ACTA ARITHMETICA
171.2 (2015)

A note on ternary purely exponential diophantine equations

by

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1. Introduction. Let a,b,c be fixed coprime positive integers with $\min\{a,b,c\}>1$, and let $m=\max\{a,b,c\}$. Further let

$$P(a,b,c)=\{(a,b,c),(b,a,c),(c,a,b),(c,b,a)\}. \tag{1.1}$$

In 1933, K. Mahler [13] used his p-adic analogue of the method of Thue–Siegel to prove that the ternary purely exponential diophantine equation

$$a^x+b^y=c^z,\qquad x,y,z\in\mathbb{N}, \tag{1.2}$$

has only finitely many solutions (x,y,z). His method is ineffective. An effective result for solutions of (1.2) was given by A. O. Gel'fond [4]. In 1999, M.-H. Le [10] proved that if $2\nmid c$, then the solutions (x,y,z) satisfy $z<\frac{2}{\pi}ab\log(2eab)$. Afterwards, N. Hirata-Kohno [6] showed that if $2\nmid c$, then $\max\{x,y,z\}<2^{288}\sqrt{abc}(\log(abc))^3$.

Throughout this paper, log is used for natural logarithm. Combining a lower bound for linear forms in two logarithms and an upper bound for the p-adic logarithms due to M. Laurent [9] and Y. Bugeaud [2], we give a better upper bound for solutions of (1.2):

Theorem 1.1. *All solutions* (x,y,z) *of* (1.2) *satisfy*

$$\max\{x,y,z\}<155000(\log m)^3. \tag{1.3}$$

It is worth noticing that under the assumption that the Masser–Oesterle abc-conjecture holds (see [5, Problem B19]), we have $\max\{x,y,z\}\ll(1+\varepsilon)\log\operatorname{rad}(abc)$ for any $\varepsilon>0$, where $\operatorname{rad}(abc)$ is the product of distinct prime divisors of abc.

As a straightforward consequence of an upper bound for the number of solutions of binary S-unit equations due to F. Beukers and H. P. Schlickewei [1], (1.2) has at most 2^{36} solutions (x,y,z). Because (1.2) has at most

2010 *Mathematics Subject Classification*: Primary 11D61.
Key words and phrases: ternary purely exponential diophantine equation, upper bound for solutions, counting solutions.

DOI: 10.4064/aa171-2-4

two solutions for the most known cases, there have been a series of conjectures concerning exact upper bounds for the number of solutions of (1.2). For instance, we have:

Conjecture 1.1 (L. Jeśmanowicz [8]). *If* (a,b,c) *is a primitive Pythagorean triple with* $a^2+b^2=c^2$, *then* (1.2) *has only one solution* $(x,y,z)=(2,2,2)$.

Conjecture 1.2 (N. Terai [14]). *If* (1.2) *has a solution* (x,y,z) *with* $\min\{x,y,z\}>1$, *then it has only one solution.*

In 1999, Z.-F.Cao [3] showed that Conjecture 1.2 is clearly false. He suggested that the condition $\max\{a,b,c\}>7$ should be added to the hypotheses of Conjecture 1.2, and used the term "Terai–Jeśmanowicz conjecture" for the resulting statement. However, M.-H. Le [11] found infinitely many counterexamples to the Terai–Jeśmanowicz conjecture. He stated

Conjecture 1.3 (M.-H. Le [11]). (1.2) *has at most one solution* (x,y,z) *with* $\min\{x,y,z\}>1$.

The above conjecture was proved for some special cases. But, in general, the problem has not been solved yet.

In this paper, using Theorem 1.1, we shall show that Conjecture 1.3 is true if a,b,c satisfy certain divisibility conditions and m is large enough.

We now introduce some notation. Let f,g be coprime positive integers with $\min\{f,g\}>1$. By Euler's theorem, we have $f^{\phi(g)}\equiv 1 \pmod g$, where ϕ is the Euler function. This implies that there exist positive integers r such that

$$f^r \equiv \pm 1 \pmod g. \tag{1.4}$$

Further, let

$$r(f,g)=\min\{r\in\mathbb{N} \mid r \text{ satisfies } (1.4)\} \tag{1.5}$$

and

$$f^{r(f,g)} \equiv \delta(f,g) \pmod g, \qquad \delta(f,g)\in\{\pm 1\}. \tag{1.6}$$

Let $g=p_1^{l_1}\dots p_k^{l_k}$ be the factorization of g, and let

$$S(g)=\{p_1^{s_1}\cdots p_k^{s_k} \mid s_i\in\mathbb{Z},\ s_i\ge 0,\ i=1,\dots,k\}. \tag{1.7}$$

For any fixed positive integer n, let $d(n,S(g))$ denote the maximal divisor of n belonging to $S(g)$.

