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On the functional properties of Bessel zeta-functions

by

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1. Introduction. Let $\theta > 0$, $\nu \in \mathbb{C}$ and s be a complex variable. Our main object is to investigate the following Dirichlet series:

$$\mathbf{J}_{\nu-1}(s;\theta) := \sum_{n=1}^{\infty} \frac{J_{\nu-1}(2\sqrt{\theta n})}{n^{s+(\nu+1)/2}}, \tag{1.1}$$

where $J_\nu(z)$ denotes the J-Bessel function (cf. [Er2, 7.2.1, (2)]). In the present paper, we describe an integral representation, a transformation formula and a power series expansion involving the Riemann zeta-function via the Bromwich integrals (Theorem 1.1). These functional properties of $\mathbf{J}_{\nu-1}(s;\theta)$ show that (1.1) is one of the artless zeta-functions. We therefore call $\mathbf{J}_{\nu-1}(s;\theta)$ the *J-Bessel zeta-function* of order $\nu-1$.

The J-Bessel zeta-function appears in the Fourier series expansion of the Poincaré series attached to $\mathrm{SL}(2,\mathbb{Z})$ by applying the inverse Mellin transform. This fact strongly suggested that $\mathbf{J}_{\nu-1}(s;\theta)$ should have a kind of functional equation. The inverse Laplace transform of Weber's first exponential integral (Lemma 2.2 below) is the key ingredient in the proof of the integral expression for $\mathbf{J}_{\nu-1}(s;\theta)$, which leads to the expected transformation formula (Theorem 1.1, (1.4)). Since (1.4) holds on the left half s-plane $\mathrm{Im}(s) < -1$, our transformation formula can be regarded as a kind of Hurwitz zeta-type functional equation. The integral expression also indicates that the J-Bessel function is a new generating function of the Riemann zeta-function, namely a power series expansion involving the Riemann zeta-function in its coefficients (Theorem 1.1, (1.5)). As one application of these results, in Section 4 below, we will give a new proof of the Fourier expansion of Poincaré series attached to $\mathrm{SL}(2,\mathbb{Z})$, which shows that the J-Bessel zeta-function plays a similar role to the Hurwitz zeta-function in the theory of (holomorphic and non-holomorphic) Eisenstein series.

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In addition, the Bessel zeta-function $\mathbf{J}_{\nu-1}(s;\theta)$ naturally inherits the recurrence relation for $J_{\nu-1}(2\sqrt{\theta n})$ (Theorem 1.1, (1.6)). Furthermore, the techniques employed in the proof of our main results allow us to deduce integral representations and transformation formulas for several new zeta-functions (cf. [No]).

We remark that Kaczorowski and Perelli [KP4] treated more general zeta-functions twisted by hypergeometric or Bessel functions. They derived meromorphic continuations of these zeta-functions via the properties of non-linear twists obtained in [KP1]–[KP3].

In this paper, we discuss the *K-Bessel zeta-function* as well, defined by the following Dirichlet series:

$$\mathbf{K}_{\nu-1}(s;\theta) := \sum_{n=1}^{\infty} \frac{K_{\nu-1}(2\sqrt{\theta n})}{n^{s+(\nu+1)/2}}. \tag{1.2}$$

Here $K_\nu(z)$ is the K-Bessel function (cf. [Er2, 7.2.2, (13)]), and (1.2) is an entire function of s for any given $\theta > 0$ and $\nu \in \mathbb{C}$.

Let $\Gamma(s)$ be the Gamma function and $F(\alpha;\gamma;z)$ be Kummer's confluent hypergeometric function of the first kind defined by

$$F(\alpha;\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\alpha)\Gamma(\gamma-\alpha)} \int_0^1 e^{zu} u^{\alpha-1}(1-u)^{\gamma-\alpha-1}\, du$$

for $\mathrm{Re}(\gamma) > \mathrm{Re}(\alpha) > 0$ (see [Er1, 6.5, (1)]), and $U(\alpha;\gamma;z)$ be the confluent hypergeometric function of the second kind defined by

$$U(\alpha;\gamma;z) = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-zu} u^{\alpha-1}(1+u)^{\gamma-\alpha-1}\, du$$

for $\mathrm{Re}(\alpha) > 0$ and $|\arg(z)| < \pi/2$ (see [Er1, 6.5, (2)]).

Throughout this paper, $\zeta(s)$ denotes the Riemann zeta-function, $\int_{-\infty}^{(0+)}$ denotes integration over a Hankel contour, starting at negative infinity on the real axis, encircling the origin with a small radius in the positive direction, and returning to the starting point, and $\int_{(c)}$ denotes an integral over the vertical straight path from $c - i\infty$ to $c + i\infty$.

