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Number of solutions of cubic Thue inequalities with positive discriminant

by

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1. Introduction. Let $F(X,Y)$ be an irreducible binary form with integer coefficients. Let r and D denote the degree and discriminant of F, respectively. Suppose that $r \geq 3$. Let k be a positive integer. In a pioneering work in 1909, Thue proved that the equation

$$F(X,Y) = k$$

has only finitely many solutions in integers x and y. Such equations are now called *Thue equations*. A pair (x,y) of integers is said to be *primitive* if $\gcd(x,y) = 1$. In 1983, Evertse [6] obtained an upper bound for the number of primitive solutions of the above equation, which depended only on r and k and was otherwise independent of F, thereby proving a conjecture of Siegel. Later, the upper bound of Evertse was greatly improved by Bombieri & Schmidt [3].

Let $N_F(k)$ denote the number of primitive integer solutions of the inequality

$$|F(X,Y)| \leq k. \tag{1}$$

Here, (x,y) and $(-x,-y)$ are counted as one solution. Therefore if (x,y) is a solution of (1) with $y \neq 0$, we can assume that $y > 0$. Hence if (x,y) is a primitive solution of (1), then either

$$\gcd(x,y) = 1, y > 0 \quad \text{or} \quad (x,y) = (1,0). \tag{2}$$

Throughout this paper, we assume (2) without further mention.

Many mathematicians have considered inequality (1) when k is small in comparison with $|D|$ and obtained bounds for $N_F(k)$ which involve only r. (See [7] and [8].) For instance, in [10, p. 253], we showed that if

$$|D(F)| \geq (127^r k^{4.41})^{r-1},$$

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then

$$N_F(k) \leq 27r.$$

In fact, if r is large, better bounds were found by Győry [7, Corollary 3 and remarks after Theorem 2]. He showed that if

$$|D(F)| \geq r^r(3.5^r k^2)^{2(r-1)/(1-\vartheta)} \quad \text{with } 0 < \vartheta < 1,$$

then

$$N_F(k) \leq 5r + \frac{r+2}{\vartheta}$$

for r sufficiently large.

We now restrict to the case $r = 3$. Further, suppose that the discriminant D of F is positive. (We refer to Wakabayashi [12] for the case of negative discriminant.) By the result of Evertse mentioned above, we know that there is an absolute constant C such that $N_F(1) \leq C$ for all such forms F. In fact, Evertse [5] developed the method of Siegel [9] and Gel'man (see [4, Chapter 5]) to show that

$$N_F(1) \leq 12.$$

To prove this, he first showed the following theorem.

THEOREM E. *Let $H(X,Y)$ be the Hessian of the form F. The number of solutions (x,y) of* (1) *with*

$$H(x,y) \geq \frac{3\sqrt{3D}}{2}k^3 \tag{3}$$

is at most 9.

(See Section 2 for the definition of Hessian.) When $k = 1$, Evertse showed that there are at most three solutions satisfying

$$H(x,y) < 3\sqrt{3D}/2.$$

Recently, using Theorem E, Akhtari [1, Theorem 1.1] showed that if k is a positive integer satisfying

$$k < \frac{(3D)^{1/4}}{2\pi}, \tag{4}$$

then (1) has at most

$$9 + \frac{\log\left(\frac{3}{8\epsilon} + \frac{1}{2}\right)}{\log 2}$$

solutions in coprime integers x and y with $y \neq 0$, where

$$\epsilon = \frac{1}{4} - \frac{\log(2\pi k)}{\log(3D)}.$$

We point out here that by following her argument of [1, p. 737], the above bound must be corrected to

$$9+3\left\lceil\frac{\log\left(\frac{3}{8\epsilon}-\frac{1}{2}\right)}{\log 2}\right\rceil.$$

(Here and elsewhere, $\lceil x\rceil$ denotes the smallest integer greater than or equal to the real number x.) If we add 1 to this corrected estimate for the possible solution $(1,0)$, Akhtari's result yields

$$\begin{aligned}(5)\qquad N_F(k) &\le 10+3\left\lceil\frac{\log\left(\frac{3}{8\epsilon}-\frac{1}{2}\right)}{\log 2}\right\rceil\\ &= 10+3\left\lceil\frac{1}{\log 2}\log\left(\frac{\log(3D)+2\log k+2\log(2\pi)}{\log(3D)-4\log k-4\log(2\pi)}\right)\right\rceil.\end{aligned}$$

In 2001, Bennett [2] used extensive computation and made the work of Evertse more precise in the case $k=1$ to show that

$$N_F(1)\le 10.$$

In fact, according to [2, Sections 8 & 9],

$$N_F(1)\le 9 \quad \text{if } D\le 10^6.$$

Since the smallest positive discriminant of an irreducible cubic form is 49, we will assume from now onwards that

$$(6)\qquad D\ge\begin{cases}10^6 & \text{if } k=1,\\ 49 & \text{if } k\ge 2.\end{cases}$$

We shall use the calculations of Bennett to obtain the following result analogous to Theorem E.

