ACTA ARITHMETICA
171.1 (2015)

The Dirichlet–Bohr radius

by

Daniel Carando (Buenos Aires), Andreas Defant (Oldenburg), Domingo García (Valencia), Manuel Maestre (Valencia) and Pablo Sevilla-Peris (Valencia)

1. Introduction. The study of absolute convergence of Dirichlet series (of the form $\sum_n a_n n^{-s}$, where s is a complex variable) led H. Bohr to relate absolute convergence to boundedness (on the right half-plane) of the holomorphic function defined by the Dirichlet series. One of his first results in this direction is the following inequality [6, Satz XIII]: for every Dirichlet series of the form $\sum_{p \text{ prime}} a_p p^{-s}$ we have

$$\sum_{p \text{ prime}} |a_p| \le \sup_{\operatorname{Re} s>0} \Big| \sum_{p \text{ prime}} a_p p^{-s} \Big|. \tag{1.1}$$

He then established [6, 7] a close relationship between Dirichlet series and power series in infinitely many variables (this relationship was presented in a modern, systematic way much later by Hedenmalm, Lindqvist and Seip [14]). Bohr then looked at holomorphic functions and proved his well known power series theorem [8]: for every holomorphic function f on the open unit disc $\mathbb{D}$ we have

$$\sum_n \left| \frac{f^{(n)}(0)}{n!} \right| \frac{1}{3^n} \le \|f\|_\infty, \tag{1.2}$$

and the number $1/3$ is optimal. As a simple consequence of the maximum modulus principle, it can be seen that for each Dirichlet series $\sum_n a_{2^n} 2^{-ns}$ we have

$$\sup_{z \in \mathbb{D}} \Big| \sum_n a_{2^n} z^n \Big| = \sup_{\operatorname{Re} s>0} \Big| \sum_n a_{2^n} 2^{-ns} \Big|.$$

Hence (1.2) can be reformulated as follows:

$$\sum_n \left| a_{2^n} \frac{1}{3^n} \right| \le \sup_{\operatorname{Re} s>0} \Big| \sum_n a_{2^n} 2^{-ns} \Big| \tag{1.3}$$

for every Dirichlet series $\sum_n a_{2^n} 2^{-ns}$.

2010 *Mathematics Subject Classification*: 11M41, 30B50, 11M36.
Key words and phrases: Dirichlet series, Bohr radius, holomorphic functions.

DOI: 10.4064/aa171-1-3

The work of Dineen and Timoney [13] renewed the interest in Bohr's power series theorem, and Boas and Khavinson [5] defined the *n-dimensional Bohr radius* K_n to be the best $0 < r < 1$ such that

$$\sum_{\alpha\in\mathbb{N}_0^n} \left|\frac{\partial^\alpha f(0)}{\alpha!}\right| r^{|\alpha|} \leq \sup_{z\in\mathbb{D}^n} \left|\sum_{\alpha\in\mathbb{N}_0^n} \frac{\partial^\alpha f(0)}{\alpha!} z^\alpha\right|$$

for every bounded, holomorphic function f on $\mathbb{D}^n$. That was the starting point of a long search for the optimal asymptotic behaviour of K_n as n grows, which was finally closed in [10] and [4] (see Section 3 for more details).

Because of the link between Dirichlet series and power series, each result in either framework has an immediate translation into the other. This is of course the case with the behaviour of K_n (a fact which is stated in more detail in Example 3.6). But, as it happens, what is natural on one side may not be as natural on the other; and while taking n variables (or, equivalently, n-dimensional spaces) is natural in the domain of holomorphic functions, for Dirichlet series we would rather take finite sums of (the first) n terms. So, inspired by the Bohr radius for holomorphic functions, our main aim in this note is to determine, for each $x \geq 2$, the best $r = r(x) \geq 0$ such that for every Dirichlet polynomial $\sum_{n\leq x} a_n n^{-s}$ of length x,

$$\sum_{n\leq x} |a_n| r^{\Omega(n)} \leq \sup_{\operatorname{Re} s>0} \left|\sum_{n\leq x} a_n n^{-s}\right|,$$

where $\Omega(n)$ denotes the number of prime divisors of $n \in \mathbb{N}$ (counted with multiplicities). We do this in our main result Theorem 2.1, which gives the asymptotically correct order of this best radius.

We then take a general point of view, and for a given subset J of $\mathbb{N}$, we define the *Dirichlet–Bohr radius* $L(J)$ of J to be the best $r = r(J) \geq 0$ such that for every Dirichlet series $\sum_{n\in J} a_n n^{-s}$ convergent on the open half-plane $[\operatorname{Re} s > 0]$, we have

$$\sum_{n\in J} |a_n| r^{\Omega(n)} \leq \sup_{\operatorname{Re} s>0} \left|\sum_{n\in J} a_n n^{-s}\right|. \tag{1.4}$$

With this, denoting by P the set of prime numbers, we can rephrase (1.1) and (1.3) as

$$L(P) = 1 \quad \text{and} \quad L(\{2^k \mid k \in \mathbb{N}\}) = 1/3. \tag{1.5}$$

Moreover, Theorem 2.1 gives the correct asymptotic order of $L(\{n \in \mathbb{N} \mid 1 \leq n \leq x\})$. We will see that, following an idea of H. Bohr based on Diophantine approximation, one can extended this study to other sets J of indices.

Finally, we mention another estimate which seems of relevance when motivating our results: For every $\varepsilon > 0$ there is $C = C(\varepsilon) \geq 1$ such that for every x and every finite Dirichlet polynomial $\sum_{n\le x} a_n n^{-s}$,

$$\sum_{n\le x} |a_n| \frac{e^{(1/\sqrt{2}-\varepsilon)\sqrt{\log n \log\log n}}}{n^{1/2}} \le C \sup_{\mathrm{Re}\, s>0} \Big|\sum_{n\le x} a_n n^{-s}\Big|. \tag{1.6}$$

This result is optimal from several different aspects, and it is the final outcome of a long series of results due to [2, 9, 10, 15, 17, 18]. Our main result, Theorem 2.1, can be considered to be a relative of (1.6).

