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Quadratic congruences on average and rational points on cubic surfaces

by

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1. Introduction. Given a (possibly singular) del Pezzo surface S defined over the field $\mathbb{Q}$ of rational numbers and containing infinitely many rational points, we would like to study the distribution of these points more precisely. We will be most interested in the cubic surface of singularity type $\mathbf{A}_5 + \mathbf{A}_1$ defined in $\mathbb{P}^3$ by

$$x_1^3 + x_2x_3^2 + x_0x_1x_2 = 0. \tag{1.1}$$

Let $H : S(\mathbb{Q}) \to \mathbb{R}$ be an anticanonical height function. The number of rational points of bounded height on S is dominated by the number of points lying on the lines on (an anticanonical model of) S. Therefore, it is more interesting to study rational points of height bounded by B on the complement U of the lines on S, i.e., the number

$$N_{U,H}(B) = \#\{\mathbf{x} \in U(\mathbb{Q}) \mid H(\mathbf{x}) \le B\}.$$

Manin's conjecture [FMT89] predicts that, as B tends to ∞,

$$N_{U,H}(B) = c_{S,H} B(\log B)^{r-1}(1 + o(1)),$$

where r is the rank of the Picard group of (a minimal desingularization of) S and $c_{S,H}$ is a positive constant for which Peyre, Batyrev and Tschinkel have given a conjectural interpretation [Pey95], [BT98b].

If S is an equivariant compactification of an algebraic group G, Manin's conjecture can be proved in certain cases. For instance, see [BT98a] for the case of toric varieties (with $G = \mathbb{G}_{\mathrm{m}}^2$), [CLT02] for the case of the additive group $G = \mathbb{G}_{\mathrm{a}}^2$ and [TT12] for certain semidirect products $G = \mathbb{G}_{\mathrm{a}} \rtimes \mathbb{G}_{\mathrm{m}}$. However, equation (1.1) defines a cubic surface that is not covered by any of these results (see [DL10], [DL15]).

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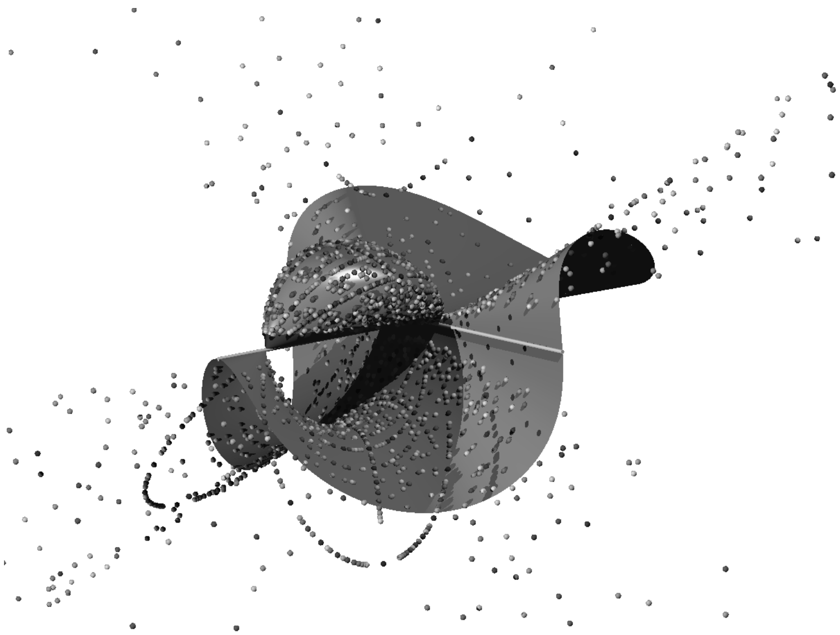

Fig. 1. Points of height at most 100 on the $\mathbf{A}_5 + \mathbf{A}_1$ cubic surface

For general surfaces S, one can approach Manin's conjecture resorting to universal torsors. Using Cox rings, a universal torsor $\mathcal{T}$ of a minimal desingularization $\widetilde{S}$ of a del Pezzo surface S of degree d can be explicitly described as an open subset of an affine variety $\operatorname{Spec}\operatorname{Cox}(\widetilde{S})$. The basic case is again the one of toric varieties [Sal98], where $\operatorname{Spec}\operatorname{Cox}(\widetilde{S}) \cong \mathbb{A}^{12-d}$ is an affine space.

The next natural case is when $\operatorname{Spec}\operatorname{Cox}(\widetilde{S}) \subset \mathbb{A}^{13-d}$ is a hypersurface defined by one *torsor equation* in the variables $\eta_1, \dots, \eta_{13-d}$. For example, for our surface of degree $d = 3$ and type $\mathbf{A}_5 + \mathbf{A}_1$, the torsor equation is

$$\eta_1\eta_{10} + \eta_2\eta_9^2 + \eta_4\eta_5^2\eta_6^4\eta_7^3\eta_8 = 0. \tag{1.2}$$

All such del Pezzo surfaces are classified in [Der14], where a detailed description of $\operatorname{Cox}(\widetilde{S})$ is also given.

The passage to a universal torsor translates the problem of counting rational points on S to the one of counting tuples $(\eta_1, \dots, \eta_{13-d})$ of integers satisfying the torsor equation and certain height and coprimality conditions.

This is basically done as follows. The coprimality conditions can be taken care of by Möbius inversions (in this introduction, we will simply ignore all auxiliary variables occurring because of this). Using a torsor equation such as (1.2), we may eliminate one variable η_{13-d} that occurs linearly in it. Fixing $\eta_1, \dots, \eta_{11-d}$, we are led to count the number of integers η_{12-d} satisfying a congruence condition modulo some integer q and lying in some range I given by the height conditions. In our example, the congruence condition is

$$\eta_2\eta_9^2 \equiv -\eta_4\eta_5^2\eta_6^4\eta_7^3\eta_8 \bmod \eta_1.$$

Note that both I and q may depend on $\eta_1, \dots, \eta_{11-d}$.

If η_{12-d} also occurs linearly in the torsor equation then the congruence is linear, so that the number of such η_{12-d} is basically $q^{-1}\operatorname{vol}(I) + E$, where $E = O(1)$. Summing this over the remaining variables $\eta_1, \dots, \eta_{11-d}$, we must estimate the main term $q^{-1}\operatorname{vol}(I)$ and show that the contribution of the error term E is negligible. The estimation of the error term of the first summation is sometimes straightforward and sometimes hard. The estimation of the main term is expected to be often straightforward using the results of [Der09, Sections 4, 5, 7] in the case of linear η_{12-d}.

However, if η_{12-d} occurs with a square power in the torsor equation (such as η_9^2 in (1.2)), the main term contains an extra factor of the shape

$$\mathcal{N}(a, q) = \#\{\varrho \mid 1 \le \varrho \le q,\ (\varrho, q) = 1,\ \varrho^2 \equiv a \bmod q\}, \tag{1.3}$$

where a and q are, basically, monomials in $\eta_1, \dots, \eta_{11-d}$ (for instance $q = \eta_1$ and $a = -\eta_2\eta_4\eta_7\eta_8$ in our example; see also [Der09, Proposition 2.4]). Our experience is that the presence of $\mathcal{N}(a, q)$ usually makes the treatment of the error term in the next summation over η_{11-d} (over some interval J) much harder.

Following the most natural order of summation (which is guided by the requirement to start with the η_i that may be the largest), a term of the shape $\mathcal{N}(a, q)$ appears in the treatment of the following singular del Pezzo surfaces (with one torsor equation):

- quartic del Pezzo surfaces of types $\mathbf{D}_5$ and $\mathbf{A}_4$,
- cubic surfaces of types $\mathbf{E}_6$, $\mathbf{D}_5$, $\mathbf{A}_5 + \mathbf{A}_1$,
- del Pezzo surfaces of degree 2 of types $\mathbf{E}_7$, $\mathbf{E}_6$, $\mathbf{D}_6 + \mathbf{A}_1$,
- del Pezzo surfaces of degree 1 of types $\mathbf{E}_8$, $\mathbf{E}_7 + \mathbf{A}_1$.

Let us sketch the effects of $\mathcal{N}(a, q)$ in the summation of the main term over η_{11-d} in an interval J. To avoid complications which are irrelevant to our issue, we replace $q^{-1}\operatorname{vol}(I)$ by 1 for the moment; this can be restored by using partial summation. If η_{11-d} occurs linearly in a, we can switch the order of the summations over ϱ and η_{11-d}. Then the summation over η_{11-d} subject to the linear congruence modulo q gives the main term $q^{-1}\operatorname{vol}(J)$ and an error term $F = O(1)$, which we must sum over ϱ subject to $1 \le \varrho \le q$ and $(\varrho, q) = 1$ and over the remaining variables $\eta_1, \dots, \eta_{10-d}$.

The most naive estimation $\sum_{\varrho=1}^{q} F = O(q)$ is usually not good enough. This problem has been approached in several different ways:

- For the quartic $\mathbf{A}_4$ case [BD09b], it is enough to obtain an extra saving by using different orders of summation over η_{11-d} and η_{10-d}, depending on their relative size.

- Alternatively, one can get an extra saving by making F explicit, improving $O(q)$ to $O(q^{1/2+\varepsilon})$ as in [BB07, Lemma 3] using Fourier series and quadratic Gauss sums, which is sufficient for the second summation for the quartic surface of type $\mathbf{D}_5$ [BB07] and for the cubic surface of type $\mathbf{E}_6$ [BBD07]; for the latter over imaginary quadratic fields, one can apply Poisson summation combined with Hua's results for exponentional sums over number fields [DF15].
- For the cubic surface of type $\mathbf{D}_5$ [BD09a], the previous two approaches are combined and slightly improved.
- For the degree 2 del Pezzo surface of type $\mathbf{E}_7$ [BB13], the first two summations over η_{11-d}, η_{12-d} are treated simultaneously.

Furthermore, Manin's conjecture is true for some smooth and singular del Pezzo surfaces of degree greater than or equal to 3 for which the factor $\mathcal{N}(a,q)$ does not appear, in particular for certain singular cubic surfaces of types $2\mathbf{A}_2 + \mathbf{A}_1$ [LB12] and $\mathbf{D}_4$ [LB14].

However, for other cases such as the cubic surface S of type $\mathbf{A}_5 + \mathbf{A}_1$, different ideas seem to be needed. In our approach, the main novelty is that we get cancellation effects from summation over ϱ, several variables η_i occurring linearly in a and, most importantly, a variable η_1 occurring in q, while using the trivial $O(1)$-bound for F. This is done in Section 2, using the Pólya–Vinogradov bound for character sums and Heath-Brown's large sieve for real character sums [HB95].

In what follows, for $X > 0$, the notation $x \sim X$ indicates that $X < x \leq 2X$. Let $K_2, K_4, K_7, K_8, Q \geq 1/2$ and $K = K_2K_4K_7K_8$. Applied to the cubic surface of type $\mathbf{A}_5 + \mathbf{A}_1$, the most basic case of our result gives the asymptotic formula

$$\sum_{\substack{\eta_i \sim K_i \\ i=2,4,7,8}} \sum_{\eta_1 \sim Q} \mathcal{N}(-\eta_2\eta_4\eta_7\eta_8, \eta_1) = cKQ + O(K^{1-\delta}Q(\log Q)^{1+\varepsilon}) \tag{1.4}$$

for some explicit $c, \delta > 0$ and for any fixed $\varepsilon > 0$.

