{C}1 \otimes \mathcal{M}$ and the trace κ, we have $\tilde{\mu}(\tilde{\pi}(x)) = \mu(x)$, it can be assumed that the von Neumann algebra $\mathcal{M}$ is nonatomic.

Suppose that a.e. $0 \downarrow_n f_n \le \mu(x)$. By Proposition 1.2, there exists a $*$-isomorphism V from $S([0, \tau(\mathbf{1})), m)$ into $S(\mathcal{M}, \tau)$ such that $V(\mu(x)) = |x|$ and $\mu(V(f)) = \mu(f)$ for all $f \in S([0, \tau(\mathbf{1})), m)$. Therefore $0 \downarrow_n V(f_n) \le V(\mu(x)) = |x|$ and since x is order continuous, $\|f_n\|_E = \|V(f_n)\|_{E(\mathcal{M}, \tau)} \to 0$. ∎

The following convergence result is known under the stronger assumption that the whole space E is order continuous [6, Proposition 1.1].

PROPOSITION 2.4. *Let E be strongly symmetric. For an order continuous element $x \in E(\mathcal{M}, \tau)$ and $\{x_n\} \subset E(\mathcal{M}, \tau)$, the following conditions are equivalent:*

  (i) $\|x - x_n\|_{E(\mathcal{M}, \tau)} \to 0$,
  (ii) $\|\mu(x) - \mu(x_n)\|_E \to 0$ *and* $x_n \xrightarrow{\tau} x$.

Proof. Since $\mu(x) - \mu(x_n) \prec \mu(x - x_n)$, $n \in \mathbb{N}$, and the embedding of $E(\mathcal{M}, \tau)$ in $S(\mathcal{M}, \tau)$ is continuous, (i) implies (ii).

Suppose now that (ii) holds. Note that it is enough to show that there exists a subsequence that satisfies (i), since this implies that every subsequence has a subsequence which satisfies (i). If $x_n - x \xrightarrow{\tau} 0$, then by [18, Chapter II, Lemma 5.15], passing to a subsequence of $\{x_n\}$, there exists a sequence $\{p_j\} \subset \mathcal{P}(\mathcal{M})$ satisfying $p_j \uparrow \mathbf{1}$, $\tau(p_j^\perp) \to 0$ as $j \to \infty$ and $\|(x - x_n)p_j\|_{\mathcal{M}} \to 0$ as $n \to \infty$ for all $j \in \mathbb{N}$.

For all projections p in $E(\mathcal{M}, \tau)$ and all $n, j \in \mathbb{N}$ we have

$$(2.1) \quad \|(x - x_n)p\|_{E(\mathcal{M}, \tau)} \le \|(x - x_n)p_jp\|_{E(\mathcal{M}, \tau)} + \|(x - x_n)p_j^\perp p\|_{E(\mathcal{M}, \tau)}$$
$$\le \|(x - x_n)p_j\|_{\mathcal{M}}\|p\|_{E(\mathcal{M}, \tau)} + \|\mu(x)\chi_{[0, \tau(p_j^\perp))} + \mu(x_n)\chi_{[0, \tau(p_j^\perp))}\|_E$$
$$\le \|(x - x_n)p_j\|_{\mathcal{M}}\|p\|_{E(\mathcal{M}, \tau)} + \|\mu(x_n)\chi_{[0, \tau(p_j^\perp))} - \mu(x)\chi_{[0, \tau(p_j^\perp))}\|_E$$
$$+ 2\|\mu(x)\chi_{[0, \tau(p_j^\perp))}\|_E$$
$$\le \|(x - x_n)p_j\|_{\mathcal{M}}\|p\|_{E(\mathcal{M}, \tau)} + \|\mu(x_n) - \mu(x)\|_E + 2\|\mu(x)\chi_{[0, \tau(p_j^\perp))}\|_E.$$

Consider first the case when $x_n \in S_0(\mathcal{M}, \tau)$ for all $n \in \mathbb{N}$. Since x is order continuous, also $x \in S_0(\mathcal{M}, \tau)$. Set $q(x) = s(x) \vee s(x^*)$ and $q(x_n) = s(x_n) \vee s(x_n^*)$, $n \in \mathbb{N}$, and define $p = \vee_{n=1}^{\infty} q(x_n) \vee q(x)$. Then the trace τ_p is σ-finite on the von Neumann algebra $\mathcal{M}_p$. Moreover, for all $n \in \mathbb{N}$, $\|x_p - (x_n)_p\|_{E(\mathcal{M}_p, \tau_p)} = \|p(x - x_n)p\|_{E(\mathcal{M}, \tau)} = \|x - x_n\|_{E(\mathcal{M}, \tau)}$, $\mu^{\tau_p}((x_n)_p) = \mu(x_n)$, $\mu^{\tau_p}(x_p) = \mu(x)$ and $\mu^{\tau_p}(x_p - (x_n)_p) = \mu(x - x_n)$, where μ^{τ_p} is the singular value function computed with respect to the trace τ_p and $\mathcal{M}_p$. Therefore we can assume that the trace τ is σ-finite. Consider the nonatomic von Neumann algebra $\mathbb{C}\mathbb{1} \otimes \mathcal{M}$ and the $*$-isomorphism $\tilde{\pi} : x \mapsto \mathbb{1} \overline{\otimes} x$ from $S(\mathcal{M}, \tau)$ onto $S(\mathbb{C}\mathbb{1} \otimes \mathcal{M}, \kappa)$. Since $\tilde{\pi}$ preserves the singular value function, it can also be assumed that the von Neumann algebra $\mathcal{M}$ is nonatomic.

Let $\{e_k\} \subset \mathcal{P}(\mathcal{M})$ be such that $e_k \uparrow_k \mathbb{1}$ and $\tau(e_k) < \infty$ for all $k \in \mathbb{N}$. By Proposition 1.2, for each $n \in \mathbb{N}$ there exists a $*$-isomorphism V_n from $S([0, \tau(\mathbb{1})), m)$ into $S(\mathcal{M}, \tau)$ such that $V_n(\mu(x_n)) = |x_n|$ and $\mu(V_n(f)) = \mu(f)$ for all $f \in S([0, \tau(\mathbb{1})), m)$. Let $q_{nk} = V_n(\chi_{[0, \tau(e_k))})$, $n, k \in \mathbb{N}$. Clearly, each q_{nk} is a projection with $\tau(q_{nk}) = \tau(e_k) < \infty$, and therefore $q_{nk} \in E(\mathcal{M}, \tau)$. Since $|x_n q_{nk}^\perp| = ||x_n| q_{nk}^\perp|$ and $q_{nk}^\perp = \mathbb{1} - q_{nk} = V_n(\chi_{[0, \tau(\mathbb{1}))}) - V_n(\chi_{[0, \tau(e_k))}) = V_n(\chi_{[\tau(e_k), \tau(\mathbb{1}))})$, for all $n, k \in \mathbb{N}$ we have

$$
\begin{aligned}
(2.2) \quad \|x_n q_{nk}^\perp\|_{E(\mathcal{M}, \tau)} &= \||x_n| q_{nk}^\perp\|_{E(\mathcal{M}, \tau)} \\
&= \|V_n(\mu(x_n))V_n(\chi_{[\tau(e_k), \tau(\mathbb{1}))})\|_{E(\mathcal{M}, \tau)} = \|\mu(x_n)\chi_{[\tau(e_k), \tau(\mathbb{1}))}\|_E \\
&\leq \|\mu(x_n)\chi_{[\tau(e_k), \tau(\mathbb{1}))} - \mu(x)\chi_{[\tau(e_k), \tau(\mathbb{1}))}\|_E \|\mu(x)\chi_{[\tau(e_k), \tau(\mathbb{1}))}\|_E \\
&\leq \|\mu(x_n) - \mu(x)\|_E + \|\mu(x)\chi_{[\tau(e_k), \tau(\mathbb{1}))}\|_E.
\end{aligned}
$$

Let $\epsilon > 0$. Since x and $\mu(x)$ are order continuous by Proposition 2.3, $\|x e_k^\perp\|_{E(\mathcal{M}, \tau)} \to 0$, $\|\mu(x)\chi_{[\tau(e_k), \tau(\mathbb{1}))}\|_E \to 0$ and $\|\mu(x)\chi_{[0, \tau(p_j^\perp))}\|_E \to 0$ as $k, j \to \infty$. Therefore there exist $k_0, j_0 \in \mathbb{N}$ such that $\|x e_{k_0}^\perp\|_{E(\mathcal{M}, \tau)} \leq \epsilon/5$, $\|\mu(x)\chi_{[\tau(e_{k_0}), \tau(\mathbb{1}))}\|_E \leq \epsilon/5$ and $\|\mu(x)\chi_{[0, \tau(p_{j_0}^\perp))}\|_E \leq \epsilon/20$. Observe that $\|q_{nk}\|_{E(\mathcal{M}, \tau)} = \|V_n(\chi_{[0, \tau(e_k))})\|_{E(\mathcal{M}, \tau)} = \|\chi_{[0, \tau(e_k))}\|_E = \|e_k\|_{E(\mathcal{M}, \tau)}$ and $\mathbb{1} = q_{nk} + q_{nk}^\perp = q_{nk} + q_{nk}^\perp(e_k + e_k^\perp) = q_{nk} + q_{nk}^\perp e_k + q_{nk}^\perp e_k^\perp$, $n, k \in \mathbb{N}$. This, combined with inequalities (2.1) and (2.2), implies that for all $n \in \mathbb{N}$,

$$
\begin{aligned}
\|x - x_n\|_{E(\mathcal{M}, \tau)} &\leq \|(x - x_n)q_{nk_0}\|_{E(\mathcal{M}, \tau)} + \|(x - x_n)q_{nk_0}^\perp e_{k_0}\|_{E(\mathcal{M}, \tau)} \\
&\quad + \|x q_{nk_0}^\perp e_{k_0}^\perp\|_{E(\mathcal{M}, \tau)} + \|x_n q_{nk_0}^\perp e_{k_0}^\perp\|_{E(\mathcal{M}, \tau)} \\
&\leq \|(x - x_n)q_{nk_0}\|_{E(\mathcal{M}, \tau)} + \|(x - x_n)e_{k_0}\|_{E(\mathcal{M}, \tau)} \\
&\quad + \|x e_{k_0}^\perp\|_{E(\mathcal{M}, \tau)} + \|x_n q_{nk_0}^\perp\|_{E(\mathcal{M}, \tau)} \\
&\leq 2\|(x - x_n)p_{j_0}\|_{\mathcal{M}} \|e_{k_0}\|_{E(\mathcal{M}, \tau)} + 3\|\mu(x_n) - \mu(x)\|_E + 3\epsilon/5.
\end{aligned}
$$