Theorem 1.1. *Let $F(X,Y)$ be an irreducible binary cubic form with integer coefficients and positive discriminant D. Let k be a positive integer. Then there are at most six solutions (x,y) of (1) with*

$$(7)\qquad H(x,y)\ge 1.8(3D)^{5/6}k^4.$$

Complementing Theorem 1.1, we show the next result.

Theorem 1.2. *Let $F(X,Y)$ be an irreducible binary cubic form with integer coefficients and positive discriminant D. Let k be a positive integer satisfying (4). Then the number of solutions (x,y) of (1) with $y\ne 0$ and*

$$H(x,y)<1.8(3D)^{5/6}k^4$$

is at most

$$3\left\lceil\frac{1}{\log 2}\log\left(\frac{5\log(3D)+12\log k+2.13}{3\log(3D)-12\log k-5.56}\right)\right\rceil.$$

As an immediate consequence of Theorems 1.1, 1.2 and including the possible solution $(1,0)$, we get the following corollary.

Corollary 1.3. *Let $F(X,Y)$ be an irreducible binary cubic form with integer coefficients and positive discriminant D. Let k be a positive integer satisfying* (4)*. Then*

$$N_F(k) \le 7 + 3\left\lceil \frac{1}{\log 2}\log\left(\frac{5\log(3D)+12\log k+2.13}{3\log(3D)-12\log k-5.56}\right)\right\rceil. \tag{8}$$

Remark 1.4. (i) Note that $N_F(1) \le 10$, thus retrieving Bennett's result.

(ii) We now show that the upper bound for $N_F(k)$ in (8) is better than the bound in (5). Put

$$\Upsilon = \frac{5\log(3D)+12\log k+2.13}{3\log(3D)-12\log k-5.56}, \qquad \chi = \frac{\log(3D)+2\log k+3.67}{\log(3D)-4\log k-7.35}.$$

Now

$$\Upsilon = \frac{5\log(3D)+12\log k+2.13}{3\log(3D)+6\log k+1.05}\cdot\frac{3\log(3D)+6\log k+1.05}{3\log(3D)-12\log k-5.56} \le 2\chi.$$

By (8), we have

$$N_F(k) \le 7+3\left\lceil\frac{\log\Upsilon}{\log 2}\right\rceil \le 7+3\left\lceil 1+\frac{\log\chi}{\log 2}\right\rceil \le 10+3\left\lceil\frac{\log\chi}{\log 2}\right\rceil,$$

proving the claim.

(iii) Our proof of Theorem 1.1 depends on the method given in the papers of Evertse and Bennett. They assumed that there were at least four solutions of (1) related to any pair of resolvent forms (see Section 2 for definition). This enabled them to get a good gap principle and derive a contradiction. To obtain Theorem 1.1, we assume that there are at least three solutions of (1) related to any pair of resolvent forms. To get a good gap principle, we need to assume that (7) is valid, which is weaker than (3). In fact, if k satisfies (4), then it is enough to assume that

$$H(x,y) \ge 1.8(3D)^{5/6}k^{7/2},$$

and this leads to an estimate

$$N_F(k) \le 7 + 3\left\lceil \frac{1}{\log 2}\log\left(\frac{5\log(3D)+9\log k+2.13}{3\log(3D)-12\log k-5.56}\right)\right\rceil \tag{9}$$

in Corollary 1.3. It is possible to carry this method further to deduce that there are at most three primitive solutions (x,y) of (1) satisfying

$$H(x,y) \ge 3(3D)^{5/3}k^6.$$

This yields

$$N_F(k) \le 4 + 3\left\lceil \frac{1}{\log 2}\log\left(\frac{10\log(3D)+24\log k+5.22}{3\log(3D)-12\log k-5.56}\right)\right\rceil.$$

One can easily see that this bound is no better than (9). Thus it may not be possible to significantly improve (9) by this method.

Corollary 1.3 readily yields an estimate for the number of all integer solutions (primitive and non-primitive) to (1). When $k = 1$, all the solutions are primitive. So we assume that $k \geq 2$.

COROLLARY 1.5. *Suppose that*

$$2\pi k = (3D)^{1/4-\delta} \qquad \text{with } 0.1 < \delta < 1/4.$$

Then the number of integer solutions to (1) *is at most* $18k^{1/3}$.

REMARK 1.6. In [11], Thunder considered the inequality (1) for any positive integer k and discriminant of F positive or negative. Using a different method, he showed that the number of integer solutions is at most

$$9 + \frac{16k^{2/3}}{|D|^{1/6}} + \frac{2008k^{1/2}}{|D|^{1/12}} + 3156k^{1/3}.$$

Under the assumptions of Corollary 1.5, the above bound can be majorized by $3770k^{1/3}$.