1.1. Notation. As already mentioned, $\Omega(n)$ denotes, for $n \in \mathbb{N}$, the number of prime divisors of n, counted with their multiplicity. We denote by $(p_n)_n$ the sequence of prime numbers. The set of multiindices α that are eventually 0 is denoted by $\mathbb{N}_0^{(\mathbb{N})}$. For $\alpha = (\alpha_1, \ldots, \alpha_k, 0, 0, \ldots)$ we write $p^\alpha = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_k$.

Along this note π denotes the prime counting function, i.e., $\pi(x)$ is the number of prime numbers less than or equal to x.

Given two real functions f and g we write $f(x) \ll g(x)$ if there exists a constant $C > 0$ such that $f(x) \le Cg(x)$ for every x. If $f(x) \ll g(x)$ and $g(x) \ll f(x)$, we write $f(x) \approx g(x)$.

For each N we denote by $H_\infty(\mathbb{D}^N)$ the space of bounded, holomorphic functions on $\mathbb{D}^N$. If $f \in H_\infty(\mathbb{D}^N)$ and $\alpha \in \mathbb{N}_0^N$, we write $c_\alpha(f) = \partial^\alpha f(0)/\alpha!$, the αth coefficient of the monomial expansion.

2. Main result. For any $x \ge 2$, we write

$$L(x) = L(\{n \in \mathbb{N} \mid 1 \le n \le x\}),$$

where L is defined in (1.4), and call this number the *xth Dirichlet–Bohr radius.* The main result of this note then reads as follows.

THEOREM 2.1. *We have*

$$L(x) \approx \frac{\sqrt[4]{\log x}}{x^{1/8}}.$$

In particular, there is a universal constant $C > 0$ such that

$$\sum_{n\le x} |a_n| \bigg(\frac{C\sqrt[4]{\log x}}{x^{1/8}}\bigg)^{\Omega(n)} \le \sup_{\mathrm{Re}\, s>0} \Big|\sum_{n\le x} a_n n^{-s}\Big|$$

for every $x \ge 2$ and every Dirichlet polynomial $\sum_{n\le x} a_n n^{-s}$.

The rest of this section is devoted to the proof of this result.

2.1. Reduction I. We start with a device which reduces the estimation of the Dirichlet–Bohr radii $L(x)$ to the estimation of their *homogeneous* *parts* $L_m(x)$ which we are going to define now. For $x \geq 2$ define the finite-dimensional Banach space

$$\mathcal{H}_\infty^{(x)} := \Big\{ D = \sum_{n=1}^{\infty} a_n n^{-s} \;\Big|\; a_n \neq 0 \text{ only if } n \leq x \Big\},$$

$$\|D\|_\infty := \sup_{t\in\mathbb{R}} \Big| \sum_{n\leq x} a_n \frac{1}{n^{it}} \Big| = \sup_{\operatorname{Re} s>0} \Big| \sum_{n\leq x} a_n \frac{1}{n^s} \Big|,$$

together with its closed subspace

$$\mathcal{H}_\infty^{(x,m)} := \Big\{ \sum_{n=1}^{\infty} a_n n^{-s} \;\Big|\; a_n \neq 0 \text{ only if } n \leq x \text{ and } \Omega(n) = m \Big\}.$$

Then

$$L(x) = \sup \Big\{ 0 \leq r \leq 1 \;\Big|\; \forall D \in \mathcal{H}_\infty^{(x)} : \sum_{n\leq x} |a_n| r^{\Omega(n)} \leq \|D\|_\infty \Big\},$$

and therefore for $m \in \mathbb{N}$ we define the *m-homogeneous xth Dirichlet–Bohr radius* by

$$L_m(x) := \sup \Big\{ 0 \leq r \leq 1 \;\Big|\; \forall D \in \mathcal{H}_\infty^{(x,m)} : \sum_{n\leq x} |a_n| \leq r^{-m} \|D\|_\infty \Big\}. \tag{2.1}$$

The following result is the announced *reduction theorem.*

Proposition 2.2. *With the previous notation,*

$$\tfrac{1}{3} \inf_m L_m(x) \leq L(x) \leq \inf_m L_m(x) \quad \textit{for all } x \geq 2.$$

We start with a reformulation in terms of holomorphic functions. Note that if $n = p^\alpha$ and $1 \leq n \leq x$ then clearly α has at most the first $\pi(x)$ coordinates different from zero; in other words $\alpha \in \mathbb{N}_0^{\pi(x)}$. Then by Bohr's fundamental lemma (see [18]) we know that for every Dirichlet polynomial $\sum_{n\leq x} a_n n^{-s}$ we have

$$\sup_{t\in\mathbb{R}} \Big| \sum_{n\leq x} a_n n^{-it} \Big| = \sup_{z\in\mathbb{D}^{\pi(x)}} \Big| \sum_{\substack{\alpha\in\mathbb{N}_0^{\pi(x)} \\ 1\leq p^\alpha\leq x}} a_{p^\alpha} z^\alpha \Big|. \tag{2.2}$$

With this identity in mind we define the Banach space

$$H_\infty^{(x)} := \{ f \in H_\infty(\mathbb{D}^{\pi(x)}) \mid c_\alpha(f) \neq 0 \text{ only if } p^\alpha \leq x \}$$

(the norm clearly given by the right side of (2.2)) and its closed subspace

$$H_\infty^{(x,m)} := \{ f \in H_\infty(\mathbb{D}^{\pi(x)}) \mid c_\alpha(f) \neq 0 \text{ only if } p^\alpha \leq x \text{ and } |\alpha| = m \}.$$

