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Generalizations to monotonicity for uniform convergence of double sine integrals over $\overline{\mathbb{R}}^2_+$

by

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Abstract. We investigate the convergence behavior of the family of double sine integrals of the form

$$\int_0^\infty\int_0^\infty f(x,y)\sin ux\sin vy\,dx\,dy,\quad \text{where}\quad (u,v)\in\mathbb{R}^2_+:=\mathbb{R}_+\times\mathbb{R}_+,$$

$\mathbb{R}_+ := (0,\infty)$, and $f:\mathbb{R}^2_+\to\mathbb{C}$ is a locally absolutely continuous function satisfying certain generalized monotonicity conditions. We give sufficient conditions for the uniform convergence of the remainder integrals $\int_{a_1}^{b_1}\int_{a_2}^{b_2}$ to zero in $(u,v)\in\mathbb{R}^2_+$ as $\max\{a_1,a_2\}\to\infty$ and $b_j>a_j\ge 0$, $j=1,2$ (called uniform convergence in the regular sense). This implies the uniform convergence of the partial integrals $\int_0^{b_1}\int_0^{b_2}$ in $(u,v)\in\mathbb{R}^2_+$ as $\min\{b_1,b_2\}\to\infty$ (called uniform convergence in Pringsheim's sense). These sufficient conditions are the best possible in the special case when $f(x,y)\ge 0$.

1. Introduction: Convergence of double integrals over $\overline{\mathbb{R}}^2_+$. Let $\phi:\overline{\mathbb{R}}^2_+\to\mathbb{C}$ be a locally integrable function over $\overline{\mathbb{R}}^2_+$ in Lebesgue's sense, in symbols: $\phi\in L^1_{\rm loc}(\overline{\mathbb{R}}^2_+)$, where $\overline{\mathbb{R}}_+:=[0,\infty)$. By definition, the double integral

$$\int_0^\infty\int_0^\infty \phi(x,y)\,dx\,dy \tag{1.1}$$

is said to *converge in Pringsheim's sense* if the *partial integrals*

$$I(\phi;b_1,b_2):=\int_0^{b_1}\int_0^{b_2}\phi(x,y)\,dx\,dy,\quad (b_1,b_2)\in\mathbb{R}^2_+,$$

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converge to a finite limit as $b_1, b_2 \to \infty$ independently of each other, in symbols: $\min\{b_1, b_2\} \to \infty$.

By the *Cauchy convergence criterion*, a necessary and sufficient condition for the convergence of the double integral (1.1) in Pringsheim's sense is that for every $\varepsilon > 0$ there exists $b_0 = b_0(\varepsilon)$ such that

$$|I(\phi; b_1, b_2) - I(\phi; b_3, b_4)| < \varepsilon \quad \text{if } \min\{b_1, b_2, b_3, b_4\} > b_0.$$

It follows from the convergence in Pringsheim's sense that the *remainder integrals* satisfy

$$\begin{aligned}(1.2)\qquad & \int_{a_1}^{b_1}\int_{a_2}^{b_2} \phi(x,y)\,dx\,dy \\ &= I(\phi; b_1, b_2) - I(\phi; a_1, b_2) - I(\phi; a_2, b_1) + I(\phi; a_1, a_2) \to 0 \\ &\qquad\qquad \text{as } \min\{a_1, a_2\} \to \infty,\ b_j > a_j > 0,\ j = 1, 2.\end{aligned}$$

By definition, the double integral (1.1) is said to *converge in the regular sense* if

$$(1.3)\qquad \int_{a_1}^{b_1}\int_{a_2}^{b_2} \phi(x,y)\,dx\,dy \to 0 \quad \text{as } \max\{a_1, a_2\} \to \infty,\ b_j > a_j \geq 0,\ j = 1, 2$$

(cf. (1.2)). Condition (1.3) is equivalent to the joint fulfillment of the two conditions

$$\int_{a_1}^{b_1}\int_{0}^{b_2} \phi(x,y)\,dx\,dy \to 0 \quad \text{as } b_1 > a_1 \to \infty \text{ and } b_2 > 0 \text{ is arbitrary,}$$

and

$$\int_{0}^{b_1}\int_{a_2}^{b_2} \phi(x,y)\,dx\,dy \to 0 \quad \text{as } b_2 > a_2 \to \infty \text{ and } b_1 > 0 \text{ is arbitrary.}$$

It follows immediately that if the double integral (1.1) converges in the regular sense, then it converges in Pringsheim's sense. The converse implication is not true. For example, set

$$\phi(x,y) := \begin{cases} (-1)^{[x]}[1 + x/2] & \text{for } (x,y) \in [0,\infty) \times [0,1), \\ (-1)^{1+[x]}[1 + x/2] & \text{for } (x,y) \in [0,\infty) \times [1,2), \\ 0 & \text{for } (x,y) \in [2,\infty) \times (2,\infty), \\ \phi(y,x) & \text{for } (x,y) \in [0,2) \times [2,\infty), \end{cases}$$

where $[\cdot]$ means the integer part of a real number. Since

$$I(\phi; b_1, b_2) = 0 \quad \text{whenever } \min\{b_1, b_2\} \geq 2,$$

the double integral (1.1) converges to zero in Pringsheim's sense. On the other hand, the double integral (1.1) cannot converge in the regular sense, since

$$\int_{2k-2}^{2k-1}\int_{0}^{1}\phi(x,y)\,dx\,dy = k, \quad k = 1,2,\ldots.$$

In the special case when ϕ is Lebesgue integrable over the whole quadrant $\overline{\mathbb{R}}^2_+$, the double integral (1.1) converges in the regular sense, and its limit in Pringsheim's sense is equal to the Lebesgue integral of ϕ over $\overline{\mathbb{R}}^2_+$.

We note that the notions of the 'convergence of a double integral over $\overline{\mathbb{R}}^2_+$ in Pringsheim's sense or in the regular sense' are the nondiscrete versions of the notions of the 'convergence of a double series $\sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} a_{k\ell}$ of complex numbers in Pringsheim's sense or in the regular sense', respectively. For details, see [10, Vol. 2, pp. 300–302], [3] and [6].