Since $\|(x - x_n)p_{j_0}\|_{\mathcal{M}} \to 0$ and $\|\mu(x_n) - \mu(x)\|_E \to 0$ as $n \to \infty$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $\|(x - x_n)p_{j_0}\|_{\mathcal{M}} \leq \epsilon/(10\|e_{k_0}\|_{E(\mathcal{M}, \tau)})$

and $\|\mu(x_n) - \mu(x)\|_E \leq \epsilon/15$. Consequently, $\|x - x_n\|_{E(\mathcal{M},\tau)} \leq \epsilon$ for all $n \geq N$.

Suppose now that for some $n \in \mathbb{N}$, $\mu_\infty(x_n) > 0$. Then $\mu(x_n) > c\chi_{[0,\tau(\mathbf{1}))}$ for some constant $c > 0$ and therefore $\mu(\mathbf{1}) = \chi_{[0,\tau(\mathbf{1}))} \in E$, which implies that $\mathbf{1} \in E(\mathcal{M},\tau)$. By (2.1), it follows that

$$\|x - x_n\|_{E(\mathcal{M},\tau)} \leq \|(x - x_n)p_j\|_{\mathcal{M}}\|\mathbf{1}\|_{E(\mathcal{M},\tau)} + \|\mu(x_n) - \mu(x)\|_E$$
$$+ 2\|\mu(x)\chi_{[0,\tau(p_j^\perp))}\|_E$$

for all $n, j \in \mathbb{N}$. Therefore similarly to the previous case, one can show that $\|x - x_n\|_{E(\mathcal{M},\tau)} \to 0$ as $n \to \infty$, and the claim follows. ∎

THEOREM 2.5. *Let E be fully symmetric and x be an order continuous element of $E(\mathcal{M},\tau)$. If the singular value function $\mu(x)$ is a MLUR point of B_{E_0} then x is a MLUR point of $B_{E_0(\mathcal{M},\tau)}$.*

Proof. Let $\mu(x) \in S_{E_0}$ be a MLUR point of B_{E_0}, and suppose that we have $\|2x - y_n - z_n\|_{E(\mathcal{M},\tau)} \to 0$, $\{y_n\}, \{z_n\} \subset B_{E_0(\mathcal{M},\tau)}$. By Lemma 2.1,

$$\|\mu(x) - \mu(y_n)\|_E \to 0 \quad \text{and} \quad \|\mu(x) - \mu(z_n)\|_E \to 0.$$

Also

$$\|2x - (y_n + x)/2 - (z_n + x)/2\|_{E(\mathcal{M},\tau)} = \|x - y_n/2 - z_n/2\|_{E(\mathcal{M},\tau)} \to 0,$$

with $(y_n + x)/2$, $(z_n + x)/2$ in the unit ball of $E_0(\mathcal{M},\tau)$. Again, referring to Lemma 2.1 we get

$$\|\mu(x) - \mu((y_n + x)/2)\|_E \to 0 \quad \text{and} \quad \|\mu(x) - \mu((z_n + x)/2)\|_E \to 0.$$

Now, Lemma 2.2 yields $y_n \xrightarrow{\tau} x$ and $z_n \xrightarrow{\tau} x$. Thus $\|x - y_n\|_{E(\mathcal{M},\tau)} \to 0$ and $\|x - z_n\|_{E(\mathcal{M},\tau)} \to 0$ by Proposition 2.4, and x is a MLUR point of $B_{E_0(\mathcal{M},\tau)}$. ∎

Next, we want to establish that if $x \in S_{E(\mathcal{M},\tau)}$ is strongly extreme, then $\mu(x)$ is a strongly extreme point of B_E. We will need the following elementary lemma.

LEMMA 2.6. *Let $x \in S(\mathcal{M},\tau)$. If $n(x) \sim n(x^*)$, then there exists an isometry w such that $x = w|x|$.*

Proof. Suppose that $n(x) \sim n(x^*)$, that is, $n(x) = v^*v$ and $n(x^*) = vv^*$, where v is a partial isometry. Let $x = u|x|$ be a polar decomposition of x, that is, u is a partial isometry with $\operatorname{Ker} u = \operatorname{Ker} x$. Set $w = u + v$. We claim that w is an isometry, that is, $w^*w = \mathbf{1}$. To see this, note first that since $\operatorname{Ker} u^* = \operatorname{Ker} x^*$, $u^*vv^* = u^*n(x^*) = 0$. Thus $|v^*u|^2 = u^*vv^*u = 0$ and $v^*u = 0$. Hence

$$w^*w = (u^* + v^*)(u + v) = u^*u + u^*v + v^*u + v^*v$$
$$= u^*u + (v^*u)^* + v^*u + v^*v = s(x) + n(x) = \mathbf{1},$$

proving that w is an isometry.

Since v^*v is a projection on $\operatorname{Ker}^\perp v$ and $0 = (|x|n(x))^* = n(x)|x| = v^*v|x|$, it follows that $\operatorname{Range}|x| \subset \operatorname{Ker} v$. Consequently, $v|x| = 0$. Therefore

$$w|x| = (u+v)|x| = u|x| + v|x| = u|x| = x,$$

and the proof is complete. ∎

THEOREM 2.7. *Suppose that the trace τ on $\mathcal{M}$ is σ-finite. If x is a MLUR point of $B_{E(\mathcal{M},\tau)}$ then $\mu(x)$ is a MLUR point of B_E and either*

(i) $\mu_\infty(x) = 0$, *or*
(ii) $n(x)\mathcal{M}n(x^*) = 0$ *and* $|x| \geq \mu_\infty(x)s(x)$.

Proof. Recall that there exists a $*$-isomorphism $\tilde{\pi} : x \mapsto \mathbb{1} \overline{\otimes} x$ from $S(\mathcal{M},\tau)$ onto $S(\mathbb{C}\mathbb{1} \otimes \mathcal{M},\kappa)$, which is also an isometry from $E(\mathcal{M},\tau)$ onto $E(\mathbb{C}\mathbb{1} \otimes \mathcal{M},\kappa)$. Hence x is a MLUR point in $E(\mathcal{M},\tau)$ if and only if $\mathbb{1} \overline{\otimes} x$ is a MLUR point in $E(\mathbb{C}\mathbb{1} \otimes \mathcal{M},\kappa)$. Since also $\tilde{\mu}(\mathbb{1} \overline{\otimes} x) = \mu(x)$, where $\tilde{\mu}$ is the singular value function computed with respect to the trace κ and the nonatomic von Neumann algebra $\mathbb{C}\mathbb{1} \otimes \mathcal{M}$, without loss of generality we can assume that the von Neumann algebra $\mathcal{M}$ is nonatomic.

Suppose that x is a MLUR point of the unit ball in $E(\mathcal{M},\tau)$. Since every strongly extreme point is extreme, conditions (i) and (ii) are satisfied by the well known criterion on extreme points in $B_{E(\mathcal{M},\tau)}$ [5]. It remains to show that $\mu(x)$ is a MLUR point of B_E.

Assume first that $\tau(s(x)) = \infty$. By Proposition 1.1(2) and by (ii), we have $\mu(|x|-\mu_\infty(x)s(x)) = \mu(x)-\mu_\infty(x)$. Consequently, $\mu_\infty(|x|-\mu_\infty(x)s(x)) = 0$ and $|x| - \mu_\infty(x)s(x) \in S_0^+(\mathcal{M},\tau)$. Also, in view of $s(x)|x| = |x|s(x)$ we have

$$s(x)(|x| - \mu_\infty(x)s(x)) = (|x| - \mu_\infty(x)s(x))s(x) = |x| - \mu_\infty(x)s(x).$$

Hence Proposition 1.3 applied to the element $|x| - \mu_\infty(x)s(x)$ with $p = s(x)$ yields a $*$-isomorphism W from $S([0,\infty),m)$ into $s(x)S(\mathcal{M},\tau)s(x)$ such that

$$W(\mu(|x|-\mu_\infty(x)s(x))) = |x| - \mu_\infty(x)s(x) \quad \text{and} \quad \mu(W(f)) = \mu(f)$$

for all $f \in S([0,\infty),m)$. As $W(1) = s(x)$, where $1 = \chi_{[0,\infty)}$, it follows that

$$|x| - \mu_\infty(x)s(x) = W(\mu(|x|-\mu_\infty(x)s(x))) = W(\mu(x)-\mu_\infty(x))$$
$$= W(\mu(x)) - \mu_\infty(x)W(1) = W(\mu(x)) - \mu_\infty(x)s(x),$$

and consequently $W(\mu(x)) = |x|$.