2. Preliminaries. We refer to [5] for the ensuing facts on cubic forms. Write

$$F(X,Y) = aX^3 + bX^2Y + cXY^2 + dY^3.$$

The *quadratic covariant*, or *Hessian*, and the *cubic covariant* of F are defined as

$$H(X,Y) = -\frac{1}{4}\left(\frac{\partial^2 F}{\partial X^2}\frac{\partial^2 F}{\partial Y^2} - \left(\frac{\partial^2 F}{\partial X \partial Y}\right)^2\right),$$

$$G(X,Y) = \frac{\partial F}{\partial X}\frac{\partial H}{\partial Y} - \frac{\partial F}{\partial Y}\frac{\partial H}{\partial X},$$

respectively. It can be checked that

$$H(X,Y) = AX^2 + BXY + CY^2$$

where

$$A = b^2 - 3ac, \quad B = bc - 9ad \quad \text{and} \quad C = c^2 - 3bd.$$

Further,

$$B^2 - 4AC = -3D,$$

where D is the discriminant of F, and

$$4H(X,Y)^3 = G(X,Y)^2 + 27DF(X,Y)^2. \tag{10}$$

The form F is said to be *reduced* if

$$C \geq A \geq |B|.$$

Since every cubic form of positive discriminant is $\mathrm{GL}(2,\mathbb{Z})$-equivalent to a reduced form and $N_{F_1}(k) = N_{F_2}(k)$ for equivalent forms F_1 and F_2, we can assume that F is reduced.

Let $\sqrt{-3D}$ be a fixed choice of the square root of $-3D$, let $M = \mathbb{Q}(\sqrt{-3D})$ and let $\mathfrak{O}_M$ denote the ring of integers in M. Then

$$\mathfrak{O}_M = \left\{\frac{m+n\sqrt{-3D}}{2} : m,n \in \mathbb{Z},\ m \equiv nD \pmod 2\right\}.$$

Set

$$U(X,Y) = \frac{G(X,Y) + 3\sqrt{-3D}\,F(X,Y)}{2},$$
$$V(X,Y) = \frac{G(X,Y) - 3\sqrt{-3D}\,F(X,Y)}{2}.$$

Observe that $U(X,Y)$ and $V(X,Y)$ are cubic forms in $M[X,Y]$ having no common factor. Also, the corresponding coefficients of $U(X,Y)$ and $V(X,Y)$ are complex conjugates. The relation (10) implies that

$$U(X,Y)V(X,Y) = H(X,Y)^3.$$

Hence

$$U(X,Y) = \xi(X,Y)^3, \qquad V(X,Y) = \eta(X,Y)^3,$$

where $\xi(X,Y)$ and $\eta(X,Y)$ are linear forms whose corresponding coefficients are complex conjugates. Further,

$$\begin{aligned} \xi(X,Y)^3 - \eta(X,Y)^3 &= 3\sqrt{-3D}\,F(X,Y) \\ \xi(X,Y)^3 + \eta(X,Y)^3 &= G(X,Y) \\ \xi(X,Y)\eta(X,Y) &= H(X,Y) \\ \frac{\xi(X,Y)}{\xi(1,0)}, \frac{\eta(X,Y)}{\eta(1,0)} &\in M[X,Y]. \end{aligned} \tag{11}$$

Therefore for all integers x and y, we have

$$|\xi(x,y)|^2 = |\eta(x,y)|^2 = |H(x,y)|. \tag{12}$$

A pair (ξ,η) of forms satisfying the properties (11) is called a pair of *resolvent forms*. If (ξ,η) is such a pair, then there are precisely two others, namely, $(\rho\xi,\rho^2\eta)$ and $(\rho^2\xi,\rho\eta)$, where ρ is a primitive cube root of unity. A pair (x,y) of integers is said to be *related to a pair of resolvent forms* (ξ,η) if

$$\left|1 - \frac{\eta(x,y)}{\xi(x,y)}\right| = \min_{0\le\ell\le 2}\left|e^{2\ell\pi i/3} - \frac{\eta(x,y)}{\xi(x,y)}\right|. \tag{13}$$

We need the following lemma.