2. Main results: Uniform convergence of double sine integrals.

We investigate the uniform convergence of the family of *double sine integrals*

$$\int_0^\infty\int_0^\infty f(x,y)\sin ux\sin vy\,dx\,dy, \quad (u,v)\in\mathbb{R}^2_+, \tag{2.1}$$

where $f:\mathbb{R}^2_+\to\mathbb{C}$ is a Lebesgue measurable function.

In order to ensure the existence of the partial integrals

$$I_{uv}(f;b_1,b_2) := \int_0^{b_1}\int_0^{b_2} f(x,y)\sin ux\sin vy\,dx\,dy, \quad b_1,b_2>0, \tag{2.2}$$

we always assume that for all $(b_1,b_2)\in\mathbb{R}^2_+$,

$$xyf(x,y)\in L^1_{\mathrm{loc}}(\overline{\mathbb{R}}^2_+), \quad \text{that is,} \quad \int_0^{b_1}\int_0^{b_2} xy|f(x,y)|\,dx\,dy<\infty. \tag{2.3}$$

We note that the double sine integral (2.1) is the nondiscrete version of the double sine series $\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} a_{k\ell}\sin ku\sin \ell v$, where $\{a_{k\ell}\}$ is a double sequence of complex numbers (for details see [4]). Historically, the first result in this topic is due to Chaundy and Jolliffe [2], who proved the following theorem: If $\{a_k : k = 1,2,\ldots\}$ is a decreasing sequence of nonnegative numbers, then the sine series $\sum_{k=1}^{\infty} a_k \sin ku$ converges uniformly in $u\in\mathbb{R}_+$ if and only if $ka_k\to 0$ as $k\to\infty$.

In what follows, we always assume that the function f occurring in (2.1) is *locally absolutely continuous* on $\mathbb{R}^2_+$, in symbols: $f\in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}^2_+)$, by which we mean the following: the partial derivatives $f_x := \partial f/\partial x$ and $f_y := \partial f/\partial y$ exist everywhere on $\mathbb{R}^2_+$, and f can be recovered from them in the usual way:

for all $b_j > a_j > 0$, $j = 1, 2$, we have

$$\int_{a_1}^{b_1} f_x(x,y)\,dx = f(b_1,y) - f(a_1,y), \quad y > 0, \tag{2.4}$$

$$\int_{a_2}^{b_2} f_y(x,y)\,dy = f(x,b_2) - f(x,a_2), \quad x > 0; \tag{2.5}$$

furthermore, the mixed partial derivatives f_{xy}, f_{yx} exist and $f_{xy} = f_{yx}$ almost everywhere on $\mathbb{R}^2_+$, and f_x, f_y can be recovered from them: for all $b_j > a_j > 0$, $j = 1, 2$, we have

$$\int_{a_1}^{b_1} f_{xy}(x,y)\,dx = f_y(b_1,y) - f_y(a_1,y), \quad y > 0, \tag{2.6}$$

$$\int_{a_2}^{b_2} f_{xy}(x,y)\,dy = f_x(x,b_2) - f_x(x,a_2), \quad x > 0. \tag{2.7}$$

It follows immediately that for all $b_j > a_j > 0$, $j = 1, 2$, we have

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} f_{xy}(x,y)\,dx\,dy = f(b_1,b_2) - f(a_1,b_2) - f(b_1,a_2) + f(a_1,a_2).$$

We refer to [1] and the references in it for the definition and basic properties of absolute continuity of functions in two variables.

Our next definition is motivated by the analogous one in the case of single sine integrals introduced in [7]; the latter was inspired by the corresponding discrete definition in the case of single trigonometric series (see [9]). A function $f \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}^2_+)$ is said to be of *mean value bounded variation*, in symbols: $f \in \mathrm{MVBVF}(\mathbb{R}^2_+)$, if there exist constants C and $\lambda \geq 2$, depending only on f, such that for all $a_1, y > 0$ we have

$$\int_{a_1}^{2a_1} |f_x(x,y)|\,dx \leq \frac{C}{a_1} \int_{\lambda^{-1}a_1}^{\lambda a_1} |f(x,y)|\,dx; \tag{2.8}$$

for all $x, a_2 > 0$ we have

$$\int_{a_2}^{2a_2} |f_y(x,y)|\,dy \leq \frac{C}{a_2} \int_{\lambda^{-1}a_2}^{\lambda a_2} |f(x,y)|\,dy; \tag{2.9}$$

and for all $a_1, a_2 > 0$ we have

$$\int_{a_1}^{2a_1}\int_{a_2}^{2a_2} |f_{xy}(x,y)|\,dx\,dy \leq \frac{C}{a_1 a_2} \int_{\lambda^{-1}a_1}^{\lambda a_1}\int_{\lambda^{-1}a_2}^{\lambda a_2} |f(x,y)|\,dx\,dy. \tag{2.10}$$

In our first main result, we give sufficient conditions for the uniform convergence of the double sine integrals (2.1) in the regular sense.

Theorem 1. *Assume the function* $f : \mathbb{R}^2_+ \to \mathbb{C}$ *satisfies condition* (2.3) *and belongs to the class* $\mathrm{MVBVF}(\mathbb{R}^2_+)$. *If for all* $x, y > 0$ *we have*

$$xyf(x,y) \to 0 \quad \textit{as } \max\{x,y\} \to \infty, \tag{2.11}$$

and for all $b_1, b_2 > 0$ *we have*

$$\frac{1}{b_1 b_2} \int_0^{b_1} \int_0^{b_2} xy|f(x,y)|\,dx\,dy \to 0 \quad \textit{as } \max\{b_1, b_2\} \to \infty, \tag{2.12}$$

then the double sine integrals (2.1) *converge in the regular sense uniformly in* $(u,v) \in \mathbb{R}^2_+$.

We note that if the product $xyf(x,y)$ is bounded on $\mathbb{R}^2_+$, then condition (2.3) is satisfied and condition (2.11) clearly implies (2.12).

In our next theorem, we show that in case $f(x,y) \geq 0$ condition (2.11) is necessary for the uniform convergence of the double sine integrals (2.1) in the regular sense.