Let $\|\mu(x) + \lambda f_n\|_E \to 1$ for $\lambda = \pm 1$, where $f_n \in B_E$, $n \in \mathbb{N}$. Clearly,

$$\lim_n \left\| |x| + \lambda W(f_n) \right\|_{E(\mathcal{M},\tau)} = \lim_n \|W(\mu(x)) + \lambda W(f_n)\|_{E(\mathcal{M},\tau)}$$
$$= \lim_n \|\mu(x) + \lambda f_n\|_E = 1.$$

Let $x = u|x|$ be a polar decomposition of x. Since for $\lambda = \pm 1$,

$$\overline{\lim_n} \|x + \lambda u W(f_n)\|_{E(\mathcal{M},\tau)} = \overline{\lim_n} \|u|x| + \lambda u W(f_n)\|_{E(\mathcal{M},\tau)}$$
$$\leq \lim_n \big\||x| + \lambda W(f_n)\big\|_{E(\mathcal{M},\tau)} = 1$$

and $\|x\|_{E(\mathcal{M},\tau)} = 1$, by Lemma 1.5 we have

$$\lim_n \|x + \lambda u W(f_n)\|_{E(\mathcal{M},\tau)} = 1, \quad \lambda = \pm 1.$$

Using the assumption that x is a strongly extreme point, we get the convergence $\|u W(f_n)\|_{E(\mathcal{M},\tau)} \to 0$. Recall that $u^*u = s(x)$. Hence, also $\|s(x)W(f_n)\|_{E(\mathcal{M},\tau)} \to 0$. But the image of the isomorphism W is contained in $s(x)S(\mathcal{M},\tau)s(x)$, where the unit element is $s(x)$. Therefore $s(x)W(f_n) = W(f_n)$ and consequently

$$\|f_n\|_E = \|\mu(f_n)\|_E = \|\mu(W(f_n))\|_E = \|W(f_n)\|_{E(\mathcal{M},\tau)} \to 0.$$

This concludes the proof in the case when $\tau(s(x)) = \infty$.

Suppose now that $\tau(s(x)) < \infty$. Thus $\mu_\infty(x) = 0$. Let $x = u|x|$ be a polar decomposition of x. Since $s(x) = u^*u$ and $s(x^*) = uu^*$, we have $s(x) \sim s(x^*)$ and $\tau(s(x^*)) = \tau(s(x)) < \infty$. Hence $s(x)$ and $s(x^*)$ are finite, equivalent projections in $\mathcal{M}$ and by [34, Chapter 5, Proposition 1.38], $n(x) \sim n(x^*)$. Therefore by Lemma 2.6 there exists an isometry w such that $x = w|x|$.

Let $\|\mu(x) + \lambda f_n\|_E \to 1$ for $\lambda = \pm 1$, where $f_n \in B_E$, $n \in \mathbb{N}$. Proposition 1.2, applied to the operator $|x|$, implies the existence of a $*$-isomorphism V from $S([0,\alpha), m)$ into $S(\mathcal{M},\tau)$ such that $V(\mu(x)) = |x|$ and $\mu(V(f)) = \mu(f)$ for all $f \in S([0,\alpha), m)$. Note that

$$\lim_n \big\||x| \pm V(f_n)\big\|_{E(\mathcal{M},\tau)} = \lim_n \|V(\mu(x)) \pm V(f_n)\|_{E(\mathcal{M},\tau)}$$
$$= \lim_n \|\mu(x) \pm f_n\|_E = 1.$$

Since w is an isometry, for $\lambda = \pm 1$ we have

$$\lim_n \|x + \lambda w V(f_n)\|_{E(\mathcal{M},\tau)} = \lim_n \big\|w|x| + \lambda w V(f_n)\big\|_{E(\mathcal{M},\tau)}$$
$$= \lim_n \big\||x| + \lambda V(f_n)\big\|_{E(\mathcal{M},\tau)} = 1.$$

Now as x is a MLUR point of $B_{E(\mathcal{M},\tau)}$, we get $\|w V(f_n)\|_{E(\mathcal{M},\tau)} \to 0$. Hence

$$\lim_n \|f_n\|_E = \lim_n \|V(f_n)\|_{E(\mathcal{M},\tau)} = \lim_n \|w V(f_n)\|_{E(\mathcal{M},\tau)} = 0,$$

which proves that $\mu(x)$ is a MLUR point of B_E, and ends the proof. ∎

The next corollary combines the results of Theorems 2.5 and 2.7. It follows immediately from the well known fact that any MLUR space E is order continuous [29].

COROLLARY 2.8. *Let E be a fully symmetric function space on $[0, \alpha)$, $\alpha = \tau(\mathbf{1})$, and $\mathcal{M}$ be a von Neumann algebra with a faithful, normal, σ-finite trace τ.*

(1) *Let x be an order continuous element of $E(\mathcal{M}, \tau)$. Then $\mu(x)$ is a MLUR point of B_{E_0} if and only if x is a MLUR point of $B_{E_0(\mathcal{M},\tau)}$.*

(2) *E is a MLUR space if and only if $E(\mathcal{M}, \tau)$ is a MLUR space.*

Before proving the next theorem, we shall need a version of Lemma 2.1 for a symmetric sequence space E and the unitary matrix space C_E. The proof of the atomic variant of Lemma 2.1 can be conducted in the same way as for symmetric function spaces, replacing singular value functions with sequences of singular numbers.

THEOREM 2.9. *Let $E \subset c_0$ be a fully symmetric sequence space. Then C_E is a MLUR space if and only if E is a MLUR space.*

Proof. Since E is isometrically embedded in C_E [2, Proposition 1.1], if C_E is a MLUR space then so is E. As explained in the preliminary section, $C_E = G(B(H), \text{tr})$ and $\|x\|_{C_E} = \|x\|_{G(B(H),\text{tr})}$ for any compact operator x. Therefore proceeding as in the proof of Theorem 2.5, we can show that if E is a MLUR space then so is C_E. Indeed, let $s(x) = \{s_n(x)\}_{n=1}^{\infty}$ be a MLUR point of B_E, and suppose that $\|2x - y_n - z_n\|_{C_E} \to 0$, where $\{y_n\}, \{z_n\} \subset B_{C_E}$. By Lemma 2.1 for the symmetric sequence space E,

$$\|s(x) - s(y_n)\|_E \to 0, \quad \|s(x) - s((y_n + x)/2)\|_E \to 0,$$
$$\|s(x) - s(z_n)\|_E \to 0, \quad \|s(x) - s((z_n + x)/2)\|_E \to 0.$$

Then, since $\mu(x) = \sum_{i=1}^{\infty} s_i(x)\chi_{[i-1,i)}$ and $\mu(y_n) = \sum_{i=1}^{\infty} s_i(y_n)\chi_{[i-1,i)}$ for all $n \in \mathbb{N}$, we have

$$\|\mu(x) - \mu(y_n)\|_G = \left\|\sum_{i=1}^{\infty}(s_i(x) - s_i(y_n))\chi_{[i-1,i)}\right\|_G = \|s(x) - s(y_n)\|_E \to 0.$$

Similarly, one can show that $\|\mu(x) - \mu((y_n + x)/2)\|_G \to 0$, $\|\mu(x) - \mu(z_n)\|_G \to 0$ and $\|\mu(x) - \mu((z_n + x)/2)\|_G \to 0$. Now, by Lemma 2.2 it follows that $y_n \xrightarrow{\text{tr}} x$ and $z_n \xrightarrow{\text{tr}} x$. Consequently, by Proposition 2.4 applied to the symmetric space $G(B(H), \text{tr})$ it follows that

$$\|x - y_n\|_{C_E} = \|x - y_n\|_{G(B(H),\text{tr})} \to 0 \text{ and } \|x - z_n\|_{C_E} = \|x - z_n\|_{G(B(H),\text{tr})} \to 0,$$

proving that x is a MLUR point of B_{C_E}. Therefore if E is a MLUR space then so is C_E. ∎

3. Complex extreme points and complex rotundity. The main result of this section, Theorem 3.11, gives a criterion for an operator x to be a complex extreme point of $B_{E(\mathcal{M},\tau)}$. This criterion is analogous to the characterization of extreme points obtained in [5, 18]. We will need several

auxiliary results. The first two lemmas describe elementary characteristics of extreme points.

LEMMA 3.1. *Let* $x, y \in B_{E(\mathcal{M}, \tau)}$ *and suppose that* $\mu_t(x) \leq \mu_t(y)$ *for all* $t \in [0, \infty)$. *If there exists* $t_0 > 0$ *such that* $\mu_{t_0}(x) < \mu_{t_0}(y)$ *then* $\mu(x)$ *is not a* $\mathbb{C}$-*extreme point of* B_E.

Proof. By the assumption $\mu(x), \mu(y) \in B_E$ and for all $t > 0$, $\mu_t(x) \leq \mu_t(y)$. Suppose that $\mu(x)$ is $\mathbb{C}$-extreme. Then by Theorem 1.4(1), $\mu(x)$ is an UM point in its real part E_r. If $\mu_{t_0}(x) < \mu_{t_0}(y)$ for some $t_0 > 0$, then by the right continuity of the singular value function, there exists a set A of positive measure such that $\mu_t(x) < \mu_t(y)$ for every $t \in A$. By the upper monotonicity of $\mu(x)$ we get

$$1 = \|\mu(x)\|_E < \|\mu(y)\|_E,$$

contradicting the fact that $\mu(y) \in B_E$. ∎

LEMMA 3.2. *Let* $x \in S(\mathcal{M}, \tau)$ *be a self-adjoint operator. Then* $x \in \mathbb{C}$-$\mathrm{ext}(B_{E_h(\mathcal{M}, \tau)})$ *if and only if* $x \in \mathbb{C}$-$\mathrm{ext}(B_{E(\mathcal{M}, \tau)})$.

