

The largest prime factor of $X^3 + 2$

by

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1. Introduction. Given an irreducible polynomial f over \mathbb{Z} , whose values have no fixed prime divisor, it is conjectured that there are infinitely many n for which $f(n)$ is prime. This problem appears to be extremely difficult when $\deg f \geq 2$, so it is natural to look for weaker statements which can be proven. For example, rather than insisting that $f(n)$ is prime, one can ask only that it has a large prime factor. We therefore consider lower bounds for $P(x; f)$, the largest prime which divides

$$\prod_{n \leq x} f(n).$$

The best result for polynomials of arbitrary degree is due to Tenenbaum [11] who showed that

$$P(x; f) \gg x \exp((\log x)^A)$$

for any $A < 2 - \log 4$. However, for low degree polynomials one can do much better, gaining a power of x . The first result of this kind is due to Hooley [9] who showed that

$$P(x; X^2 + 1) \gg x^{11/10},$$

which was subsequently improved by Deshouillers and Iwaniec [4] to

$$P(x; X^2 + 1) \gg x^{1.202}.$$

Hooley also considered the case of the cubic polynomial $X^3 + 2$. In [10] he showed that, assuming certain bounds for short Kloosterman sums,

$$P(x; X^3 + 2) \gg x^{1+1/30}.$$

An unconditional result of this form was established by Heath-Brown [8], albeit with a considerably smaller exponent than $1/30$. Specifically, let $\varpi = 10^{-303}$ and suppose that X is sufficiently large. Heath-Brown proved that, for a positive proportion of the integers $n \in (X, 2X]$, the largest prime factor

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of $n^3 + 2$ exceeds $X^{1+\varpi}$. In this work we will show that this conclusion holds with a larger value of ϖ .

THEOREM 1.1. *Let $\varpi = 10^{-52}$ and suppose that X is sufficiently large. Then, for at least a positive proportion of the integers $n \in (X, 2X]$, $n^3 + 2$ has a prime factor in excess of $X^{1+\varpi}$. In particular,*

$$P(x; X^3 + 2) \gg x^{1+\varpi}.$$

This will be proven by refining one part of Heath-Brown's argument. Lemma 2.3 gives a lower bound for a weighted sum, in which the weights depend exponentially on the number of prime factors of $n^3 + 2$ which exceed $X^{1/321}$, and we require a lower bound for the number of nonzero terms. Heath-Brown achieves this by taking the maximum of the weight, which is extremely large and therefore gives a very small value of ϖ . Our improvement comes from observing that there are relatively few terms in the sum for which the weight is large and therefore a more efficient treatment can be given.

The key ingredient in our work is the estimation of the quantity $T(h, \delta)$, defined to be the number of $n \in (X, 2X]$ for which $n^3 + 2$ has at least h prime factors, when counted with multiplicities, which exceed X^δ . Our estimates will be of the form

$$T(h, \delta) \leq X(c(h, \delta) + o(1))$$

as $X \rightarrow \infty$, where $c(h, \delta)$ are constants which we must give explicitly. We will prove two results of this form, Lemmas 3.2 and 4.2. The latter is sharper but requires a significant amount of computation to use. We will apply our results with $\delta = 1/321$ and $133 \leq h \leq 963$. Roughly speaking, the important fact is that for these values our bound on $c(h, \delta)$ is smaller than 2^{-h} . We will show that Lemma 3.2 yields such a bound when h is greater than a certain large multiple of $\log(1/\delta)$. This means that the exponential dependence on δ , described by Heath-Brown in [8], has been removed. However, our interest is in the particular value $\delta = 1/321$, so such asymptotic results should be treated with caution as the size of the implied constants can be more significant.

We conjecture that the true asymptotic size of $T(h, \delta)$ should be of the above form but with smaller values of $c(h, \delta)$. In fact, we believe that the proportion of n counted by $T(h, \delta)$ should be asymptotic to the proportion of $n \in (X^3, 2X^3]$ having at least h prime factors in excess of X^δ . This latter quantity can be computed: see Billingsley [1] or Donnelly and Grimmett [6] for details. When working with the values $n^3 + 2$ we are constrained by the available level of distribution information. Specifically, we can only give an asymptotic formula for

$$\#\{n \in (X, 2X] : d \mid (n^3 + 2)\}$$

when $d \leq X$, and there is thus no hope of us establishing the asymptotic for $T(h, \delta)$. Instead, we give upper bounds by finding sets \mathcal{D} of $d \leq x$, such that any value of $n^3 + 2$ counted by $T(h, \delta)$ is divisible by at least one $d \in \mathcal{D}$. Our methods could easily be extended to handle irreducible polynomials of arbitrary degree, or much more general sets having some level of distribution. Naturally, the results for higher degree or lower level of distribution would be weaker.