Lemma 2.1 ([2, Lemma 5.1]). *Let F be an irreducible, reduced binary cubic form with positive discriminant D and Hessian H. Then for all integers x, y with $y \neq 0$, we have*

$$H(x,y) \ge \sqrt{3D}/2.$$

LEMMA 2.2. *Let (x,y), $y\neq 0$, be a solution of* (1) *related to (ξ,η). Put*

$$(\mu,\nu)=\begin{cases}(1,\pi/3) & \text{if } |\xi(x,y)|\geq 1.8^{1/2}(3D)^{5/12}k^2,\\ (3\sqrt{2}/\pi,1.1) & \text{if (4) holds.}\end{cases}$$

Then

$$\left|1-\frac{\eta(x,y)^3}{\xi(x,y)^3}\right|<\mu \quad \text{and} \quad \left|1-\frac{\eta(x,y)}{\xi(x,y)}\right|<\frac{\nu\sqrt{3D}\,k}{|\xi(x,y)|^3}.$$

Proof. Lemma 2.1 and equation (12) imply that

$$|\xi(x,y)|\geq (3D)^{1/4}/\sqrt{2}. \tag{14}$$

If (4) holds, then using (11) and (6) we obtain

$$\left|1-\frac{\eta(x,y)^3}{\xi(x,y)^3}\right|\leq\frac{3\sqrt{3D}\,k}{|\xi(x,y)|^3}\leq\frac{3\sqrt{3D}\,k\times 2^{3/2}}{(3D)^{3/4}}<\frac{3\sqrt{2}}{\pi}.$$

Similarly, if $|\xi(x,y)|\geq 1.8^{1/2}(3D)^{5/12}k^2$, then

$$\left|1-\frac{\eta(x,y)^3}{\xi(x,y)^3}\right|\leq\frac{3}{1.8^{3/2}(3D)^{3/4}k^5}<1.$$

This proves the first assertion of the lemma. Let

$$\theta=\arg(\eta(x,y)/\xi(x,y)).$$

Since $|\eta(x,y)/\xi(x,y)|=1$ and equation (13) holds, we have

$$\arg(\eta(x,y)^3/\xi(x,y)^3)=3\theta.$$

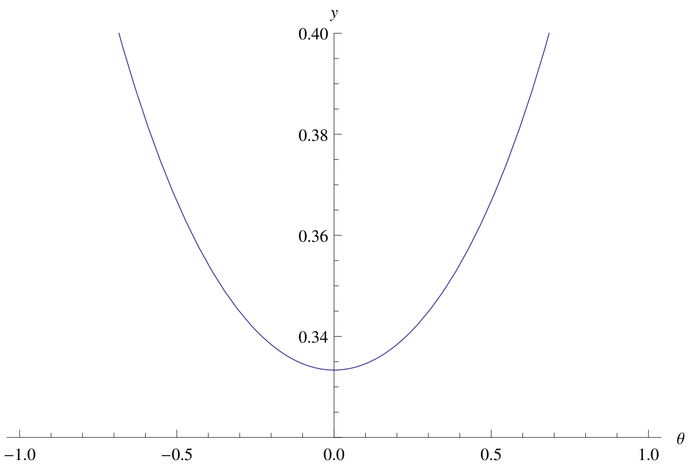

Fig. 1. $y=\frac{|\theta|}{\sqrt{2-2\cos(3\theta)}}$

In view of the fact that

$$2-2\cos(3\theta)=\left|1-\frac{\eta(x,y)^3}{\xi(x,y)^3}\right|^2<\mu^2,$$

we get

$$|\theta|<\begin{cases}\pi/9 & \text{if } |\xi(x,y)|\ge 1.8^{1/2}(3D)^{5/12}k^2,\\ 0.495 & \text{if (4) holds.}\end{cases}$$

Now

$$\left|1-\frac{\eta(x,y)}{\xi(x,y)}\right|\le|\theta|=\frac{|\theta|}{\sqrt{2-2\cos(3\theta)}}\left|1-\frac{\eta(x,y)^3}{\xi(x,y)^3}\right|<\frac{\nu}{3}\left|1-\frac{\eta(x,y)^3}{\xi(x,y)^3}\right|.$$

(See also Figure 1 for an approximate value of ν.) Using (11), we obtain the second assertion of the lemma. ∎

3. Gap principle. Let (x_1,y_1), (x_2,y_2) be two distinct solutions of (1) related to (ξ,η) with $|\xi(x_2,y_2)|\ge|\xi(x_1,y_1)|$. In this section, we will establish certain results regarding the *gaps* between such solutions. For $i=1,2$, we denote $\xi(x_i,y_i)$ by ξ_i and $\eta(x_i,y_i)$ by η_i. Since the determinant of the linear transformation $(x,y)\to(\xi,\eta)$ is $\pm\sqrt{-3D}$, we have

$$\xi_1\eta_2-\xi_2\eta_1=\pm\sqrt{-3D}(x_1y_2-x_2y_1).$$

This implies that

$$\sqrt{3D}\le|\xi_1\eta_2-\xi_2\eta_1|\le|\xi_1|\,|\xi_2|\left(\left|1-\frac{\eta_1}{\xi_1}\right|+\left|1-\frac{\eta_2}{\xi_2}\right|\right).$$

By Lemma 2.2, we obtain

$$\sqrt{3D}<\nu k|\xi_1|\,|\xi_2|\sqrt{3D}\left(\frac{1}{|\xi_1|^3}+\frac{1}{|\xi_2|^3}\right).$$

Thus

$$|\xi_1|^3+|\xi_2|^3>\frac{1}{\nu k}|\xi_1\xi_2|^2 \tag{15}$$

implying that

$$|\xi_2|>\frac{1}{2\nu k}|\xi_1|^2.$$

We obtain a better gap principle in the following lemma.