Proof. It is enough to show one implication. Suppose that $x \in S_{E(\mathcal{M}, \tau)}$, $x = x^*$ and $x \in \mathbb{C}$-$\mathrm{ext}(()B_{E_h(\mathcal{M}, \tau)})$. Let $x + \lambda y \in B_{E(\mathcal{M}, \tau)}$, $\lambda = \pm 1, \pm i$, where $y \in B_{E(\mathcal{M}, \tau)}$. Setting $y_1 = (y^* + y)/2$ and $y_2 = (y - y^*)/(2i)$, we get $y = y_1 + iy_2$, where both y_1 and y_2 are self-adjoint. Note that for all $\lambda = \pm 1, \pm i$,

$$\begin{aligned}
\|x + \lambda y_1\|_{E(\mathcal{M}, \tau)} &= \|x + \lambda(y + y^*)/2\|_{E(\mathcal{M}, \tau)} \\
&\leq \|x + \lambda y\|_{E(\mathcal{M}, \tau)}/2 + \|x + \lambda y^*\|_{E(\mathcal{M}, \tau)}/2 \\
&= \|x + \lambda y\|_{E(\mathcal{M}, \tau)}/2 + \|x + \overline{\lambda} y\|_{E(\mathcal{M}, \tau)}/2 \leq 1,
\end{aligned}$$

and

$$\begin{aligned}
\|x + \lambda y_2\|_{E(\mathcal{M}, \tau)} &= \|x + \lambda(y - y^*)/(2i)\|_{E(\mathcal{M}, \tau)} \\
&\leq \|x - \lambda iy\|_{E(\mathcal{M}, \tau)}/2 + \|x + \lambda iy^*\|_{E(\mathcal{M}, \tau)}/2 \\
&= \|x - \lambda iy\|_{E(\mathcal{M}, \tau)}/2 + \|x - \overline{\lambda} iy\|_{E(\mathcal{M}, \tau)}/2 \leq 1.
\end{aligned}$$

As x is a $\mathbb{C}$-extreme point, it follows that $y_1 = y_2 = 0$ and so $y = 0$. ∎

LEMMA 3.3. *For any* $x \in S(\mathcal{M}, \tau)$, $n(x)S(\mathcal{M}, \tau)n(x^*) = 0$ *whenever* $n(x)\mathcal{M}n(x^*) = 0$.

Proof. Suppose that $n(x)\mathcal{M}n(x^*) = 0$ and let $y \in S(\mathcal{M}, \tau)$. Recall that if a is a closed linear operator with the domain $\mathcal{D}(a) = H$, then by the Closed Graph Theorem, $a \in B(H)$. Furthermore, if a is a bounded linear operator affiliated with $\mathcal{M}$, that is, $ba = ab$ for all $b \in \mathcal{M}'$, then $a \in (\mathcal{M}')' = \mathcal{M}$. Since $e^{|y|}[0, n](H) \subset \mathcal{D}(|y|) = \mathcal{D}(y)$, we have $ye^{|y|}[0, n] \in B(H)$ and by the τ-measurability of y, $ye^{|y|}[0, n]$ is affiliated with $\mathcal{M}$. Hence for all $n \in \mathbb{N}$,

$ye^{|y|}[0,n] \in \mathcal{M}$ and by assumption $n(x)ye^{|y|}[0,n]n(x^*) = 0$. We will now show that

$$n(x)ye^{|y|}[0,n]n(x^*) \xrightarrow{\tau} n(x)yn(x^*) \quad \text{as } n \to \infty.$$

By the τ-measurability of $|y|$ there exists $n_1 \in \mathbb{N}$ for which $\tau(e^{|y|}(n_1, \infty)) < \infty$. Since

$$e^{|y|}(n_1, \infty) - e^{|y|}(n, \infty) \uparrow e^{|y|}(n_1, \infty),$$

by the normality of trace τ it follows that

$$\tau(e^{|y|}(n_1, \infty) - e^{|y|}(n, \infty)) \to \tau(e^{|y|}(n_1, \infty)),$$

where for $n \geq n_1$,

$$\tau(e^{|y|}(n_1, \infty) - e^{|y|}(n, \infty)) = \tau(e^{|y|}(n_1, \infty)) - \tau(e^{|y|}(n, \infty)).$$

Thus $\tau(e^{|y|}(n, \infty)) \to 0$ as $n \to \infty$. Consequently, for all $t \in \mathbb{R}^+$,

$$\mu_t(n(x)ye^{|y|}[0,n]n(x^*) - n(x)yn(x^*)) = \mu_t(n(x)ye^{|y|}(n, \infty)n(x^*))$$
$$\leq \mu_t(ye^{|y|}(n, \infty)) = \mu_t(y)\chi_{[0,\tau(e^{|y|}(n,\infty)))}(t) \to 0.$$

It follows that $n(x)ye^{|y|}[0,n]n(x^*) \xrightarrow{\tau} n(x)yn(x^*)$ [19, Lemma 3.4], and since $n(x)ye^{|y|}[0,n]n(x^*) = 0$ for all $n \in \mathbb{N}$, the claim follows. ∎

LEMMA 3.4. *Let* $x \in S(\mathcal{M}, \tau)$. *Then* $|x| \geq \mu_\infty(x)s(x)$ *if and only if* $|x^*| \geq \mu_\infty(x)s(x^*)$.

Proof. Suppose that $|x| \geq \mu_\infty(x)s(x)$. Let $x = u|x|$ be a polar decomposition of x. Then $u|x|u^* = |x^*|$ and $|x^*| \geq \mu_\infty(x)us(x)u^*$. Indeed,

$$\langle |x^*|\xi, \xi\rangle = \langle u|x|u^*\xi, \xi\rangle = \langle |x|u^*\xi, u^*\xi\rangle \geq \mu_\infty(x)\langle s(x)u^*\xi, u^*\xi\rangle$$
$$= \mu_\infty(x)\langle us(x)u^*\xi, \xi\rangle$$

for any ξ in the domain of $|x^*|$. Applying now the well known equalities $s(x) = u^*u$ and $s(x^*) = uu^*$, we get $us(x)u^* = uu^*uu^* = s(x^*)$, and so $|x^*| \geq \mu_\infty(x)s(x^*)$.

Conversely, if $|x^*| \geq \mu_\infty(x)s(x^*)$, then by the above argument $|x| = |(x^*)^*| \geq \mu_\infty(x)s(x)$. ∎

We shall need the following results, in particular Corollary 3.6, to prove that x is a complex extreme point whenever $\mu(x)$ is.

LEMMA 3.5. *Let* $x \in S(\mathcal{M}, \tau)$ *and* $x \geq \mu_\infty(x)\mathbf{1}$. *If* $\mu(x) \in \mathbb{C}\text{-ext}(B_E)$ *then* $x \in \mathbb{C}\text{-ext}(B_{E(\mathcal{M}, \tau)})$.

Proof. Suppose that $\mu(x) \in \mathbb{C}\text{-ext}(B_E)$ and $x \pm y$, $x \pm iy$ belong to $B_{E(\mathcal{M}, \tau)}$ for some $y \in B_{E(\mathcal{M}, \tau)}$. In view of Lemma 3.2, we can assume that y is a self-adjoint operator. Now by Proposition 1.1(4), for all $t > 0$,

$$\mu_t(x) \leq \mu_t(x + iy).$$

Since $\mu(x) \in \mathbb{C}\text{-ext}(B_E)$ and $\mu(x + iy) \in B_E$, by Lemma 3.1 it follows that for all $t > 0$,

$$\mu_t(x) = \mu_t(x + iy).$$

Then Proposition 1.1(5) implies that $y = 0$, and the claim follows. ∎

COROLLARY 3.6. *If $\mu(x) \in \mathbb{C}\text{-ext}(B_E)$ and $|x| \geq \mu_\infty(x)s(x)$, then $|x| + \mu_\infty(x)n(x) \in \mathbb{C}\text{-ext}(B_{E(\mathcal{M},\tau)})$. Consequently, if $\mu(x) \in \mathbb{C}\text{-ext}(B_E)$ and $\mu_\infty(x) = 0$, then $|x| \in \mathbb{C}\text{-ext}(B_{E(\mathcal{M},\tau)})$.*

Proof. This follows immediately from Lemma 3.5, since $|x| \geq \mu_\infty(x)s(x)$ implies that $|x| + \mu_\infty(x)n(x) \geq \mu_\infty(x)\mathbf{1}$, and by Proposition 1.1(1), we have $\mu(|x| + \mu_\infty(x)n(x)) = \mu(x)$. ∎

We are now ready for our first main claim in this section.

THEOREM 3.7. *An element $x \in S_{E(\mathcal{M},\tau)}$ is a $\mathbb{C}$-extreme point of $B_{E(\mathcal{M},\tau)}$ whenever $\mu(x)$ is a $\mathbb{C}$-extreme point of B_E and one of the following conditions holds:*

(i) $\mu_\infty(x) = 0$,
(ii) $n(x)\mathcal{M}n(x^*) = 0$ *and* $|x| \geq \mu_\infty(x)s(x)$.

Proof. Suppose that $\mu(x)$ is a $\mathbb{C}$-extreme point and $x \pm y$, $x \pm iy$ belong to $B_{E(\mathcal{M},\tau)}$ for some $y \in B_{E(\mathcal{M},\tau)}$. Let $x = u|x|$ and $x^* = u^*|x^*|$ be polar decompositions of x and x^*, respectively. Since $\operatorname{Ker} u = \operatorname{Ker} x$ and $\operatorname{Ker} u^* = \operatorname{Ker} x^*$, we have $un(x) = u^*n(x^*) = 0$. Hence $x = u(|x| + \mu_\infty(x)n(x))$ and $x^* = u^*(|x^*| + \mu_\infty(x)n(x^*))$. Thus $|x| + \mu_\infty(x)n(x) = u^*x$ and $|x^*| + \mu_\infty(x)n(x^*) = ux^*$, and so

$$|x| + \mu_\infty(x)n(x) + \lambda u^*y, |x^*| + \mu_\infty(x)n(x^*) + \lambda uy^* \in B_{E(\mathcal{M},\tau)}$$

for all $\lambda = \pm 1, \pm i$.