Identifying Dirichlet polynomials $\sum_{n\leq x} a_n n^{-s}$ with functions $\sum_{\alpha\in\mathbb{N}_0^{\pi(x)},\, 1\leq p^\alpha\leq x} a_{p^\alpha} z^\alpha$ we then obtain the isometric equalities

$$\mathcal{H}_\infty^{(x)} = H_\infty^{(x)} \quad \text{and} \quad \mathcal{H}_\infty^{(x,m)} = H_\infty^{(x,m)},$$

and this in turn shows that

$$L(x) = \sup\Big\{0 \le r \le 1 \;\Big|\; \forall f \in H_\infty^{(x)} : \sum_{\substack{\alpha\in\mathbb{N}_0^{\pi(x)}\\ 1\le p^\alpha\le x}} |c_\alpha(f)| r^{|\alpha|} \le \|f\|_\infty\Big\} \tag{2.3}$$

and

$$L_m(x) = \sup\Big\{0 \le r \le 1 \;\Big|\; \forall f \in H_\infty^{(x,m)} : \sum_{\substack{1\le p^\alpha\le x\\ |\alpha|=m}} |c_\alpha(f)| \le r^{-m}\|f\|_\infty\Big\}. \tag{2.4}$$

Proof of Proposition 2.2. The upper estimate is obvious, and to prove the lower estimate we follow [11, Section 2]. Fix $f \in H_\infty^{(x)}$ with $\|f\|_\infty \le 1$, and write its m-homogeneous part as

$$f_m(\omega) = \sum_{\substack{1\le p^\alpha\le x\\ |\alpha|=m}} c_\alpha(f)\omega^\alpha, \qquad \omega \in \mathbb{D}^{\pi(x)};$$

obviously, $f_m \in H_\infty^{(x,m)}$, and using the Cauchy inequalities we see that $\|f_m\|_\infty \le 1$ for all m. We now fix some $z_0 \in \mathbb{D}^{\pi(x)}$ and $\theta \in \mathbb{T}$ such that $|c_0(f)| = \theta c_0(f)$, and define

$$\begin{aligned} &g : \mathbb{D} \to \mathbb{C}, \quad g(\omega) := f(\omega z_0) = \sum_{m=1}^{\infty} f_m(z_0)\omega^m,\\ &h : \mathbb{D} \to \mathbb{C}, \quad h := 1 - \theta g. \end{aligned}$$

Since $\|g\|_\infty \le 1$, we have $\operatorname{Re} h \ge 0$ on $\mathbb{D}$, and by Carathéodory's theorem (for an elementary proof, see [1, Lemma 1.1]) we have, for all m,

$$|f_m(z_0)| = \frac{h^{(m)}(0)}{m!} \le 2\operatorname{Re} h(0) = 2(1 - |c_0(f)|). \tag{2.5}$$

We now take some $r < \inf_m L_m(x)$. Then for all $z \in \mathbb{D}^{\pi(x)}$ and all m we have, by (2.4) and (2.5),

$$\sum_{\substack{1\le p^\alpha\le x\\ |\alpha|=m}} \left| c_\alpha(f)\left(\frac{r}{3}z\right)^\alpha \right| \le \frac{1}{3^m}\|f_m\|_\infty \le \frac{1}{3^m}2(1 - |c_0(f)|),$$

and hence for all $z \in (r/3)\mathbb{D}^{\pi(x)}$,

$$\sum_{1\le p^\alpha\le x} |c_\alpha(f)z^\alpha| \le |c_0(f)| + \sum_{m=1}^{\infty} \frac{1}{3^m}2(1 - |c_0(f)|) = 1.$$

The conclusion now follows from (2.3). ∎

2.2. The tool. The following proposition is our main tool—an elaboration of a result due to Balasubramanian, Calado and Queffélec [2, Theorem 1.4] (see also [12, Theorem 4.2]).

PROPOSITION 2.3. *Let $m \geq 2$ and $\kappa > 1$. There exists $C(\kappa) > 0$ such that for every m-homogeneous Dirichlet polynomial $D = \sum_{n \leq x} a_n n^{-s}$ we have*

$$\sum_{n \leq x} |a_n| \frac{(\log n)^{(m-1)/2}}{n^{(m-1)/(2m)}} \leq C(\kappa) m^{m-1} (2\kappa)^m \|D\|_\infty.$$

Our proof follows from a careful analysis of the original proof of [2], which allows us to obtain the constant $C(\kappa)m^{m-1}(2\kappa)^m$, smaller than the original one. Since this fact is essential for our purpose, we add the proof for completeness. Every m-homogeneous polynomial in n variables admits two possible representations:

$$P(z) = \sum_{\substack{\alpha \in \mathbb{N}^n \\ |\alpha| = m}} c_\alpha z^\alpha = \sum_{1 \leq j_1 \leq \cdots \leq j_m \leq n} c_{j_1,\ldots,j_m} z_{j_1} \cdot \ldots \cdot z_{j_m} \quad \text{for } z \in \mathbb{C}^n.$$

We need the following lemma [10, p. 492] (see also [12, Lemma 4.3] or [3, Lemma 2.6]).

LEMMA 2.4. *Let $n \geq 1$, $m \geq 1$ and $\kappa > 1$. Then there exists $C(\kappa) > 0$ such that for every m-homogeneous polynomial P on $\mathbb{C}^n$ we have*

$$\sum_{j_m=1}^{n} \Big(\sum_{1 \leq j_1 \leq \cdots \leq j_m} |c_{j_1,\ldots,j_m}|^2 \Big)^{1/2} \leq C(\kappa) \big(2\kappa\big)^m \sup\{|P(z)| : z \in \mathbb{D}^n\}.$$