Our result shall be compared with the work of Heath-Brown [HB03, Section 5]. In order to obtain an upper bound for $N_{U,H}(B)$ in the case of Cayley's cubic surface, Heath-Brown proved that the left-hand side of (1.4) is $\ll KQ$. However, to obtain an asymptotic formula for $N_{U,H}(B)$ for the cubic surface defined by (1.1), we need an asymptotic formula for the left-hand side of (1.4), but also for the more complicated expression Σ defined in (2.7).

Comparing the proof of the asymptotic formula for Σ stated in Theorem 2.1 and its application in Section 3.4 with Heath-Brown's work, we notice that our result involves several extra difficulties. In particular, we have to isolate the main term, work out the case of even q, include a weight function and some additional parameters, and finally work with ranges for η_1 depending on the remaining variables. This latter task is the main difficulty and its resolution requires some extra tools such as Perron's formula.

It is also interesting to note that we essentially manage to remove the factor $\mathcal{N}(a,q)$ from the main term of the first summation in Lemma 3.3, so that we can continue the proof just as in the case of linear η_{11-d} in the torsor equation.

As an application of our general estimate for the average number of solutions of our quadratic congruence, we prove Manin's conjecture for the cubic surface S of singularity type $\mathbf{A}_5 + \mathbf{A}_1$ defined by (1.1). The complement of the lines is $U = S \setminus \{x_1 = 0\}$. We use the anticanonical height function defined by $H(\mathbf{x}) = \max\{|x_0|, \dots, |x_3|\}$ for $\mathbf{x} = (x_0 : \dots : x_3)$, where $(x_0, \dots, x_3) \in \mathbb{Z}^4$ is such that $(x_0, \dots, x_3) = 1$. See Section 3.1 for more information on the geometry of S. Besides Theorem 2.1, our main result is as follows.

THEOREM 1.1. *Let $\varepsilon > 0$ be fixed. As B tends to ∞, we have the estimate*

$$N_{U,H}(B) = c_{S,H} B (\log B)^6 + O(B(\log B)^{5+\varepsilon}),$$

where

$$c_{S,H} = \frac{1}{172800} \omega_\infty \prod_p \left(1 - \frac{1}{p}\right)^7 \left(1 + \frac{7}{p} + \frac{1}{p^2}\right),$$

$$\omega_\infty = \int_{0 \le |(x_1x_2)^{-1}(x_1^3 + x_2x_3^2)|, |x_1|, x_2, |x_3| \le 1} \frac{1}{x_1 x_2} \, dx_1 \, dx_2 \, dx_3.$$

We will check in Section 3.6 that this agrees with Manin's conjecture and that the constant $c_{S,H}$ is the one predicted by Peyre, Batyrev and Tschinkel.

2. Quadratic congruences on average. As explained in the introduction, our motivation to study quadratic congruences in this section is their appearance in proofs of Manin's conjecture.

2.1. Counting solutions of quadratic congruences. To evaluate the main term of the first summation over a variable occurring non-linearly in the torsor equation (such as η_9 in (1.2) in our example; see Lemma 3.2 below for the result of the first summation in our case and [Der09, Proposition 2.4] for the result in a more general situation), we need to count solutions of quadratic congruences on average. To this end, we consider the following general situation.

Throughout, for $X > 0$, we use the notation $x \sim X$ to indicate that $X < x \le 2X$. Let $b \in \mathbb{Z} \setminus \{0\}$, $k \in \mathbb{Z}_{>0}$ with $(k, b) = 1$, $r \in \mathbb{Z}_{>0}$ with $r \ge 2$ and $K_1, \dots, K_r, Q, V$ be positive real numbers. We assume that Φ is a continuous real-valued function defined on $(K_1, 2K_1] \times \cdots \times (K_r, 2K_r] \times (0, Q]$ which satisfies

$$0 \leq \Phi \leq V \tag{2.1}$$

and, in each of the variables, can be divided into finitely many continuously differentiable and monotone pieces whose number is bounded by an absolute constant. We further assume that Q^- and Q^+ are continuous real-valued functions defined on $(K_1, 2K_1] \times \cdots \times (K_r, 2K_r]$ such that

$$0 < Q^- \leq Q^+ \leq Q. \tag{2.2}$$

Moreover, for any given $i \in \{1, \ldots, r\}$, for $x_j \sim K_j$ with $j \in \{1, \ldots, r\} \setminus \{i\}$, and for $0 < y \leq Q$, we assume that the set

$$\begin{aligned} \mathcal{A}_i&(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_r, y) \\ &= \{x_i \sim K_i \mid Q^-(x_1, \ldots, x_r) < y \leq Q^+(x_1, \ldots, x_r)\} \end{aligned} \tag{2.3}$$

is the union of finitely many intervals whose number is bounded by an absolute constant. Throughout, for brevity, we write

$$K = 2^{r+1} K_1 \cdots K_r, \qquad Q^\pm = Q^\pm(a_1, \ldots, a_r), \tag{2.4}$$

$$\mathcal{A}_i(y) = \mathcal{A}_i(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_r, y). \tag{2.5}$$

Finally, for any integer $n \in \mathbb{Z}_{>0}$, we set

$$\operatorname{rad}(n) = \prod_{p \mid n} p. \tag{2.6}$$

Our goal is to evaluate asymptotically the expression

$$\Sigma = \sum_{a_1 \sim K_1} \cdots \sum_{a_r \sim K_r} \sum_{Q^- < q \leq Q^+} \Phi(a_1, \ldots, a_r, q) \mathcal{N}(-a_1 \cdots a_r b, kq), \tag{2.7}$$

where $\mathcal{N}(-a_1 \cdots a_r b, kq)$ is defined in (1.3).

We begin by splitting Σ into a main term and an error term. Let $kq = 2^{v(kq)} h$, where $v(\ell)$ is the 2-adic valuation of $\ell \in \mathbb{Z}_{>0}$ and h is odd. Thus, for any $n \in \mathbb{Z}$, we have

$$\sum_{\varrho^2 \equiv n \bmod kq} 1 = \Big(\sum_{\varrho^2 \equiv n \bmod 2^{v(kq)}} 1 \Big) \Big(\sum_{\varrho^2 \equiv n \bmod h} 1 \Big). \tag{2.8}$$

In the following, for $j \geq 0$, we set

$$\left\{ \frac{n}{2^j} \right\} = \sum_{\substack{\varrho \bmod 2^j \\ \varrho^2 \equiv n \bmod 2^j}} 1.$$

It is well-known that if $(n,2^j)=1$, then

$$\left\{\frac{n}{2^j}\right\} = \begin{cases} 1 & \text{if } j=0, \\ 1 & \text{if } n\equiv 1 \bmod 2 \text{ and } j=1, \\ 2 & \text{if } n\equiv 1 \bmod 4 \text{ and } j=2, \\ 4 & \text{if } n\equiv 1 \bmod 8 \text{ and } j\geq 3, \\ 0 & \text{otherwise.} \end{cases} \tag{2.9}$$

Moreover, if h is odd and $(n,h)=1$, then

$$\sum_{\varrho^2\equiv n \bmod h} 1 = \sum_{d\mid h}\mu^2(d)\left(\frac{n}{d}\right). \tag{2.10}$$

The equalities (2.8)–(2.10) imply that if $(a_1\cdots a_r b, kq)=1$, then

$$\mathcal{N}(-a_1\cdots a_r b, kq) = \left\{\frac{-a_1\cdots a_r b}{2^{v(kq)}}\right\}\sum_{\substack{d\mid kq\\(d,2)=1}}\mu^2(d)\left(\frac{-a_1\cdots a_r b}{d}\right). \tag{2.11}$$

If $(a_1\cdots a_r b, kq)\neq 1$, then $\mathcal{N}(-a_1\cdots a_r b, kq)=0$. Therefore, we deduce that we can write

$$\Sigma = M + E, \tag{2.12}$$

where the main term M is defined by

$$M = \sum_{a_1\sim K_1}\cdots\sum_{\substack{a_r\sim K_r\\(a_1\cdots a_r b,kq)=1}}\sum_{Q^-<q\leq Q^+}\Phi(a_1,\ldots,a_r,q)\left\{\frac{-a_1\cdots a_r b}{2^{v(kq)}}\right\}, \tag{2.13}$$

and the error term E is defined by

$$E = \sum_{a_1\sim K_1}\cdots\sum_{\substack{a_r\sim K_r\\(a_1\cdots a_r b,kq)=1}}\sum_{Q^-<q\leq Q^+}\Phi(a_1,\ldots,a_r,q)\left\{\frac{-a_1\cdots a_r b}{2^{v(kq)}}\right\} \times \sum_{\substack{d\mid kq\\ d>1\\(d,2)=1}}\mu^2(d)\left(\frac{-a_1\cdots a_r b}{d}\right). \tag{2.14}$$

In the following sections, we estimate the error term by generalizing the method used by Heath-Brown [HB03, Section 5]. We shall not evaluate the main term any further since this is not needed in our application. Our result is as follows.

THEOREM 2.1. *Let $\varepsilon>0$ be fixed. Set $L=\log(2+Q)$. Then*

$$\Sigma - M \ll E',$$

where

$$E' = VK^{1/2+\varepsilon}QL^{\varepsilon}(K^{1/2-1/2r}\operatorname{rad}(k)^{1/4} + |b|^{\varepsilon}2^{(1+\varepsilon)\omega(k)} + 2^{\omega(k)}L).$$

The term Σ is not exactly the one that we need in our application. Let Σ' be defined like Σ in (2.7), but with some additional coprimality conditions included, namely

$$(2.15)\quad \Sigma' = \mathop{\sum_{\substack{a_1\sim K_1\\(a_1,t_1)=1}} \cdots \sum_{\substack{a_r\sim K_r\\(a_r,t_r)=1}}}_{(a_i,a_j)=1,\,1\le i<j\le r} \sum_{\substack{Q^-<q\le Q^+\\(q,u)=1}} \Phi(a_1,\dots,a_r,q)\mathcal{N}(-a_1\cdots a_r b, kq),$$

where $t_1,\dots,t_r,u\in\mathbb{Z}_{>0}$. Accordingly, we set

$$(2.16)\quad M' = \mathop{\sum_{\substack{a_1\sim K_1\\(a_1,t_1)=1}} \cdots \sum_{\substack{a_r\sim K_r\\(a_r,t_r)=1}}}_{(a_i,a_j)=1,\,1\le i<j\le r} \sum_{\substack{Q^-<q\le Q^+\\(q,u)=1\\(a_1\cdots a_r b,kq)=1}} \Phi(a_1,\dots,a_r,q)\left\{\frac{-a_1\cdots a_r b}{2^{v(kq)}}\right\}.$$

Removing the additional coprimality conditions using Möbius inversions, we shall deduce from Theorem 2.1 the following asymptotic formula for Σ'.