In 2009, M.-H. Le [12] proved that if (a,b,c) is a primitive Pythagorean triple such that $a^2+b^2=c^2$, $c>4\cdot 10^9$ and $d((a^{r(a,c)}-\delta(a,c))/c,S(c))=1$, then Conjecture 1.1 is true. By using Theorem 1.1, we will prove a more general result:

THEOREM 1.2. *If there exists a triple $(A,B,C)\in P(a,b,c)$ such that*

$$C>\max\{2,m^{\varepsilon_1}\},\qquad 1\ge\varepsilon_1>0, \tag{1.8}$$

$$d\left(\frac{1}{C}(A^{r(A,C)}-\delta(A,C)),S(C)\right)\le C^{1-\varepsilon_2},\qquad 1\ge\varepsilon_2>0, \tag{1.9}$$

and

$$m>(\rho(2\log\rho)^6)^{1/\varepsilon}, \tag{1.10}$$

where $\rho=(155000/\varepsilon^3)^2$ and $\varepsilon=\varepsilon_1\varepsilon_2$, then (1.2) has at most one solution (x,y,z) with $\min\{x,y,z\}>1$.

2. Proof of Theorem 1.1

LEMMA 2.1. *Let $\alpha_1,\alpha_2,\beta_1,\beta_2$ be positive integers with $\min\{\alpha_1,\alpha_2\}\ge 3$. Further let $\Lambda=\beta_1\log\alpha_1-\beta_2\log\alpha_2$. If $\Lambda\ne 0$, then*

$$\log|\Lambda|>-46.81(\log\alpha_1)(\log\alpha_2)B^2, \tag{2.1}$$

where

$$B=\log 3+\max\left\{\log 3, 0.43+\log\left(\frac{\beta_1}{\log\alpha_2}+\frac{\beta_2}{\log\alpha_1}\right)\right\}. \tag{2.2}$$

Proof. Since $[\mathbb{Q}(\alpha_1,\alpha_2):\mathbb{Q}]/[\mathbb{R}(\alpha_1,\alpha_2):\mathbb{R}]=1$, by [9, Theorem 2] we have

$$\log|\Lambda|\ge -CA_1A_2\left(h+\frac{\lambda}{\delta}\right)^2-\sqrt{\omega\theta}\left(h+\frac{\lambda}{\delta}\right)-\log\left(C'A_1A_2\left(h+\frac{\lambda}{\delta}\right)^2\right), \tag{2.3}$$

where $A_1,A_2,C,C',h,\rho,\lambda,\delta$ and θ are real numbers such that

$$\delta=\tfrac12(1+2\mu-\mu^2),\quad 1\ge\mu\ge 1/3,\quad \lambda=\delta\log\rho,\quad \rho>1, \tag{2.4}$$

$$A_i\ge\max\{1,(\rho+1)\log\alpha_i\},\quad i=1,2,\quad A_1A_2\ge\lambda^2, \tag{2.5}$$

$$h\ge\max\left\{\frac{\log 2}{2},\lambda,1.81+\log\left(\frac{\beta_1}{A_2}+\frac{\beta_2}{A_1}\right)\right\}, \tag{2.6}$$

$$H=\frac{h}{\lambda}+\frac{1}{\delta}, \tag{2.7}$$

$$\omega=2\left(1+\sqrt{1+\frac{1}{4H^2}}\right),\qquad \theta=\frac{1}{2H}+\sqrt{1+\frac{1}{4H^2}}, \tag{2.8}$$

$$C=\frac{\mu}{\lambda^3\delta}\left(\frac{\omega}{6}+\frac12\sqrt{\frac{\omega^2}{9}+\frac{8\lambda\omega^{5/4}\theta^{1/4}}{3\sqrt{A_1A_2}H^{1/2}}+\frac43\left(\frac{1}{A_1}+\frac{1}{A_2}\right)\frac{\lambda\omega}{H}}\right)^2, \tag{2.9}$$

$$C'=\sqrt{\frac{C\delta\omega\theta}{\lambda^3\mu}}.$$

We choose $\mu = 1$ and $\lambda = \log 3$. By (2.4), we have $\delta = 1$ and $\rho = 3$. Since $\min\{\alpha_1, \alpha_2\} \geq 3$, by (2.5) we may take

$$A_i = 4\log\alpha_i, \quad i = 1, 2. \tag{2.10}$$

Further, by (2.6)–(2.10), we may choose

$$h = \max\left\{\log 3, 0.43 + \log\left(\frac{\beta_1}{\log\alpha_2} + \frac{\beta_2}{\log\alpha_1}\right)\right\}, \quad H \geq 2, \tag{2.11}$$

$$\omega \leq 4.07, \quad \theta \leq 1.29, \quad C \leq 2.77, \quad C' \leq 3.32. \tag{2.12}$$