By the estimates of the J-Bessel function (Proposition 3.1 below), the Dirichlet series (1.1) converges absolutely in the region $\mathrm{Re}(s) > 0$ when $\mathrm{Re}(\nu) > 1/2$, and also in the region $\mathrm{Re}(s) > \lfloor 3/2 - \mathrm{Re}(\nu) \rfloor/2 - 1$ when $\mathrm{Re}(\nu) \le 1/2$. When ν is an integer, $\mathbf{J}_{\nu-1}(s;\theta)$ converges absolutely for $\mathrm{Re}(s) > (1-\nu)/2$.

Theorem 1.1. *Let $\nu \in \mathbb{C}$ and $\theta > 0$. The J-Bessel zeta-function has an integral representation*

$$(1.3)\qquad \mathbf{J}_{\nu-1}(s;\theta) = \frac{\theta^{s+(\nu+1)/2}\Gamma(-s)}{2\pi i\Gamma(\nu)} \int_{-\infty}^{(0+)} \frac{u^s e^{\theta u}}{1-e^{\theta u}} F(-s;\nu;-u^{-1})\,du,$$

which provides a meromorphic continuation to the whole s-plane. Further, the transformation formula

$$(1.4)\qquad \mathbf{J}_{\nu-1}(s;\theta) = \frac{\theta^{(\nu-1)/2}\Gamma(-s)}{\Gamma(\nu)} \sum_{n=-\infty,\, n\neq 0}^{\infty} (2\pi i n)^s F\biggl(-s;\nu;\frac{-\theta}{2\pi i n}\biggr)$$

holds for $\mathrm{Re}(s) < -1$, *and the power series expansion*

$$(1.5)\qquad \mathbf{J}_{\nu-1}(s;\theta) = \sum_{m=0}^{\infty} \frac{\theta^{(\nu-1)/2}}{\Gamma(\nu+m)m!}\zeta(s+1-m)(-\theta)^m,$$

holds for $s \in \mathbb{C} \setminus \{0,1,2,\ldots\}$. *The J-Bessel zeta-function also satisfies the following recurrence formula:*

$$(1.6)\qquad \mathbf{J}_{\nu-1}(s;\theta) + \mathbf{J}_{\nu+1}(s-1;\theta) = \frac{\nu}{\sqrt{\theta}}\mathbf{J}_{\nu}(s;\theta).$$

Remark. The power series expression (1.5) is a generalization of Ramanujan's formula [Ra] (the binomial type power series)

$$(1.7)\qquad \zeta(s,1+x) = \sum_{m=0}^{\infty} \frac{\Gamma(s+m)}{\Gamma(s)m!}\zeta(s+m)(-x)^m$$

for $|x| < 1$ and $s \in \mathbb{C}\setminus\{1\}$. Here $\zeta(s,x)$ is the Hurwitz zeta-function. An exponential type series was found by Chowla and Hawkins [CH], and Gauss' hypergeometric type and Kummer's confluent hypergeometric type series were introduced by Katsurada [Ka1]. In [Ka2, Theorem 5.1], some Dirichlet series whose coefficients involve hypergeometric functions were introduced as generating functions of $\zeta(s)$. Boudjelkha [Bo] provided functional equations and power series expansions of a kind of Bessel series via the Schläfli–Sonine integral representations. For related results and generalizations of Ramanujan's formula (1.7), we refer to [SC].

Theorem 1.2. *For every complex* ν *and* $\theta > 0$, *the K-Bessel zeta-function is an entire function of s and has an integral representation*

$$(1.8)\qquad \mathbf{K}_{\nu-1}(s;\theta) = \frac{\theta^{s+(\nu+1)/2}}{4\pi i}\Gamma(-s)\Gamma(-s-\nu+1) \int_{-\infty}^{(0+)} \frac{u^s e^{\theta u}}{1-e^{\theta u}} U(-s;\nu;u^{-1})\,du,$$

which provides the transformation formula

$$\text{(1.9)}\qquad \mathbf{K}_{\nu-1}(s;\theta) = \frac{\theta^{(\nu-1)/2}}{2}\Gamma(-s)\Gamma(-s-\nu+1)\sum_{n=-\infty,\, n\neq 0}^{\infty}(2\pi in)^s U\Big(-s;\nu;\frac{\theta}{2\pi in}\Big)$$

for $\mathrm{Re}(s) < \min\{-1, -\nu\}$.

2. Preliminary results. In this section, we establish some Fourier–Mellin integrals of confluent hypergeometric functions, which are equivalent to the inverse integral transforms of Weber's first exponential-type integrals (cf. [Wa, 13.3]). Formally, equalities (2.3) and (2.4) in Lemma 2.2 below are derived by the inverse Laplace transforms of Bessel functions (cf. [Er1, 6.10, (8), (9)]). We examine these integral transforms directly starting from the Mellin–Barnes integrals of Bessel functions and confluent hypergeometric functions.