Proof of Proposition 2.3. We begin by fixing a Dirichlet polynomial

$$D = \sum_{n \leq x} a_n n^{-s} \in \mathcal{H}_\infty^{(x,m)}.$$

Now we define the following m-homogeneous polynomial in $\pi(x)$ variables:

$$P(z) = \sum_{1 \leq j_1 \leq \cdots \leq j_m \leq \pi(x)} c_{j_1,\ldots,j_m} z_{j_1} \cdot \ldots \cdot z_{j_m}, \qquad z \in \mathbb{C}^{\pi(x)},$$

where $c_{j_1,\ldots,j_m} = a_n$ if $1 \leq n = p_{j_1} \cdots p_{j_m} \leq x$ and 0 otherwise. Then

$$\begin{aligned} \sum_{n \leq x} |a_n| \frac{(\log n)^{(m-1)/2}}{n^{(m-1)/(2m)}} &= \sum_{1 \leq j_1 \leq \cdots \leq j_m \leq \pi(x)} |c_{j_1,\ldots,j_m}| \frac{(\log(p_{j_1} \cdots p_{j_m}))^{(m-1)/2}}{(p_{j_1} \cdots p_{j_m})^{(m-1)/(2m)}} \\ &\leq \sum_{j_m=1}^{\pi(x)} \frac{(m \log p_{j_m})^{(m-1)/2}}{p_{j_m}^{(m-1)/(2m)}} \sum_{1 \leq j_1 \leq \cdots \leq j_{m-1} \leq j_m} \frac{|c_{j_1,\ldots,j_m}|}{(p_{j_1} \cdots p_{j_{m-1}})^{(m-1)/(2m)}} \end{aligned}$$

$$\le \sum_{j_m=1}^{\pi(x)} \frac{(m\log p_{j_m})^{(m-1)/2}}{p_{j_m}^{(m-1)/(2m)}} \Big(\sum_{1\le j_1\le\cdots\le j_{m-1}\le j_m} |c_{j_1,\dots,j_m}|^2\Big)^{1/2} \times \Big(\sum_{1\le j_1\le\cdots\le j_{m-1}\le j_m} \frac{1}{(p_{j_1}\cdots p_{j_{m-1}})^{(m-1)/m}}\Big)^{1/2},$$

where the last step follows from the Cauchy–Schwarz inequality. To bound the last factor, we now use the fact that for $0<\alpha<1$ (see [16, Satz 4.2, p. 22]),

$$\sum_{p\le x} p^{-\alpha} \ll \frac{1}{1-\alpha}\,\frac{x^{1-\alpha}}{\log x}.$$

By taking $\alpha=(m-1)/m$, we get

$$\Big(\sum_{1\le j_1\le\cdots\le j_{m-1}\le j_m} \frac{1}{(p_{j_1}\cdots p_{j_{m-1}})^{(m-1)/m}}\Big)^{1/2} \le \Big(\sum_{j\le j_m}\Big(\frac{1}{p_j}\Big)^{(m-1)/m}\Big)^{(m-1)/2} \ll \Big(m\frac{p_{j_m}^{1/m}}{\log p_{j_m}}\Big)^{(m-1)/2}.$$

Hence

$$\sum_{n\le x}|a_n|\frac{(\log n)^{(m-1)/2}}{n^{(m-1)/(2m)}} \ll m^{m-1}\sum_{j_m=1}^{\pi(x)} \frac{(\log p_{j_m})^{(m-1)/2}}{p_{j_m}^{(m-1)/(2m)}}\Big(\frac{p_{j_m}^{1/m}}{\log p_{j_m}}\Big)^{(m-1)/2} \times \Big(\sum_{1\le j_1\le\cdots\le j_{m-1}\le j_m} |c_{j_1,\dots,j_m}|^2\Big)^{1/2}.$$

Finally, by Lemma 2.4 and (2.2), there exists $C(\kappa)>0$ such that

$$\sum_{n\le x}|a_n|\frac{(\log n)^{(m-1)/2}}{n^{(m-1)/(2m)}} \le C(\kappa)m^{m-1}(2\kappa)^m\|P\| = C(\kappa)m^{m-1}(2\kappa)^m\|D\|_\infty. \ \blacksquare$$

2.3. Proofs

Proof of the lower estimate in Theorem 2.1. We fix some $x\ge 2$. By Proposition 2.2 we only have to control each m-homogeneous part, $L_m(x)$. Note first that if $1\le n\le x$ is such that $\Omega(n)=m$ then $2^m\le n\le x$, which gives $m\le \frac{\log x}{\log 2}$. Therefore $\mathcal{H}_\infty^{x,m}=\{0\}$, and hence $L_m(x)=1$ for every $m>\frac{\log x}{\log 2}$. Thus

$$\tfrac{1}{3}\min_{1\le m\le \frac{\log x}{\log 2}} L_m(x) \;\le\; L_x. \tag{2.6}$$

By (1.5) we have $L_1(x)=1$ for every x. We then fix $m\ge 2$, and observe that for every $D=\sum_{n\le x}a_n n^{-s}\in\mathcal{H}_\infty^{(x,m)}$ we have $a_1=a_2=a_3=0$. By

Proposition 2.3, for each $\kappa > 1$ there exists $C(\kappa) > 0$ such that

$$\sum_{n\le x} |a_n| = \sum_{4\le n\le x} |a_n| \le \sum_{4\le n\le x} |a_n|(\log n)^{(m-1)/2}$$
$$\le C(\kappa) m^{m-1}(2\kappa)^m x^{(m-1)/(2m)}\|D\|_\infty.$$

This, using (2.1), gives

$$m^{-1}x^{-(m-1)/(2m^2)} \ll (C(\kappa)m^{m-1}(2\kappa)^m x^{(m-1)/(2m)})^{-1/m} \le L_m(x).$$

But the sequence $(x^{-(m-1)/(2m^2)})_{m=2}^\infty$ is increasing to 1 (recall that $x \ge 2$). This implies that for all $m \ge 3$,

$$m^{-1}x^{-1/9} \ll L_m(x),$$

and hence for all $3 \le m \le \frac{\log x}{\log 2}$,

$$\frac{\sqrt[4]{\log x}}{x^{1/8}} \ll \frac{\log 2}{\log x}\frac{1}{x^{1/9}} \ll L_m(x). \tag{2.7}$$

We finish our argument by handling the case $m = 2$. We observe first that

$$f(t) = \frac{\sqrt{\log t}}{t^{1/4}} = e^{g(t)} \quad \text{with} \quad g(t) = \frac{1}{2}\log\log t - \frac{1}{4}\log t, \qquad t \ge 2.$$