Theorem 2. *Assume the function* $f : \mathbb{R}^2_+ \to \overline{\mathbb{R}}_+$ *satisfies condition* (2.3) *and belongs to the class* $\mathrm{MVBVF}(\mathbb{R}^2_+)$. *If the double sine integrals* (2.1) *converge in the regular sense uniformly in* $(u,v) \in \mathbb{R}^2_+$, *then condition* (2.11) *holds true.*

The following corollary is an immediate consequence of Theorems 1 and 2.

Corollary 1. *Assume the function* $f : \mathbb{R}^2_+ \to \overline{\mathbb{R}}_+$ *satisfies conditions* (2.3) *and* (2.12), *and belongs to the class* $\mathrm{MVBVF}(\mathbb{R}^2_+)$. *Then the double sine integrals* (2.1) *converge in the regular sense uniformly in* $(u,v) \in \mathbb{R}^2_+$ *if and only if condition* (2.11) *is satisfied.*

Our next definition is also motivated by [7] and [8]. A function $f \in \mathrm{AC}_{\mathrm{loc}}(\mathbb{R}^2_+)$ is said to be of *non-onesided bounded variation*, in symbols: $f \in \mathrm{NBVF}(\mathbb{R}^2_+)$, if there exists a constant C, depending only on f, such that

$$\int_{a_1}^{2a_1} |f_x(x,y)|\,dx \leq C(|f(a_1,y)| + |f(2a_1,y)|), \qquad a_1, y > 0, \tag{2.13}$$

$$\int_{a_2}^{2a_2} |f_y(x,y)|\,dy \leq C(|f(x,a_2)| + |f(x,2a_2)|), \qquad x, a_2 > 0, \tag{2.14}$$

$$(2.15)\qquad \int_{a_1}^{2a_1}\int_{a_2}^{2a_2} |f_{xy}(x,y)|\,dx\,dy$$
$$\le C(|f(a_1,a_2)|+|f(2a_1,a_2)|+|f(a_1,2a_2)|+|f(2a_1,2a_2)|),\qquad a_1,a_2>0.$$

We say that a function $f:\mathbb{R}^2_+\to\mathbb{R}$ is *monotonically decreasing* if it is decreasing in each variable and, in addition, for all $x_2>x_1>0$ and $y_2>y_1>0$,

$$f(x_1,y_1)-f(x_2,y_1)-f(x_1,y_2)+f(x_2,y_2)\ge 0.$$

We note that in the literature there exist other variations of the term 'monotonically decreasing' (see, e.g., [5]).

It is clear that if $f:\mathbb{R}^2_+\to\overline{\mathbb{R}}_+$ belongs to the class $\mathrm{AC_{loc}}(\mathbb{R}^2_+)$, then f is monotonically decreasing if and only if

$$f_x(x,y)\le 0,\quad f_y(x,y)\le 0\quad \text{and}\quad f_{xy}(x,y)\ge 0\quad \text{almost everywhere;}$$

and in this case f clearly belongs to the class $\mathrm{NBVF}(\mathbb{R}^2_+)$.

Our second main result is formulated in the following

THEOREM 3. *If the function* $f:\mathbb{R}^2_+\to\mathbb{C}$ *belongs to the class* $\mathrm{NBVF}(\mathbb{R}^2_+)$, *then it also belongs to the class* $\mathrm{MVBVF}(\mathbb{R}^2_+)$. *The converse implication is not true.*

The next corollary is an immediate consequence of Corollary 1 and Theorem 3.

COROLLARY 2. *Assume the function* $f\in\mathrm{AC_{loc}}(\mathbb{R}^2_+)$ *is monotonically decreasing and satisfies conditions* (2.3) *and* (2.12). *Then the double sine integrals* (2.1) *converge in the regular sense uniformly in* $(u,v)\in\mathbb{R}^2_+$ *if and only if condition* (2.11) *is satisfied.*

For example, the function

$$f(x,y):=(x+1)^\alpha(y+1)^\beta,\quad -2<\alpha,\beta<-1,$$

is monotonically decreasing and satisfies each of the conditions (2.3), (2.11) and (2.12). Thus, in this case the double sine integrals (2.1) converge in the regular sense uniformly in $(u,v)\in\mathbb{R}^2_+$.

We note that Theorems 1–3 and Corollaries 1–2 above may be considered as extensions of the analogous ones in [7, Theorems 2 and 3] from single to double sine integrals.

It is clear that under the conditions of Theorem 1, the double sine integrals (2.1) converge in Pringsheim's sense also uniformly in $(u,v)\in\mathbb{R}^2_+$. Similarly to the proof of Theorem 2 in Section 4 below, the following theorem can be easily proved, which gives a necessary condition for the uniform convergence of the double sine integrals (2.1) in Pringsheim's sense.

THEOREM 4. *Assume the function* $f : \mathbb{R}^2_+ \to \overline{\mathbb{R}}_+$ *satisfies condition* (2.3) *and belongs to the class* $\mathrm{MVBVF}(\mathbb{R}^2_+)$. *If the double sine integrals* (2.1) *converge in Pringsheim's sense, or only the remainder integrals* (*cf.* (1.2))

$$\int_{a_1}^{b_1}\int_{a_2}^{b_2} f(x,y)\sin ux \sin vy\,dx\,dy, \quad b_j > a_j > 0,\ j = 1,2,$$

converge to 0 *as* $\min\{a_1,a_2\} \to \infty$ *uniformly in* $(u,v) \in \mathbb{R}^2_+$, *then*

$$xyf(x,y) \to 0 \quad \text{as } \min\{x,y\} \to \infty. \tag{2.16}$$

However, all our attempts have failed so far to modify the steps in the proof of Theorem 1 in order to guarantee the uniform convergence of the double sine integrals (2.1) in Pringsheim's sense for all $(u,v) \in \mathbb{R}^2_+$ under the conditions of Theorem 1 with (2.11) replaced by the weaker (2.16).