It is probable that significant further improvement to the value of ϖ should be possible. In particular, by optimising Heath-Brown's arguments one might hope for improvements to Lemma 2.3. By a more careful choice of the various parameters it should be possible to get a result for a larger value of δ and/or a better lower bound for the sum. In addition, the results of this paper might be improved, either by giving better combinatorial constructions or simply by giving better estimates for the integrals needed for Lemmas 3.2 and 4.2.

In a recent work [3], Dartyge has extended Heath-Brown's methods to handle the quartic polynomial $X^4 - X^2 + 1$. She showed that for sufficiently large x ,

$$P(x; X^4 - X^2 + 1) \geq x^{1+\varpi}$$

with $\varpi = 10^{-26531}$. It seems probable that our methods could be used to give a considerable improvement to this exponent. Dartyge's arguments have been extended by de la Bretèche [2] to a larger set of quartic polynomials.

2. Heath-Brown's approach. As in [8] we will work with the subset of algebraic integers in $\mathbb{Q}(\sqrt[3]{2})$ given by

$$\mathcal{A} = \{n + \sqrt[3]{2} : X < n \leq 2X\}.$$

By the next lemma, the elements of \mathcal{A} are composed entirely of first degree prime ideals, so Theorem 1.1 will follow if we can show that a positive proportion of them have a prime ideal factor P with $N(P) \geq X^{1+\varpi}$. If I is an ideal we let

$$\mathcal{A}_I = \{\alpha \in \mathcal{A} : I \mid \alpha\}$$

and

$$\rho(I) = \#\{n \pmod{N(I)} : n \equiv \sqrt[3]{2} \pmod{I}\}.$$

Heath-Brown gives the following result which describes the function ρ .

LEMMA 2.1 ([8, Lemma 1]). *If $n \equiv \sqrt[3]{2} \pmod{I}$ is solvable with a rational integer n , then I is composed of first degree prime ideals only. Moreover, I cannot be divisible by two distinct prime ideals of the same norm, nor by P_2^2 or P_3^2 , where P_2 and P_3 are the primes above 2 and 3 respectively. In all other cases the congruence is solvable, and we have $\rho(I) = 1$. Moreover, if*

I is an ideal for which $\rho(I) = 1$, then for any $m \in \mathbb{Z}$, we have $I \mid m$ if and only if $N(I) \mid m$.

If we define

$$\log^{(1)}(n^3 + 2) = \sum_{\substack{P^e \parallel (n + \sqrt[3]{2}) \\ N(P) \leq 3X}} \log N(P^e),$$

then Heath-Brown proves the following.

LEMMA 2.2 ([8, Lemma 2]). *Suppose that $\alpha, \delta > 0$ and we can find at least αX elements $n + \sqrt[3]{2} \in \mathcal{A}$ for which*

$$(1) \quad \log^{(1)}(n^3 + 2) \geq (1 + \delta) \log X.$$

The number of $n + \sqrt[3]{2} \in \mathcal{A}$ which have a prime ideal factor P with $N(P) \geq X^{1+\alpha\delta/2}$ is then at least $(\delta\alpha^2 + o(1))X$.

As shown by Heath-Brown, a sufficient condition for (1) is that $n + \sqrt[3]{2}$ has an ideal factor $J = KL$ with

$$(2) \quad X^{1+\delta} < N(KL) \leq X^{1+2\delta},$$

$$(3) \quad X^{3\delta} < N(K) \leq X^{4\delta}.$$

It therefore remains to find a lower bound for the proportion of elements of \mathcal{A} divisible by such an ideal. We will let K run over the set \mathcal{K} of degree 1 prime ideals satisfying (3). We will then let L run over a set $\mathcal{L}(K)$ of ideals which satisfy (2).