Since

$$g'(t) = \frac{1}{2t}\frac{2-\log t}{2\log t},$$

we see that f is strictly decreasing for $t > e^2$. Then the sequence $(\sqrt{\log n}/n^{1/4})$ is strictly decreasing for $n \ge 8$. Thus there exists $A > 0$ such that for every $2 \le n \le x$ we have

$$\frac{\sqrt{\log x}}{x^{1/4}} \le A\frac{\sqrt{\log n}}{n^{1/4}}.$$

Applying again Proposition 2.3 we see that for every $D \in \mathcal{H}_\infty^{(x,2)}$,

$$\frac{\sqrt{\log x}}{x^{1/4}}\sum_{n\le x}|a_n| \le AC(\kappa)8\kappa^2\|D\|_\infty,$$

and hence

$$\frac{\sqrt[4]{\log x}}{x^{1/8}} \ll L_2(x).$$

This equation combined with (2.7) and (2.6) proves the lower estimate. ∎

Proof of the upper estimate in Theorem 2.1. By Proposition 2.2 it suffices to show that there is a constant $C > 0$ such that for all x,

$$L_2(x) \le C\frac{\sqrt[4]{\log x}}{x^{1/8}}. \tag{2.8}$$

According to (2.1), fix some x and assume that $r > 0$ satisfies

$$\sum_{n\le x} |a_n| \le r^{-2} \sup_{t\in\mathbb{R}} \Big|\sum_{n\le x} a_n n^{it}\Big| \tag{2.9}$$

for every Dirichlet polynomial $\sum_{n\le x} a_n n^{-s} \in \mathcal{H}_\infty^{(x,2)}$. We choose q to be the largest natural number $\le \pi(\sqrt{x})/2$. Consider the $q\times q$ matrix $(a_{nk})_{n,k}$ defined by $a_{nk} = e^{2\pi i nk/q}$ (sometimes called the *Fourier matrix*). Then it is well known (by a straightforward calculation) that for all n, k we have $|a_{nk}| = 1$ and $\sum_l a_{ln}\overline{a}_{lk} = q\delta_{nk}$.

We define the Dirichlet series

$$\sum_{n,k=1}^{q} a_{nk} \frac{1}{(p_n p_{q+k})^s} \in \mathcal{H}_\infty^{(x,2)}.$$

Note that for every $1 \le n, k \le q$ we have $p_n p_{q+k} \le p_{2q}^2 \le p_{\pi(\sqrt{x})}^2 \le x$, and the Dirichlet series indeed belongs to $\mathcal{H}_\infty^{(x,2)}$. Obviously,

$$\sum_{n,k=1}^{q} |a_{nk}| = q^2.$$

On the other hand, for every $t \in \mathbb{R}$ we have

$$\begin{aligned}
\Big|\sum_{n,k=1}^{q} a_{nk} p_n^{it} p_{q+k}^{it}\Big| &\le q^{1/2}\Big(\sum_k \Big|\sum_n a_{nk} p_n^{it}\Big|^2\Big)^{1/2} \\
&= q^{1/2}\Big(\sum_k \sum_{n_1,n_2} a_{kn_1}\overline{a_{kn_2}} p_{n_1}^{it} p_{n_2}^{-it}\Big)^{1/2} = q^{1/2}\Big(\sum_{n_1,n_2} p_{n_1}^{it} p_{n_2}^{-it} \sum_k a_{kn_1}\overline{a_{kn_2}}\Big)^{1/2} \\
&= q^{1/2}\Big(\sum_{n_1,n_2} p_{n_1}^{it} p_{n_2}^{-it} q\delta_{n_1,n_2}\Big)^{1/2} = q\Big(\sum_n |p_n^{it}|^2\Big)^{1/2} \le q^{3/2}.
\end{aligned}$$

Then by (2.9) we conclude that $q^2 \le r^{-2} q^{3/2}$. But from the prime number theorem we deduce that there is a (universal) constant $C > 0$ such that $\sqrt{x}/\log x \le Cq$, and therefore

$$r \le C\frac{\sqrt[4]{\log x}}{x^{1/8}}.$$

Clearly, this gives the desired estimate (2.8). ∎

3. Dirichlet–Bohr radii. The main goal of the previous section was to find the correct asymptotic order of the Dirichlet–Bohr radius $L(\{n \in \mathbb{N} \mid 1 \le n \le x\})$.

Analysing the ideas of our proof, in the coming subsection we show how to reduce the study of the Dirichlet–Bohr radii $L(J)$ for index sets to the study of Bohr radii for holomorphic functions in infinitely many variables with lacunary monomial coefficients. Finally, we consider a series of old and new examples.

3.1. Reduction II. Let Λ be a subset of $\mathbb{N}_0^{(\mathbb{N})}$. Consider the Banach space

$$H_\infty^\Lambda(B_{c_0}) := \{f \in H_\infty(B_{c_0}) \mid c_\alpha(f) \neq 0 \text{ only if } \alpha \in \Lambda\},$$

where as usual $H_\infty(B_{c_0})$ denotes the Banach space of all bounded holomorphic (= Fréchet differentiable) functions on the open unit ball B_{c_0} of the Banach space c_0 of all null sequences.

Now, the *Bohr radius* $K(\Lambda)$ is defined to be the best $r = r(\Lambda) \geq 0$ such that for every $f \in H_\infty^\Lambda(B_{c_0})$ we have

$$\sum_{\alpha \in \Lambda} |c_\alpha(f)| r^{|\alpha|} \leq \|f\|_\infty.$$

Note that, with this notation, the classical Bohr radius K_n is just $K(\mathbb{N}_0^n)$.