Let $B = \log 3 + h$. Since $B \geq 2\log 3$, by (2.10)–(2.12) we have

$$\frac{\sqrt{\omega\theta}\, B}{(\log\alpha_1)(\log\alpha_2)B^2} < 0.87, \quad \frac{\log(C'A_1A_2B^2)}{(\log\alpha_1)(\log\alpha_2)B^2} < 0.99. \tag{2.13}$$

Thus, by (2.3), (2.11) and (2.13), we obtain (2.1) and (2.2) immediately. The lemma is proved.

Lemma 2.2. *Let α_1, α_2 be odd integers with $\min\{|\alpha_1|, |\alpha_2|\} \geq 3$, and let β_1, β_2 be positive integers. Further let $\Lambda' = \alpha_1^{\beta_1} - \alpha_2^{\beta_2}$. If $\Lambda' \neq 0$ and $\alpha_1 \equiv 1 \pmod 4$, then*

$$\operatorname{ord}_2 \Lambda' \leq 208(\log|\alpha_1|)(\log|\alpha_2|)B'^2,$$

where $\operatorname{ord}_2 \Lambda'$ is the order of 2 in $|\Lambda'|$, and

$$B' = \max\left\{10, 0.04 + \log\left(\frac{\beta_1}{\log|\alpha_2|} + \frac{\beta_2}{\log|\alpha_1|}\right)\right\}.$$

Proof. This is the special case of [2, Theorem 2] for $p = 2$ and $y_1 = y_2 = 1$.

Lemma 2.3. *If (x, y, z) is a solution of (1.2) such that $2 \mid \min\{a^{2x}, b^{2y}\}$ and $\min\{a^{2x}, b^{2y}\} < c^z$, then*

$$\max\{x, y, z\} < 15000(\log m)^2. \tag{2.14}$$

Proof. By the symmetry of a^x and b^y in (1.2), it suffices to prove the lemma for $2 \mid a$ and $a^{2x} < c^z$. Then we have $2 \nmid bc$, $\min\{b, c\} \geq 3$, $b^y > a^x$ and $2b^y > c^z$. Therefore, by (1.2),

$$\begin{aligned} z\log c = \log(b^y + a^x) &= y\log b + \frac{2a^x}{2b^y + a^x}\sum_{j=0}^{\infty}\frac{1}{2j+1}\left(\frac{a^x}{2b^y + a^x}\right)^{2j} \\ &< y\log b + \frac{4a^x}{2b^y + a^x} < y\log b + \frac{4a^x}{c^z} < y\log b + \frac{4}{c^{z/2}}. \end{aligned} \tag{2.15}$$

Let $(\alpha_1, \alpha_2, \beta_1, \beta_2) = (c, b, z, y)$ and $\Lambda = \beta_1\log\alpha_1 - \beta_2\log\alpha_2$. By (2.15), we have $0 < \Lambda < 4/c^{z/2}$ and

$$\log 4 - \log\Lambda > \frac{z}{2}\log c. \tag{2.16}$$

Since $\min\{b,c\} \geq 3$, by Lemma 2.1 we get

$$\log \Lambda > -46.81(\log c)(\log b)B^2, \tag{2.17}$$

where

$$B = \log 3 + \max\left\{\log 3, 0.43 + \log\left(\frac{z}{\log b} + \frac{y}{\log c}\right)\right\}. \tag{2.18}$$

If $\log 3 \geq 0.43 + \log(z/\log b + y/\log c)$, then $3 > 1.53(z/\log b + y/\log c) > 1.53z/\log b$ and

$$z < 2\log b. \tag{2.19}$$

Since $x\log a < y\log b < z\log c$, by (2.19) we get $\max\{x,y,z\} < 2(\log m)^2$ and (2.14) holds.

If $\log 3 < 0.43 + \log(z/\log b + y/\log c)$, then from (2.16)–(2.18),

$$\log 4 + 46.81(\log c)(\log b)\left(\log 3 + 0.43 + \log\left(\frac{z}{\log b} + \frac{y}{\log c}\right)\right)^2 > \frac{z}{2}\log c. \tag{2.20}$$

Further, since $z/\log b > y/\log c$, we can easily verify that $\log 3 + 0.43 + \log\frac{2z}{\log b} > \log 3 + 0.43 + \log(z/\log b + y/\log c) > 2$ and $\log 4/(\log b \log c) \leq \log 4/(\log 2\log 3) < 2$. Then, by (2.20),

$$189.24\left(\log 3 + 0.43 + \log\frac{2z}{\log b}\right)^2 > \frac{4\log 4}{(\log b)(\log c)} + 187.24\left(\log 3 + 0.43 + \log\left(\frac{z}{\log b} + \frac{y}{\log c}\right)\right)^2 > \frac{2z}{\log b}. \tag{2.21}$$