Corollary 2.2. *Let $\varepsilon>0$ be fixed. Then*

$$\Sigma' - M' \ll (1+\varepsilon)^{\omega(t_1)+\cdots+\omega(t_r)+\omega(u)}E'.$$

Remark 2.3. Theorem 2.1 and Corollary 2.2 remain true if the left half-open q-summation interval $(Q^-,Q^+]$ is replaced by an arbitrary interval $\mathcal{I}(Q^-,Q^+)$ (left half-open, right half-open, open, closed) with endpoints Q^- and Q^+. The proof is the same, with the relevant summation intervals being altered accordingly.

Theorem 2.1 and Corollary 2.2 trivially hold if $K_i<1/2$ for some i in $\{1,\dots,r\}$ or $Q<1$ since in this case we have $\Sigma=M=0$. Therefore, we shall assume that $K_i\ge 1/2$ for any $i\in\{1,\dots,r\}$ and $Q\ge 1$ throughout the proofs of these results. Recalling the definition (2.4) of K, we note that $K\ge 2$.

2.2. Application of the Pólya–Vinogradov bound I. Let us write $d=fg$, where $g=(d,k)$. It follows that $(f,k/g)=1$, and so the condition $d\,|\,kq$ is equivalent to $f\,|\,q$. Thus, we can write $q=ef$. Let us set

$$Q^-(e,g)=\max\{1/g,Q^-/e\},\qquad Q^+(e)=Q^+/e.$$

Reordering the summations and noting that $\mu^2(fg)=1$ if and only if $(f,g)=1$ and $\mu^2(f)=\mu^2(g)=1$, we can rewrite the error term E defined in (2.14) as

$$(2.17)\qquad E=\sum_{\substack{g|k\\(g,2)=1}}\mu^2(g)\sum_{\substack{e\le Q\\(e,b)=1}}E(e,g),$$

where

(2.18)

$$E(e,g)=\sum_{\substack{a_1\sim K_1 \\ (a_1\cdots a_r,ke)=1}}\cdots\sum_{a_r\sim K_r}\left\{\frac{-a_1\cdots a_r b}{2^{v(ke)}}\right\}\sum_{\substack{Q^-(e,g)<f\le Q^+(e) \\ (f,2k)=1}}\Phi(a_1,\dots,a_r,ef) \times \mu^2(f)\left(\frac{-a_1\cdots a_r b}{fg}\right).$$

In the following sections, we will estimate $E(e,g)$ in three different ways. We start with an application of the Pólya–Vinogradov bound for character sums. Pulling in the summation over a_1, we get

$$E(e,g)=\sum_{\substack{a_2\sim K_2 \\ (a_2\cdots a_r,ke)=1}}\cdots\sum_{a_r\sim K_r}\sum_{\substack{1/g<f\le Q/e \\ (f,2k)=1}}\mu^2(f)\left(\frac{-a_2\cdots a_r b}{fg}\right) \times\sum_{h=1}^{8}\left\{\frac{-ha_2\cdots a_r b}{2^{v(ke)}}\right\}\sum_{\substack{a_1\in\mathcal{A}_1(ef) \\ a_1\equiv h \bmod 8 \\ (a_1,ke)=1}}\Phi(a_1,\dots,a_r,ef)\left(\frac{a_1}{fg}\right), \tag{2.19}$$

where $\mathcal{A}_1(ef)$ is defined in (2.3) and (2.5). In the following, we estimate the innermost sum over a_1 under the assumption $\mu^2(fg)=1$. Using partial summation and the assumptions on Φ in Section 2.1 (in particular, (2.1)), we get

$$\sum_{\substack{a_1\in\mathcal{A}_1(ef) \\ a_1\equiv h \bmod 8 \\ (a_1,ke)=1}}\Phi(a_1,\dots,a_r,ef)\left(\frac{a_1}{fg}\right)\ll V\sup_{L_1<L_2}\Bigg|\sum_{\substack{L_1<a_1\le L_2 \\ a_1\in\mathcal{A}_1(ef) \\ a_1\equiv h \bmod 8 \\ (a_1,ke)=1}}\left(\frac{a_1}{fg}\right)\Bigg|. \tag{2.20}$$

Removing the coprimality condition $(a_1,ke)=1$ using a Möbius inversion, we obtain

$$\sum_{\substack{L_1<a_1\le L_2 \\ a_1\in\mathcal{A}_1(ef) \\ a_1\equiv h \bmod 8 \\ (a_1,ke)=1}}\left(\frac{a_1}{fg}\right)=\sum_{d\mid ke}\mu(d)\left(\frac{d}{fg}\right)\sum_{\substack{L_1/d<a\le L_2/d \\ da\in\mathcal{A}_1(ef) \\ da\equiv h \bmod 8}}\left(\frac{a}{fg}\right). \tag{2.21}$$

Recalling the assumption that $\mathcal{A}_1(ef)$ is the union of finitely many intervals whose number is bounded by an absolute constant, we deduce from the Pólya–Vinogradov bound for character sums that

$$\sum_{\substack{L_1/d<a\le L_2/d \\ da\in\mathcal{A}_1(ef) \\ da\equiv h \bmod 8}}\left(\frac{a}{fg}\right)\ll f^{1/2}g^{1/2}\log(fg), \tag{2.22}$$

where we note that fg is not a perfect square since $fg>1$ and $\mu^2(fg)=1$.

Combining (2.19)–(2.22), we get

$$E(e,g) \ll VK_2\cdots K_rQ^{3/2}e^{-3/2}g^{1/2}\log(2gQe^{-1})2^{\omega(ke)}.$$

Similarly, for every $i\in\{1,\dots,r\}$, we obtain

$$E(e,g) \ll V\frac{K_1\cdots K_r}{K_i}Q^{3/2}e^{-3/2}g^{1/2}\log(2gQe^{-1})2^{\omega(ke)}.$$

Hence, on taking K_i as the maximum of $K_1,\dots,K_r$, it follows that

$$E(e,g) \ll VK^{1-1/r}Q^{3/2}e^{-3/2}g^{1/2}\log(2gQe^{-1})2^{\omega(ke)}, \tag{2.23}$$

where K is defined in (2.4).

2.3. Application of the Pólya–Vinogradov bound II. In this section, we set $a=a_1\cdots a_r$. Alternatively, we may use the Pólya–Vinogradov bound to treat the innermost sum over f in (2.18) non-trivially if $-ab$ is not a perfect square, which we assume in the following. Using partial summation and the bound (2.1), we deduce

$$\begin{aligned}\sum_{\substack{Q^-(e,g)<f\le Q^+(e)\\(f,2k)=1}} \Phi(a_1,\dots,a_r,ef)\mu^2(f)\left(\frac{-ab}{fg}\right)\\ \ll V\sup_{Q^-(e,g)\le F_1<F_2\le Q^+(e)}\Bigg|\sum_{\substack{F_1<f\le F_2\\(f,2k)=1}}\mu^2(f)\left(\frac{-ab}{f}\right)\Bigg|.\end{aligned} \tag{2.24}$$

Using the well-known formula

$$\mu^2(f)=\sum_{d^2|f}\mu(d)$$

and writing $f=d^2\tilde f$, we get

$$\sum_{\substack{F_1<f\le F_2\\(f,2k)=1}}\mu^2(f)\left(\frac{-ab}{f}\right)=\sum_{\substack{d\le F_2^{1/2}\\(d,2abk)=1}}\mu(d)\sum_{\substack{F_1/d^2<\tilde f\le F_2/d^2\\(\tilde f,2k)=1}}\left(\frac{-ab}{\tilde f}\right). \tag{2.25}$$

Removing the coprimality condition $(\tilde f,k)=1$ using a Möbius inversion, we obtain

$$\sum_{\substack{F_1/d^2<\tilde f\le F_2/d^2\\(\tilde f,2k)=1}}\left(\frac{-ab}{\tilde f}\right)=\sum_{\substack{\tilde d|k\\(\tilde d,2)=1}}\mu(\tilde d)\left(\frac{-ab}{\tilde d}\right)\sum_{\substack{F_1/(d^2\tilde d)<f'\le F_2/(d^2\tilde d)\\(f',2)=1}}\left(\frac{-ab}{f'}\right). \tag{2.26}$$

The Pólya–Vinogradov bound gives

$$\sum_{\substack{F_1/(d^2\tilde d)<f'\le F_2/(d^2\tilde d)\\(f',2)=1}}\left(\frac{-ab}{f'}\right)\ll(a|b|)^{1/2}\log(2a|b|), \tag{2.27}$$

where we recall our assumption that $-ab$ is not a perfect square.

Let $E'(e,g)$ be the contribution to $E(e,g)$ of those $a_1,\dots,a_r$ for which $-ab$ is not a perfect square. Then, combining (2.2) and (2.24)–(2.27), we get

$$E'(e,g) \ll VK^{3/2}Q^{1/2}e^{-1/2}|b|^{1/2}\log(K|b|)2^{\omega(k)}. \tag{2.28}$$

The remaining contribution $E^{\square}(e,g)$ of perfect squares $-ab$ is trivially calculated to be

$$E^{\square}(e,g) \ll VK^{1/2+\varepsilon}Qe^{-1}. \tag{2.29}$$

Combining (2.28) and (2.29), we obtain

$$E(e,g) \ll VK^{3/2}Q^{1/2}e^{-1/2}|b|^{1/2}\log(K|b|)2^{\omega(k)} + VK^{1/2+\varepsilon}Qe^{-1}. \tag{2.30}$$

2.4. Application of Heath-Brown's large sieve. Finally, we will make use of Heath-Brown's large sieve for real character sums to bound $E(e,g)$. Set

$$u_f = \Phi(a_1,\dots,a_r,ef)\mu^2(f)\left(\frac{-a_1\cdots a_r b}{fg}\right).$$

To make the summation ranges independent, we first remove the summation condition $Q^-(e,g) < f \le Q^+(e)$ using Perron's formula, getting

(2.31)

$$\sum_{\substack{Q^-(e,g)<f\le Q^+(e)\\ (f,2k)=1}} u_f = \frac{1}{2\pi i}\int_{c-iT}^{c+iT}\Big(\sum_{\substack{1\le f\le Q/e\\ (f,2k)=1}} u_f f^{-s}\Big)(Q^+(e)^s - Q^-(e,g)^s)\,\frac{ds}{s} + O\left(V + \frac{VQ\log 2Q}{eT}\right)$$

where we have set $c = 1/\log 2Q$ and used (2.1). Set

$$\begin{aligned} T &= 2Q(\log 2Q)e^{-1},\\ A(a_1,\dots,a_r;s) &= (Q^+(e)^s - Q^-(e,g)^s\left\{\frac{-a_1\cdots a_r b}{2^{v(ke)}}\right\}\left(\frac{-a_1\cdots a_r b}{g}\right),\\ B(f;s) &= f^{-s}\mu^2(f)(-b/f) \end{aligned}$$

and

$$I(s) = \sum_{\substack{a_1\sim K_1\\ }}\cdots\sum_{\substack{a_r\sim K_r\\ (a_1\cdots a_r,ke)=1}}\sum_{\substack{1\le f\le Q/e\\ (f,2)=1}} \Phi(a_1,\dots,a_r,ef)A(a_1,\dots,a_r;s)B(f;s)\left(\frac{a_1\cdots a_r}{f}\right).$$