In view of the assumption (i) or (ii) and Lemma 3.4, $|x| \geq \mu_\infty(x)s(x)$ and $|x^*| \geq \mu_\infty(x)s(x^*)$. Since $\mu(x) = \mu(x^*)$ is a $\mathbb{C}$-extreme point, Corollary 3.6 implies that $|x| + \mu_\infty(x)n(x)$ and $|x^*| + \mu_\infty(x)n(x^*)$ are complex extreme points of $B_{E(\mathcal{M},\tau)}$. Therefore $u^*y = uy^* = 0$. Hence $s(x^*)y = uu^*y = 0$ and $ys(x) = yu^*u = 0$, since $(yu^*u)^* = u^*uy^* = 0$. It follows that $y = (s(x^*) + n(x^*))y(s(x) + n(x)) = n(x^*)yn(x)$.

If (ii) is satisfied, then by Lemma 3.3, $n(x^*)S(\mathcal{M},\tau)n(x) = 0$ and consequently $y = n(x^*)yn(x) = 0$.

Consider now the case when (i) holds true, that is, $\mu_\infty(x) = 0$. Then

$$|x + \lambda y|^2 = |x + \lambda n(x^*)yn(x)|^2 = (x + \lambda n(x^*)yn(x))^*(x + \lambda n(x^*)yn(x))$$
$$= (x^* + \overline{\lambda}n(x)y^*n(x^*))(x + \lambda n(x^*)yn(x))$$
$$= x^*x + \lambda x^*n(x^*)yn(x) + \overline{\lambda}n(x)y^*n(x^*)x + n(x)y^*n(x^*)n(x^*)yn(x)$$
$$= |x|^2 + |n(x^*)yn(x)|^2.$$

Also

$$(|x| + |y|)^2 = (|x| + |n(x^*)yn(x)|)^2$$
$$= |x|^2 + |x|\,|n(x^*)yn(x)| + |n(x^*)yn(x)||x| + |n(x^*)yn(x)|^2.$$

Let $n(x^*)yn(x) = v|n(x^*)yn(x)|$ be a polar decomposition of $n(x^*)yn(x)$. Then $|n(x^*)yn(x)| = v^*n(x^*)yn(x) = n(x)y^*n(x^*)v$, and so $|x|\,|n(x^*)yn(x)| = u^*xn(x)y^*n(x^*)v = 0$. Also $|n(x^*)yn(x)|\,|x| = 0$. Hence

$$|x + \lambda y| = |x| + |y| \quad \text{for } \lambda = \pm 1, \pm i,$$

and so $|x| + |y| \in B_{E(\mathcal{M},\tau)}$. Since $\mu(|x| + |y|) \geq \mu(x)$, Lemma 3.1 implies that $\mu(|x|+|y|) = \mu(x)$. By Proposition 1.1(3), if $|y| \neq 0$ then $\mu_s(|x|+|y|) > \mu_s(|x|)$ for some $s > 0$. Thus $|y| = 0$, and consequently $y = 0$. This concludes the proof in the case when $\mu_\infty(x) = 0$. ∎

To show the converse statement we will need the next two lemmas.

LEMMA 3.8. *If x is a $\mathbb{C}$-extreme point of $B_{E(\mathcal{M},\tau)}$ then $|x| \geq \mu_\infty(x)s(x)$.*

Proof. Suppose that $\mu_\infty(x) > 0$. Let $x = u|x|$ be a polar decomposition of x. Fix $0 < \epsilon < 1$ and consider the operators

$$a_\pm = |x| \pm \epsilon|x|e^{|x|}[0,\beta], \qquad b_\pm = |x| \pm i\epsilon|x|e^{|x|}[0,\beta],$$

where $\beta = \frac{1}{1+\epsilon}\mu_\infty(x)$. Clearly,

$$a_- = |x|e^{|x|}(\beta,\infty) + (1-\epsilon)|x|e^{|x|}[0,\beta],$$
$$a_+ = |x|e^{|x|}(\beta,\infty) + (1+\epsilon)|x|e^{|x|}[0,\beta].$$

Hence $0 \leq a_- \leq |x|$, and so $\mu(a_-) \leq \mu(x)$. Consider now the real-valued function $f(t) = (1+\epsilon)t\chi_{[0,\beta]} + t\chi_{(\beta,\infty)}$, $t \geq 0$. We have $a_+ = f(|x|)$, and it follows that

$$e^{a_+}(\lambda,\infty) = e^{|x|}(f^{-1}(\lambda,\infty)) = \begin{cases} e^{|x|}\left(\dfrac{1}{1+\epsilon}\lambda,\infty\right) & \text{if } \lambda < \mu_\infty(x), \\ e^{|x|}(\lambda,\infty) & \text{if } \lambda \geq \mu_\infty(x), \end{cases}$$

for all $\lambda \geq 0$. Since $\mu_\infty(x) = \inf\{s \geq 0 : \tau(e^{|x|}(s,\infty)) < \infty\}$, we have $\tau(e^{a_+}(\lambda,\infty)) = \infty$ for all $\lambda < \mu_\infty(x)$. Thus for any $t > 0$,

$$\mu_t(a_+) = \inf\{\lambda : \tau(e^{a_+}(\lambda,\infty)) \leq t\} = \inf\{\lambda \geq \mu_\infty(x) : \tau(e^{|x|}(\lambda,\infty)) \leq t\}$$
$$= \mu_t(x),$$

which yields $\mu(a_+) = \mu(x)$. Now observe that

$$|b_-|^2 = |b_+|^2 = b_-^*b_- = b_+^*b_+ = |x|^2 + \epsilon^2|x|^2e^{|x|}[0,\beta]$$
$$\leq |x|^2 + \epsilon^2|x|^2e^{|x|}[0,\beta] + 2\epsilon|x|^2e^{|x|}[0,\beta] = (|x| + \epsilon|x|e^{|x|}[0,\beta])^2 = a_+^2.$$

Hence

$$\mu^2(b_-) = \mu^2(b_+) = \mu(|b_+|^2) = \mu(|b_-|^2) \le \mu^2(a_+) = \mu^2(x)$$

and $\mu(b_+) = \mu(b_-) \le \mu(x)$.

Thus $|x| + \lambda\epsilon|x|e^{|x|}[0,\beta] \in B_{E(\mathcal{M},\tau)}$, and so $x + \lambda\epsilon u|x|e^{|x|}[0,\beta] \in B_{E(\mathcal{M},\tau)}$ for all $\lambda = \pm 1, \pm i$. As x is a $\mathbb{C}$-extreme point, $u|x|e^{|x|}[0,\beta] = 0$. But $u^*u|x| = |x|$ and therefore

$$|x|e^{|x|}[0,\beta] = |x|e^{|x|}\left[0, \frac{1}{1+\epsilon}\mu_\infty(x)\right] = 0 \quad \text{for every } 0 < \epsilon < 1.$$

Since ϵ can be arbitrarily small, it follows that $|x|e^{|x|}[0,\mu_\infty(x)) = 0$. Hence $0 = \int_0^{\mu_\infty(x)} \lambda\, de^{|x|}(\lambda)$, which implies that $e^{|x|}(0,\mu_\infty(x)) = 0$. Therefore $s(x) = e^{|x|}(0,\infty) = e^{|x|}[\mu_\infty(x),\infty)$, and finally

$$|x| = \int_{[\mu_\infty(x),\infty)} \lambda\, de^{|x|}(\lambda) \ge \mu_\infty(x)e^{|x|}[\mu_\infty(x),\infty) = \mu_\infty(x)s(x). \quad \blacksquare$$

LEMMA 3.9. *If x is a $\mathbb{C}$-extreme point of $B_{E(\mathcal{M},\tau)}$ then $n(x)\mathcal{M}n(x^*) = 0$ or $\mu_\infty(x) = 0$.*

Proof. Assume for a contradiction that $n(x)\mathcal{M}n(x^*) \ne 0$ and $\mu_\infty(x) > 0$, while x is a complex extreme point. By [34, Chapter 5, Lemma 1.7] there exist nonzero projections $p, q \in \mathcal{P}(\mathcal{M})$ such that $p \le n(x)$, $q \le n(x^*)$ and $p \sim q$, that is, there exists a partial isometry v such that $p = v^*v$ and $q = vv^*$.

Let $x = u|x|$ be a polar decomposition of x. Set $w = u + v$. We claim that w is a partial isometry and $x = w|x|$. Indeed, since $\operatorname{Ker} u^* = \operatorname{Ker} x^*$ we have $u^*n(x^*) = 0$ and $u^*q = 0$. Now $0 = u^*qu = u^*vv^*u = (v^*u)^*(v^*u) = |v^*u|^2$. Therefore $v^*u = 0$ and $u^*v = 0$. Hence

$$|w|^2 = |u+v|^2 = (u^* + v^*)(u+v) = u^*u + u^*v + v^*u + v^*v$$
$$= u^*u + v^*v = s(x) + p,$$

and thus w is a partial isometry, since w^*w is a projection. Now, since $0 = (|x|n(x)p)^* = (|x|p)^* = p|x| = v^*v|x|$ and v^*v is a projection onto $\operatorname{Ker}^\perp v$, it follows that $\operatorname{Range}|x| \subset \operatorname{Ker} v$, and therefore $v|x| = 0$. Hence $x1 = u|x| = u|x| + v|x| = w|x|$.