3. Auxiliary results

LEMMA 1. *Assume the function* $f : \mathbb{R}^2_+ \to \mathbb{C}$ *satisfies condition* (2.3) *and belongs to the class* $\mathrm{MVBVF}(\mathbb{R}^2_+)$. *If condition* (2.11) *is satisfied, then for all* $a_1, y > 0$ *we have*

$$a_1 y \int_{a_1}^{\infty} |f_x(x,y)|\,dx \to 0 \quad \text{as } \max\{a_1,y\} \to \infty; \tag{3.1}$$

for all $x, a_2 > 0$ *we have*

$$x a_2 \int_{a_2}^{\infty} |f_y(x,y)|\,dy \to 0 \quad \text{as } \max\{x,a_2\} \to \infty; \tag{3.2}$$

and for all $a_1, a_2 > 0$ *we have*

$$a_1 a_2 \int_{a_1}^{\infty}\int_{a_2}^{\infty} |f_{xy}(x,y)|\,dx\,dy \to 0 \quad \text{as } \max\{a_1,a_2\} \to \infty. \tag{3.3}$$

Proof. By (2.11), for every $\varepsilon > 0$ there exists $x_0 = x_0(\varepsilon) > 0$ such that for all $x, y > 0$,

$$xy|f(x,y)| < \varepsilon \quad \text{if } \max\{x,y\} > x_0. \tag{3.4}$$

(i) Let $\max\{\lambda^{-1}a_1, y\} > x_0$. By (3.4) and (2.8), we estimate as follows:

$$a_1 y \int_{a_1}^{\infty} |f_x(x,y)|\,dx = a_1 y \sum_{k=0}^{\infty} \int_{2^k a_1}^{2^{k+1}a_1} |f_x(x,y)|\,dx$$

$$\begin{aligned}
&\le a_1 y \sum_{k=0}^{\infty} \frac{C}{2^k a_1} \int_{\lambda^{-1}2^k a_1}^{\lambda 2^k a_1} |f(x,y)|\,dx \le Cy \sum_{k=0}^{\infty} \frac{1}{2^k} \int_{\lambda^{-1}2^k a_1}^{\lambda 2^k a_1} \frac{\varepsilon}{xy}\,dx \\
&= 2C\varepsilon(\ln\lambda) \sum_{k=0}^{\infty} \frac{1}{2^k} = 4C(\ln\lambda)\varepsilon \quad \text{if } \max\{\lambda^{-1}a_1, y\} > x_0.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, (3.1) is proved.

(ii) In a similar way, (3.2) is proved by making use of (3.4) and (2.9).

(iii) Let $\max\{\lambda^{-1}a_1, \lambda^{-1}a_2\} > x_0$. By (3.4) and (2.10),

$$\begin{aligned}
&a_1 a_2 \int_{a_1}^{\infty}\int_{a_2}^{\infty} |f_{xy}(x,y)|\,dx\,dy \\
&\quad = a_1 a_2 \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \int_{2^k a_1}^{2^{k+1}a_1} \int_{2^\ell a_2}^{2^{\ell+1}a_2} |f_{xy}(x,y)|\,dx\,dy \\
&\quad \le \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{C}{2^k 2^\ell} \int_{\lambda^{-1}2^k a_1}^{\lambda 2^k a_1} \int_{\lambda^{-1}2^\ell a_2}^{\lambda 2^\ell a_2} |f(x,y)|\,dx\,dy \\
&\quad \le \sum_{k=0}^{\infty}\sum_{\ell=0}^{\infty} \frac{C}{2^k 2^\ell} \int_{\lambda^{-1}2^k a_1}^{\lambda 2^k a_1} \int_{\lambda^{-1}2^\ell a_2}^{\lambda 2^\ell a_2} \frac{\varepsilon}{xy}\,dx\,dy \\
&\quad = 4C\varepsilon(\ln\lambda)^2 \sum_{k=0}^{\infty}\frac{1}{2^k} \sum_{\ell=0}^{\infty}\frac{1}{2^\ell} = 16C(\ln\lambda)^2\varepsilon \quad \text{if } \max\{a_1, a_2\} > \lambda x_0.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, (3.3) is proved. ■

LEMMA 2. *Assume the function $g : \mathbb{R}_+ \to \mathbb{C}$ is locally absolutely continuous on $\mathbb{R}_+$ and such that $xg(x) \in L^1_{\mathrm{loc}}(\overline{\mathbb{R}}_+)$. If there exists a constant C such that for every $a > 0$,*

$$\int_a^{2a} |g'(x)|\,dx \le C(|g(a)| + |g(2a)|),$$

then for every $a > 0$,

$$\int_a^{2a} |g'(x)|\,dx \le \frac{4C}{a} \int_{a/4}^{4a} |g(x)|\,dx.$$

The proof of Lemma 2 is contained in the proof of [7, Theorem 3].

LEMMA 3. *Assume the function $f : \mathbb{R}^2_+ \to \overline{\mathbb{R}}_+$ belongs to the class* $\mathrm{MVBVF}(\mathbb{R}^2_+)$. *Then for all $a, b > 0$ we have*

$$abf(a,b) \le (8C+2) \int_{a/2\lambda}^{\lambda a} \int_{b/2\lambda}^{\lambda b} f(x,y)\,dx\,dy, \tag{3.5}$$

where the constants C *and* λ *are from the definition of the class* $\mathrm{MVBVF}(\mathbb{R}^2_+)$.

Proof. By (2.4), for all $a \le s \le 2a$ and $t > 0$ we have

$$f(s,t) - f(a,t) = \int_a^s f_x(x,t)\,dx.$$

Using (2.8) and the fundamental theorem of calculus, we have

$$\begin{aligned} f(a,t) &\le \int_a^s |f_x(x,t)|\,dx + f(s,t) \\ &\le \int_{s/2}^{s} |f_x(x,t)|\,dx + f(s,t) \le \frac{2C}{s} \int_{s/2\lambda}^{\lambda s/2} f(x,t)\,dx + f(s,t) \\ &\le \frac{2C}{s} \int_{a/2\lambda}^{\lambda a} f(x,t)\,dx + f(s,t), \quad a \le s \le 2a \text{ and } t > 0. \end{aligned} \tag{3.6}$$

Integrating both sides of (3.6) with respect to s over the interval $[a,2a]$ gives

$$af(a,t) \le 2C \int_{a/2\lambda}^{\lambda a} f(x,t)\,dx + \int_a^{2a} f(x,t)\,dx, \quad t > 0. \tag{3.7}$$