Lemma 3.1. *Let (x_1,y_1), (x_2,y_2) be two solutions of* (1) *related to (ξ,η) with $|\xi_2|\ge|\xi_1|$ and $y_1,y_2\ne0$. Then*

$$|\xi_2|\ge\frac{\tau}{k}|\xi_1|^2,$$

where

$$\tau=\begin{cases}0.95 & \text{if } |\xi(x,y)|\ge 1.8^{1/2}(3D)^{5/12}k^2,\\ 0.89 & \text{if (4) holds.}\end{cases}$$

Proof. Let

$$\phi = k|\xi_2|\,|\xi_1|^{-2} \quad \text{and} \quad h(z) = z^3 - \frac{1}{\nu}z^2 + \left(\frac{k}{|\xi_1|}\right)^3.$$

From equation (15), we have $h(\phi) > 0$. Observe that $h(0) > 0$. Using (6) and (14), respectively, we obtain

$$\frac{k}{|\xi_1|} < \begin{cases} 0.047 & \text{if } |\xi(x,y)| \geq 1.8^{1/2}(3D)^{5/12}k^2, \\ 1/\sqrt{2}\pi & \text{if (4) holds.} \end{cases}$$

Hence $h(2/(3\nu)) < 0$. Further $h(z)$ assumes a local maximum at $z = 0$ and a local minimum at $z = 2/(3\nu)$. Hence $h(z)$ has two positive zeros, say ϕ_1 and ϕ_2, with $\phi_1 < \phi_2$; and $h(z)$ is negative for $\phi_1 < z < \phi_2$ and positive for $0 < z < \phi_1$, $z > \phi_2$. If $\phi \leq \phi_1$ then we must have $h(k/|\xi_1|) > 0$ as $\phi \geq k/|\xi_1|$. But $h(k/|\xi_1|) < 0$. Therefore $\phi > \phi_1$. This, together with the fact that $h(\phi) > 0$, implies that $\phi > \phi_2$. Since $h(\tau) < 0$, we have $\phi \geq \tau$. ∎

4. Proof of Theorem 1.2. Let (x, y) be a solution of (1) with $y \neq 0$ and

$$|\xi(x,y)|^2 = H(x,y) < 1.8(3D)^{5/6}k^4. \tag{16}$$

Enumerate the solutions (x, y) of (1) with $y \neq 0$ as $(x_1, y_1), (x_2, y_2), \ldots$ with $\xi_i = \xi(x_i, y_i)$ and

$$|\xi_1| \leq |\xi_2| \leq \cdots.$$

Applying inductively the gap principle stated in Lemma 3.1 and using (14), we obtain

$$|\xi_t| \geq \left(\frac{0.89}{k}\right)^{2^{t-1}-1} |\xi_1|^{2^{t-1}} \geq \left(\frac{0.89}{k}\right)^{2^{t-1}-1} \left(\frac{(3D)^{1/4}}{\sqrt{2}}\right)^{2^{t-1}} \qquad \text{for all } t \geq 1.$$

Let t be the least integer such that

$$\left(\frac{0.89}{k}\right)^{2^{t-1}-1} \left(\frac{(3D)^{1/4}}{\sqrt{2}}\right)^{2^{t-1}} \geq 1.8^{1/2}(3D)^{5/12}k^2. \tag{17}$$

Then there are at most $t - 1$ solutions (x, y) of (1) with $y \neq 0$ and satisfying (16). Taking logarithms twice in (17), we obtain the assertion of the theorem. ∎

5. Padé approximation. In this section, we introduce some auxiliary polynomials used in the proof of Theorem 1.1. Let α, β and γ be complex numbers. The standard hypergeometric function $F(\alpha, \beta, \gamma, z)$ is represented as

$$F(\alpha,\beta,\gamma,z) = 1 + \sum_{n=1}^{\infty} \frac{\alpha(\alpha+1)\cdots(\alpha+n-1)\beta(\beta+1)\ldots(\beta+n-1)}{\gamma(\gamma+1)\ldots(\gamma+n-1)n!} z^n.$$

Let r be a positive integer and let $g \in \{0,1\}$. Define

$$A_{r,g} = \binom{2r-g}{r} F\Big(-\frac{1}{3} - r + g, -r, -2r+g, z\Big)$$
$$B_{r,g} = \binom{2r-g}{r-g} F\Big(\frac{1}{3} - r, -r+g, -2r+g, z\Big).$$

See [5, Lemma 3] for the proof of the following lemma on Padé approximation.