First, we quote Barnes' integral representations of the modified Bessel functions.

LEMMA 2.1. *The J-Bessel function has an integral representation*

$$\text{(2.1)}\qquad J_{\nu-1}(Z) = \frac{1}{2\pi i}\int_{(c_1)} \frac{\Gamma(-w)}{\Gamma(w+\nu)}\Big(\frac{Z}{2}\Big)^{2w+\nu-1}\,dw$$

for $Z > 0$ and $(1-\mathrm{Re}(\nu))/2 < c_1 < 0$, and the K-Bessel function has an expression

$$\text{(2.2)}\qquad K_{\nu-1}(Z) = \frac{1}{4\pi i}\int_{(c_2)} \Gamma(w)\Gamma(w-\nu+1)\Big(\frac{Z}{2}\Big)^{-2w+\nu-1}\,dw$$

for $Z > 0$ and $c_2 > \max\{0, \mathrm{Re}(\nu)-1\}$.

Proof. Noting that the pair

$$Z^{-\nu}J_\nu(Z), \qquad \frac{2^{s-\nu-1}\Gamma(s/2)}{\Gamma(\nu+1-s/2)} \qquad (0 < \mathrm{Re}(s) < \nu + 3/2)$$

is a pair of Mellin transforms (cf. [Ti1, (7.9.1)]), we obtain the integral (2.1) taking $s/2 = -w$ and replacing ν by $\nu-1$. Under the condition $(1-\mathrm{Re}(\nu))/2 < c_1 < 0$, the integral (2.1) is absolutely convergent. Similarly,

$$Z^{-\nu}K_\nu(Z), \qquad 2^{s-\nu-2}\Gamma(s/2)\Gamma(s/2-\nu) \qquad (\mathrm{Re}(s) > \max\{0, 2\mathrm{Re}(\nu)\})$$

is a pair of Mellin transforms (cf. [Ti1, (7.9.11)]). Taking $s/2 = w$ and replacing ν by $\nu-1$, we obtain the representation (2.2). ∎

The following lemma is crucial in the proof of our main results.

LEMMA 2.2. *Let $T > 0$. The integral representations*

$$T^{\alpha-(\gamma+1)/2} J_{\gamma-1}(2\sqrt{T}) = \frac{\Gamma(\alpha)}{2\pi i \Gamma(\gamma)} \int_{-\infty}^{(0+)} u^{-\alpha} e^{Tu} F(\alpha;\gamma;-u^{-1})\, du, \tag{2.3}$$

$$\begin{aligned} &T^{\alpha-(\gamma+1)/2} K_{\gamma-1}(2\sqrt{T}) \\ &\qquad = \frac{\Gamma(\alpha)\Gamma(\alpha-\gamma+1)}{4\pi i} \int_{-\infty}^{(0+)} u^{-\alpha} e^{Tu} U(\alpha;\gamma;u^{-1})\, du \end{aligned} \tag{2.4}$$

hold for $(\alpha,\gamma) \in \mathbb{C}^2$, and represent entire functions both of α and of γ.

Proof. First, we show the following formula: for $c_3 > 0$,

$$T^{\alpha-(\gamma+1)/2} J_{\gamma-1}(2\sqrt{T}) = \frac{\Gamma(\alpha)}{2\pi i \Gamma(\gamma)} \int_{(c_3)} u^{-\alpha} e^{Tu} F(\alpha;\gamma;-u^{-1})\, du. \tag{2.3b}$$

Temporarily, we assume $1 - \mathrm{Re}(\alpha) < 0$ and $1/2 + \mathrm{Re}(\alpha) - \mathrm{Re}(\gamma) < 0$. Kummer's confluent hypergeometric function has an integral representation

$$F(\alpha;\gamma;z) = \frac{1}{2\pi i} \frac{\Gamma(\gamma)}{\Gamma(\alpha)} \int_{(c_4)} \frac{\Gamma(-w)\Gamma(\alpha+w)}{\Gamma(\gamma+w)} (-z)^w\, dw \tag{2.5}$$

for $-\pi/2 < \arg(-z) < \pi/2$ and $-\mathrm{Re}(\alpha) < c_4 < 0$ (cf. [Er1, 6.5, (4)]). Taking $z = u^{-1}$ and substituting (2.5) into the integrand of (2.3b), we observe that the right side of (2.3b) is equal to

$$\begin{aligned} &\frac{1}{(2\pi i)^2} \int_{(c_3)} e^{Tu} u^{-\alpha} \int_{(c_4)} \frac{\Gamma(-w)\Gamma(\alpha+w)}{\Gamma(\gamma+w)} u^{-w}\, dw\, du \\ &\qquad = \frac{1}{(2\pi i)^2} \int_{(c_4)} \frac{\Gamma(-w)\Gamma(\alpha+w)}{\Gamma(\gamma+w)} \int_{(c_3)} e^{Tu} u^{-(\alpha+w)}\, du\, dw. \end{aligned} \tag{2.6}$$