Making use of (2.5) and (2.9), an analogous argument yields

$$bf(s,b) \le 2C \int_{b/2\lambda}^{\lambda b} f(s,y)\,dy + \int_b^{2b} f(s,y)\,dy, \quad s > 0. \tag{3.8}$$

Next, by making use of (2.6) (or (2.7)) and (2.10), a double version of the above argument gives the following: for all $a \le s \le 2a$ and $b \le t \le 2b$ we have

$$\begin{aligned} f(a,b) &= \int_a^s \int_b^t f_{xy}(x,y)\,dx\,dy + f(a,t) + f(s,b) - f(s,t) \\ &\le \int_a^s \int_b^t |f_{xy}(x,y)|\,dx\,dy + f(a,t) + f(s,b) \\ &\le \int_{s/2}^{s} \int_{t/2}^{t} |f_{xy}(x,y)|\,dx\,dy + f(a,t) + f(s,b) \end{aligned} \tag{3.9}$$

$$\leq \frac{4C}{st} \int_{s/2\lambda}^{\lambda s/2} \int_{t/2\lambda}^{\lambda t/2} f(x,y)\,dx\,dy + f(a,t) + f(s,b)$$

$$\leq \frac{4C}{ab} \int_{a/2\lambda}^{\lambda a} \int_{b/2\lambda}^{\lambda b} f(x,y)\,dx\,dy + f(a,t) + f(s,b).$$

Integrating both sides of (3.9) with respect to $s \in [a, 2a]$ and $t \in [b, 2b]$, we find that

$$abf(a,b) \leq 4C \int_{a/2\lambda}^{\lambda a} \int_{b/2\lambda}^{\lambda b} f(x,y)\,dx\,dy + a\int_{b}^{2b} f(a,t)\,dt + b\int_{a}^{2a} f(s,b)\,ds. \tag{3.10}$$

Combining inequalities (3.7), (3.8) and (3.10) yields

$$\begin{aligned} abf(a,b) \leq 4C &\int_{a/2\lambda}^{\lambda a} \int_{b/2\lambda}^{\lambda b} f(x,y)\,dx\,dy \\ &+ 2C \int_{a/2\lambda}^{\lambda a} \int_{b}^{2b} f(x,t)\,dx\,dt + \int_{a}^{2a}\int_{b}^{2b} f(x,t)\,dx\,dt \\ &+ 2C \int_{a}^{2a} \int_{b/2\lambda}^{\lambda b} f(s,y)\,ds\,dy + \int_{a}^{2a}\int_{b}^{2b} f(s,y)\,ds\,dy, \end{aligned}$$

whence (3.5) follows immediately, due to the fact that $\lambda \geq 2$. ∎

4. Proofs of Theorems 1–4

Proof of Theorem 1. Let an arbitrary $\varepsilon > 0$ be given. By conditions (2.11) and (2.12), there exists $b_0 = b_0(\varepsilon) > 0$ such that for all $x, y > 0$ we have

$$xy|f(x,y)| < \varepsilon \quad \text{if } \max\{x,y\} > b_0, \tag{4.1}$$

and for all $b_1, b_2 > 0$ we have

$$\frac{1}{b_1 b_2} \int_0^{b_1}\int_0^{b_2} xy|f(x,y)|\,dx\,dy < \varepsilon \quad \text{if } \max\{b_1,b_2\} > b_0. \tag{4.2}$$

Furthermore, by (3.1)–(3.3) in Lemma 1, there exists $x_0 = x_0(\varepsilon) > 0$ such that

$$a_1 y \int_{a_1}^{\infty} |f_x(x,y)|\,dx < \varepsilon \quad \text{if } \max\{a_1,y\} > x_0, \tag{4.3}$$

$$x a_2 \int_{a_2}^{\infty} |f_y(x,y)|\,dy < \varepsilon \quad \text{if } \max\{x,a_2\} > x_0, \tag{4.4}$$

$$a_1a_2\int_{a_1}^{\infty}\int_{a_2}^{\infty}|f_{xy}(x,y)|\,dx\,dy<\varepsilon \quad \text{if } \max\{a_1,a_2\}>x_0, \tag{4.5}$$

where $\min\{a_1,a_2,x,y\}>0$.

Let $y_0:=\max\{b_0,x_0\}$. We claim that for all $(u,v)\in\mathbb{R}^2_+$ we have

$$|I_{uv}(f;a_1,b_1;a_2,b_2)|:=\Big|\int_{a_1}^{b_1}\int_{a_2}^{b_2}f(x,y)\sin ux\sin vy\,dx\,dy\Big|<16\varepsilon \tag{4.6}$$

$$\text{whenever } \max\{a_1,a_2\}>y_0 \text{ and } b_j>a_j\geq 0,\ j=1,2.$$

To justify this claim, we will distinguish nine cases (i)–(ix). By Fatou's lemma, we may assume that $a_1,a_2>0$.

Case (i): $a_1<b_1\leq 1/u$ and $a_2<b_2\leq 1/v$. By (4.2), we have

$$\begin{aligned}|I_{uv}(f;a_1,b_1;a_2,b_2)|&\leq uv\int_{a_1}^{b_1}\int_{a_2}^{b_2}xy|f(x,y)|\,dx\,dy\\&\leq uv\int_0^{1/u}\int_0^{1/v}xy|f(x,y)|\,dx\,dy<\varepsilon,\end{aligned}$$

which is (4.6) with ε in place of 16ε.