Lemma 5.1.

(i) *There exists a power series $F_{r,g}(z)$ such that for all complex numbers z with $|z| < 1$, we have*

$$A_{r,g}(z) - (1-z)^{1/3} B_{r,g}(z) = z^{2r+1-g} F_{r,g}(z), \tag{18}$$

$$|F_{r,g}(z)| \le \frac{\binom{r-g+1/3}{r+1-g}\binom{r-1/3}{r}}{\binom{2r+1-g}{r}} \frac{1}{(1-|z|)^{(2r+1-g)/2}}. \tag{19}$$

(ii) *For all complex numbers z with $|1-z| \le 1$, we have*

$$|A_{r,g}(z)| \le \binom{2r-g}{r}. \tag{20}$$

(iii) *Let z be a non-zero complex number and let $h \in \{0,1\}$. Then*

$$A_{r,0}(z)B_{r+h,1}(z) - A_{r+h,1}(z)B_{r,0}(z) \neq 0.$$

Let $C_{r,g}$ denote the greatest common divisor of the numerators of the coefficients of $A_{r,g}$. Note that $C_{r,g}$ is also the greatest common divisor of the numerators of the coefficients of $B_{r,g}$. See Table 1 in the next section for the values of $C_{r,g}$ for some choices of r and g.

6. Proof of Theorem 1.1. In this section, we will show that there are at most two solutions (x,y) of (1) with

$$|\xi(x,y)|^2 = H(x,y) \ge 1.8(3D)^{5/6}k^4 \tag{21}$$

which are related to a given pair of resolvent forms. Since there are exactly three pairs of resolvent forms, Theorem 1.1 will follow.

Assume that there are three solutions (x_{-1},y_{-1}), (x_0,y_0), (x_1,y_1) of (1) satisfying (21) and related to (ξ,η). As before, we denote $\xi(x_i,y_i)$ by ξ_i and $\eta(x_i,y_i)$ by η_i for $i=-1,0,1$. Set

$$z_0 = 1-\eta_0^3/\xi_0^3, \quad \Sigma_{r,g} = \frac{\eta_1}{\xi_1}A_{r,g}(z_0) - \frac{\eta_0}{\xi_0}B_{r,g}(z_0), \quad \Lambda_{r,g} = \frac{1}{C_{r,g}}\xi_0^{3r+1-g}\xi_1\Sigma_{r,g}.$$

From (6), (11) and Lemma 3.1, it follows that

$$0 < |z_0| < 10^{-6}. \tag{22}$$

Hence for any primitive cube root of unity ρ, we have

$$|1-(1-z_0)^{1/3}| < |1-\rho(1-z_0)^{1/3}|.$$

This, together with (13), implies that

$$\eta_0/\xi_0 = (1-z_0)^{1/3}. \tag{23}$$

It follows from the proof of [5, Lemma 5] that

$$\varLambda_{r,0} \in \sqrt{-3D}\mathbb{Z} \quad \text{and} \quad \varLambda_{r,1}^3 \in \mathfrak{O}_M \setminus \mathbb{Z}.$$

Therefore if $\varLambda_{r,g} \neq 0$, we have

$$|\varLambda_{r,0}| \geq \sqrt{3D} \quad \text{and} \quad |\varLambda_{r,1}^3| \geq \sqrt{3D}/2,$$

i.e.

$$|\varLambda_{r,g}| \geq 2^{-g/3}(3D)^{1/2-g/3}. \tag{24}$$

LEMMA 6.1. *Let*

$$\begin{aligned}
c_1'(r,g) &= \frac{1}{C_{r,g}}\binom{2r}{r}\frac{\pi}{3}2^{-2g/3},\\
c_2'(r,g) &= \frac{1}{C_{r,g}}\frac{\binom{r-g+1/3}{r+1-g}\binom{r-1/3}{r}}{\binom{2r+1-g}{r}}2^{g/3}(3.001)^{2r+1-g},\\
\varXi(r,g) &= c_1'(r,g)(3D)^{g/3}|\xi_0|^{3r+1-g}|\xi_1|^{-2}k,\\
\varPi(r,g) &= c_2'(r,g)(3D)^{r-g/6}|\xi_1|\,|\xi_0|^{-(3r+2-2g)}k^{2r+1-g}.
\end{aligned}$$