The following result extends (2.3) to arbitrary index sets. Note that the proof of (2.3) was based on Bohr's fundamental lemma (2.2). We need, then, an extension of this. Inspired by an idea of Bohr and based on the fundamental theorem of arithmetic we consider the bijection

$$\mathfrak{b} : \mathbb{N}_0^{(\mathbb{N})} \to \mathbb{N}, \qquad \mathfrak{b}(\alpha) = p^\alpha.$$

We now denote by $\mathcal{H}_\infty$ all Dirichlet series $\sum_n a_n n^{-s}$ defining a bounded holomorphic function on $[\operatorname{Re} s > 0]$; this vector space together with the sup norm on $[\operatorname{Re} s > 0]$ forms a Banach space. By [14, Lemma 2.3 and Theorem 3.1] (a fact also essentially due to Bohr [6]) there is a unique isometric and linear bijection Φ from $H_\infty(B_{c_0})$ onto $\mathcal{H}_\infty$ such that $\Phi(z^\alpha) = n^{-s}$ with $\mathfrak{b}(\alpha) = n$:

$$H_\infty(B_{c_0}) = \mathcal{H}_\infty.$$

Using this general principle and a simple translation argument from Dirichlet series into holomorphic functions and vice versa, we obtain the following result.

Proposition 3.1. *For each set $J \subset \mathbb{N}$ and $\Lambda \subset \mathbb{N}_0^{(\mathbb{N})}$ with $J = \mathfrak{b}(\Lambda)$,*

$$K(\Lambda) = L(J).$$

Our next device reduces the estimation of the Dirichlet–Bohr radii of a given index set J to the estimation of the Dirichlet–Bohr radii of certain parts of J. Given $J \subseteq \mathbb{N}$ and $n, m \in \mathbb{N}$, the *n-dimensional kernel of J* is defined to be

$$J(n) = \{k \in J \mid \forall j > n : p_j \nmid k\},$$

and its *m-homogeneous kernel* is

$$J[m] = \{k \in J \mid \Omega(k) = m\}.$$

Note that when $J = \mathbb{N}$, the n-dimensional kernel consists of all the natural numbers that factor into the first n primes, and the m-homogeneous kernel consists of those which have precisely m prime divisors (counted with multiplicities). In other words,

$$\mathbb{N}(n) = \{p_1^{\alpha_1} \cdots p_n^{\alpha_n} \mid \alpha \in \mathbb{N}_0^n\},$$
$$\mathbb{N}[m] = \{p_1^{\alpha_1} \cdots p_k^{\alpha_k} \cdots \mid \alpha_1 + \cdots + \alpha_k + \cdots = m\}.$$

Then clearly $J(n) = J \cap \mathbb{N}(n)$ and $J[m] = J \cap \mathbb{N}[m]$. We also have

$$\mathfrak{b}^{-1}(J(n)) = \{\alpha \in \mathbb{N}_0^n \mid p^\alpha \in J\},$$
$$\mathfrak{b}^{-1}(J[m]) = \{\alpha \in \mathbb{N}_0^{(\mathbb{N})} \mid p^\alpha \in J \text{ with } |\alpha| = m\}.$$

In particular, $\mathfrak{b}^{-1}(\mathbb{N}(n)) = \mathbb{N}_0^n$ and $\mathfrak{b}^{-1}(\mathbb{N}[m]) = \{\alpha \in \mathbb{N}_0^{(\mathbb{N})} \mid |\alpha| = m\}$. Let us finally observe that

$$\mathbb{N}(n)[m] = \{p_1^{\alpha_1} \cdots p_n^{\alpha_n} \mid \alpha \in \mathbb{N}_0^n \text{ and } \alpha_1 + \cdots + \alpha_n = m\} = \mathbb{N}[m](n),$$

and from this $J(n)[m] = J \cap \mathbb{N}(n)[m] = J \cap \mathbb{N}[m](n) = J[m](n)$ for every $J \subseteq \mathbb{N}$ and all n, m.

We can now exhibit our announced reduction device.

PROPOSITION 3.2. *Let J be a subset of $\mathbb{N}$. Then*

(i) $L(J) = \inf_n L(J(n))$;
(ii) $\frac{1}{3} \inf_m L(J[m]) \le L(J) \le \inf_m L(J[m])$.

Proof. The proof of (ii) is a word for word copy of the proof of Proposition 2.2. The argument for (i) is easy after a translation to holomorphic functions via Proposition 3.1. ∎

Of course, (i) and (ii) can be combined to show that the infimum of each of the double sequences $(L(J[m](n)))_{m,n}$ and $(L(J(n)[m]))_{m,n}$ equals $L(J)$ up to the constant $1/3$.

3.2. Examples. We first recover, in this systematic language, the fundamental examples (1.5) that were already mentioned in the introduction.

EXAMPLE 3.3.

(i) $L(\mathbb{N}[1]) = L(\{p \mid p \text{ prime}\}) = 1$;
(ii) $L(\mathbb{N}(1)) = L(\{2^k \mid k \in \mathbb{N}\}) = 1/3$.

We remark that (i) is nothing else than Bohr's inequality (1.1), whereas (ii) is just a reformulation via Proposition 3.1 of Bohr's power series theorem (1.2) (see also (1.3)). Basically, these and the one in the following example are the only precise values of Dirichlet–Bohr radii we know.

EXAMPLE 3.4. $L(\{p_\ell^k \mid k, \ell \in \mathbb{N}\}) = 1/3$.

This turns out to be an immediate consequence of the following more general result. Given a subset A of $\mathbb{N}$, we will denote its cardinality by $|A|$.

Proposition 3.5. *Let P_k, $k \in \mathbb{N}$, be disjoint sets of primes such that*

$$n = \max_k |P_k| < \infty.$$

Define J_{P_k} to be the set of all natural numbers which are finite products of primes in P_k, that is,

$$J_{P_k} = \{p^\alpha \mid \alpha_j = 0 \text{ if } p_j \notin P_k\}.$$

Then

$$L\Big(\bigcup_k J_{P_k}\Big) = L(\mathbb{N}(n)).$$

Clearly, Example 3.4 is an immediate consequence of this result: set $P_k = \{p_k\}$ (the kth prime) and apply Example 3.3 together with Proposition 3.5.