Note that $\big||x| + \lambda\mu_\infty(x)n(x)\big| = |x| + \mu_\infty(x)n(x)$ for all $\lambda = \pm 1, \pm i$, since $|x|n(x) = n(x)|x| = 0$. Furthermore, by Proposition 1.1(1), we have $\mu(|x| + \mu_\infty(x)n(x)) = \mu(x)$. Hence $|x| + \lambda\mu_\infty(x)n(x) \in B_{E(\mathcal{M},\tau)}$ for all $\lambda = \pm 1, \pm i$. Moreover,

$$\mu(x + \lambda\mu_\infty(x)wn(x)) = \mu(w|x| + \lambda\mu_\infty(x)wn(x))$$
$$\le \mu(|x| + \lambda\mu_\infty(x)n(x)) = \mu(x),$$

which implies that $x + \lambda \mu_\infty(x) wn(x) \in B_{E(\mathcal{M},\tau)}$ for all $\lambda = \pm 1, \pm i$. The assumption that x is a $\mathbb{C}$-extreme point yields $\mu_\infty(x) wn(x) = 0$. But $\mu_\infty(x) > 0$ and therefore $wn(x) = 0$. Since $x = u|x|$, we know that $\operatorname{Ker} u = \operatorname{Ker} x$, and so $un(x) = 0$. Hence $0 = wn(x) = (u + v)n(x) = vn(x)$. But then

$$p = pn(x) = v^* vn(x) = 0,$$

which contradicts the fact that $p \neq 0$. ∎

THEOREM 3.10. *Suppose that the trace τ on $\mathcal{M}$ is σ-finite. If x is a complex extreme point of $B_{E(\mathcal{M},\tau)}$ then $\mu(x)$ is a complex extreme point of B_E and either*

(i) $\mu_\infty(x) = 0$, *or*
(ii) $n(x)\mathcal{M}n(x^*) = 0$ *and* $|x| \geq \mu_\infty(x)s(x)$.

Proof. Suppose that x is a $\mathbb{C}$-extreme point of the unit ball in $E(\mathcal{M},\tau)$. Since $\tilde{\pi} : x \mapsto \mathbb{1} \overline{\otimes} x$ is a $*$-isometry from $E(\mathcal{M},\tau)$ onto $E(\mathbb{C}\mathbb{1} \otimes \mathcal{M}, \kappa)$, clearly $\mathbb{1} \overline{\otimes} x$ is a $\mathbb{C}$-extreme point in $E(\mathbb{C}\mathbb{1} \otimes \mathcal{M}, \kappa)$, where $\mathbb{C}\mathbb{1} \otimes \mathcal{M}$ is a nonatomic von Neumann algebra. Moreover, $\tilde{\pi}$ preserves the singular value function, and therefore we can assume that the von Neumann algebra $\mathcal{M}$ is nonatomic.

Consider first the case $\tau(s(x)) = \infty$. By Lemma 3.8, $|x| \geq \mu_\infty(x)s(x)$. Also $\mu(|x| - \mu_\infty(x)s(x)) = \mu(x) - \mu_\infty(x)$ by Proposition 1.1(2), and consequently $\mu_\infty(|x| - \mu_\infty(x)s(x)) = 0$. Clearly,

$$s(x)(|x| - \mu_\infty(x)s(x)) = (|x| - \mu_\infty(x)s(x))s(x) = |x| - \mu_\infty(x)s(x).$$

Proposition 1.3 applied to the operator $|x| - \mu_\infty(x)s(x)$ and $p = s(x)$ yields a $*$-isomorphism W from $S([0,\infty), m)$ into $s(x)S(\mathcal{M},\tau)s(x)$ such that

$$W(\mu(|x| - \mu_\infty(x)s(x))) = |x| - \mu_\infty(x)s(x) \quad \text{and} \quad \mu(W(f)) = \mu(f)$$

for all $f \in S([0,\infty), m)$. Since $W(1) = s(x)$,

$$|x| - \mu_\infty(x)s(x) = W(\mu(|x| - \mu_\infty(x)s(x))) = W(\mu(x) - \mu_\infty(x))$$
$$= W(\mu(x)) - \mu_\infty(x)W(1) = W(\mu(x)) - \mu_\infty(x)s(x),$$

and consequently $W(\mu(x)) = |x|$.

Let now $\mu(x) + \lambda f \in B_E$ for all $\lambda = \pm 1, \pm i$, where $f \in B_E$. Since W is an isometry, $\big\||x| + \lambda W(f)\big\|_{E(\mathcal{M},\tau)} = \|W(\mu(x)) + \lambda W(f)\|_{E(\mathcal{M},\tau)} \leq 1$ for $\lambda = \pm 1, \pm i$. Let $x = u|x|$ be a polar decomposition of x. Clearly,

$$x + \lambda uW(f) = u|x| + \lambda uW(f) \in B_{E(\mathcal{M},\tau)}, \quad \lambda = \pm 1, \pm i.$$

Since x is a $\mathbb{C}$-extreme point, $uW(f) = 0$. Recall that $u^*u = s(x)$. Hence $s(x)W(f) = 0$. Note also that $W(f) \in s(x)S(\mathcal{M},\tau)s(x)$. Therefore $W(f) = s(x)W(f) = 0$, and consequently $f = 0$.

Consider now the case when $\tau(s(x)) < \infty$, and hence $\mu_\infty(x) = 0$. By Proposition 1.2 applied to $|x|$, there exists a $*$-isomorphism V from

$S([0, \alpha), m)$ into $S(\mathcal{M}, \tau)$ such that $V(\mu(x)) = |x|$ and $\mu(V(f)) = \mu(f)$ for all $f \in S([0, \alpha), m)$. Furthermore, since $s(x) \sim s(x^*)$, we have $\tau(s(x^*)) = \tau(s(x)) < \infty$ and it follows from [34, Chapter 5, Proposition 1.38] that $n(x) \sim n(x^*)$. Hence by Lemma 2.6, there exists an isometry w such that $x = w|x|$.

Let $\mu(x) + \lambda f \in B_E$, $\lambda = \pm 1, \pm i$ and $f \in B_E$. Then

$$\|x + \lambda w V(f)\|_{E(\mathcal{M}, \tau)} = \big\|w|x| + \lambda w V(f)\big\|_{E(\mathcal{M}, \tau)} = \big\||x| + \lambda V(f)\big\|_{E(\mathcal{M}, \tau)}$$
$$= \|V(\mu(x)) + \lambda V(f)\|_{E(\mathcal{M}, \tau)} = \|\mu(x) + \lambda f\|_E \leq 1.$$

Now by the assumption that x is a $\mathbb{C}$-extreme point, $wV(f) = 0$. Since both w and V are injective, $f = 0$. ∎

We finish this section with a complete characterization of $\mathbb{C}$-extreme points in $B_{E(\mathcal{M}, \tau)}$. The first result is an immediate consequence of Theorems 3.7 and 3.10.

THEOREM 3.11. *Let E be a symmetric space on $[0, \alpha)$, $\alpha = \tau(\mathbf{1})$, and $\mathcal{M}$ be a semifinite von Neumann algebra with a faithful, normal, σ-finite trace τ. An operator x is a complex extreme point of $B_{E(\mathcal{M}, \tau)}$ if and only if $\mu(x)$ is a complex extreme point of B_E and one of the following, not mutually exclusive, conditions holds:*

(i) $\mu_\infty(x) = 0$,
(ii) $n(x)\mathcal{M}n(x^*) = 0$ *and* $|x| \geq \mu_\infty(x)s(x)$.

Since $\mu_\infty(x) = 0$ for every $x \in E_0$, we immediately obtain the following corollary.

COROLLARY 3.12. *Let E be a symmetric space on $[0, \alpha)$, $\alpha = \tau(\mathbf{1})$ and $\mathcal{M}$ be a semifinite von Neumann algebra with a faithful, normal, σ-finite trace τ. Then E_0 is complex rotund if and only if $E_0(\mathcal{M}, \tau)$ is complex rotund.*

THEOREM 3.13. *Let $E \subset c_0$ be a symmetric sequence space. Then C_E is complex rotund if and only if E is complex rotund.*

Proof. Since E is isometrically embedded in C_E [2], the claim that E is a complex rotund space if C_E has this property is immediate. As $C_E = G(B(H), \mathrm{tr})$ and $\|x\|_{C_E} = \|x\|_{G(B(H), \mathrm{tr})}$ for any compact operator x, the proof of the converse implication is analogous to the proof of Theorem 3.7. ∎

4. Complex local uniform rotund points and complex local uniform rotundity. In this section we study the relations between the complex local uniform rotundity of the symmetric function space E and the complex local uniform rotundity of the corresponding symmetric space $E(\mathcal{M}, \tau)$ of measurable operators.

THEOREM 4.1. *Let E be strongly symmetric and x be an order continuous element of $E(\mathcal{M}, \tau)$. If $\mu(x)$ is a $\mathbb{C}$-LUR point of B_{E_0} then x is a $\mathbb{C}$-LUR point of $B_{E_0(\mathcal{M}, \tau)}$.*

Proof. Let $x \in S_{E_0(\mathcal{M}, \tau)}$ and suppose that $\mu(x)$ is a $\mathbb{C}$-LUR point of B_{E_0}.