As in [8] we will use a lower-bound sieve to restrict to ideals L which have no prime ideal factor of small norm. Specifically, we let λ_d denote the lower-bound linear sieve of level $X^{3\delta}$ (see for example Friedlander and Iwaniec [7, Chapter 12]) and

$$Q = \prod_{p < X^\delta} p,$$

the product being restricted to primes which split in $\mathbb{Q}(\sqrt[3]{2})$. We consider the sum

$$S = \sum_{K \in \mathcal{K}} \sum_{L \in \mathcal{L}(K)} \left(\sum_{d \mid (Q, N(L))} \lambda_d \right) \# \mathcal{A}_{KL}.$$

By definition of the sieve we have

$$\begin{aligned} S &\leq \sum_{K \in \mathcal{K}} \sum_{\substack{L \in \mathcal{L}(K) \\ (N(L), Q) = 1}} \# \mathcal{A}_{KL} \\ &= \sum_{n + \sqrt[3]{2} \in \mathcal{A}} \#\{(K, L) : K \in \mathcal{K}, L \in \mathcal{L}(K), (N(L), Q) = 1, KL \mid (n + \sqrt[3]{2})\}. \end{aligned}$$

It follows that any $n + \sqrt[3]{2}$ counted with positive weight in S must satisfy (1). The remainder of the work [8] is concerned with the estimation of the sum S from below. The following lemma summarises the conclusion.

LEMMA 2.3. *Let $\delta = 1/321$. The sets of ideals $\mathcal{L}(K)$ can then be chosen in such a way that (2) is satisfied and*

$$S \geq (9.2 \times 10^{-8} + o(1))X.$$

Let

$$W(n + \sqrt[3]{2}) = \#\{(K, L) : K \in \mathcal{K}, L \in \mathcal{L}(K), (N(L), Q) = 1, KL \mid (n + \sqrt[3]{2})\}.$$

We wish to give a lower bound for

$$\#\{n + \sqrt[3]{2} \in \mathcal{A} : W(n + \sqrt[3]{2}) > 0\}$$

whereas the last lemma gives an estimate for

$$\sum_{n + \sqrt[3]{2} \in \mathcal{A}} W(n + \sqrt[3]{2}).$$

Heath-Brown's approach is to estimate the maximum of $W(n + \sqrt[3]{2})$ and use the fact that

$$\#\{n + \sqrt[3]{2} \in \mathcal{A} : W(n + \sqrt[3]{2}) > 0\} \geq \frac{1}{\max W(n)} \sum_{n + \sqrt[3]{2} \in \mathcal{A}} W(n).$$

This is inefficient since the maximum is exponentially large in terms of δ but it is only achieved on a very low density subset of \mathcal{A} . Our improvement is therefore to show that terms with large $W(n + \sqrt[3]{2})$ give a small contribution to S .

We let $\Omega_\delta(n + \sqrt[3]{2})$ be the number of prime ideals P , counted with multiplicities, for which $P \mid (n + \sqrt[3]{2})$ and $N(P) \geq X^\delta$. Observe that, when X is large enough, we have

$$\Omega_\delta(n + \sqrt[3]{2}) \leq [3/\delta].$$

For any given $n + \sqrt[3]{2}$, the number of prime ideals $P \mid (n + \sqrt[3]{2})$ with $N(P) \geq X^{3\delta}$, and therefore the number of choices for K , may be bounded by $\min(\Omega_\delta(n + \sqrt[3]{2}), [1/\delta])$. For each K the number of possible L is at most $2^{\Omega_\delta(n + \sqrt[3]{2})}$, so we may conclude that

$$W(n + \sqrt[3]{2}) \leq \min(\Omega_\delta(n + \sqrt[3]{2}), [1/\delta]) 2^{\Omega_\delta(n + \sqrt[3]{2})}.$$