The interchange of the order of integration is justified by the absolute convergence of the double integrals in (2.6) when $1 - \mathrm{Re}(\alpha) < 0$ and $1/2 + \mathrm{Re}(\alpha) - \mathrm{Re}(\gamma) < c_4$ due to Stirling's formula. The u-integral on the right side of (2.6) is evaluated by using Laplace's integral representation (cf. [WW, 12.22, ex. 1])

$$\frac{1}{2\pi i} \int_{(c_3)} e^u u^{-s}\, du = \frac{1}{\Gamma(s)},$$

for a positive constant c_3 and $\mathrm{Re}(s) > 1$. Hence the right side of (2.6) is equal to

$$\frac{T^{\alpha-1}}{2\pi i} \int_{(c_4)} \frac{\Gamma(-w)}{\Gamma(\gamma+w)} T^w\, dw. \tag{2.7}$$

Applying Barnes' representation (2.1) of the J-Bessel function to (2.7), we obtain (2.3b) under the assumptions $1-\mathrm{Re}(\alpha)<0$ and $1/2+\mathrm{Re}(\alpha)-\mathrm{Re}(\gamma)<0$. Because the confluent hypergeometric series $F(\alpha;\gamma;z)$ converges for all finite z and defines an entire function of z, we may modify the integration path in (2.3b) to Hankel's contour in (2.3) under the assumption $1-\mathrm{Re}(\alpha)<0$.

The analytic continuation is established by the fact that $F(\alpha;\gamma;z)/\Gamma(\gamma)$ is entire both in α and in γ (cf. [Er1, 6.7.1]). Hence, the right side of (2.3) is a meromorphic function of α and an entire function of γ. The holomorphy in α is ascertained by the left side of (2.3).

The proof of (2.4) is similar. Substituting the integral representation of $U(\alpha;\gamma;z)$ (cf. [Er1, 6.5, (5)]) into the integrand of (2.4), we exchange the order of the integrals. Applying Hankel's integral (cf. [Er1, 1.6, (2)]) and Barnes' integral representation (2.2) of the K-Bessel function, we obtain (2.4) for $\mathrm{Re}(\alpha)>\max\{0,\mathrm{Re}(\gamma)-1\}$. The analytic continuation is carried out similarly to the proof of (2.3) by applying the connection formula (cf. [Er1, 6.5, (7)]) to the integrand of (2.4). ∎

3. Proofs of theorems. First, to determine the domain of absolute convergence of the J-Bessel zeta-function (1.1), we show some estimations of $J_{\nu-1}(Z)$, which are derived from Poisson's integral and Bessel's representation of the J-Bessel function (for ν real, these estimates are described in [Wa, 3.31], [Er2, 7.3.2] and [Ol, §2, ex. 9.6]).

PROPOSITION 3.1. *Let $Z=X+iY\in\mathbb{C}$. For given $\nu\in\mathbb{C}$, there are positive constants C_1, C_2 and C_3 depending only on ν which satisfy*

$$|J_{\nu-1}(Z)|\le C_1e^{|Y|}|Z^{\nu-1}| \tag{3.1}$$

for $\mathrm{Re}(\nu)>1/2$, *and*

$$|J_{\nu-1}(Z)|\le e^{|Y|}\{C_2|Z^{\nu+N_0-1}|+C_3|Z^{\nu+N_0+1}|\} \tag{3.2}$$

for $\mathrm{Re}(\nu)\le 1/2$. *Here $N_0=\lfloor 3/2-\mathrm{Re}(\nu)\rfloor$ denotes the positive integer such that* $1/2<\mathrm{Re}(\nu)+N_0\le 3/2$.