CASE 1. Let $x \geq 0$ and $\{y_n\} \subset B_{E_h(\mathcal{M}, \tau)}$, $\{y_n\} \subset E_0(\mathcal{M}, \tau)$, be such that $\|x + iy_n\|_{E(\mathcal{M}, \tau)} \to 1$. By Proposition 1.1(4), $\mu_t(x + iy_n) \geq \mu_t(x)$ for all $t > 0$. Since $\mu(x)$ is an ULUM point in E_0 by Theorem 1.4(2), we have

$$(4.1) \qquad \|\mu(x) - \mu(x + iy_n)\|_E \to 0.$$

Also $\mu_t(x + iy_n/2) \geq \mu_t(x)$ for all $t > 0$ and $\|x + iy_n/2\|_{E(\mathcal{M}, \tau)} \to 1$. The latter follows from the inequality $1 \leq \|x + iy_n/2\|_{E(\mathcal{M}, \tau)} \leq \|x + iy_n\|_{E(\mathcal{M}, \tau)}/2 + \|x\|_{E(\mathcal{M}, \tau)}/2 \to 1$ as $n \to \infty$. Again, using the fact that $\mu(x)$ is an ULUM point we conclude that

$$(4.2) \qquad \|\mu(x) - \mu(x + iy_n/2)\|_E \to 0.$$

Denote $a_n = x + iy_n$. From (4.2) it follows that $\|\mu(x) - \mu((x + a_n)/2)\|_E \to 0$, and from (4.1) that $\|\mu(x) - \mu(a_n)\|_E \to 0$. Applying now Lemma 2.2 we get $a_n \xrightarrow{\tau} x$. Finally, Proposition 2.4 implies that $\|y_n\|_{E(\mathcal{M}, \tau)} = \|x - a_n\|_{E(\mathcal{M}, \tau)} \to 0$.

CASE 2. Let $\|x + \lambda y_n\|_{E(\mathcal{M}, \tau)} \to 1$ for $\lambda = \pm 1, \pm i$, where $x \geq 0$ and $\{y_n\} \subset B_{E_0(\mathcal{M}, \tau)}$. Recall that for any $n \in \mathbb{N}$, $\operatorname{Re} y_n = (y_n + y_n^*)/2$, $\operatorname{Im} y_n = (y_n - y_n^*)/(2i)$ are self-adjoint operators and $y_n = \operatorname{Re} y_n + i \operatorname{Im} y_n$. Note that by Proposition 1.1(4), $\mu(x + i(y_n + y_n^*)/2) \geq \mu(x)$, and thus

$$1 \leq \|x + i \operatorname{Re} y_n\|_{E(\mathcal{M}, \tau)} \leq \tfrac{1}{2}\|x + iy_n\|_{E(\mathcal{M}, \tau)} + \tfrac{1}{2}\|x + iy_n^*\|_{E(\mathcal{M}, \tau)}$$
$$= \tfrac{1}{2}\|x + iy_n\|_{E(\mathcal{M}, \tau)} + \tfrac{1}{2}\|x - iy_n\|_{E(\mathcal{M}, \tau)}.$$

Thus $\|x + i \operatorname{Re} y_n\|_{E(\mathcal{M}, \tau)} \to 1$, and by Case 1, $\|\operatorname{Re} y_n\|_{E(\mathcal{M}, \tau)} \to 0$. Similarly, the inequality

$$1 \leq \|x + i \operatorname{Im} y_n\|_{E(\mathcal{M}, \tau)} \leq \tfrac{1}{2}\|x + y_n\|_{E(\mathcal{M}, \tau)} + \tfrac{1}{2}\|x - y_n^*\|_{E(\mathcal{M}, \tau)}$$
$$= \tfrac{1}{2}\|x + y_n\|_{E(\mathcal{M}, \tau)} + \tfrac{1}{2}\|x - y_n\|_{E(\mathcal{M}, \tau)}$$

implies that $\|x + i \operatorname{Im} y_n\|_{E(\mathcal{M}, \tau)} \to 1$, and consequently $\|\operatorname{Im} y_n\|_{E(\mathcal{M}, \tau)} \to 0$. Therefore $\|y_n\|_{E(\mathcal{M}, \tau)} \to 0$.

CASE 3. Suppose now that x is an arbitrary element of $S_{E_0(\mathcal{M}, \tau)}$, $\{y_n\} \subset B_{E_0(\mathcal{M}, \tau)}$ and $\|x + \lambda y_n\|_{E(\mathcal{M}, \tau)} \to 1$ for all $\lambda = \pm 1, \pm i$. Let $x = u|x|$ be a polar decomposition of x. Then for all $\lambda = \pm 1, \pm i$, $\||x| + \lambda u^* y_n\|_{E(\mathcal{M}, \tau)} = \|u^* x + \lambda u^* y_n\|_{E(\mathcal{M}, \tau)} \leq \|x + \lambda y_n\|_{E(\mathcal{M}, \tau)}$, and so $\overline{\lim}_n \||x| + \lambda u^* y_n\|_{E(\mathcal{M}, \tau)} \leq 1$. By Lemma 1.5, it follows that

$$\lim_n \||x| + \lambda u^* y_n\|_{E(\mathcal{M}, \tau)} = 1.$$

Similarly, using the polar decomposition $x^* = u^*|x^*|$ of x^*, one can show that

$$\lim_n \big\| |x^*| + \lambda u y_n^* \big\|_{E(\mathcal{M},\tau)} = 1.$$

Since $\mu(|x|) = \mu(|x^*|)$ is a $\mathbb{C}$-LUR point, by Case 2 we conclude that

$$\|u^* y_n\|_{E(\mathcal{M},\tau)} \to 0 \quad \text{and} \quad \|u y_n^*\|_{E(\mathcal{M},\tau)} = \|y_n u^*\|_{E(\mathcal{M},\tau)} \to 0.$$

Hence, in view of $s(x) = u^* u$ and $s(x^*) = u u^*$ we have

$$\|s(x^*) y_n\|_{E(\mathcal{M},\tau)} \to 0 \quad \text{and} \quad \|y_n s(x)\|_{E(\mathcal{M},\tau)} \to 0.$$

It is also easy to check that

$$|x + \lambda n(x^*) y_n n(x)| = \big\| |x| + \lambda |n(x^*) y n(x)| \big\|.$$

Combining the above and the equality $\mathbf{1} = n(x^*) + s(x^*) = n(x) + s(x)$, we get

$$\begin{aligned}
\big\| |x| + \lambda |n(x^*) y_n n(x)| \big\|_{E(\mathcal{M},\tau)} &= \|x + \lambda n(x^*) y_n n(x)\|_{E(\mathcal{M},\tau)} \\
&\leq \|x + \lambda y_n\|_{E(\mathcal{M},\tau)} + \|s(x^*) y_n s(x)\|_{E(\mathcal{M},\tau)} \\
&\quad + \|s(x^*) y_n n(x)\|_{E(\mathcal{M},\tau)} \\
&\quad + \|n(x^*) y_n s(x)\|_{E(\mathcal{M},\tau)} \to 1.
\end{aligned}$$

Again, Lemma 1.5 yields $\lim_n \big\| |x| + \lambda |n(x^*) y_n n(x)| \big\|_{E(\mathcal{M},\tau)} = 1$, and by the first case, $\|n(x^*) y_n n(x)\|_{E(\mathcal{M},\tau)} \to 0$. Hence

$$\|y_n\|_{E(\mathcal{M},\tau)}$$
$$\leq \|n(x^*) y_n n(x)\|_{E(\mathcal{M},\tau)} + 2\|s(x^*) y_n\|_{E(\mathcal{M},\tau)} + \|y_n s(x)\|_{E(\mathcal{M},\tau)} \to 0,$$

and the proof is complete. ■

THEOREM 4.2. *Suppose that the trace τ on $\mathcal{M}$ is σ-finite. If x is a $\mathbb{C}$-LUR point of $B_{E(\mathcal{M},\tau)}$ then $\mu(x)$ is a $\mathbb{C}$-LUR point of B_E and either*

(i) *$\mu_\infty(x) = 0$, or*
(ii) *$n(x)\mathcal{M}n(x^*) = 0$ and $|x| \geq \mu_\infty(x)s(x)$.*

Proof. Since every $\mathbb{C}$-LUR point is a $\mathbb{C}$-extreme point, by Theorem 3.10, (i) or (ii) is satisfied. The fact that $\mu(x)$ is a $\mathbb{C}$-LUR point whenever x is $\mathbb{C}$-LUR can be proved analogously to the corresponding statement about strongly extreme points in Theorem 2.7, replacing $\lambda = \pm 1$ with $\lambda = \pm 1, \pm i$. ■

COROLLARY 4.3. *Let E be an order continuous symmetric space on $[0,\alpha)$, $\alpha = \tau(\mathbf{1})$, and $\mathcal{M}$ be a von Neumann algebra with a faithful, normal, σ-finite trace τ. Then E is a $\mathbb{C}$-LUR space if and only if $E(\mathcal{M},\tau)$ is a $\mathbb{C}$-LUR space.*

REMARK 4.4. Note that if E is order continuous then $E \subset E_0$, and the norm on E is strongly symmetric.

THEOREM 4.5. *Let $E \subset c_0$ be an order continuous symmetric sequence space. Then C_E is a $\mathbb{C}$-LUR space if and only if E is a $\mathbb{C}$-LUR space.*

Proof. Since E is isometrically embedded in C_E, it follows that if C_E is a $\mathbb{C}$-LUR space then so is E. Now $C_E = G(B(H), \mathrm{tr})$ and $\|x\|_{C_E} = \|x\|_{G(B(H),\mathrm{tr})}$ for any compact operator x. Hence the proof of the converse implication is analogous to the proof of Theorem 4.1. $\blacksquare$

5. $\mathbb{C}$-LUR and $\mathbb{C}$-MLUR properties. Let us discuss here the notions of complex strongly extreme points and complex midpoint locally uniformly rotund spaces. One can define a modulus of complex strong extremality analogously to the modulus of strong extremality in the real case, introduced by C. Finet in [20]. Let $(X, \|\cdot\|)$ be a Banach space over the field of complex numbers. For $x \in S_X$ and $\epsilon > 0$, the *modulus of complex strong extremality* at x is the number

$$\Delta(x, \epsilon) = \inf\{1 - |\lambda| : \exists y, \|y\| > \epsilon, \|\lambda x \pm y\| \leq 1 \text{ and } \|\lambda i x \pm y\| \leq 1\}.$$

The element $x \in S_X$ is said to be a $\mathbb{C}$-*MLUR* point of B_X, or a *complex strongly extreme point* of B_X, if $\Delta(x, \epsilon) > 0$ for any $\epsilon > 0$. A Banach space X is said to be *complex midpoint locally uniformly rotund*, or a $\mathbb{C}$-MLUR space, if every element of S_X is a $\mathbb{C}$-MLUR point.