Let H be a parameter to be chosen later. We have

$$\begin{aligned} & \#\{n + \sqrt[3]{2} \in \mathcal{A} : W(n + \sqrt[3]{2}) > 0\} \\ & \geq \#\{n + \sqrt[3]{2} \in \mathcal{A} : W(n + \sqrt[3]{2}) > 0, \Omega_\delta(n + \sqrt[3]{2}) \leq H\} \end{aligned}$$

$$\begin{aligned}
&\geq \frac{2^{-H}}{\min(H, [1/\delta])} \sum_{\substack{n+\sqrt[3]{2} \in \mathcal{A} \\ \Omega_\delta(n+\sqrt[3]{2}) \leq H}} W(n+\sqrt[3]{2}) \\
&= \frac{2^{-H}}{\min(H, [1/\delta])} \left(\sum_{n+\sqrt[3]{2} \in \mathcal{A}} W(n+\sqrt[3]{2}) - \sum_{\substack{n+\sqrt[3]{2} \in \mathcal{A} \\ \Omega_\delta(n+\sqrt[3]{2}) > H}} W(n+\sqrt[3]{2}) \right).
\end{aligned}$$

It therefore remains to give an upper bound for

$$\sum_{\substack{n+\sqrt[3]{2} \in \mathcal{A} \\ \Omega_\delta(n+\sqrt[3]{2}) > H}} W(n+\sqrt[3]{2}).$$

This is at most

$$\begin{aligned}
&\sum_{h>H} \min(h, [1/\delta]) 2^h \#\{n+\sqrt[3]{2} \in \mathcal{A} : \Omega_\delta(n+\sqrt[3]{2}) = h\} \\
&\leq \sum_{h>H} \min(h, [1/\delta]) 2^h T(h, \delta)
\end{aligned}$$

where

$$T(h, \delta) = \#\{n+\sqrt[3]{2} \in \mathcal{A} : \Omega_\delta(n+\sqrt[3]{2}) \geq h\}.$$

Since $n+\sqrt[3]{2}$ is composed of first degree prime ideals, we may take norms to deduce that $\Omega_\delta(n+\sqrt[3]{2})$ is equal to the number of rational prime factors $p|n^3+2$, counted with multiplicities, for which $p \geq X^\delta$. We will denote this latter quantity by $\Omega_\delta(n^3+2)$, so that

$$T(h, \delta) = \#\{n : X < n \leq 2X, \Omega_\delta(n^3+2) \geq h\}.$$

3. First estimate for $T(h, \delta)$. We wish to bound the number of n in $(X, 2X]$ for which n^3+2 has at least h prime factors, when counted with multiplicities, which exceed X^δ . Specifically, for a fixed real $\delta > 0$ and integer h , we aim to find an explicit constant $c(h, \delta)$ such that

$$(4) \quad T(h, \delta) \leq X(c(h, \delta) + o(1)).$$

We will achieve this by constructing a set \mathcal{D} of integers with the following properties:

1. If $n \in (X, 2X]$ is such that $\Omega_\delta(n^3+2) \geq h$ then n^3+2 is divisible by some $d \in \mathcal{D}$.
2. If $d \in \mathcal{D}$ then $d = O(X)$.
3. If $d \in \mathcal{D}$ then all the prime factors of d exceed X^δ .

Given such a set \mathcal{D} we have

$$T(h, \delta) \leq \sum_{d \in \mathcal{D}} A_d$$

where

$$A_d = \#\{n \in (X, 2X] : n^3 + 2 \equiv 0 \pmod{d}\}.$$

Defining the arithmetic function ν by

$$\nu(d) = \#\{n \pmod{d} : n^3 + 2 \equiv 0 \pmod{d}\}$$

we get

$$A_d = \frac{X\nu(d)}{d} + O(\nu(d)).$$

By the Chinese Remainder Theorem, ν is multiplicative. In addition, for any prime power p^e we have $\nu(p^e) \ll 1$. Since any $d \in \mathcal{D}$ has at most $[1/\delta]$ prime factors, we conclude that $\nu(d) \ll_\delta 1$ and therefore that

$$A_d = \frac{X\nu(d)}{d} + O_\delta(1).$$

The cardinality of \mathcal{D} cannot exceed the total number of integers up to $O(X)$ with no prime factor smaller than X^δ . This latter quantity is $O_\delta(X/\log X)$, so we conclude that

$$T(h, \delta) \leq X \sum_{d \in \mathcal{D}} \frac{\nu(d)}{d} + O_\delta(X/\log X).$$