If the order is an integer, then

$$|J_m(Z)|\le e^{|Y|}\qquad (m\in\mathbb{Z}). \tag{3.3}$$

Proof. Starting from Poisson's integral representation

$$J_{\nu-1}(Z)=\frac{(Z/2)^{\nu-1}}{\pi^{1/2}\Gamma(\nu-1/2)}\int_{-\pi/2}^{\pi/2}\exp(iZ\sin\theta)\cos^{2(\nu-1)}\theta\,d\theta,$$

which holds for $\mathrm{Re}(\nu) > 1/2$ (cf. [Er2, 7.12, (6)]), we see that

$$|J_{\nu-1}(Z)| \le \frac{e^{|Y|}|(Z/2)^{\nu-1}|}{\pi^{1/2}|\Gamma(\nu-1/2)|}\int_{-\pi/2}^{\pi/2} |\cos^{2(\nu-1)}\theta|\,d\theta$$
$$= \frac{e^{|Y|}|(Z/2)^{\nu-1}|\Gamma(\mathrm{Re}(\nu)-1/2)}{|\Gamma(\nu-1/2)|\Gamma(\mathrm{Re}(\nu)+1)}.$$

Here we have used an integral expression for the beta-function:

$$B(z_1,z_2) = \frac{\Gamma(z_1)\Gamma(z_2)}{\Gamma(z_1+z_2)} = 2\int_0^{\pi/2} (\sin\theta)^{2z_1-1}(\cos\theta)^{2z_2-1}\,d\theta$$

for $\mathrm{Re}(z_j) > 0$ $(j = 1, 2)$ (cf. [Er1, 1.5.1, (19)]). Taking $C_1 = |2^{1-\nu}|\Gamma(\mathrm{Re}(\nu)-1/2)/|\Gamma(\nu-1/2)|\Gamma(\mathrm{Re}(\nu)+1)$, we obtain the assertion of (3.1).

For $\mathrm{Re}(\nu) \le 1/2$, we apply (3.1) to the recurrence formula (cf. [Wa, 3.2 (1)], [Er2, 7.2.8, (56)])

$$J_{\nu-1}(Z) + J_{\nu+1}(Z) = \frac{2\nu}{Z}J_\nu(Z), \tag{3.4}$$

and achieve the estimate (3.2) by induction.

The inequality (3.3) is a consequence of Bessel's integral representation for $J_m(Z)$ (cf. [Er2, 7.3.1, (2)]), which holds for integer m:

$$J_m(Z) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \exp(-im\theta + iZ\sin\theta)\,d\theta.$$

Thus, the proof of Proposition 3.1 is complete. ■

Now, we prove our main theorems.

Proof of Theorem 1.1. For $z \in \mathbb{C}$, we define the function

$$\mathbf{J}_{\nu-1}(z,s;\theta) := \sum_{n=1}^{\infty} \frac{J_{\nu-1}(2\sqrt{\theta n})}{n^{s+(\nu+1)/2}} z^n. \tag{3.5}$$

By Proposition 3.1, for given $\theta > 0$ and $(\nu, s) \in \mathbb{C}^2$, the right side of (3.5) converges absolutely when $|z| < 1$, and by Lemma 2.2, (3.5) is equal to

$$\frac{\theta^{s+(\nu+1)/2}\Gamma(-s)}{2\pi i\Gamma(\nu)} \sum_{n=1}^{\infty} \int_{-\infty}^{(0+)} z^n u^s e^{\theta n u} F(-s;\nu;-u^{-1})\,du. \tag{3.6}$$

Here, we let r be the small radius around the origin on the path of the integral above. Then the interchange of summation and integration is justified when $|ze^{\theta u}| < 1$. Accordingly, (3.6) is equal to

$$\frac{\theta^{s+(\nu+1)/2}\Gamma(-s)}{2\pi i\Gamma(\nu)} \int_{-\infty}^{(0+)} \frac{zu^s}{e^{-\theta u} - z} F(-s;\nu;-u^{-1})\,du. \tag{3.7}$$

The integral above converges absolutely if $z \neq e^{-\theta u}$ for $u \in U_r = \{u \in \mathbb{C} \mid u \in (-\infty, -r) \text{ or } |u| = r\}$, hence (3.7) represents a holomorphic function of z and a meromorphic function of s when $z \neq e^{-\theta u}$ with $u \in U_r$. Therefore, (3.7) gives an analytic continuation of $\mathbf{J}_{\nu-1}(z, s; \theta)$ in both z and s. In particular, $\mathbf{J}_{\nu-1}(z, s; \theta)$ is holomorphic at $z = 1$, which provides the equality (1.3) in Theorem 1.1.