If $\varSigma_{r,g} \neq 0$, *then*

$$\varXi(r,g) + \varPi(r,g) > 1.$$

Proof. Using (23), (18), (20), (19), Lemma 2.2 and (22), we obtain

$$\begin{aligned}
|\varLambda_{r,g}| &= \frac{1}{C_{r,g}}|\xi_0|^{3r+1-g}|\xi_1|\left|\left(\frac{\eta_1}{\xi_1}-1\right)A_{r,g}(z_0) + z_0^{2r+1-g}F_{r,g}(z_0)\right|\\
&\leq \frac{1}{C_{r,g}}|\xi_0|^{3r+1-g}|\xi_1|\left(\binom{2r-g}{r}\left|1-\frac{\eta_1}{\xi_1}\right|\right.\\
&\qquad\qquad\left. + \frac{\binom{r-g+1/3}{r+1-g}\binom{r-1/3}{r}}{\binom{2r+1-g}{r}}\frac{|z_0|^{2r+1-g}}{(1-|z_0|)^{(2r+1-g)/2}}\right)\\
&\leq \frac{1}{C_{r,g}}|\xi_0|^{3r+1-g}|\xi_1|\left(\binom{2r-g}{r}\frac{\pi}{3}\frac{\sqrt{3D}\,k}{|\xi_1|^3}\right.\\
&\qquad\qquad\left. + \frac{\binom{r-g+1/3}{r+1-g}\binom{r-1/3}{r}}{\binom{2r+1-g}{r}}\left(\frac{3\sqrt{3D}\,k}{(1-10^{-6})^{1/2}|\xi_0|^3}\right)^{2r+1-g}\right)\\
&< 2^{-g/3}(3D)^{1/2-g/3}\big(c_1'(r,g)(3D)^{g/3}|\xi_0|^{3r+1-g}|\xi_1|^{-2}k\\
&\qquad\qquad + c_2'(r,g)(3D)^{r-g/6}|\xi_1|\,|\xi_0|^{-(3r+2-2g)}k^{2r+1-g}\big).
\end{aligned}$$

Now the lemma follows from (24). ∎

For $1 \le r \le 7$ and for certain choices of g, let $c_1(r,g)$ and $c_2(r,g)$ be defined by Table 1. For $r \ge 8$, set

$$c_1(r,g) = 4^r/\sqrt{r} \quad \text{and} \quad c_2(r,g) = (2.252)^r/\sqrt{r}.$$

Table 1

(r,g)	$C_{r,g}$	$c_1(r,g)$	$c_2(r,g)$	(r,g)	$C_{r,g}$	$c_1(r,g)$	$c_2(r,g)$
$(1,1)$	1	1.32	1.3	$(5,0)$	28	9.425	0.7
$(1,0)$	2	1.048	0.7	$(6,1)$	14	43.54	2.4
$(2,0)$	1	6.284	2.4	$(6,0)$	14	69.116	2.7
$(3,0)$	20	1.048	0.3	$(7,1)$	4	566.017	16.9
$(4,0)$	5	14.661	1.8	$(7,0)$	88	40.841	0.9

The values of $C_{r,g}$ and $c_2(r,g)$ are exactly as in [2]. The values of $c_1(r,g)$ are different due to the gap principle in Lemma 3.1. We will now prove that

$$c_1'(r,g) \le c_1(r,g) \quad \text{and} \quad c_2'(r,g) \le c_2(r,g). \tag{25}$$

(Then for these choices of r and g, we can use Lemma 6.1 with $c_1'(r,g)$ and $c_2'(r,g)$ replaced by $c_1(r,g)$ and $c_2(r,g)$, respectively.) It can be easily checked that (25) holds for the choices of r and g listed in Table 1. Further, by [2, eq. (6.7)], we have

$$\binom{2r}{r} < \frac{4^r}{\sqrt{\pi r}}$$

for all positive integers r. Therefore

$$c_1'(r,g) < \frac{4^r}{\sqrt{\pi r}}\frac{\pi}{3} < \frac{4^r}{\sqrt{r}} \qquad \forall r \in \mathbb{N}.$$

By [2, proof of Lemma 6.2, last inequality],

$$c_2'(r,g) < \frac{3.001\sqrt{3}}{\pi}\left(\frac{r+1}{2r+1}\right)\frac{(2.252)^r}{\sqrt{r}}.$$

Hence

$$c_2'(r,g) < (2.252)^r/\sqrt{r} \qquad \text{for } r \ge 5.$$

This completes the proof of (25).

The next two results are Lemmas 6.3 and 6.4 from [2].