Since ν is not completely multiplicative it is convenient to deal trivially with the d which are not squarefree. If $\delta > 1/2$ then, for sufficiently large x , all $d \in \mathcal{D}$ are squarefree. Otherwise

$$\begin{aligned} \sum_{\substack{d \in \mathcal{D} \\ \mu(d)=0}} \frac{\nu(d)}{d} &\ll \sum_{\substack{d \in \mathcal{D} \\ \mu(d)=0}} \frac{1}{d} \leq \sum_{X^\delta \leq p \ll \sqrt{X}} \sum_{\substack{d \in \mathcal{D} \\ p^2 | d}} \frac{1}{d} \\ &\leq \sum_{X^\delta \leq p \ll \sqrt{X}} \sum_{\substack{d \leq X \\ p^2 | d}} \frac{1}{d} = \sum_{X^\delta \leq p \ll \sqrt{X}} \sum_{d \leq X/p^2} \frac{1}{dp^2} \\ &\ll \log X \sum_{X^\delta \leq p \ll \sqrt{X}} \frac{1}{p^2} \ll X^{-\delta} \log X. \end{aligned}$$

Therefore

$$T(h, \delta) \leq X \left(\sum_{\substack{d \in \mathcal{D} \\ \mu(d) \neq 0}} \frac{\nu(d)}{d} + o(1) \right).$$

The next lemma gives a construction of a suitable set \mathcal{D} .

LEMMA 3.1. *Suppose $\delta \in (0, 1)$ and $h \geq 3$. Let $k = [h/3]$ and let $\mathcal{D} = \mathcal{D}(h, \delta)$ be the set of integers $d = p_1 \dots p_k$ with*

$$X^\delta \leq p_1 \leq \dots \leq p_k \quad \text{and} \quad p_1 \dots p_{k-1} p_k^{h-k+1} \leq 9X^3.$$

The set \mathcal{D} then satisfies the above three hypotheses.

Proof. If $d \in \mathcal{D}$ then, by the assumptions on its factorisation,

$$d^3 = p_1^3 \dots p_k^3 \leq p_1 \dots p_{k-1} p_k^{2k+1} \leq p_1 \dots p_{k-1} p_k^{h-k+1} \ll X^3$$

and thus $d \ll X$. By construction, the prime factors of d all exceed X^δ , and therefore we have verified the second and third hypotheses.

Suppose $n \in (X, 2X]$ and $\Omega_\delta(n^3 + 2) \geq h$. Write

$$X^\delta \leq p_1 \leq \dots \leq p_h$$

for the smallest h prime factors of $n^3 + 2$ which are at least X^δ . Then

$$p_1 \dots p_{k-1} p_k^{h-k+1} \leq p_1 \dots p_h \leq n^3 + 2 \leq 9X^3$$

so that $d = p_1 \dots p_k \in \mathcal{D}$. This verifies the first hypothesis. ■

We must now estimate, for the \mathcal{D} constructed in the last lemma, the sum

$$\sum_{\substack{d \in \mathcal{D} \\ \mu(d) \neq 0}} \frac{\nu(d)}{d}.$$

We begin with the well-known estimate (see for example Diamond and Halberstam [5, Proposition 10.1])

$$\sum_{p \leq x} \frac{\nu(p) \log p}{p} = \log x + O(1).$$

If $d \in \mathcal{D}$ with $\mu(d) \neq 0$ then $\nu(d) = \nu(p_1) \dots \nu(p_k)$, so we may repeatedly apply partial integration and summation to deduce that

$$\sum_{\substack{d \in \mathcal{D} \\ \mu(d) \neq 0}} \frac{\nu(d)}{d} = \int_T \frac{dt_1 \dots dt_k}{t_1 \dots t_k (\log t_1) \dots (\log t_k)} + o(1)$$

where

$$T = \{(t_1, \dots, t_k) \in \mathbb{R}^k : X^\delta \leq t_1 \leq \dots \leq t_k, t_1 \dots t_{k-1} t_k^{h-k+1} \leq 9X^3\}.$$

We then make the substitution $t_i = X^{s_i}$ to deduce that

$$\sum_{\substack{d \in \mathcal{D} \\ \mu(d) \neq 0}} \frac{\nu(d)}{d} = \int_{R(h, \delta)} \frac{ds_1 \dots ds_k}{s_1 \dots s_k} + o(1)$$

where

$$R(h, \delta) = \{(s_1, \dots, s_k) \in \mathbb{R}^k : \delta \leq s_1 \leq \dots \leq s_k, s_1 + \dots + s_{k-1} + (h - k + 1)s_k \leq 3\}.$$

Note that one first obtains the condition

$$s_1 + \dots + s_{k-1} + (h - k + 1)s_k \leq 3 + \frac{\log 9}{\log X}$$

but the error in removing the $\frac{\log 9}{\log X}$ is $o(1)$.