Next, we deform the Hankel contour (1.3) so as to prove the transformation formula (1.4). Let R be a sufficiently large integer and C_R be the integration path that starts at negative infinity on the real axis, encircles the origin with radius $2\pi(R + 1/2)/\theta$ in the positive direction, and returns to the starting point. In transforming the integration path, the contour C_R passes simple poles at $u = 2\pi i n/\theta$ ($n = \pm 1, \pm 2, \ldots, \pm R$) with residues $(-1/\theta)(2\pi i n/\theta)^s F(-s; \nu; -\theta/2\pi i n)$. Then, by the residue theorem, we have

$$\text{(3.8)} \qquad \frac{1}{2\pi i} \int_{-\infty}^{(0+)} \frac{u^s}{e^{-\theta u} - 1} F(-s; \nu; -1/u)\, du =$$

$$\frac{1}{\theta} \sum_{n=-R,\, n\neq 0}^{R} \left(\frac{2\pi i n}{\theta}\right)^s F\left(-s; \nu; -\frac{\theta}{2\pi i n}\right) + \frac{1}{2\pi i} \int_{C_R} \frac{u^s}{e^{-\theta u} - 1} F(-s; \nu; -1/u)\, du.$$

Because $F(-s; \nu; Z)$ is holomorphic at $Z = 0$, $F(-s; \nu; -1/u)$ is bounded on C_R, and also $(e^{-\theta u} - 1)^{-1}$ is bounded except on neighborhoods of the points where $\theta u = 2\pi i m$ ($m \in \mathbb{Z}$). Hence, there exists a positive constant A depending only on R such that

$$\left| \frac{u^s}{e^{-\theta u} - 1} F(-s; \nu; -1/u) \right| \leq A R^{\mathrm{Re}(s)} e^{\pi\, \mathrm{Im}(s)}$$

on the contour C_R. Taking $R \to \infty$ in (3.8), we obtain

$$\begin{aligned} &\frac{1}{2\pi i} \int_{-\infty}^{(0+)} \frac{u^s}{e^{-\theta u} - 1} F(-s; \nu; -1/u)\, du \\ &\qquad = \frac{1}{\theta} \sum_{n=-\infty,\, n\neq 0}^{\infty} \left(\frac{2\pi i n}{\theta}\right)^s F\left(-s; \nu; -\frac{\theta}{2\pi i n}\right) \end{aligned}$$

for $\mathrm{Re}(s) < -1$, which provides the transformation formula (1.4).

In order to prove (1.5), we start with Kummer's series

$$\text{(3.9)} \qquad F\left(-s; \nu; -\frac{1}{u}\right) \cdot \frac{\Gamma(-s)}{\Gamma(\nu)} = \sum_{m=0}^{\infty} \frac{\Gamma(-s + m)}{\Gamma(\nu + m)\Gamma(m + 1)} \left(-\frac{1}{u}\right)^m,$$

which converges absolutely for any $\nu \in \mathbb{C}$ and $s \in \mathbb{C} \setminus \{0, 1, 2, \ldots\}$ (cf. [Er1, 6.1, (1)], [Ol, Chap. 7, §9, (9.04)]). Substituting (3.9) into the integrand in (1.3) and exchanging the order of integration and summation, we see that

$\mathbf{J}_{\nu-1}(s;\theta)$ is equal to

$$\frac{\theta^{s+(\nu+1)/2}}{2\pi i}\sum_{m=0}^{\infty}\frac{(-1)^m\Gamma(-s+m)}{\Gamma(\nu+m)\Gamma(m+1)}\int_{-\infty}^{(0+)}\frac{u^{s-m}e^{\theta u}}{1-e^{\theta u}}\,du. \tag{3.10}$$

By employing the integral representation of the Riemann zeta-function (see [Ti2, (2.4.2)]),

$$\zeta(s)=\frac{e^{-\pi is}\Gamma(1-s)}{2\pi i}\int_{+\infty}^{(0+)}\frac{v^{s-1}e^{-v}}{1-e^{-v}}\,dv,$$

with s replaced by $s-m+1$ and changing the variable according to $v=e^{i\pi}\theta u$, we have

$$\zeta(s-m+1)=\frac{\theta^{s-m+1}\Gamma(-s+m)}{2\pi i}\int_{-\infty}^{(0+)}\frac{u^{s-m}e^{\theta u}}{1-e^{\theta u}}\,du. \tag{3.11}$$

Substituting (3.11) into (3.10), we obtain the power series expansion (1.5).

The formula (1.6) is a direct consequence of the recurrence relation (3.4). This completes the proof of Theorem 1.1. ∎

Proof of Theorem 1.2. The asymptotic expansion of the K-Bessel function (cf. [Wa, 3.71, (12)], [Er2, 7.4.1, (1)]) yields

$$K_{\nu-1}(Z)=\left(\frac{\pi}{2Z}\right)^{1/2}e^{-Z}\{1+O(Z^{-1})\} \tag{3.12}$$

for $|\arg(Z)|<3\pi/2$. Here the O-constant depends only on ν. Due to the exponential decay above, the Dirichlet series (1.2) converges absolutely over the whole complex s-plane, so $\mathbf{K}_{\nu-1}(s;\theta)$ is an entire function of s for given $\theta>0$ and $\nu\in\mathbb{C}$. The integral representation (1.8) and the transformation formula (1.9) are obtained in a similar way to Theorem 1.1, replacing the role of the J-Bessel function by the K-Bessel function. ∎