Then it follows from (2.31) that

$$E(e,g) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} I(s)\,\frac{ds}{s} + O(VK) \tag{2.32}$$
$$\ll (\log T) \sup_{-T\le t\le T} |I(c+it)| + VK$$
$$= (\log T)|I(c+it_0)| + VK$$

for a particular $t_0 \in [-T,T]$. From [HB95, Corollary 4], a version of Heath-Brown's large sieve for real character sums, we have

$$\sum_{a_1\sim K_1}\cdots\sum_{a_r\sim K_r}\sum_{\substack{1\le f\le F\\(f,2)=1}} A'(a_1,\dots,a_r)B'(f)\left(\frac{a_1\cdots a_r}{f}\right) \ll (KF^{1/2}+K^{1/2}F)(KF)^{\varepsilon} \tag{2.33}$$

whenever $A'(a_1,\dots,a_r), B'(f)\ll 1$ and $F\ge 1$, and where we note that

$$\Big|\sum_{\substack{a_1\sim K_1\quad a_r\sim K_r\\ a_1\cdots a_r=a}}\cdots\sum A'(a_1,\dots,a_r)\Big| \ll \tau_r(a) \ll a^{\varepsilon}$$

for any given $a\in\mathbb{Z}_{>0}$, with τ_r denoting the Dirichlet convolution of the constant arithmetic function equal to 1 with itself r times. Using the bound (2.33) together with partial summation in f to remove the weight function $\Phi(a_1,\dots,a_r,ef)$, we deduce that

$$|I(c+it_0)| \ll V(KQ^{1/2}e^{-1/2}+K^{1/2}Qe^{-1})(KQe^{-1})^{\varepsilon}, \tag{2.34}$$

where we take into account that

$$A(a_1,\dots,a_r;t_0)\ll 1, \qquad B(f;t_0)\ll 1.$$

Combining (2.32) and (2.34), and noting that

$$\log T = \log\frac{2Q\log 2Q}{e} = \log\left(\frac{2Q}{e}\right) + \log\log(2Q) \ll \left(\frac{Q}{e}\right)^{\varepsilon}\log^{\varepsilon}(2+Q),$$

we deduce that

$$E(e,g) \ll V(KQ^{1/2}e^{-1/2}+K^{1/2}Qe^{-1})(KQe^{-1})^{\varepsilon}\log^{\varepsilon}(2+Q). \tag{2.35}$$

2.5. Proofs of Theorem 2.1 and Corollary 2.2

Proof of Theorem 2.1. Combining the three bounds (2.23), (2.30) and (2.35), we obtain

$$E(e,g) \ll (V(KQe^{-1})^{\varepsilon}\log^{\varepsilon}(2+Q))\mathbf{m} + VK^{1/2+\varepsilon}Qe^{-1}, \tag{2.36}$$

where

$$\begin{aligned}\mathbf{m} &= \min\{K^{1-1/r}Q^{3/2}e^{-3/2}g^{1/2+\varepsilon}, K^{3/2}Q^{1/2}e^{-1/2}|b|^{1/2+\varepsilon}2^{\omega(k)},\\ &\qquad\qquad KQ^{1/2}e^{-1/2}+K^{1/2}Qe^{-1}\}\\ &\ll \min\{K^{1-1/r}Q^{3/2}e^{-3/2}g^{1/2+\varepsilon}, KQ^{1/2}e^{-1/2}\}\\ &\quad + \min\{K^{3/2}Q^{1/2}e^{-1/2}|b|^{1/2+\varepsilon}2^{\omega(k)}, K^{1/2}Qe^{-1}\}\\ &\ll (K^{1-1/r}Q^{3/2}e^{-3/2}g^{1/2+\varepsilon})^{\mu}(KQ^{1/2}e^{-1/2})^{1-\mu}\\ &\quad + (K^{3/2}Q^{1/2}e^{-1/2}|b|^{1/2+\varepsilon}2^{\omega(k)})^{\nu}(K^{1/2}Qe^{-1})^{1-\nu}\\ &\ll K^{1-\mu/r}Q^{1/2+\mu}e^{-(1/2+\mu)}g^{\mu/2+\varepsilon} + K^{1/2+\nu}Q^{1-\nu/2}e^{-(1-\nu/2)}|b|^{\nu/2+\varepsilon}2^{\nu\omega(k)}\end{aligned}$$

for any $\mu,\nu \in [0,1]$. Choosing $(\mu,\nu) = (1/2-3\varepsilon, 4\varepsilon)$, recalling (2.17) and (2.36), and summing over g and e now gives

$$\begin{aligned}E &\ll VK^{1-1/(2r)+\varepsilon}Q\,\mathrm{rad}(k)^{1/4}\log^{\varepsilon}(2+Q)\\ &\quad + VK^{1/2+4\varepsilon}Q|b|^{3\varepsilon}2^{(1+4\varepsilon)\omega(k)}\log^{\varepsilon}(2+Q) + VK^{1/2+\varepsilon}Q\log(2+Q)2^{\omega(k)}. \ \blacksquare\end{aligned}$$

Proof of Corollary 2.2. Removing all additional coprimality conditions separately using Möbius inversion, i.e., the formula

$$\sum_{d|(m,n)} \mu(d) = \begin{cases} 1 & \text{if } (m,n)=1,\\ 0 & \text{otherwise,}\end{cases}$$

we are led to

$$\begin{aligned}(2.37)\qquad \Sigma' = \sum_{\substack{(d_{\alpha,\beta})\in\mathbb{Z}_{>0}^{r(r-1)/2}\\ (1\le\alpha<\beta\le r)}} \sum_{d_1|t_1}\cdots\sum_{d_r|t_r}\sum_{d|u}\Big(\prod_{1\le i<j\le r}\mu(d_{i,j})\Big)\Big(\prod_{l=1}^{r}\mu(d_l)\Big)\\ \times \mu(d)\Sigma((d_{i,j})_{1\le i<j\le r}, d_1,\dots,d_r,d)\end{aligned}$$

with

$$\begin{aligned}(2.38)\qquad &\Sigma((d_{i,j})_{1\le i<j\le r}, d_1,\dots,d_r,d)\\ &= \sum_{a_1\sim K_1/D_1}\cdots\sum_{a_r\sim K_r/D_r}\sum_{Q^-/d<q\le Q^+/d}\Phi(a_1D_1,\dots,a_rD_r,qd)\mathcal{N}(-aDb,kdq),\end{aligned}$$

where

$$a=a_1\cdots a_r,\quad D_i=\mathrm{lcm}(d_i,d_{1,i},\dots,d_{i-1,i},d_{i,i+1},\dots,d_{i,r}),\quad D=D_1\cdots D_r.$$

Using Theorem 2.1, we obtain

$$\begin{aligned}(2.39)\qquad &\Sigma((d_{i,j})_{1\le i<j\le r}, d_1,\dots,d_r,d) - M((d_{i,j})_{1\le i<j\le r}, d_1,\dots,d_r,d)\\ &\ll V\Big(\frac{K}{D}\Big)^{1/2+\varepsilon}\frac{Q}{d}L^{\varepsilon}\\ &\quad\times\Big(\Big(\frac{K}{D}\Big)^{1/2-1/2r}d^{1/4}\,\mathrm{rad}(k)^{1/4} + |Db|^{\varepsilon}2^{(1+\varepsilon)\omega(dk)} + 2^{\omega(dk)}L\Big),\end{aligned}$$

where $L = \log(2+Q)$ and

$$M((d_{i,j})_{1\le i<j\le r}, d_1, \dots, d_r, d)$$
$$= \sum_{a_1\sim K_1/D_1} \cdots \sum_{a_r\sim K_r/D_r} \sum_{Q^-/d<q\le Q^+/d} \Phi(a_1D_1, \dots, a_rD_r, qd)\left\{\frac{-a_1\cdots a_rDb}{2^{v(kdq)}}\right\}.$$

Reverting all the Möbius inversions carried out, we find that

$$M' = \sum_{\substack{(d_{\alpha,\beta})\in\mathbb{Z}_{>0}^{r(r-1)/2}\\ (1\le\alpha<\beta\le r)}} \sum_{d_1|t_1}\cdots\sum_{d_r|t_r}\sum_{d|u}\Big(\prod_{1\le i<j\le r}\mu(d_{i,j})\Big)\Big(\prod_{l=1}^{r}\mu(d_l)\Big) \times \mu(d)M((d_{i,j})_{1\le i<j\le r}, d_1, \dots, d_r, d),$$

where M' is defined in (2.16). Summing up the error term in (2.39) over $D \le K$ and $d \le Q^-$, and noting that the number of $d_{\alpha,\beta}$'s and d_γ's such that

$$D = D_1\cdots D_r = \prod_{i=1}^{r}\operatorname{lcm}(d_i, d_{1,i}, \dots, d_{i-1,i}, d_{i,i+1}, \dots, d_{i,r})$$

is bounded by $O(D^\varepsilon)$, we get the error term claimed. ∎

3. Counting rational points on a singular cubic surface. In this part, we give a proof of Manin's conjecture (Theorem 1.1) for the singular cubic surface with $\mathbf{A}_5+\mathbf{A}_1$ singularity type. We will apply our result on quadratic congruences (Corollary 2.2).

3.1. Geometry. Our cubic surface S defined by (1.1) over the field $\mathbb{Q}$ has singularities only at $(0:0:1:0)$ of type $\mathbf{A}_1$, and at $(1:0:0:0)$ of type $\mathbf{A}_5$. It contains precisely two lines, $\{x_1=x_2=0\}$ and $\{x_1=x_3=0\}$. The complement of the lines is $U=\{\mathbf{x}\in S \mid x_1\ne 0\}$. It is rational, as one can see by projecting to $\mathbb{P}^2$ from one of the singularities.

Its minimal desingularization $\widetilde{S}$ is a blow-up of $\mathbb{P}^2$ in six points, so $\operatorname{Pic}(\widetilde{S})$ is free of rank 7. The Cox ring of $\widetilde{S}$ has been determined in [Der14]. It has ten generators $\eta_1,\dots,\eta_{10}$ satisfying (1.2). The configuration of the rational curves on $\widetilde{S}$ corresponding to the generators of $\operatorname{Cox}(\widetilde{S})$ is described by the extended Dynkin diagram in Figure 2, where each vertex corresponds to a curve E_i for η_i, and an edge indicates that two curves intersect.

3.2. Passage to a universal torsor. Let

$$\boldsymbol{\eta} = (\eta_1,\dots,\eta_{10}),\qquad \boldsymbol{\eta}' = (\eta_1,\dots,\eta_8),\qquad \boldsymbol{\eta}^{(k_1,\dots,k_8)} = \eta_1^{k_1}\cdots\eta_8^{k_8}$$

for any $(k_1,\dots,k_8)\in\mathbb{R}^8$.