We will demonstrate that the notions of $\mathbb{C}$-LUR and $\mathbb{C}$-MLUR points, and hence the notions of $\mathbb{C}$-LUR and $\mathbb{C}$-MLUR spaces, are equivalent in any complex Banach space. Consequently, in complex Banach spaces these complex properties are not distinguishable, unlike the corresponding properties LUR and MLUR [29]. We need the following lemma.

LEMMA 5.1. *An element $x \in S_X$ is a $\mathbb{C}$-MLUR point if and only for any $\{x_n\} \subset X$, $\|x + \lambda x_n\| \to 1$ for all $\lambda = \pm 1, \pm i$ implies that $\|x_n\| \to 0$.*

Proof. Suppose that x is a $\mathbb{C}$-MLUR point, that is, $\Delta(x, \epsilon) > 0$ for all $\epsilon > 0$. Let $\|x \pm x_n\| \to 1$ and $\|x \pm i x_n\| \to 1$, where $\{x_n\} \subset X$. Set

$$c_n = \max_{\lambda \in \{\pm 1, \pm i\}} \|x + \lambda x_n\|.$$

Clearly, $c_n \to 1$. If $c_n \leq 1$ for some n, then $\|x + \lambda x_n\| \leq 1$ for all $\lambda = \pm 1, \pm i$, and consequently $x_n = 0$. Indeed, suppose that $x_n \neq 0$. Then there exists an $\epsilon > 0$ such that $\|x_n\| > \epsilon$, $\|x \pm x_n\| \leq 1$ and $\|ix \pm x_n\| \leq 1$. But then $\Delta(x, \epsilon) = 0$, a contradiction. Therefore we can assume that $c_n > 1$ for all $n \in \mathbb{N}$. Clearly, for all $n \in \mathbb{N}$,

$$\|c_n^{-1} x \pm c_n^{-1} x_n\| \leq 1 \quad \text{and} \quad \|i c_n^{-1} x \pm c_n^{-1} x_n\| \leq 1.$$

Denote $\lambda_n = c_n^{-1}$, $n \in \mathbb{N}$. Then for each n, the element $a_n = c_n^{-1} x_n$ satisfies $\|\lambda_n x \pm a_n\| \leq 1$ and $\|i\lambda_n x \pm a_n\| \leq 1$. Hence $\|a_n\| \to 0$ and consequently $\|x_n\| \to 0$: if not, then there exists an $\epsilon > 0$ and a subsequence a_{n_k} such

that $\|a_{n_k}\| > \epsilon$, and since $\lambda_n \to 1$,

$$\Delta(x,\epsilon) = \inf\{1 - |\lambda| : \exists y, \|y\| > \epsilon, \|\lambda x \pm y\| \leq 1 \text{ and } \|\lambda i x \pm y\| \leq 1\} = 0.$$

To prove the converse implication, assume that $\Delta(x,\epsilon) = 0$ for some $\epsilon > 0$. Then there exists a sequence $\{\lambda_n\} \subset \mathbb{C}$ satisfying $|\lambda_n| \uparrow 1$ and for all $n \in \mathbb{N}$, there is an $x_n \in B_X$ with $\|x_n\| \geq \epsilon$ such that $\|\lambda_n x \pm x_n\| \leq 1$ and $\|i\lambda_n x \pm x_n\| \leq 1$. Therefore, for all $n \in \mathbb{N}$ we have

$$\|x \pm \lambda_n^{-1} x_n\| \leq |\lambda_n|^{-1} \quad \text{and} \quad \|x \pm i\lambda_n^{-1} x_n\| \leq |\lambda_n|^{-1},$$

and since $|\lambda_n| \to 1$, $\overline{\lim}_n \|x \pm \lambda_n^{-1} x_n\| \leq 1$ and $\overline{\lim}_n \|x \pm i\lambda_n^{-1} x_n\| \leq 1$. As $\|x\| = 1$, Lemma 1.5 yields $\lim_n \|x \pm \lambda_n^{-1} x_n\| = 1$ and $\lim_n \|x \pm i\lambda_n^{-1} x_n\| = 1$. Hence there exists a subsequence $\{\lambda_{n_k}^{-1} x_{n_k}\}$ with $\lim_k \|\lambda_{n_k}^{-1} x_{n_k}\| \neq 0$ such that $\lim_k \|x \pm \lambda_{n_k}^{-1} x_{n_k}\| = 1$ and $\lim_k \|x \pm i\lambda_{n_k}^{-1} x_{n_k}\| = 1$. ∎

Now we can state the equivalence of $\mathbb{C}$-LUR and $\mathbb{C}$-MLUR properties.

PROPOSITION 5.2. *Let* $x \in S_X$. *The following conditions are equivalent:*

(i) x *is a* $\mathbb{C}$*-LUR point of* B_X.

(ii) *For all* $\epsilon > 0$ *there exists* $\delta(\epsilon) > 0$ *such that for all* $y \in X$,

$$\sup_{\lambda = \pm 1, \pm i} \|x + \lambda y\| < 1 + \delta \quad \text{implies } \|y\| < \epsilon.$$

(iii) *For all* $\{y_n\} \subset X$, $\sup_{\lambda = \pm 1, \pm i} \|x + \lambda y_n\| \to 1$ *implies* $\|y_n\| \to 0$.

(iv) *For all* $\{y_n\} \subset X$, $\|x \pm y_n\| \to 1$ *and* $\|x \pm i y_n\| \to 1$ *implies* $\|y_n\| \to 0$.

(v) x *is a* $\mathbb{C}$*-MLUR point of* B_X.

Proof. It is clear that (ii) implies (i), (iii) implies (iv) and conditions (ii) and (iii) are equivalent. By Lemma 5.1, conditions (iv) and (v) are also equivalent. It remains to show (i)⇒(ii) and (iv)⇒(iii).

Let $\epsilon > 0$ and suppose that (i) holds. Hence for $\epsilon/2 > 0$, there exists $\delta(\epsilon) > 0$ such that for any $y \in X$, $\sup_{|\lambda| \leq 1} \|x + \lambda y\| < 1 + \delta(\epsilon)$ implies that $\|y\| < \epsilon/2$. Assume now that $\sup_{\lambda = \pm 1, \pm i} \|x + \lambda y\| < 1 + \delta$. Then for $-1 \leq c \leq 1$ and $\lambda = \pm 1, \pm i$ we have

$$\|x + c\lambda y\| \leq ((1+c)/2)\|x + \lambda y\| + ((1-c)/2)\|x - \lambda y\| < 1 + \delta.$$

Hence for all $c \in \mathbb{R}$ with $|c| \leq 1$, $\|x + cy\| < 1 + \delta$ and $\|x + ciy\| < 1 + \delta$. Let $\lambda = a + bi \in \mathbb{C}$, with $|\lambda| \leq 1$. Since $|a|, |b| \leq 1$ it follows that

$$\|x + \lambda y/2\| = \|x + ay/2 + biy/2\| \leq \|x + ay\|/2 + \|x + biy\|/2 < 1 + \delta.$$

Consequently, $\sup_{|\lambda| \leq 1} \|x + y/2\| < 1 + \delta$ and by (i), $\|y/2\| < \epsilon/2$ and so $\|y\| < \epsilon$. Therefore (ii) is satisfied.

To show (iv)⇒(iii), suppose that $\{y_n\} \subset X$ and $\sup_{\lambda = \pm 1, \pm i} \|x + \lambda y_n\| \to 1$. Then $\overline{\lim}_n \|x \pm y_n\| \leq 1$ and $\overline{\lim}_n \|x \pm iy_n\| \leq 1$. Thus by Lemma 1.5, for all $\lambda = \pm 1, \pm i$ we have $\lim_n \|x + \lambda y_n\| = 1$, and so by (iv), $\|y_n\| \to 0$. ∎

We finish with examples showing that complex uniform rotundity, complex local uniform rotundity and complex rotundity do not coincide.

EXAMPLE 5.3. (1) The space $E = (\ell_\infty, \|\cdot\|)$ equipped with the norm $\|x\| = \|x\|_\infty + \sum_{n=1}^\infty |x(n)|/2^{n-1}$ is complex rotund but not complex locally uniformly rotund. One can show easily [26] that the unit vector $2^{-1}e_1 = (2^{-1}, 0, \dots)$ is an UM point in B_{E_r}, so it is $\mathbb{C}$-extreme point in B_E, but not an ULUM point so not a $\mathbb{C}$-LUR point (Theorem 1.4).

(2) Orlicz–Lorentz spaces $\Lambda_{\varphi,w}$ are locally uniformly rotund and hence $\mathbb{C}$-LUR whenever φ is strictly convex and satisfies condition Δ_2 [4]. However if they are complex uniformly rotund then in addition to those conditions on φ, the weight w must be regular [8]. So there exist Orlicz–Lorentz spaces that are $\mathbb{C}$-LUR but not complex uniformly rotund.

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