Case (ii): $a_1<b_1\leq 1/u$ and $1/v\leq a_2<b_2$. Applying Fubini's theorem, we find that

$$\begin{aligned}|I_{uv}(f;a_1,b_1;a_2,b_2)|&=\Big|\int_{a_1}^{b_1}(\sin ux)\Big(\int_{a_2}^{b_2}f(x,y)\sin vy\,dy\Big)\,dx\Big|\\&\leq\int_{a_1}^{b_1}ux\Big|\int_{a_2}^{b_2}f(x,y)\sin vy\,dy\Big|\,dx.\end{aligned} \tag{4.7}$$

Integrating by parts with respect to y gives

$$\begin{aligned}&\Big|\int_{a_2}^{b_2}f(x,y)\sin vy\,dy\Big|\\&\quad=\Big|\Big[-f(x,y)\frac{\cos vy}{v}\Big]_{y=a_2}^{b_2}+\int_{a_2}^{b_2}f_y(x,y)\frac{\cos vy}{v}\,dy\Big|\\&\quad\leq\frac{1}{v}\Big\{|f(x,a_2)|+|f(x,b_2)|+\int_{a_2}^{b_2}|f_y(x,y)|dy\Big\}\\&\quad\leq a_2|f(x,a_2)|+b_2|f(x,b_2)+a_2\int_{a_2}^{b_2}|f_y(x,y)|\,dy.\end{aligned} \tag{4.8}$$

Combining (4.7) and (4.8), it follows from (4.1) and (4.4) that

$$
\begin{aligned}
&|I_{uv}(f;a_1,b_1;a_2,b_2)|\\
&\qquad\le u\int_{a_1}^{b_1}\Big\{xa_2|f(x,a_2)|+xb_2|f(x,b_2)|+xa_2\int_{a_2}^{b_2}|f_y(x,y)|\,dy\Big\}\,dx\\
&\qquad\le u\int_{a_1}^{b_1}3\varepsilon\,dx=3\varepsilon u(b_1-a_1)\le 3\varepsilon,
\end{aligned}
$$

which is (4.6) with 3ε in place of 16ε.

CASE (iii): $a_1<b_1\le 1/u$ and $a_2<1/v<b_2$. In view of the decomposition

$$
I_{uv}(f;a_1,b_1;a_2,b_2)=\Big\{\int_{a_1}^{b_1}\int_{a_2}^{1/v}+\int_{a_1}^{b_1}\int_{1/v}^{b_2}\Big\}f(x,y)\sin ux\sin vy\,dx\,dy, \tag{4.9}
$$

the previous Cases (i) and (ii) give (4.6) with 4ε.

CASE (iv): $1/u\le a_1<b_1$ and $a_2<b_2\le 1/v$. This is the symmetric counterpart of Case (ii), but this time in the proof we use (4.3) instead of (4.4). Thus, we have (4.6) again with 3ε.

CASE (v): $1/u\le a_1<b_1$ and $1/v\le a_2<b_2$. Making use of Fubini's theorem and integrating by parts with respect to y (cf. (4.7) and (4.8)) gives

$$
\begin{aligned}
I_{uv}(f;a_1,b_1;a_2,b_2)&=\int_{a_1}^{b_1}(\sin ux)\Big\{f(x,a_2)\frac{\cos va_2}{v}\\
&\qquad -f(x,b_2)\frac{\cos vb_2}{v}+\int_{a_2}^{b_2}f_y(x,y)\frac{\cos vy}{v}\,dy\Big\}\,dx\\
&=\frac{1}{v}\Big\{(\cos va_2)\int_{a_1}^{b_1}f(x,a_2)\sin ux\,dx-(\cos vb_2)\int_{a_1}^{b_1}f(x,b_2)\sin ux\,dx\\
&\qquad+\int_{a_2}^{b_2}(\cos vy)\Big(\int_{a_1}^{b_1}f_y(x,y)\sin ux\,dx\Big)\,dy\Big\}=:J_1+J_2+J_3,\quad\text{say.}
\end{aligned}
\tag{4.10}
$$

Integrating by parts with respect to x, we obtain

$$
\begin{aligned}
|J_1|&\le\frac{1}{v}\Big|\int_{a_1}^{b_1}f(x,a_2)\sin ux\,dx\Big|\\
&=\frac{1}{v}\Big\{f(a_1,a_2)\frac{\cos ua_1}{u}-f(b_1,a_2)\frac{\cos ub_1}{u}+\int_{a_1}^{b_1}f_x(x,a_2)\frac{\cos ux}{u}\,dx\Big\}\\
&\le\frac{1}{uv}\Big\{|f(a_1,a_2)|+|f(b_1,a_2)|+\int_{a_1}^{b_1}|f_x(x,a_2)|\,dx\Big\}.
\end{aligned}
$$

Since this time $1/u \le a_1$ and $1/v \le a_2$, it follows from (4.1) and (4.3) that

$$|J_1| \le a_1a_2|f(a_1,a_2)| + b_1a_2|f(b_1,a_2)| + a_1a_2\int_{a_1}^{b_1}|f_x(x,a_2)|\,dx < 3\varepsilon. \tag{4.11}$$

In an analogous way, we conclude that

$$\begin{aligned} |J_2| &\le \frac{1}{v}\Big|\int_{a_1}^{b_1} f(x,b_2)\sin ux\,dx\Big| \\ &\le a_1b_2|f(a_1,b_2)| + b_1b_2|f(b_1,b_2)| + a_1b_2\int_{a_1}^{b_1}|f_x(x,b_2)|\,dx < 3\varepsilon. \end{aligned} \tag{4.12}$$

Finally, applying Fubini's theorem and integrating by parts twice, we find that

$$\begin{aligned} |J_3| &\le \frac{1}{v}\int_{a_2}^{b_2}\Big|\int_{a_1}^{b_1} f_y(x,y)\sin ux\,dx\Big|\,dy \\ &= \frac{1}{v}\int_{a_2}^{b_2}\Big|\Big[-f_y(x,y)\frac{\cos ux}{u}\Big]_{x=a_1}^{b_1} + \int_{a_1}^{b_1} f_{xy}(x,y)\frac{\cos ux}{u}\,dx\Big|\,dy \\ &= \frac{1}{uv}\int_{a_2}^{b_2}\Big|f_y(a_1,y)\cos ua_1 - f_y(b_1,y)\cos ub_1 + \int_{a_1}^{b_1} f_{xy}(x,y)\cos ux\,dx\Big|\,dy \\ &\le \frac{1}{uv}\int_{a_2}^{b_2}\Big\{|f_y(a_1,y)| + |f_y(b_1,y)| + \int_{a_1}^{b_1}|f_{xy}(x,y)|\,dx\Big\}\,dy \\ &\le a_1a_2\int_{a_2}^{b_2}|f_y(a_1,y)|dy + b_1a_2\int_{a_2}^{b_2}|f_y(b_1,y)|\,dy \\ &\quad + a_1a_2\int_{a_1}^{b_1}\int_{a_2}^{b_2}|f_{xy}(x,y)|\,dx\,dy < 3\varepsilon, \end{aligned} \tag{4.13}$$

due to (4.4) and (4.5).