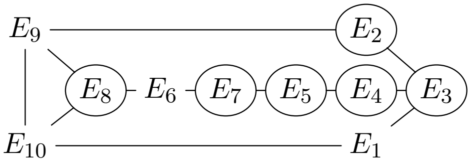

Fig. 2. Configuration of curves on $\widetilde{S}$

For $i = 1, \dots, 10$, let

$$(\mathbb{Z}_i, J_i, J_i') = \begin{cases} (\mathbb{Z}_{>0}, \mathbb{R}_{\geq 1}, \mathbb{R}_{\geq 1}), & i \in \{1, \dots, 6\}, \\ (\mathbb{Z}_{>0}, \mathbb{R}_{\geq 1}, \mathbb{R}_{\geq 0}), & i = 7, \\ (\mathbb{Z}_{\neq 0}, \mathbb{R}_{\leq -1} \cup \mathbb{R}_{\geq 1}, \mathbb{R}), & i = 8, \\ (\mathbb{Z}, \mathbb{R}, \mathbb{R}), & i \in \{9, 10\}. \end{cases} \tag{3.1}$$

In the course of our argument, we estimate summations over $\eta_i \in \mathbb{Z}_i$ by integrations over $\eta_i \in J_i$, which we enlarge to $\eta_i \in J_i'$ in (3.24).

Lemma 3.1. *We have*

$$N_{U,H}(B) = \#\{\boldsymbol{\eta} \in \mathbb{Z}_1 \times \cdots \times \mathbb{Z}_{10} \mid (3.2)\text{–}(3.6) \text{ hold}\}$$

with the torsor equation

$$\eta_1\eta_{10} + \eta_2\eta_9^2 + \eta_4\eta_5^2\eta_6^4\eta_7^3\eta_8 = 0, \tag{3.2}$$

the height condition

$$h(\boldsymbol{\eta}', \eta_9; B) = B^{-1} \max \left\{ \begin{matrix} |\eta_1^{-1}(\eta_2\eta_8\eta_9^2 + \eta_4\eta_5^2\eta_6^4\eta_7^3\eta_8^2)|, |\boldsymbol{\eta}^{(1,1,2,2,2,2,2,1)}|, \\ |\boldsymbol{\eta}^{(3,2,4,3,2,0,1,0)}|, |\boldsymbol{\eta}^{(0,1,1,1,1,1,1,1)}\eta_9| \end{matrix} \right\} \leq 1 \tag{3.3}$$

and the coprimality conditions

$$(\eta_{10}, \eta_2\eta_3\eta_4\eta_5\eta_6\eta_7) = (\eta_9, \eta_1\eta_3\eta_4\eta_5\eta_6\eta_7) = 1, \tag{3.4}$$

$$(\eta_8, \eta_1\eta_2\eta_3\eta_4\eta_5\eta_7) = 1, \tag{3.5}$$

$$\begin{aligned} (\eta_7, \eta_1\eta_2\eta_3\eta_4) &= (\eta_6, \eta_1\eta_2\eta_3\eta_4\eta_5) = (\eta_5, \eta_1\eta_2\eta_3) \\ &= (\eta_4, \eta_1\eta_2) = (\eta_1, \eta_2) = 1. \end{aligned} \tag{3.6}$$

Proof. Based on the birational projection $S \dashrightarrow \mathbb{P}^2$ from the $\mathbf{A}_5$-singularity and the structure of $\widetilde{S}$ as a blow-up of $\mathbb{P}^2$ in six points, we prove as in [DT07, Section 4] that the map

$$\psi : \boldsymbol{\eta} \mapsto (\eta_8\eta_{10}, \boldsymbol{\eta}^{(1,1,2,2,2,2,2,1)}, \boldsymbol{\eta}^{(3,2,4,3,2,0,1,0)}, \boldsymbol{\eta}^{(0,1,1,1,1,1,1,1)}\eta_9)$$

gives a bijection between the rational points on U and the set of $\boldsymbol{\eta}$ in $\mathbb{Z}_1 \times \cdots \times \mathbb{Z}_{10}$ satisfying (3.2) and the coprimality conditions encoded in the extended Dynkin diagram in Figure 2, which are (3.4)–(3.6).

We note that the coprimality conditions imply that the image of such $\boldsymbol{\eta}$ under ψ has coprime coordinates, so that the height of $\psi(\boldsymbol{\eta})$ is simply the maximum of their absolute values. Using (3.2), we eliminate η_{10} and obtain (3.3). ∎

3.3. Counting points. Recalling the definition (3.1) of J_i, let

$$\mathcal{R}(B) = \{(\boldsymbol{\eta}', \eta_9) \in J_1 \times \cdots \times J_9 \mid h(\boldsymbol{\eta}', \eta_9; B) \leq 1\}$$

be the set whose number of lattice points we want to compare with its volume (both under the torsor equation (3.2) and the coprimality conditions (3.4)–(3.6)).

Recall the definition (1.3) of $\mathcal{N}(q,a)$. Summing over η_9, with η_{10} as a dependent variable, we get:

Lemma 3.2. *We have*

$$N_{U,H}(B) = \sum_{\boldsymbol{\eta}' \in \mathbb{Z}_1 \times \cdots \times \mathbb{Z}_8} \theta_1(\boldsymbol{\eta}') V_1(\boldsymbol{\eta}'; B) + O(B(\log B)^3),$$

where

$$V_1(\boldsymbol{\eta}'; B) = \int_{(\boldsymbol{\eta}', \eta_9) \in \mathcal{R}(B)} \eta_1^{-1}\, d\eta_9 \tag{3.7}$$

and

$$\theta_1(\boldsymbol{\eta}') = \sum_{\substack{k \mid \eta_3 \\ (k, \eta_2\eta_4) = 1}} \frac{\mu(k)\varphi^*(\eta_3\eta_4\eta_5\eta_6\eta_7)}{k\varphi^*((\eta_3, k\eta_1))} \mathcal{N}(-\eta_2\eta_4\eta_7\eta_8, k\eta_1)$$

if $\boldsymbol{\eta}'$ satisfies the coprimality conditions (3.5)–(3.6), *while $\theta_1(\boldsymbol{\eta}') = 0$ otherwise.*

Proof. Essentially because Figure 2 describing the coprimality conditions and the torsor equation (3.2) have the right shape, we are in a position to apply the general result of [Der09, Proposition 2.4]. This gives the main term as above after simplifying the condition $(k, \eta_2\eta_4\eta_5\eta_6\eta_7\eta_8) = 1$ in the summation over k to $(k, \eta_2\eta_4) = 1$, which is allowed because of $k \mid \eta_3$ and (3.5)–(3.6).

The sum of the error term over all relevant $\boldsymbol{\eta}'$ is bounded by

$$\sum_{\boldsymbol{\eta}'} 2^{\omega(\eta_3)+\omega(\eta_3\eta_4\eta_5\eta_6\eta_7)+\omega(\eta_1\eta_3)} \ll \sum_{\eta_1,\ldots,\eta_7} \frac{2^{\omega(\eta_3)+\omega(\eta_3\eta_4\eta_5\eta_6\eta_7)+\omega(\eta_1\eta_3)} B}{\boldsymbol{\eta}^{(1,1,2,2,2,2,2,0)}} \ll B(\log B)^3,$$

where we use the second part of (3.3) for the summation over η_8. ∎

3.4. Application of Corollary 2.2. Using Corollary 2.2, we now want to prove that Lemma 3.2 still holds when we replace the error term by

$O(B(\log B)^{4+\varepsilon})$ and θ_1 in the main term by θ_1' with

$$\theta_1'(\boldsymbol{\eta}') = \sum_{\substack{k \mid \eta_3 \\ (k,\eta_2\eta_4)=1}} \frac{\mu(k)\varphi^*(\eta_3\eta_4\eta_5\eta_6\eta_7)}{k\varphi^*((\eta_3,k\eta_1))} \left\{ \frac{-\eta_2\eta_4\eta_7\eta_8}{2^{v(k\eta_1)}} \right\}$$

if (3.5)–(3.6) hold and $\theta_1'(\boldsymbol{\eta}') = 0$ otherwise. Hence, we want to show the following.

LEMMA 3.3. *Let $\varepsilon > 0$ be fixed. Then*

$$N_{U,H}(B) = \sum_{\boldsymbol{\eta}' \in \mathbb{Z}_1 \times \cdots \times \mathbb{Z}_8} \theta_1'(\boldsymbol{\eta}') V_1(\boldsymbol{\eta}'; B) + O(B(\log B)^{4+\varepsilon}).$$

Proof. First, we write

$$\sum_{\boldsymbol{\eta}' \in \mathbb{Z}_1 \times \cdots \times \mathbb{Z}_8} \theta_1(\boldsymbol{\eta}') V_1(\boldsymbol{\eta}'; B) = F^+(B) + F^-(B),$$

where

$$F^+(B) = \sum_{\boldsymbol{\eta}' \in \mathbb{Z}_{>0}^7 \times \mathbb{Z}_{>0}} \theta_1(\boldsymbol{\eta}') V_1(\boldsymbol{\eta}'; B),$$

$$F^-(B) = \sum_{\boldsymbol{\eta}' \in \mathbb{Z}_{>0}^7 \times \mathbb{Z}_{<0}} \theta_1(\boldsymbol{\eta}') V_1(\boldsymbol{\eta}'; B).$$

The term $F^-(B)$ can be treated similarly to $F^+(B)$. Therefore, we confine ourselves to the treatment of the latter, which we now transform in such a way that Corollary 2.2 can be applied.

For convenience, we break the summation ranges of $\eta_1, \eta_2, \eta_4, \eta_7, \eta_8$ into dyadic intervals, i.e., we write

$$F^+(B) = \sum_{\boldsymbol{\eta}'' \in \mathbb{Z}_{>0}^3} \sum_{k \mid \eta_3} \frac{\mu(k)}{k} \sum_{L_1,L_2,L_4,L_7,L_8} W(\boldsymbol{\eta}'', k, L_1, L_2, L_4, L_7, L_8), \tag{3.8}$$

where $\boldsymbol{\eta}'' = (\eta_3, \eta_5, \eta_6)$ satisfies the coprimality conditions $(\eta_3, \eta_5\eta_6) = 1 = (\eta_5, \eta_6)$, the variables $L_1, L_2, L_4, L_7, L_8 \geq 1/2$ run over powers of 2, and

$$W(\boldsymbol{\eta}'', k, L_1, L_2, L_4, L_7, L_8) = \sum_{\substack{\eta_1 \sim L_1 \\ (\eta_1,\eta_5\eta_6)=1}} \varphi^*((\eta_3, k\eta_1))^{-1} \sum_{\substack{\eta_4 \sim L_4 \\ (\eta_4,\eta_6)=1}} \sum_{\substack{\eta_7 \sim L_7 \\ (\eta_7,\eta_3\eta_4)=1}}$$
$$\varphi^*(\eta_3\eta_4\eta_5\eta_6\eta_7) \sum_{\substack{\eta_2 \sim L_2 \\ (\eta_2,\eta_4\eta_5\eta_6\eta_7)=1}} \sum_{\substack{\eta_8 \sim L_8 \\ (\eta_8,\eta_2\eta_3\eta_4\eta_5\eta_7)=1}} V_1(\boldsymbol{\eta}; B) \mathcal{N}(-\eta_2\eta_4\eta_7\eta_8, k\eta_1).$$

Here we note that the coprimality condition $(\eta_2\eta_4\eta_7\eta_8, k\eta_1) = 1$ is contained in the definition of $\mathcal{N}(-\eta_2\eta_4\eta_7\eta_8, k\eta_1)$.