Proof. Define $\Lambda_k = \mathfrak{b}^{-1}(J_{P_k}) \subset \mathbb{N}_0^{(\mathbb{N})}$. Looking at Proposition 3.1, since $\mathbb{N}_0^n = \mathfrak{b}^{-1}(\mathbb{N}(n))$, it suffices to prove that

$$K\Big(\bigcup_k \Lambda_k\Big) = K(\mathbb{N}_0^n).$$

Let $I_k = \bigcup_{\alpha\in\Lambda_k} \operatorname{supp} \alpha \subset \mathbb{N}$ be the support of Λ_k. Clearly, $n_k := |I_k| = |P_k|$ for all k. We identify $\operatorname{span}\{e_i : i \in I_k\}$ with $\mathbb{C}^{n_k}$.

By considering bounded holomorphic functions with support in any I_k of length n, we get $K(\bigcup_k \Lambda_k) \le K(\mathbb{N}_0^n)$. We have to prove now the reverse inequality

$$K(\mathbb{N}_0^n) \le K\Big(\bigcup_k \Lambda_k\Big). \tag{3.1}$$

Now, we want to show that

$$\sum_{\alpha\in\bigcup_k \Lambda_k} |a_\alpha| K(\mathbb{N}_0^n)^{|\alpha|} \le \sup_{z\in B_{c_0}} \Big|\sum_{\alpha\in\bigcup_k \Lambda_k} a_\alpha z^\alpha\Big|$$

for every function $\sum_{\alpha\in\bigcup_k \Lambda_k} a_\alpha z^\alpha \in H_\infty(B_{c_0})$. Since the Λ_k's are disjoint, we have

$$\sup_{z\in B_{c_0}} \Big|\sum_{\alpha\in\bigcup_{k=1}^N \Lambda_k} a_\alpha z^\alpha\Big| \le \sup_{z\in B_{c_0}} \Big|\sum_{\alpha\in\bigcup_k \Lambda_k} a_\alpha z^\alpha\Big|$$

for all N, and so it will be enough to show that

$$\sum_{\alpha\in\bigcup_{k=1}^N \Lambda_k} |a_\alpha| K(\mathbb{N}_0^n)^{|\alpha|} \le \sup_{z\in B_{c_0}} \Big|\sum_{\alpha\in\bigcup_{k=1}^N \Lambda_k} a_\alpha z^\alpha\Big|. \tag{3.2}$$

We proceed by induction on N. For $N = 1$, (3.2) is just a consequence of $K(\mathbb{N}_0^n) \le K(\mathbb{N}_0^{n_1}) = K(\Lambda_1)$. For the inductive step, we write

$$\sum_{\alpha\in\bigcup_{k=1}^N \Lambda_k} a_\alpha z^\alpha = a_0 + f_1(u_1) + \dots + f_N(u_N),$$

where u_k is the projection of z on the Λ_k-coordinates and

$$f_k(w) = \sum_{\substack{\alpha\in\mathbb{N}_0^{n_k}\\|\alpha|\geq 1}} a_\alpha^k w^\alpha$$

for $w \in \mathbb{C}^{n_k}$. Note that $f_k(0) = 0$ for every k. By inductive hypothesis we know that

$$|a_0| + \sum_{k=1}^{N-1} \sum_{\substack{\alpha\in\mathbb{N}_0^{n_k}\\|\alpha|\geq 1}} |a_\alpha| K(\mathbb{N}_0^n)^{|\alpha|} \leq \sup_{u_1\in\mathbb{D}^{n_1},\ldots,u_{N-1}\in\mathbb{D}^{n_{N-1}}} \Big|a_0 + \sum_{k=1}^{N-1} f_k(u_k)\Big|. \tag{3.3}$$

Fix now $u_k \in \mathbb{D}^{n_k}$ for $k = 1,\ldots,N-1$ and set $\widetilde{a}_0 = a_0 + \sum_{k=1}^{N-1} f_k(u_k)$. Since $K(\mathbb{N}_0^n) \leq K(\mathbb{N}_0^{n_N}) = K(\Lambda_N)$, we have

$$|\widetilde{a}_0| + \sum_{\substack{\alpha\in\mathbb{N}_0^{n_N}\\|\alpha|\geq 1}} |a_\alpha^N| K(\mathbb{N}_0^n)^{|\alpha|} \leq \sup_{u_N\in\mathbb{D}^{n_N}} |\widetilde{a}_0 + f_N(u_N)|,$$

which just means that

$$\begin{aligned}\Big|a_0 + \sum_{k=1}^{N-1} f_k(u_k)\Big| &+ \sum_{\substack{\alpha\in\mathbb{N}_0^{n_N}\\|\alpha|\geq 1}} |a_\alpha^N| K(\mathbb{N}_0^n)^{|\alpha|} \\ &\leq \sup_{u_N\in\mathbb{D}^{n_N}} \Big|\Big(a_0 + \sum_{k=1}^{N-1} f_k(u_k)\Big) + f_N(u_N)\Big|.\end{aligned} \tag{3.4}$$

Combining (3.3) and (3.4) we obtain (3.2). ∎

In the following results we present asymptotically correct estimates on Dirichlet–Bohr radii.

Example 3.6.