Unfortunately, it appears to be very difficult to evaluate the above integral exactly. However, since the integrand is positive we may produce an upper bound by enlarging the domain of integration. If $s_1, \dots, s_{k-1} \geq \delta$ and

$$s_1 + \dots + s_{k-1} + (h - k + 1)s_k \leq 3$$

then

$$s_k \leq \frac{3 - (k - 1)\delta}{h - k + 1}.$$

Since the integrand is symmetric under permutations of the coordinates, we may therefore deduce that

$$\begin{aligned} \int_{R(h, \delta)} \frac{ds_1 \dots ds_k}{s_1 \dots s_k} &\leq \frac{1}{k!} \int_{[\delta, (3 - (k - 1)\delta)/(h - k + 1)]^k} \frac{ds_1 \dots ds_k}{s_1 \dots s_k} \\ &= \frac{1}{k!} \left(\log \frac{3 - (k - 1)\delta}{h - k + 1} - \log \delta \right)^k. \end{aligned}$$

Combining the above we see that we have proved the following.

LEMMA 3.2. *Suppose $\delta \in (0, 1)$ and $h \geq 3$. If $k = [h/3]$ then*

$$T(h, \delta) \leq X \left(\frac{1}{k!} \left(\log \frac{3 - (k - 1)\delta}{(h - k + 1)\delta} \right)^k + o(1) \right).$$

This lemma gives (4) with

$$c(h, \delta) = \frac{1}{k!} \left(\log \frac{3 - (k - 1)\delta}{(h - k + 1)\delta} \right)^k \leq \frac{1}{k!} \left(\log \frac{9}{2h\delta} \right)^k \ll \frac{1}{\sqrt{h}} \left(\frac{e}{k} \log \frac{9}{2h\delta} \right)^k,$$

where the last inequality follows from Stirling's approximation. If we now suppose that $\delta \geq \exp(-\eta h)$, for some small $\eta > 0$, then

$$\frac{e}{k} \log \frac{9}{2h\delta} \leq \frac{e}{k} \left(\log \frac{9}{2} + \eta h - \log h \right) \ll \eta.$$

We deduce that there exists an absolute constant A such that if h is sufficiently large in terms of η , then

$$\frac{e}{k} \log \frac{9}{2h\delta} \leq A\eta$$

and hence

$$c(h, \delta) \ll \frac{1}{\sqrt{h}} (A\eta)^{[h/3]} \ll_{\eta} \frac{1}{\sqrt{h}} ((A\eta)^{1/3})^h.$$

We therefore choose η so that $(A\eta)^{1/3} < 1/2$, and deduce that when

$$\delta \geq \exp(-\eta h)$$

(that is, when $h \geq -(1/\eta) \log \delta$) we have

$$c(h, \delta) \ll 2^{-h}/\sqrt{h}.$$

We conclude that when h is larger than a certain multiple of $-\log \delta$, we have obtained a bound for $c(h, \delta)$ which is better than 2^{-h} . Recall that this is roughly the type of estimate we require for our application.

It is clear that the proof of Lemma 3.2 could be modified to handle any irreducible polynomial f over \mathbb{Z} . The quantity k would then be given by $\lceil h/\deg f \rceil$, and the resulting bound would be

$$T(h, \delta) \leq X \left(\frac{1}{k!} \left(\log \frac{\deg f - (k-1)\delta}{(h-k+1)\delta} \right)^k + o(1) \right).$$

4. Second estimate for $T(h, \delta)$. Suppose $n \in (X, 2X]$ is such that $\Omega_\delta(n^3 + 2) \geq h$ and let

$$X^\delta \leq p_1 \leq \dots \leq p_h$$

be the h smallest prime factors of $n^3 + 2$ which exceed X^δ . In the last section we counted such an n by using $p_1 \dots p_{\lceil h/3 \rceil} \mid n^3 + 2$ and $p_1 \dots p_{\lceil h/3 \rceil} \ll X$. In this section we will exploit the fact that, for many n , this product is actually much smaller than X , and therefore we can include more primes in it. Specifically, let $K \in [\lceil h/3 \rceil, h-1]$ be a parameter to be chosen later and define

$$k(n^3 + 2) = \max\{k \leq K : p_1 \dots p_k \leq 3X\}.$$

We showed in the last section that $k(n^3 + 2) \geq \lceil h/3 \rceil$ for all $n \in (x, 2x]$ such that $\Omega_\delta(n^3 + 2) \geq h$. We will choose a set \mathcal{D} which contains all the products $p_1 \dots p_{k(n^3+2)}$, and therefore we will derive inequalities satisfied by the primes in these products.