To make Corollary 2.2 applicable, it is necessary to remove the arithmetic factors $\varphi^*((\eta_3, k\eta_1))^{-1}$ and $\varphi^*(\eta_3\eta_4\eta_5\eta_6\eta_7)$. We write

$$\varphi^*((\eta_3, k\eta_1))^{-1} = \varphi^*(k \cdot (\eta_3/k, \eta_1))^{-1} = \varphi^*(k)^{-1} \prod_{\substack{p \mid (\eta_3/k, \eta_1) \\ p \nmid k}} \left(1 + \frac{1}{p-1}\right) = \varphi^*(k)^{-1} \sum_{\substack{d_1 \mid (\eta_3/k, \eta_1) \\ (d_1, k)=1}} \frac{\mu^2(d_1)}{\varphi(d_1)} \tag{3.9}$$

and

$$\begin{aligned} \varphi^*(\eta_3\eta_4\eta_5\eta_6\eta_7) &= \varphi^*(\eta_3\eta_5\eta_6) \prod_{\substack{p \mid \eta_4 \\ p \nmid \eta_3\eta_5\eta_6}} \left(1 - \frac{1}{p}\right) \prod_{\substack{\tilde{p} \mid \eta_7 \\ \tilde{p} \nmid \eta_3\eta_5\eta_6}} \left(1 - \frac{1}{\tilde{p}}\right) \\ &= \varphi^*(\eta_3\eta_5\eta_6) \sum_{\substack{d_4 \mid \eta_4 \\ (d_4, \eta_3\eta_5\eta_6)=1}} \frac{\mu(d_4)}{d_4} \sum_{\substack{d_7 \mid \eta_7 \\ (d_7, \eta_3\eta_5\eta_6)=1}} \frac{\mu(d_7)}{d_7}, \end{aligned} \tag{3.10}$$

where we use the fact that $(\eta_4, \eta_7) = 1$. Hence, we may write

$$\begin{aligned} &W(\boldsymbol{\eta}'', k, L_1, L_2, L_4, L_7, L_8) \\ &= \frac{\varphi^*(\eta_3\eta_5\eta_6)}{\varphi^*(k)} \sum_{\substack{d_1 \mid \eta_3/k \\ (d_1, \eta_5\eta_6 k)=1}} \sum_{\substack{d_4 \leq 2L_4 \\ (d_4, \eta_3\eta_5\eta_6)=1}} \sum_{\substack{d_7 \leq 2L_7 \\ (d_7, d_4\eta_3\eta_5\eta_6)=1}} \frac{\mu^2(d_1)\mu(d_4)\mu(d_7)}{\varphi(d_1) d_4 d_7} \\ &\qquad \times W(\boldsymbol{\eta}'', k, L_1, L_2, L_4, L_7, L_8, d_1, d_4, d_7), \end{aligned} \tag{3.11}$$

where

$$\begin{aligned} &W(\boldsymbol{\eta}'', k, L_1, L_2, L_4, L_7, L_8, d_1, d_4, d_7) \\ &= \sum_{\substack{\eta_1' \sim L_1/d_1 \\ (\eta_1', \eta_5\eta_6)=1}} \sum_{\substack{\eta_4' \sim L_4/d_4 \\ (\eta_4', d_7\eta_6)=1}} \sum_{\substack{\eta_7' \sim L_7/d_7 \\ (\eta_7', d_4\eta_3\eta_4')=1}} \sum_{\substack{\eta_2 \sim L_2 \\ (\eta_2, d_4d_7\eta_4'\eta_5\eta_6\eta_7')=1}} \sum_{\substack{\eta_8 \sim L_8 \\ (\eta_8, d_4d_7\eta_2\eta_3\eta_4'\eta_5\eta_7')=1}} \\ &\qquad V_1(d_1\eta_1', \eta_2, \eta_3, d_4\eta_4', \eta_5, \eta_6, d_7\eta_7', \eta_8; B)\mathcal{N}(-\eta_2\eta_4'\eta_7'\eta_8 d_4 d_7, kd_1\eta_1'). \end{aligned}$$

Now we observe that for $\eta_2, \eta_3, \eta_4', \eta_5, \eta_6, \eta_7', \eta_8 > 0$, the set

$$\{y > 0 \mid V_1(d_1 y, \eta_2, \eta_3, d_4\eta_4', \eta_5, \eta_6, d_7\eta_7', \eta_8; B) > 0\}$$

is an interval. To evaluate $W(\boldsymbol{\eta}'', k, L_1, L_2, L_4, L_7, L_8, d_1, d_4, d_7)$, we shall apply Corollary 2.2 and Remark 2.3 with

k replaced by kd_1, $\quad b = d_4d_7$, $\quad r = 4$,
$a_1 = \eta_4'$, $\quad a_2 = \eta_7'$, $\quad a_3 = \eta_2$, $\quad a_4 = \eta_8$, $\quad q = \eta_1'$,
$t_1 = d_7\eta_6$, $\quad t_2 = d_4\eta_3$, $\quad t_3 = d_4d_7\eta_5\eta_6$, $\quad t_4 = d_4d_7\eta_3\eta_5$, $\quad u = \eta_5\eta_6$,

$$K_1 = L_4/d_4, \quad K_2 = L_7/d_7, \quad K_3 = L_2, \quad K_4 = L_8, \quad Q = 2L_1/d_1,$$
$$\begin{aligned}\mathcal{I}(Q^-, Q^+) &= \mathcal{I}(Q^-(\eta_4', \eta_7', \eta_2, \eta_8), Q^+(\eta_4', \eta_7', \eta_2, \eta_8))\\ &= (L_1, 2L_1] \cap \{y > 0 \mid V_1(d_1y, \eta_2, \eta_3, d_4\eta_4', \eta_5, \eta_6, d_7\eta_7', \eta_8; B) > 0\},\end{aligned}$$
$$V = \sup_{\eta_1\sim L_1,\, \eta_2\sim L_2,\, \eta_4\sim L_4,\, \eta_7\sim L_7,\, \eta_8\sim L_8} V_1(\boldsymbol{\eta}; B),$$
$$\Phi(\eta_4', \eta_7', \eta_2, \eta_8, y) = \begin{cases} V_1(d_1y, \eta_2, \eta_3, d_4\eta_4', \eta_5, \eta_6, d_7\eta_7', \eta_8; B) & \text{if } Q^- < y < Q^+,\\ \lim\limits_{z\downarrow Q^-} V_1(d_1z, \eta_2, \eta_3, d_4\eta_4', \eta_5, \eta_6, d_7\eta_7', \eta_8; B) & \text{if } y \le Q^-,\\ \lim\limits_{z\uparrow Q^+} V_1(d_1z, \eta_2, \eta_3, d_4\eta_4', \eta_5, \eta_6, d_7\eta_7', \eta_8; B) & \text{if } y \ge Q^+.\end{cases}$$

It is easy to check that the functions Φ, Q^- and Q^+ so defined satisfy the conditions in Section 2.1. Therefore, Corollary 2.2 and Remark 2.3 give

$$\begin{aligned}(3.12)\qquad W(\boldsymbol{\eta}'', k, L_1, L_2, L_4, L_7, L_8, d_1, d_4, d_7)&\\ = M(\boldsymbol{\eta}'', k, L_1, L_2, L_4, L_7, L_8, d_1, d_4, d_7)&\\ + E(\boldsymbol{\eta}'', k, L_1, L_2, L_4, L_7, L_8, d_1, d_4, d_7),&\end{aligned}$$

where

$$\begin{aligned}(3.13)\qquad & M(\boldsymbol{\eta}'', k, L_1, L_2, L_4, L_7, L_8, d_1, d_4, d_7)\\ &= \sum_{\substack{\eta_1'\sim L_1/d_1\\ (\eta_1', \eta_5\eta_6)=1}} \sum_{\substack{\eta_4'\sim L_4/d_4\\ (\eta_4', d_7\eta_6)=1}} \sum_{\substack{\eta_7'\sim L_7/d_7\\ (\eta_7', d_4\eta_3\eta_4')=1}} \sum_{\substack{\eta_2\sim L_2\\ (\eta_2, d_4d_7\eta_4'\eta_5\eta_6\eta_7')=1\\ (\eta_2\eta_4'\eta_7'\eta_8d_4d_7, kd_1\eta_1')=1}} \sum_{\substack{\eta_8\sim L_8\\ (\eta_8, d_4d_7\eta_3\eta_4'\eta_5\eta_7')=1}}\\ &\qquad V_1(d_1\eta_1', \eta_2, \eta_3, d_4\eta_4', \eta_5, \eta_6, d_7\eta_7', \eta_8; B)\left\{\frac{\eta_2\eta_4'\eta_7'\eta_8d_4d_7}{2^{v(kd_1\eta_1')}}\right\}\end{aligned}$$

and

$$\begin{aligned}(3.14)\qquad & E(\boldsymbol{\eta}'', k, L_1, L_2, L_4, L_7, L_8, d_1, d_4, d_7)\\ &\ll \sup_{\eta_i\sim L_i} V_1(\boldsymbol{\eta}; B)\cdot\big(L_1(L_2L_4L_7L_8)^{7/8+\varepsilon}d_1^{-3/4}(d_4d_7)^{-7/8}k^{1/4}\\ &\qquad + L_1(L_2L_4L_7L_8)^{1/2+4\varepsilon}d_1^{-1}(d_4d_7)^{-1/2}(\log 4L_1)2^{(1+4\varepsilon)\omega(kd_1)}\big)\\ &\qquad \times (1+\varepsilon)^{\omega(\eta_3)+\omega(\eta_5)+\omega(\eta_6)}\log^{\varepsilon}(4L_1).\end{aligned}$$

Summing these contributions over k, L_i and d_i, we deduce from (3.8) and (3.11)–(3.14) that

$$(3.15)\qquad F^+(B) = M^+(B) + E^+(B),$$

where

$$M^+(B) = \sum_{\boldsymbol{\eta}'' \in \mathbb{Z}_{>0}^3} \varphi^*(\eta_3\eta_5\eta_6) \sum_{k \mid \eta_3} \frac{\mu(k)}{k\varphi^*(k)} \sum_{L_1,L_2,L_4,L_7,L_8} \sum_{\substack{d_1 \mid \eta_3/k \\ (d_1,\eta_5\eta_6 k)=1}} \sum_{\substack{d_4 \le 2L_4 \\ (d_4,\eta_3\eta_5\eta_6)=1}} \sum_{\substack{d_7 \le 2L_7 \\ (d_7, d_4\eta_3\eta_5\eta_6)=1}} \frac{\mu^2(d_1)\mu(d_4)\mu(d_7)}{\varphi(d_1)d_4 d_7} M(\boldsymbol{\eta}'', k, L_1, L_2, L_4, L_7, L_8, d_1, d_4, d_7)$$

and

$$E^+(B) \ll \sum_{\boldsymbol{\eta}'' \in \mathbb{Z}_{>0}^3} (1+\varepsilon)^{\omega(\eta_3)+\omega(\eta_5)+\omega(\eta_6)} \sum_{L_1,L_2,L_4,L_7,L_8} \mathbf{L} \sup_{\eta_i \sim L_i} V_1(\boldsymbol{\eta}; B), \tag{3.16}$$

where we have set

$$\mathbf{L} = L_1(L_2L_4L_7L_8)^{8/9}(\log 4L_1)^{1+\varepsilon}.$$

Reverting the decompositions of the arithmetic functions in (3.9) and (3.10), combining the η_1-, η_2-, η_4-, η_7- and η_8-ranges, and noting that if $k \mid \eta_3$ then the conditions $(\eta_2\eta_4\eta_7\eta_8, k\eta_1) = 1$ and $(k, \eta_2\eta_4) = 1$ are equivalent, we simplify the main term $M^+(B)$ to

$$M^+(B) = \sum_{\boldsymbol{\eta}' \in \mathbb{Z}_{>0}^7 \times \mathbb{Z}_{>0}} \theta_1'(\boldsymbol{\eta}') V_1(\boldsymbol{\eta}'; B), \tag{3.17}$$

where $\theta_1'(\boldsymbol{\eta}')$ is defined before the statement of the lemma.