4. Relation to the Poincaré series. Let $m\in\mathbb{Z}_{>0}$ and let $H=\{z\in\mathbb{C}\mid \mathrm{Im}(z)>0\}$ be the complex upper half-plane. We denote $\gamma(z)=(az+b)/(cz+d)$ for $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})$, and use the notation $\mathrm{e}(z)=\exp(2\pi iz)$. Let $k\geq 4$ be an integer, and define the mth *Poincaré series* attached to $\mathrm{SL}_2(\mathbb{Z})$ of weight k by

$$P_m^k(z):=(-1)^k\sum_{\{c,d\}}\frac{\mathrm{e}(m\gamma(z))}{(cz+d)^k}. \tag{4.1}$$

Here the summation is taken over $\gamma=\left(\begin{smallmatrix}*&*\\c&d\end{smallmatrix}\right)$, a complete system of representatives of $\left\{\left(\begin{smallmatrix}*&*\\0&*\end{smallmatrix}\right)\in\mathrm{SL}_2(\mathbb{Z})\right\}\backslash\mathrm{SL}_2(\mathbb{Z})\cong\{(c,d)\in\mathbb{Z}^2\mid \gcd(c,d)=1,\ c>0$ or $c=0,\ d=1\}$. In this section, we describe the reconstruction (4.1) from the Fourier expansion of $P_m^k(z)$ via the transformation formula for $\mathbf{J}_{\nu-1}(s;\theta)$.

Proposition 4.1. *Let $\mu > 0$. For all positive integers k,*

$$(4.2)\qquad 2\pi(-1)^{k/2}\sum_{n=1}^{\infty}\left(\frac{n}{\mu}\right)^{(k-1)/2} J_{k-1}(4\pi\sqrt{\mu n})\mathrm{e}(nz) = (-1)^k\sum_{n=-\infty}^{\infty}\frac{\mathrm{e}(-\mu/(z+n))}{(z+n)^k}.$$

Proof. First, we apply the Mellin inversion integral transformation to the left side of (4.2):

$$e^{-Z} = \frac{1}{2\pi i}\int_{(c)}\Gamma(w)Z^{-w}\,dw,$$

which holds for any $c > 0$ and $Z \in \mathbb{C}$ such that $|\arg(Z)| < \pi/2$. We choose the branch so that $-\alpha = e^{-\pi i}\alpha$ and assume $(k+1)/2 < c$. Then we may interchange the order of summation and integration under the condition $(k+1)/2 < c$, and observe that the left side of (4.2) is equal to

$$(4.3)\qquad \frac{2\pi(-1)^{k/2}\mu^{(1-k)/2}}{2\pi i}\int_{(c)}\Gamma(w)\left(\frac{2\pi z}{i}\right)^{-w}\mathbf{J}_{k-1}(w-k;4\pi^2\mu)\,dw.$$

We shift the integration path (c) so that $c \to +\infty$, and substitute (1.5) into (4.3). Then the interchange of integration and summation is justified by Stirling's formula and the order of $\zeta(\sigma+it)$ for $\sigma < 0$ (cf. [Ti2, (5.1.1)]). Shifting back each integration path to $C_l = (k+l-1/2)$ and counting the residues coming from the simple poles of $\zeta(1-k-l+w)$ at $w = k+l$, we find that (4.3) is equal to

$$(4.4)\qquad \frac{(2\pi i)^k}{2\pi i}\sum_{l=0}^{\infty}\int_{C_l}\Gamma(w)(2\pi e^{-\pi i/2}z)^{-w}\frac{(4\pi^2 e^{\pi i}\mu)^l\zeta(1-k-l+w)}{\Gamma(k+l)\Gamma(l+1)}\,dw + \sum_{l=0}^{\infty}\left(\frac{-1}{z}\right)^k\frac{(-2\pi i\mu/z)^l}{\Gamma(l+1)}.$$

In the above integrands, we apply the functional equation of the Riemann zeta-function

$$\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty} n^{-s} = (e^{\pi i s}+1)\zeta(s) = \frac{(2\pi i)^s}{\Gamma(s)}\zeta(1-s)$$

if $\arg(n) = -\pi$ for $n < 0$, and change the variable w to $-w$ to obtain

$$(4.5)\qquad \frac{1}{2\pi i}\sum_{\substack{n=-\infty\\ n\neq 0}}^{\infty}\sum_{l=0}^{\infty}\frac{(2\pi i\mu)^l n^{-k-l}}{\Gamma(l+1)}\int_{C_l'}\frac{\Gamma(-w)\Gamma(k+l+w)}{\Gamma(k+l)}\left(\frac{e^{-\pi i}z}{n}\right)^w dw + \sum_{l=0}^{\infty}\left(\frac{-1}{z}\right)^k\frac{(-2\pi i\mu/z)^l}{\Gamma(l+1)}.$$

Here, we use the notation $C_l' = (-k-l+1/2)$.