Combining (4.10)–(4.13) gives (4.6) with 9ε.

CASE (vi): $1/u \le a_1 < b_1$ and $a_2 < 1/v < b_2$. By the decomposition in Case (iii) (see (4.9)), the previous Cases (iv) and (v) give (4.6) with 12ε.

CASE (vii): $a_1 < 1/u < b_1$ and $a_2 < b_2 \le 1/v$. This is the symmetric counterpart of Case (iii). Thus, we have (4.6) again with 4ε.

CASE (viii): $a_1 < 1/u < b_1$ and $1/v \le a_2 < b_2$. This is the symmetric counterpart of Case (vi). Thus, we have (4.6) again with 12ε.

Case (ix): $a_1 < 1/u < b_1$ and $a_2 < 1/v < b_2$. Again by the decomposition in Case (iii), the previous Cases (vii) and (viii) give (4.6) with 16ε as stated.

To sum up, we have proved (4.6) for all $(u,v) \in \mathbb{R}^2_+$. Since $\varepsilon > 0$ is arbitrary, the proof of Theorem 1 is complete.

Proof of Theorem 2. Given arbitrary $a, b > 0$, set

$$u := \frac{\pi}{2\lambda a} \quad \text{and} \quad v := \frac{\pi}{2\lambda b},$$

where λ is from the definition of the class $\mathrm{MVBVF}(\mathbb{R}^2_+)$. Clearly, for all x in the interval $a/2\lambda \le x \le \lambda a$ we have

$$\frac{\pi}{4\lambda^2} \le ux \le \frac{\pi}{2};$$

and analogously, for all $b/2\lambda \le y \le \lambda b$ we have

$$\frac{\pi}{4\lambda^2} \le vy \le \frac{\pi}{2}.$$

Using the nonnegativity of f and applying Lemma 3 yields

$$\begin{aligned} &\int_{a/2\lambda}^{\lambda a} \int_{b/2\lambda}^{\lambda b} f(x,y) \sin ux \sin vy \, dx\, dy \\ &\ge \left(\sin \frac{\pi}{4\lambda^2}\right)^2 \int_{a/2\lambda}^{\lambda} \int_{b/2\lambda}^{\lambda b} f(x,y)\, dx\, dy \ge \frac{1}{8C+2}\left(\sin\frac{\pi}{4\lambda^2}\right)^2 abf(a,b). \end{aligned} \tag{4.14}$$

By assumption, (1.3) is satisfied with $\phi(x,y) := f(x,y) \sin ux \sin vy$ uniformly in $(u,v) \in \mathbb{R}^2_+$. Consequently, the integral on the left-hand side in (4.14) converges to zero as $\max\{a,b\} \to \infty$. A fortiori, for all $a, b > 0$ we have

$$abf(a,b) \to 0 \quad \text{as } \max\{a,b\} \to \infty.$$

This proves (2.11). The proof of Theorem 2 is complete. ∎

Proof of Theorem 3. (i) If $f \in \mathrm{NBVF}(\mathbb{R}^2_+)$, then conditions (2.13)–(2.15) are satisfied. By (2.15), for all $s, t > 0$ we have

$$\begin{aligned} &\int_{s/2}^{2s} \int_{t/2}^{2t} |f_{xy}(x,y)|\, dx\, dy \\ &= \left\{ \int_{s/2}^{s} \int_{t/2}^{t} + \int_{s}^{2s} \int_{t/2}^{t} + \int_{s/2}^{s} \int_{t}^{2t} + \int_{s}^{2s} \int_{t}^{2t} \right\} |f_{xy}(x,y)|\, dx\, dy \end{aligned} \tag{4.15}$$

$$\begin{aligned}
&\le \{|f(s/2,t/2)|+|f(s,t/2)|+|f(s/2,t||+|f(s,t)|\\
&\quad +|f(s,t/2)|+|f(2s,t/2)|+|f(s,t)|+|f(2s,t)|\\
&\quad +|f(s/2,t)|+|f(s,t)|+|f(s/2,2t)|+|f(s,2t)|\\
&\quad +|f(s,t)|+|f(2s,t)|+|f(s,2t)|+|f(2s,2t)|\}\\
&= C\{|f(s/2,t/2)|+2|f(s,t/2)|+2|f(s/2,t)|+4|f(s,t)|\\
&\quad +|f(2s,t/2)|+|f(s/2,2t)|+2|f(2s,t)|+2|f(s,2t)|+|f(2s,2t)|\}\\
&=: CA(s,t), \quad \text{say.}
\end{aligned}$$

Integrating both sides of this inequality with respect to s and t over $[3a/2,2a]$ and $[3b/2,2b]$, respectively, gives

$$\begin{aligned}
(4.16)\qquad &\int_{3a/2}^{2a}\int_{3b/2}^{2b}\Big\{\int_{s/2}^{2s}\int_{t/2}^{2t}|f_{xy}(x,y)|dx\,dy\Big\}\,ds\,dt\\
&\qquad\qquad \le \int_{3a/2}^{2a}\int_{3b/2}^{2b} A(s,t)\,ds\,dt, \quad a,b\ge 0.
\end{aligned}$$

Now, we observe that if $3a/2 \le s \le 2a$, then $s/2 \le a$ and $2s \ge 3a$; and analogous inequalities hold for t if $3b/2 \le t \le 2b$; that is,

$$\begin{aligned}
[s/2,2s] &\supset [a,2a] \quad \text{whenever} \quad s\in[3a/2,2a],\\
[t/2,2t] &\supset [b,2b] \quad \text{whenever} \quad t\in[3b/2,2b].
\end{aligned}$$