(1) $\lim_n \frac{L(\mathbb{N}(n))}{\sqrt{(\log n)/n}} = 1$;

(2) there is a constant $C > 1$ such that

$$\begin{aligned} \frac{1}{C}\Big(\frac{m}{n}\Big)^{(m-1)/(2m)} &\leq L((\mathbb{N}(n))[m]) \leq C\Big(\frac{m}{n}\Big)^{(m-1)/(2m)} && \text{for } n \geq m,\\ 1/C &\leq L((\mathbb{N}(n))[m]) \leq 1 && \text{for } n < m.\end{aligned}$$

Both results follow from Proposition 3.1 and their counterparts for Bohr radii:

$$\lim_n \frac{K(\mathbb{N}_0^n)}{\sqrt{(\log n)/n}} = 1$$

and

$$\frac{1}{C}\left(\frac{m}{n}\right)^{(m-1)/(2m)} \leq K(\{\alpha \in \mathbb{N}_0^n \mid |\alpha| = m\}) \leq C\left(\frac{m}{n}\right)^{(m-1)/(2m)} \quad \text{for } n \geq m,$$

$$1/C \leq K(\{\alpha \in \mathbb{N}_0^n \mid |\alpha| = m\}) \leq 1 \quad \text{for } n < m.$$

The first formula is due to Bayart, Pellegrino and Seoane-Sepúlveda [4], who improved an earlier result from [10]. The lower estimate in the second result follows from [10], and the upper one is a consequence of the Kahane–Salem–Zygmund inequality (or [11, Lemma 2.1 and (4.4)]). It would be of interest to know the precise values of $L(\mathbb{N}(n))$, $L(\mathbb{N}[m])$ and $L((\mathbb{N}(n))[m])$ for all/some $n, m > 1$.

Example 3.6 combined with Proposition 3.2 yields the following.

Example 3.7.

(1) $L(\mathbb{N}) = 0$;
(2) $L(\mathbb{N}[m]) = 0$ for all $m > 1$.

Acknowledgements. The first author was supported by CONICET-PIP 0624, UBACyT 20020130100474BA and ANPCyT PICT 2011-1456. The third, fourth and fifth authors were supported by MINECO MTM2014-57838-C2-2-P, and the third and fourth also by Prometeo II/2013/013. The fifth author was also partially supported by UPV-SP20120700.

We thank Daniel Galicer, Martín Mansilla and Santiago Muro for valuable comments.

References

[1] L. Aizenberg, *Generalization of Carathéodory's inequality and the Bohr radius for multidimensional power series*, in: Selected Topics in Complex Analysis, Oper. Theory Adv. Appl. 158, Birkhäuser, Basel, 2005, 87–94.

[2] R. Balasubramanian, B. Calado and H. Queffélec, *The Bohr inequality for ordinary Dirichlet series*, Studia Math. 175 (2006), 285–304.

[3] F. Bayart, A. Defant, L. Frerick, M. Maestre and P. Sevilla-Peris, *Multipliers of Dirichlet series and monomial series expansions of holomorphic functions in infinitely many variables*, arXiv:1405.7205 (2014).

[4] F. Bayart, D. Pellegrino and J. B. Seoane-Sepúlveda, *The Bohr radius of the n-dimensional polydisk is equivalent to $\sqrt{(\log n)/n}$*, Adv. Math. 264 (2014), 726–746.

[5] H. P. Boas and D. Khavinson, *Bohr's power series theorem in several variables*, Proc. Amer. Math. Soc. 125 (1997), 2975–2979.

[6] H. Bohr, *Über die Bedeutung der Potenzreihen unendlich vieler Variabeln in der Theorie der Dirichletschen Reihen $\sum \frac{a_n}{n^s}$*, Nachr. Ges. Wiss. Göttingen Math.-Phys. Kl. 1913, 441–488.

[7] H. Bohr, *Über die gleichmäßige Konvergenz Dirichletscher Reihen*, J. Reine Angew. Math. 143 (1913), 203–211.

[8] H. Bohr, *A theorem concerning power series*, Proc. London Math. Soc. (2) 13 (1914), 1–5.

[9] R. de la Bretèche, *Sur l'ordre de grandeur des polynômes de Dirichlet*, Acta Arith. 134 (2008), 141–148.

[10] A. Defant, L. Frerick, J. Ortega-Cerdà, M. Ounaïes and K. Seip, *The Bohnenblust–Hille inequality for homogeneous polynomials is hypercontractive*, Ann. of Math. (2) 174 (2011), 485–497.

[11] A. Defant, D. García and M. Maestre, *Bohr's power series theorem and local Banach space theory*, J. Reine Angew. Math. 557 (2003), 173–197.

[12] A. Defant, U. Schwarting and P. Sevilla-Peris, *Estimates for vector valued Dirichlet polynomials*, Monatsh. Math. 175 (2014), 89–116.

[13] S. Dineen and R. M. Timoney, *Absolute bases, tensor products and a theorem of Bohr*, Studia Math. 94 (1989), 227–234.

[14] H. Hedenmalm, P. Lindqvist and K. Seip, *A Hilbert space of Dirichlet series and systems of dilated functions in* $L^2(0,1)$, Duke Math. J. 86 (1997), 1–37.

[15] S. V. Konyagin and H. Queffélec, *The translation* $1/2$ *in the theory of Dirichlet series*, Real Anal. Exchange 27 (2001/02), 155–175.

[16] K. Prachar, *Primzahlverteilung*, Springer, Berlin, 1957.

[17] H. Queffélec, *H. Bohr's vision of ordinary Dirichlet series; old and new results*, J. Anal. 3 (1995), 43–60.

[18] H. Queffélec and M. Queffélec, *Diophantine Approximation and Dirichlet Series*, Harish-Chandra Research Inst. Lecture Notes 2, Hindustan Book Agency, New Delhi, 2013.

Daniel Carando
Departamento de Matemática
Facultad de Ciencias Exactas y Naturales
Universidad de Buenos Aires
Pab I, Ciudad Universitaria
1428, Buenos Aires, Argentina
and
IMAS – CONICET
E-mail: dcarando@dm.uba.ar

Domingo García, Manuel Maestre
Departamento de Análisis Matemático
Universidad de Valencia
Doctor Moliner 50
46100 Burjasot (Valencia), Spain
E-mail: domingo.garcia@uv.es
manuel.maestre@uv.es

Andreas Defant
Institut für Mathematik
Universität Oldenburg
D-26111 Oldenburg, Germany
E-mail: defant@mathematik.uni-oldenburg.de

Pablo Sevilla-Peris
Instituto Universitario de Matemática Pura y Aplicada
Universitat Politècnica de València
Valencia, Spain
E-mail: psevilla@mat.upv.es

Received on 16.8.2014
and in revised form on 15.1.2015 (7900)