Firstly, keeping the above notation, we observe that we must have

$$p_1 \dots p_{k(n)-1} p_{k(n)}^{h-k(n)+1} \leq 9X^3.$$

Secondly, if $k(n) < K$ then

$$p_1 \dots p_{k(n)+1} \geq 3X.$$

However, we know that

$$p_1 \dots p_{k(n)} p_{k(n)+1}^{h-k(n)} \leq 9X^3$$

so that

$$p_{k(n)+1} \leq \left(\frac{9X^3}{p_1 \dots p_{k(n)}} \right)^{\frac{1}{h-k(n)}}.$$

Therefore

$$p_1 \dots p_{k(n)} \geq 3X \left(\frac{9X^3}{p_1 \dots p_{k(n)}} \right)^{-\frac{1}{h-k(n)}},$$

which simplifies to

$$(p_1 \dots p_{k(n)})^{\frac{h-k(n)-1}{h-k(n)}} \gg_{h,k(n)} X^{\frac{h-k(n)-3}{h-k(n)}}.$$

It follows, since $k(n) < h - 1$, that

$$(5) \quad p_1 \cdots p_{k(n)} \gg_{h,k(n)} X^{\frac{h-k(n)-3}{h-k(n)-1}}.$$

We can now construct a suitable set \mathcal{D} .

LEMMA 4.1. *Suppose $\delta \in (0, 1)$ and $h \geq 3$. For an integer $k \in [[h/3], K]$ let \mathcal{D}_k be the set of $d = p_1 \cdots p_k \leq 3X$ with*

$$X^\delta \leq p_1 \leq \cdots \leq p_k, \quad p_1 \cdots p_{k-1} p_k^{h-k+1} \leq 9X^3$$

and

$$p_1 \cdots p_{k(n)} \gg X^{\frac{h-k(n)-3}{h-k(n)-1}}.$$

In the final condition the implied constants are equal to those from (5). The condition is omitted if $k = K$. The set

$$\mathcal{D} = \bigcup_{k=[h/3]}^K \mathcal{D}_k$$

then satisfies the three conditions from the start of the previous section.

We must now estimate

$$\sum_{\substack{d \in \mathcal{D} \\ \mu(d) \neq 0}} \frac{\nu(d)}{d} = \sum_{k=[h/3]}^K \sum_{\substack{d \in \mathcal{D}_k \\ \mu(d) \neq 0}} \frac{\nu(d)}{d}.$$

Converting the sums to integrals, as in the last section, we show that this is equal to

$$\sum_{k=[h/3]}^K I(h, \delta, k) + o(1)$$

where

$$I(h, \delta, k) = \int_{R(h, \delta, k)} \frac{ds_1 \cdots ds_k}{s_1 \cdots s_k}$$

and

$$R(h, \delta, k) = \left\{ s_1, \dots, s_k \in \mathbb{R}^k : \delta \leq s_1 \leq \cdots \leq s_k, s_1 + \cdots + s_k \leq 1, \right. \\ \left. s_1 + \cdots + s_{k-1} + (h - k + 1)s_k \leq 3, s_1 + \cdots + s_k \geq \frac{h - k - 3}{h - k - 1} \right\}.$$

As in the lemma, the final inequality is omitted when $k = K$.