Finally, we show that $E^+(B)$ is an error term. To estimate V_1, an application of [Der09, Lemma 5.1] gives

$$V_1(\boldsymbol{\eta}'; B) \ll \min\left\{ \frac{B^{1/2}}{\eta_1^{1/2}\eta_2^{1/2}|\eta_8|^{1/2}}, \frac{B}{\boldsymbol{\eta}^{(0,1/2,0,1/2,1,2,3/2,3/2)}} \right\} \tag{3.18}$$

$$\ll \frac{B^{2/3}}{|\boldsymbol{\eta}^{(1/3,1/2,0,1/6,1/3,1/2,2/3,5/6)}|} \tag{3.19}$$

$$= \frac{B}{|\boldsymbol{\eta}^{(1,1,1,1,1,1,1,1)}|} \left(\frac{B}{|\boldsymbol{\eta}^{(1,1,2,2,2,2,2,1)}|} \frac{B}{|\boldsymbol{\eta}^{(3,2,4,3,2,0,1,0)}|} \right)^{-1/6}, \tag{3.20}$$

where (3.19) is the weighted average of the two parts of (3.18), and (3.20) indicates how the second and third parts of the height condition (3.3) will be used below when summing over η_6, η_7. Set

$$\mathbf{L}' = L_1(L_2L_4L_8)^{8/9}(\log 4L_1)^{1+\varepsilon}.$$

Then, starting from (3.16), we see that

$$\begin{aligned}
E^+(B) &\ll \sum_{L_1,L_2,L_4,L_7,L_8} \mathbf{L} \sup_{\eta_i\sim L_i}\Bigg(\sum_{\eta_3,\eta_5,\eta_6} \frac{(1+\varepsilon)^{\omega(\eta_3)+\omega(\eta_5)+\omega(\eta_6)}B^{2/3}}{|\boldsymbol{\eta}^{(1/3,1/2,0,1/6,1/3,1/2,2/3,5/6)}|}\Bigg)\\
&\ll \sum_{L_1,L_2,L_4,L_8} \mathbf{L}' \sup_{\eta_i\sim L_i}\Bigg(\sum_{\eta_3,\eta_5,\eta_6,\eta_7} \frac{(1+\varepsilon)^{\omega(\eta_3)+\omega(\eta_5)+\omega(\eta_6)}B^{2/3}}{|\boldsymbol{\eta}^{(1/3,1/2,0,1/6,1/3,1/2,2/3,5/6)}|}\Bigg)\\
&\ll \sum_{L_1,L_2,L_4,L_8} \mathbf{L}' \sup_{\eta_i\sim L_i}\Bigg(\sum_{\eta_3,\eta_5} \frac{(1+\varepsilon)^{\omega(\eta_3)+\omega(\eta_5)}B(\log B)^{\varepsilon}}{|\eta_1\eta_2\eta_3\eta_4\eta_5\eta_8|}\Bigg)\\
&\ll \sum_{L_1,L_2,L_4,L_8} \mathbf{L}' \sup_{\eta_i\sim L_i} \frac{B(\log B)^{2+3\varepsilon}}{|\eta_1\eta_2\eta_4\eta_8|}\\
&\ll \sum_{L_1,L_2,L_4,L_7} \frac{B(\log B)^{2+4\varepsilon}(\log 4L_1)}{(L_2L_4L_8)^{1/9}}\\
&\ll B(\log B)^{4+4\varepsilon}.
\end{aligned}$$

Combining this with (3.15) and (3.17), and treating $F^-(B)$ similarly to $F^+(B)$, we obtain the desired result. ■

3.5. Completion of the proof of Theorem 1.1. It remains to evaluate the main term in Lemma 3.3 asymptotically. To this end, we would like to apply [Der09, Proposition 4.3]. We note that $\theta_1'(\boldsymbol{\eta}')$ is not of the form considered in [Der09, Section 7] because of the extra 2-adic factor. However, this factor turns out to be 1 on average, and the remaining part of $\theta_1'(\boldsymbol{\eta}')$ has the necessary properties. As in [Der09, Definition 3.7], $\mathcal{A}(\theta_1'(\boldsymbol{\eta}'),\eta_8)$ denotes the average size of θ_1' when summed over η_8.

Lemma 3.4. *We have* $\theta_1'(\boldsymbol{\eta}')\in\Theta_{2,8}(C)$ [Der09, Definition 4.2] *for some* $C\in\mathbb{R}_{\geq 0}$, *with*

$$\mathcal{A}(\theta_1'(\boldsymbol{\eta}'),\eta_8)=\theta_2(\eta_1,\dots,\eta_7)=\prod_p \theta_{2,p}(I_p(\eta_1,\dots,\eta_7))\in\Theta_{4,7}'(2)$$

[Der09, Definition 7.8], *where* $I_p(\eta_1,\dots,\eta_7)=\{i\in\{1,\dots,7\}\mid p\mid\eta_i\}$ *and*

$$\theta_{2,p}(I)=\begin{cases}1, & I=\emptyset,\\ 1-1/p, & I=\{1\},\{2\},\{6\},\\ (1-1/p)^2, & \begin{aligned}I=\{4\},\{5\},\{7\},\{1,3\},\{2,3\},\\ \{3,4\},\{4,5\},\{5,7\},\{6,7\},\end{aligned}\\ (1-1/p)(1-2/p), & I=\{3\},\\ 0, & \textit{otherwise}.\end{cases}$$

Proof. We will see that

$$\sum_{0<\eta_8\le t} \theta_1'(\boldsymbol{\eta}') = t\theta_2(\eta_1,\dots,\eta_7) + O(2^{\omega(\eta_1\eta_2\eta_3\eta_4\eta_5\eta_7)+\omega(\eta_3)}), \tag{3.21}$$

where

$$\theta_2(\eta_1,\dots,\eta_7) = \sum_{\substack{k\mid\eta_3\\ (k,\eta_2\eta_4)=1}} \frac{\mu(k)\varphi^*(\eta_3\eta_4\eta_5\eta_6\eta_7)}{k\varphi^*((\eta_3,k\eta_1))}\varphi^*(\eta_1\eta_2\eta_3\eta_4\eta_5\eta_7)$$

if (3.6) holds and $\theta_2(\eta_1,\dots,\eta_7)=0$ otherwise.

We observe that $\theta_1'(\boldsymbol{\eta}') \in \Theta_{1,8}(3,\eta_8)$ [Der09, Definition 3.8] since we have $\theta_1'(\boldsymbol{\eta}') \ll \prod_{i=1}^8(\varphi^*(\eta_i))^2 \in \Theta_{0,8}(0)$ [Der09, Definition 3.2] by [Der09, Example 3.3], and because $\theta_1'(\boldsymbol{\eta}')$ as a function in η_8 lies in $\Theta_0(0)$ [Der09, Definition 3.7] by (3.21), and because its average is

$$\theta_2(\eta_1,\dots,\eta_7) \ll \prod_{i=1}^7(\varphi^*(\eta_i))^2 \in \Theta_{0,7}(0)$$

as before, and because the error term is $\ll \prod_{i=1}^7 4^{\omega(\eta_i)} \in \Theta_{0,7}(3)$ also as in [Der09, Example 3.3].

Furthermore, we see that $\theta_2(\eta_1,\dots,\eta_7)$ has the form of [Der09, Definition 7.8], and a computation shows that its local factors $\theta_{2,p}$ are as in our statement, so $\theta_2(\eta_1,\dots,\eta_7) \in \Theta_{4,7}'(2)$, and $\theta_2(\eta_1,\dots,\eta_7) \in \Theta_{2,7}(C)$ for some $C \ge 3$ by [Der09, Corollary 7.9]. In total, this shows $\theta_1'(\boldsymbol{\eta}') \in \Theta_{2,8}(C)$ [Der09, Definition 4.2].

It remains to prove (3.21). If (3.6) does not hold, both sides of (3.21) are 0. Otherwise,

$$\sum_{0<\eta_8\le t}\theta_1'(\boldsymbol{\eta}') = \sum_{\substack{k\mid\eta_3\\ (k,\eta_2\eta_4)=1}} \frac{\mu(k)\varphi^*(\eta_3\eta_4\eta_5\eta_6\eta_7)}{k\varphi^*((\eta_3,k\eta_1))} \sum_{\substack{0<\eta_8\le t\\ (3.5)}} \left\{\frac{-\eta_2\eta_4\eta_7\eta_8}{2^{v(k\eta_1)}}\right\}.$$

We must show that the inner sum over η_8 is $t\varphi^*(\eta_1\cdots\eta_5\eta_7)+O(2^{\omega(\eta_1\cdots\eta_5\eta_7)})$. Let $n=\min\{v(k\eta_1),3\}$. If $n=0$, this holds by Möbius inversion. If $n>0$, (3.6) implies that η_2,η_4,η_7 are odd. Then the inner sum equals (with $\overline{-\eta_2\eta_4\eta_7}$ being the multiplicative inverse of $-\eta_2\eta_4\eta_7$ mod 2^n)

$$\sum_{\substack{0<\eta_8\le t\\ (\eta_8,\eta_1\cdots\eta_5\eta_7)=1\\ \eta_8\equiv\overline{-\eta_2\eta_4\eta_7} \bmod 2^n}} 2^{n-1} = \sum_{l\mid\eta_1\cdots\eta_5\eta_7}\mu(l) \sum_{\substack{0<\eta_8'\le t/l\\ l\eta_8'\equiv\overline{-\eta_2\eta_4\eta_7} \bmod 2^n}} 2^{n-1}.$$