Consequently, we conclude that

$$\begin{aligned}
(4.17)\qquad &\int_{3a/2}^{2a}\int_{3b/2}^{2b}\Big\{\int_{s/2}^{2s}\int_{t/2}^{2t}|f_{xy}(x,y)|\,dx\,dy\Big\}\,ds\,dt\\
&\ge \int_{3a/2}^{2a}\int_{3b/2}^{2b}\Big\{\int_{a}^{2a}\int_{b}^{2b}|f_{xy}(x,y)|\,dx\,dy\Big\}\,ds\,dt = \frac{ab}{4}\int_{a}^{2a}\int_{b}^{2b}|f_{xy}(x,y)|\,dx\,dy.
\end{aligned}$$

On the other hand, by (4.15) we can estimate as follows (integrating by substitution):

$$\begin{aligned}
(4.18)\qquad &\int_{3a/2}^{2a}\int_{3b/2}^{2b} A(s,t)\,ds\,dt\\
&= C\Big\{4\int_{3a/4}^{a}\int_{3b/4}^{b}|f(s_1,t_1)|\,ds_1\,dt_1 + 4\int_{3a/2}^{2a}\int_{3b/4}^{b}|f(s,t_1)|\,ds\,dt_1\\
&\quad + 4\int_{3a/4}^{a}\int_{3b/2}^{2b}|f(s_1,t)\,ds_1\,dt + 4\int_{3a/2}^{2a}\int_{3b/2}^{2b}|f(s,t)|\,ds\,dt
\end{aligned}$$

$$
\begin{aligned}
&+\int_{3a}^{4a}\int_{3b/4}^{b}|f(s_1,t_1)|\,ds_1\,dt_1+\int_{3a/4}^{a}\int_{3b}^{4b}|f(s_1,t_1)|\,ds_1\,dt_1\\
&+\int_{3a/2}^{2a}\int_{3b}^{4b}|f(s,t_1)|\,ds\,dt_1+\int_{3a}^{4a}\int_{3b/2}^{2b}|f(s_1,t)|\,ds_1\,dt\\
&+\frac{1}{4}\int_{3a}^{4a}\int_{3b}^{4b}|f(s_1,t_1)|\,ds_1\,dt_1\Big\}\\
\le{}& 4C\int_{3a/4}^{4a}\int_{3b/4}^{4b}|f(s,t)|\,ds\,dt,
\end{aligned}
$$

where we exploited the fact that the integrals in the braces $\{\cdot\}$ above are over disjoint domains.

Combining (4.16)–(4.18) yields

$$
\frac{ab}{4}\int_{a}^{2a}\int_{b}^{2b}|f_{xy}(x,y)|\,dx\,dy\le 4C\int_{3a/4}^{4a}\int_{3b/4}^{4b}|f(s,t)|\,ds\,dt,
$$

which is equivalent to (2.10) with $16C$ in place of C and $\lambda := 4$.

As to the fulfillment of conditions (2.8) and (2.9), we apply Lemma 2 for $g(x) := f(x,y)$ where $y>0$ is fixed, then for $g(y) := f(x,y)$ where $x>0$ is fixed. As a result, we conclude that (2.8) follows from (2.13), while (2.9) follows from (2.14), with $4C$ in place of C and $\lambda := 4$ in both cases.

(ii) Define

$$
g(x) := \frac{1}{1+x}\sin\Big(\frac{\pi}{\ln 2}\ln x\Big),\qquad x\in\mathbb{R}_+.
$$

Clearly, $g(x)\to 0$ as $x\to\infty$, $xg(x)\in L^1_{\text{loc}}(\overline{\mathbb{R}}_+)$, and

$$
g'(x)=\frac{\pi}{\ln 2}\,\frac{1}{x(1+x)}\cos\Big(\frac{\pi}{\ln 2}\ln x\Big)-\frac{1}{(1+x)^2}\sin\Big(\frac{\pi}{\ln 2}\ln x\Big).
$$

It follows that $g\in \mathrm{AC}_{\text{loc}}(\mathbb{R}_+)$. Since

$$
g(2^k)=\frac{1}{1+2^k}\sin k\pi=0,\qquad k=1,2,\dots,
$$

g cannot belong to the class $\mathrm{NBVF}(\mathbb{R}_+)$.

On the other hand, we claim that g belongs to $\mathrm{MVBVF}(\mathbb{R}_+)$. Indeed, for any $a>0$, an elementary argument gives

$$
\begin{aligned}
&\int_a^{2a} |g'(x)|\,dx \\
&\quad \le \frac{\pi}{\ln 2}\frac{1}{a}\int_a^{2a}\frac{1}{1+x}\left|\cos\left(\frac{\pi}{\ln 2}\ln x\right)\right| dx + \frac{1}{a}\int_a^{2a}\frac{1}{1+x}\left|\sin\left(\frac{\pi}{\ln 2}\ln x\right)\right| dx \\
&\quad = \frac{\pi}{\ln 2}\frac{1}{a}\int_a^{2a}\frac{1}{1+x}\left|\sin\left(\frac{\pi}{\ln 2}\ln x+\frac{\pi}{2}\right)\right| dx + \frac{1}{a}\int_a^{2a}|g(x)|\,dx \\
&\quad = \frac{\pi}{\ln 2}\frac{1}{a}\int_{\sqrt{2}a}^{2\sqrt{2}a}\frac{1}{\sqrt{2}+u}\left|\sin\left(\frac{\pi}{\ln 2}\ln u\right)\right| du + \frac{1}{a}\int_a^{2a}|g(x)|\,dx \\
&\quad \le C\int_{\lambda^{-1}a}^{\lambda a}|g(u)|\,du, \quad \text{where} \quad u := \sqrt{2}x,\ C := \frac{\pi}{\ln 2}+1,\ \lambda := 2\sqrt{2}.
\end{aligned}
$$

Now, define

$$f(x,y) := g(x)g(y), \qquad (x,y)\in \mathbb{R}^2_+.$$

It is easy to check that $f \notin \mathrm{NBVF}(\mathbb{R}^2_+)$, but $f \in \mathrm{MVBVF}(\mathbb{R}^2_+)$.

The proof of Theorem 3 is complete. ■

Proof of Theorem 4. It runs along the same lines as the proof of Theorem 2, with the modification that this time $\min\{a,b\} \to \infty$ (instead of $\max\{a,b\} \to \infty$).

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