Unfortunately, the resulting integrals are now even harder to deal with than those occurring in the previous section. We are therefore forced to weaken many of the constraints on R_k in order to reach an integral which

can be evaluated. In particular the inequality

$$s_1 + \cdots + s_k \leq 1$$

will be ignored, and

$$s_1 + \cdots + s_{k-1} + (h - k + 1)s_k \leq 3$$

will be replaced by the weaker inequality

$$s_k \leq \frac{3 - (k - 1)\delta}{h - k + 1}.$$

As in the previous section we therefore deduce that

$$I(h, \delta, K) \leq \frac{1}{K!} \left(\log \frac{3 - (K - 1)\delta}{(h - K + 1)\delta} \right)^K.$$

If $k < K$ then we handle the constraint

$$s_1 + \cdots + s_k \geq \frac{h - k - 3}{h - k - 1}$$

by multiplying the integrand by

$$\exp\left(\alpha \left(s_1 + \cdots + s_k - \frac{h - k - 3}{h - k - 1}\right)\right)$$

for some $\alpha > 0$. This factor is always positive and it exceeds 1 in the region of interest. Consequently,

$$\begin{aligned} I(h, \delta, k) &\leq \frac{1}{k!} \int_{\left[\delta, \frac{3 - (k-1)\delta}{h-k+1}\right]^k} \exp\left(\alpha \left(s_1 + \cdots + s_k - \frac{h - k - 3}{h - k - 1}\right)\right) \frac{ds_1 \dots ds_k}{s_1 \dots s_k} \\ &= \frac{\exp\left(-\alpha \frac{h-k-3}{h-k-1}\right)}{k!} \left(\int_{\delta}^{3 - (k-1)\delta/h-k+1} \exp(\alpha s) \frac{ds}{s} \right)^k. \end{aligned}$$

The final integral cannot be expressed using elementary functions, but it is an exponential integral which can easily be evaluated by a computer. The value of α should be chosen to minimise the expression.

The conclusion of this section is the following estimate for $T(h, \delta)$.

LEMMA 4.2. *Suppose $\delta \in (0, 1)$ and $h \geq 3$. For any integer K in $[[h/3], h - 1]$ and any positive reals α_k ,*

$$\begin{aligned} T(h, \delta) &\leq X \left(\sum_{k=[h/3]}^{K-1} \frac{\exp\left(-\alpha_k \frac{h-k-3}{h-k-1}\right)}{k!} \left(\int_{\delta}^{\frac{3 - (k-1)\delta}{h-k+1}} \exp(\alpha_k s) \frac{ds}{s} \right)^k \right. \\ &\quad \left. + \frac{1}{K!} \left(\log \frac{3 - (K - 1)\delta}{(h - K + 1)\delta} \right)^K + o(1) \right). \end{aligned}$$

In practice we have found that when applying this lemma the largest contribution to the sum comes from $k = [h/3]$. We therefore take K sufficiently large so that the bound on $I(h, \delta, K)$ is smaller than this; little can be gained by taking a larger K .

5. Proof of Theorem 1.1. Recall that we need to estimate

$$\sum_{h>H} \min(h, [1/\delta])2^h T(h, \delta)$$

when $\delta = 1/321$. This sum is not infinite as $T(h, \delta) = 0$ for $h > 963$. We begin by using Lemma 3.2 to obtain

$$\sum_{h \geq 190} \min(h, [1/\delta])2^h T(h, \delta) \leq X(9.2 \times 10^{-10} + o(1)).$$

For smaller h we use Lemma 4.2 to obtain

$$\sum_{h=133}^{189} \min(h, [1/\delta])2^h T(h, \delta) \leq X(3.6 \times 10^{-8} + o(1)).$$

Specifically, this is obtained using the value $K = [h/3] + 20$ and optimising the choices of α_k in each instance. We therefore choose $H = 132$ and conclude that

$$\sum_{h>H} \min(h, [1/\delta])2^h T(h, \delta) \leq X(3.7 \times 10^{-8} + o(1)).$$

We deduce from this that

$$\begin{aligned} \#\{n + \sqrt[3]{2} \in \mathcal{A} : W(n) > 0\} &\geq \frac{2^{-132}}{132} X(9.2 \times 10^{-8} - 3.7 \times 10^{-8} + o(1)) \\ &\geq X(7.7 \times 10^{-50} + o(1)). \end{aligned}$$

We may therefore apply Lemma 2.2 with $\delta = 1/321$ and $\alpha = 7.7 \times 10^{-50}$ to deduce that Theorem 1.1 holds when $\varpi \leq \alpha\delta/2 = 1.2 \times 10^{-52}$, so in particular it holds when $\varpi \leq 10^{-52}$. ■

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