If l is even, the congruence is never fulfilled, so the inner sum over η_8' is 0. If l is odd, the inner sum over η_8' is $\frac{2^{n-1}t}{2^n l}+O(1)=\frac{t}{2l}+O(1)$. In total, the inner sum over η_8 is

$$\sum_{\substack{l \mid \eta_1 \cdots \eta_5 \eta_7 \\ 2 \nmid l}} \frac{\mu(l)}{2l} t + O(2^{\omega(\eta_1 \cdots \eta_5 \eta_7)}) = \frac{1}{2} t \prod_{\substack{p \mid \eta_1 \cdots \eta_5 \eta_7 \\ p \neq 2}} \left(1 - \frac{1}{p}\right) + O(2^{\omega(\eta_1 \cdots \eta_5 \eta_7)})$$
$$= \varphi^*(\eta_1 \cdots \eta_5 \eta_7) t + O(2^{\omega(\eta_1 \cdots \eta_5 \eta_7)}),$$

since $n > 0$ implies that $\eta_1 \eta_3$ is even. Summing the error term over k only gives another factor $2^{\omega(\eta_3)}$. ∎

Because of (3.20) and Lemma 3.4, we are in a position to apply [Der09, Proposition 4.3], getting

$$\sum_{\boldsymbol{\eta}' \in \mathbb{Z}_1 \times \cdots \times \mathbb{Z}_8} \theta_1'(\boldsymbol{\eta}') V_1(\boldsymbol{\eta}'; B) = c_0 V_0(B) + O(B (\log B)^5 (\log \log B)^2) \tag{3.22}$$

with

$$V_0(B) = \int_{\boldsymbol{\eta}'} V_1(\boldsymbol{\eta}'; B)\, d\boldsymbol{\eta}' = \int_{(\boldsymbol{\eta}', \eta_9) \in \mathcal{R}(B)} \eta_1^{-1}\, d\eta_9\, d\boldsymbol{\eta}',$$
$$c_0 = \mathcal{A}(\theta_1'(\boldsymbol{\eta}'), \eta_8, \ldots, \eta_1) = \mathcal{A}(\theta_2(\eta_1, \ldots, \eta_7), \eta_7, \ldots, \eta_1) = \prod_p \omega_p,$$

whose local factors can be computed from the presentation of θ_2 in Lemma 3.4 by [Der09, Corollary 7.10] as

$$\omega_p = \left(1 - \frac{1}{p}\right)^7 \left(1 + \frac{7}{p} + \frac{1}{p^2}\right). \tag{3.23}$$

Recall the definition (3.1) of J_i'. We define

$$\mathcal{R}_1'(B) = \left\{ (\eta_1, \ldots, \eta_6) \in J_1' \times \cdots \times J_6' \;\middle|\; \begin{array}{l} \boldsymbol{\eta}^{(3,2,4,3,2,0,0,0)} \leq B, \\ \boldsymbol{\eta}^{(5,3,6,4,2,-2,0,0)} \geq B \end{array} \right\},$$
$$\mathcal{R}_2'(\eta_1, \ldots, \eta_6; B) = \{ (\eta_7, \eta_8, \eta_9) \in J_7' \times J_8' \times J_9' \mid h(\boldsymbol{\eta}', \eta_9; B) \leq 1 \},$$
$$\mathcal{R}'(B) = \left\{ (\boldsymbol{\eta}', \eta_9) \in \mathbb{R}^9 \;\middle|\; \begin{array}{l} (\eta_1, \ldots, \eta_6) \in \mathcal{R}_1'(B), \\ (\eta_7, \eta_8, \eta_9) \in \mathcal{R}_2'(\eta_1, \ldots, \eta_6; B) \end{array} \right\},$$
$$V_0'(B) = \int_{(\boldsymbol{\eta}', \eta_9) \in \mathcal{R}'(B)} \eta_1^{-1}\, d\eta_9\, d\boldsymbol{\eta}',$$

where the definition of $\mathcal{R}_1'(B)$ is inspired by the description in (3.26) of the polytope whose volume is $\alpha(S)$.

We claim that

$$V_0(B) = V_0'(B) + O(B (\log B)^5). \tag{3.24}$$

Comparing the definitions, in particular J_i and J_i' for $i \in \{6, 8\}$, we see that we must remove the conditions $\eta_6 \geq 1$ and $|\eta_8| \geq 1$ and add the two conditions from the definition of $\mathcal{R}_1'(B)$, all with a sufficiently small error term. We do this in four steps as in [Der09, Lemma 8.7]; the order is important:

(1) Add $\boldsymbol{\eta}^{(3,2,4,3,2,0,0,0)} \le B$: This does not change anything because this condition follows from $\eta_7 \ge 1$ and $\boldsymbol{\eta}^{(3,2,4,3,2,0,1,0)} \le B$ by (3.3).

(2) Add $\boldsymbol{\eta}^{(5,3,6,4,2,-2,0,0)} \ge B$: Using [Der09, Lemma 5.1(3)] for the integration over η_7, η_9, we see that the error term is
$$\ll \int \frac{B^{5/6}}{|\boldsymbol{\eta}^{(1/6,1/2,0,1/3,2/3,4/3,0,7/6)}|}\, d(\eta_1, \dots, \eta_6, \eta_8).$$
Using the opposite of our new condition for the integration over η_6 together with $1 \le \eta_1, \dots, \eta_5 \le B$ and $|\eta_8| \ge 1$, we see that this is $\ll B(\log B)^5$.

(3) Remove $|\eta_8| \ge 1$: Using [Der09, Lemma 5.1(1)] for the integration over η_9, we see that the error term is
$$\ll \int \frac{B^{1/2}}{\eta_1^{1/2}\eta_2^{1/2}|\eta_8|^{1/2}}\, d\boldsymbol{\eta}'.$$
Using $|\eta_8| \le 1$, and $\boldsymbol{\eta}^{(3,2,4,3,2,0,1,0)} \le B$ for η_7, and $\boldsymbol{\eta}^{(5,3,6,4,2,-2,0,0)} \ge B$ for η_6, and finally $1 \le \eta_1, \dots, \eta_5 \le B$, we see that this is $\ll B(\log B)^5$.

(4) Remove $\eta_7 \ge 1$: Using [Der09, Lemma 5.1(2)] for the integration over η_8, η_9, we see that the error is
$$\ll \int \frac{B^{3/4}}{\boldsymbol{\eta}^{(1/4,1/2,0,1/4,1/2,1,3/4,0)}}\, d(\eta_1, \dots, \eta_7).$$
Using $0 \le \eta_7 \le 1$, and $\boldsymbol{\eta}^{(3,2,4,3,2,0,0,0)} \le B$ for η_5, and finally $1 \le \eta_1, \dots, \eta_4, \eta_6 \le B$, we see that this is $\ll B(\log B)^5$.

Next, we claim as in [Der09, Lemma 8.6] that
$$V_0'(B) = \alpha(S)\omega_\infty B(\log B)^6. \tag{3.25}$$
Indeed, substituting
$$x_2 = B^{-1}\boldsymbol{\eta}^{(3,2,4,3,2,0,1,0)}, \quad x_1 = B^{-1}\boldsymbol{\eta}^{(1,1,2,2,2,2,2,1)}, \quad x_3 = B^{-1}\boldsymbol{\eta}^{(0,1,1,1,1,1,1,1)}\eta_9$$
into ω_∞ as in Theorem 1.1, where $\eta_1, \dots, \eta_6$ should be regarded as parameters and η_7, η_8, η_9 as the new integration variables, we see that
$$\frac{B\omega_\infty}{\eta_1 \cdots \eta_6} = \int_{(\eta_7,\eta_8,\eta_9)\in\mathcal{R}_2'(\eta_1,\dots,\eta_6;B)} \eta_1^{-1}\, d(\eta_7, \eta_8, \eta_9).$$
Finally, we see that
$$\alpha(S)(\log B)^6 = \int_{\mathcal{R}_1'(B)} \frac{1}{\eta_1 \cdots \eta_6}\, d(\eta_1, \dots, \eta_6)$$
by substituting $\eta_i = B^{t_i}$ into $\alpha(S) = \mathrm{vol}(P') = \int_{\mathbf{t}\in P'} d\mathbf{t}$ (see (3.26) below).

Combining Lemma 3.3 with (3.22)–(3.25) completes the proof of Theorem 1.1.

3.6. Compatibility with Manin's conjecture. As the rank of $\mathrm{Pic}(\widetilde{S})$ is equal to 7 (see Section 3.1), the exponent of $\log B$ in Theorem 1.1 is as predicted by Manin's conjecture. By [Pey95], [BT98b], we have conjecturally $c_{S,H} = \alpha(S) \cdot \omega_H(S)$.

We have

$$\alpha(S) = \frac{\alpha(S_0)}{\#W(\mathbf{A}_5) \cdot \#W(\mathbf{A}_1)} = \frac{1}{180 \cdot 6! \cdot 2!} = \frac{1}{172800}$$

by [Der07, Table 1] and [DJT08, Theorem 1.3], where S_0 is a split smooth cubic surface. Since

$$\begin{aligned}[-K_{\widetilde{S}}] &= [3E_1 + 2E_2 + 4E_3 + 3E_4 + 2E_5 + E_7],\\ [E_8] &= [2E_1 + E_2 + 2E_3 + E_4 - 2E_6 - E_7],\end{aligned}$$

we also have $\alpha(S) = \mathrm{vol}(P) = \mathrm{vol}(P')$, where

$$\begin{aligned}(3.26)\qquad P &= \left\{(t_1, \dots, t_7) \in \mathbb{R}^7_{\geq 0} \;\middle|\; \begin{matrix} 3t_1 + 2t_2 + 4t_3 + 3t_4 + 2t_5 + t_7 = 1, \\ 2t_1 + t_2 + 2t_3 + t_4 - 2t_6 - t_7 \geq 0 \end{matrix}\right\} \\ &\cong P' = \left\{(t_1, \dots, t_6) \in \mathbb{R}^6_{\geq 0} \;\middle|\; \begin{matrix} 3t_1 + 2t_2 + 4t_3 + 3t_4 + 2t_5 \leq 1, \\ 5t_1 + 3t_2 + 6t_3 + 4t_4 + 2t_5 - 2t_6 \geq 1 \end{matrix}\right\}.\end{aligned}$$

Furthermore,

$$\omega_H(S) = \omega_\infty \prod_p \left(1 - \frac{1}{p}\right)^7 \omega_p, \quad \text{where} \quad \omega_p = \frac{\#\widetilde{S}(\mathbb{F}_p)}{p^2} = 1 + \frac{7}{p} + \frac{1}{p^2},$$

because the minimal desingularization $\widetilde{S}$ of S is a blow-up of $\mathbb{P}^2$ (which has $p^2 + p + 1$ points over $\mathbb{F}_p$) in six points (each replacing one point by an exceptional divisor containing $\#\mathbb{P}^1(\mathbb{F}_p) = p + 1$ points over $\mathbb{F}_p$).

We check using the techniques of [Pey95], [BT98b] that ω_∞ is as in Theorem 1.1 since the Leray form of $\widetilde{S}$ is

$$\omega_L(\widetilde{S}) = (x_1 x_2)^{-1}\, dx_1\, dx_2\, dx_3$$

(where $x_1 x_2$ is the derivative of (1.1) with respect to x_0) and by writing x_0 in terms of x_1, x_2, x_3 using the defining equation (1.1).

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