Next, we employ the Mellin–Barnes formula for binomial functions:

$$(1+Z)^{-a}=\frac{1}{2\pi i}\int_{C}\frac{\Gamma(-w)\Gamma(a+w)}{\Gamma(a)}Z^{w}\,dw$$

for $a\in\mathbb{C}$ and $Z\in\mathbb{C}$. Here $|\arg(Z)|\le\pi-\delta$ for some positive constant δ, and the integration path C is taken from $-i\infty$ to $i\infty$ so as to separate the poles of $\Gamma(a+w)$ and $\Gamma(-w)$ (cf. [WW, 14.51]). By taking $a=k+l$ and $Z=-z/n$ in the above formula, we see that (4.5) is equal to

$$\sum_{n=-\infty}^{\infty}\frac{1}{(n-z)^{k}}\sum_{l=0}^{\infty}\frac{1}{\Gamma(l+1)}\left(\frac{2\pi i\mu}{n-z}\right)^{l},$$

which is equivalent to the right side of (4.2). This completes the proof of Proposition 4.1. ∎

By the definition of $P_m^k(z)$, we see that

$$P_m^k(z)=(-1)^k\mathrm{e}(mz)+(-1)^k\sum_{\substack{(c,d)\in\mathbb{Z}^2,\,c>0\\ \gcd(c,d)=1}}\frac{\mathrm{e}(m\gamma(z))}{(cz+d)^k}. \tag{4.6}$$

In the standard way, we rearrange the d-sum into an n-sum of finite d-sums modulo c, and apply Proposition 4.1 with $\mu=m/c^2$ and z replaced by $z+d/c$. Thus, we achieve the following:

Theorem (Fourier series expansion of the Poincaré series).

$$P_m^k(z)=(-1)^k\mathrm{e}(mz)+(-1)^{k/2}2\pi\sum_{n=1}^{\infty}\left(\frac{n}{m}\right)^{(k-1)/2}\times\sum_{c=1}^{\infty}\frac{1}{c}K_c(m,n)J_{k-1}\left(\frac{4\pi}{c}\sqrt{mn}\right)\mathrm{e}(nz).$$

Here, the Kloosterman sum is defined as follows:

$$K_c(m,n)=\sum_{\substack{d\bmod c\\ \gcd(c,d)=1}}\mathrm{e}\left(\frac{m\bar{d}+nd}{c}\right)\qquad(\bar{d}d\equiv 1\bmod c).$$

Remark. As is well-known, the equality (4.2), from right to left, can be shown by using the Fourier transform. Our procedure in the proof of Proposition 4.1 is different.

It is also natural to try to derive some linear relations or expressions for the Poincaré series $P_m^k(z)$ via the recurrence formula for $\mathbf{J}_{\nu-1}(s;\theta)$. In fact, this is possible in a way. After adapting (1.6) of Theorem 1.1 to (4.3), we employ formula (1.5) for $\mathbf{J}_{k+1}$ (resp. $\mathbf{J}_k$) as in the proof of Proposition 4.1. For the resulting integrals, we apply the Mellin–Barnes formula for binomial functions and arrange the summations by using Kummer's relation (cf. [Er1, 6.3, (7)]). Finally, we arrive at the formula

$$\begin{aligned} P_m^k(z) &= (-1)^k \mathrm{e}(mz) \\ &\quad + \sum_{\substack{(c,d)\in\mathbb{Z}^2,\, c>0 \\ \gcd(c,d)=1}} \left\{ (-1)^k \frac{\mathrm{e}(m\gamma(z))}{(cz+d)^k} F\left(1; k+1; \frac{2\pi i m}{c(cz+d)}\right) \right. \\ &\quad \left. + (-1)^k \frac{2\pi i}{k+1} \cdot \frac{m}{c} \cdot \frac{\mathrm{e}(m\gamma(z))}{(cz+d)^{k+1}} F\left(1; k+2; \frac{2\pi i m}{c(cz+d)}\right) \right\}. \end{aligned}$$

This is an expression of the Poincaré series $P_m^k(z)$. However, this equality itself can be proved directly by using the following (trivial) relation:

$$\frac{-Z}{k+1} \cdot F(1; k+2; Z) + F(1; k+1; Z) = 1.$$

Namely, the equation above is equivalent to a consequence of the recurrence formula for $\mathbf{J}_{\nu-1}(s;\theta)$.

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