LEMMA 6.2. *$\Sigma_{r,g} \ne 0$ for $(r,g) = (1,1),(1,0),(2,0),(3,0),(4,0),(5,0)$.*

LEMMA 6.3. *Let r be a positive integer and let $h \in \{0,1\}$. Then at least one of $\Sigma_{r,0}$ and $\Sigma_{r+h,1}$ is non-zero.*

The final step. Note that

$$|\xi_0| \ge \frac{0.95}{k}|\xi_{-1}|^2 \ge 1.71(3D)^{5/6}k^3. \tag{26}$$

We say that *property* $P[a_1, a_2, a_3, a_4]$ holds if

$$|\xi_1| > \frac{a_1|\xi_0|^{a_2}}{(3D)^{a_3}k^{a_4}}.$$

By Lemma 3.1, $P[0.95, 2, 0, 1]$ holds. We shall show that

$$P[2.6^{-r}, 3r+2, r, 2r+1] \text{ holds for all } r \geq 1. \tag{27}$$

From this, by (26), we deduce that $P[1, 0, -r, -7r]$ holds for any $r \geq 1$. This is not possible. Thus there can be at most two solutions related to (ξ, η).

Now we prove (27). In the calculations below, (6), (25) and the values of $c_1(r, g)$, $c_2(r, g)$ will be repeatedly used. First we take $(r, g) = (1, 1)$. Then $P[0.95, 2, 0, 1]$ and (26) yield

$$\Xi(1,1) \leq \frac{1.32}{0.95^2}\frac{(3D)^{1/3}}{|\xi_0|}k^3 \leq \frac{1.32}{0.95^2 \times 1.71(3D)^{1/2}} < 0.0706.$$

Hence it follows from Lemmas 6.2 and 6.1 that

$$\Pi(1,1) > 0.9294,$$

which shows that $P[0.7149, 3, 5/6, 2]$ holds. We abbreviate this as

$$\{P[0.95, 2, 0, 1], (1,1)\} \to P[0.7149, 3, 5/6, 2].$$

Next, we fix $(r, g) = (1, 0)$. Then arguing as above, we get

$$\{P[0.7149, 3, 5/6, 2], (1,0)\} \to P[0.4267, 5, 1, 3].$$

Proceeding thus we get the following sequence:

$$\begin{aligned}&\{P[0.95, 2, 0, 1], (1,1)\} \to \{P[0.7149, 3, 5/6, 2], (1,0)\}\\ &\to \{P[0.4267, 5, 1, 3], (2,0)\} \to \{P[0.3573, 8, 2, 5], (3,0)\}\\ &\to \{P[3.333, 11, 3, 7], (4,0)\} \to \{P[0.555, 14, 4, 9], (5,0)\} \to P[1.428, 17, 5, 11].\end{aligned}$$

Hence, (27) holds for $2 \leq r \leq 5$. We now proceed by induction. Suppose that (27) holds for some $r \geq 5$. Then we fix (r, g) as $(r+1, 0)$. Suppose $\Sigma_{r+1,0} \neq 0$. Then we argue as earlier to get

$$\Xi(r+1, 0) < 0.001,$$

and hence

$$\Pi(r+1, 0) > 0.999,$$

giving

$$|\xi_1| \geq \frac{0.999\sqrt{r+1}}{2.252^{r+1}}\frac{|\xi_0|^{3r+5}}{(3D)^{r+1}k^{2r+3}} \geq \frac{|\xi_0|^{3r+5}}{2.6^{r+1}(3D)^{r+1}k^{2r+3}}.$$

Thus we have

$$\{P[2.6^{-r}, 3r+2, r, 2r+1], (r+1, 0)\} \to P[2.6^{-(r+1)}, 3r+5, r+1, 2r+3],$$

proving the claim. If $\varSigma_{r+1,0} = 0$ then, by Lemma 6.3, both $\varSigma_{r+1,1}$ and $\varSigma_{r+2,1}$ are non-zero. First we fix (r, g) as $(r + 1, 1)$. Then we get

$$\{P[2.6^{-r}, 3r + 2, r, 2r + 1], (r + 1, 1)\} \to P[2.252^{-r}, 3r + 3, r + 5/6, 2r + 2].$$

Now we fix (r, g) as $(r + 2, 1)$. Then

$$\{P[2.252^{-r}, 3r+3, r+5/6, 2r+2], (r+2, 1)\} \to P[2.6^{-(r+1)}, 3r+5, r+1, 2r+3].$$

This completes the induction. ∎

7. Proof of Corollary 1.5. It is easy to see that the number of integer solutions to (1) is

$$\leq \sum_{d=1}^{k^{1/3}} 10 + \frac{3}{\log 2} \log\left(\frac{5\log(3D) + 12\log(k/d^3) + 2.13}{3\log(3D) - 12\log(k/d^3) - 5.56}\right).$$

Since $2\pi k = (3D)^{1/4-\delta}$ and $\delta > 0.1$, this can be estimated by

$$10k^{1/3} + \frac{3k^{1/3}}{\log 2} \log\left(\frac{(8 - 12\delta)\log(3D) - 12\log(2\pi) + 2.13}{12\delta\log(3D) + 12\log(2\pi) - 5.56}\right) \leq 18k^{1/3}. \ ∎$$

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