Let $f(t) = t - 189.24(\log 3 + 0.43 + \log t)^2$. Notice that $f(30000) > 0$, $f'(t) = 1 - 378.48(\log 3 + 0.43 + \log t)/t$ and $f'(t) > 0$ for $t \geq 30000$. We deduce from (2.21) that $2z/\log b < 30000$ and

$$z < 15000\log b. \tag{2.22}$$

Since $x\log a < y\log b < z\log c$, by (2.22) we obtain (2.14). Thus, the lemma is proved.

Proof of Theorem 1.1. Let (x,y,z) be a solution of (1.2). We first consider the case of $2 \mid a$. By Lemma 2.3, if $a^{2x} < c^z$, then (1.3) holds. Therefore, we may assume that $a^{2x} > c^z$. Then

$$x < \frac{z\log c}{\log a}, \qquad y < \frac{z\log c}{\log b}, \tag{2.23}$$

$$z < \frac{2x\log a}{\log c}. \tag{2.24}$$

By (2.23) and (2.24), if $x = 1$, then (1.3) clearly holds. We may assume that $x > 1$. This implies that $c^z - b^y \equiv 0 \pmod 4$. Let

$$(\alpha_1, \alpha_2) = \begin{cases} ((-1)^{(c-1)/2}c, b) & \text{if } 2 \mid z, \\ (c, b) & \text{if } 2 \nmid z \text{ and } 2 \mid y, \\ ((-1)^{(c-1)/2}c, (-1)^{(c-1)/2}b) & \text{if } 2 \nmid yz, \end{cases} \tag{2.25}$$

$$(\beta_1, \beta_2) = (z, y),$$

$$\Lambda' = \alpha_1^{\beta_1} - \alpha_2^{\beta_2}. \tag{2.26}$$

By (1.2), (2.25) and (2.26), we have $\alpha_1 \equiv 1 \pmod 4$, $|\Lambda'| = a^x$ and $\mathrm{ord}_2 \Lambda' \geq x$. Therefore, by Lemma 2.2,

$$x \leq 208(\log c)(\log b)\left(\max\left\{10, 0.04 + \log\left(\frac{z}{\log b} + \frac{y}{\log c}\right)\right\}\right)^2. \tag{2.27}$$

If $10 \geq 0.04 + \log(z/\log b + y/\log c)$, then $9.96 > \log(z/\log b)$ and

$$z < 21200 \log b. \tag{2.28}$$

From (2.23) and (2.28), we conclude that (1.3) holds.

If $10 < 0.04 + \log(z/\log b + y/\log c)$, then from (2.23), (2.24) and (2.27) we get

$$\begin{aligned} \frac{z \log c}{2 \log a} < x &\leq 208(\log c)(\log b)\left(0.04 + \log\left(\frac{z}{\log b} + \frac{y}{\log c}\right)\right)^2 \\ &< 208(\log c)(\log b)\left(0.04 + \log \frac{2z}{\log b}\right)^2. \end{aligned} \tag{2.29}$$

Let $t = 2z/\log b$. We see from (2.29) that

$$t < 832(\log a)(0.04 + \log t)^2. \tag{2.30}$$

Let $f(t) = t - 832(\log a)(0.04 + \log t)^2$ and $t_0 = 310000(\log a)^2$. Notice that $a \geq 2$, $f(t_0) > 0$ and $f'(t) > 0$ for $t \geq t_0$. Then we deduce from (2.30) that $t < 310000(\log a)^2$ and

$$z < 155000(\log a)^2(\log b). \tag{2.31}$$

Thus, by (2.23) and (2.31), the conclusion holds if $2 \mid a$.

If $2 \mid b$, by the symmetry of a and b in (1.2), we can use the same method as in the proof of the case $2 \mid a$.

Finally, assume that $2 \mid c$. By (2.23), (1.3) holds if $z = 1$. Therefore, we assume that $z > 1$. Then $a^x + b^y \equiv 0 \pmod 4$ and at least one of x and y is odd. Let

$$(\alpha_1, \alpha_2) = \begin{cases} (-a, b) & \text{if } 2 \nmid x \text{ and } 2 \mid y, \\ ((-1)^{(a-1)/2}a, -b) & \text{if } 2 \mid x \text{ and } 2 \nmid y, \\ ((-1)^{(a-1)/2}a, (-1)^{(a-1)/2}b) & \text{if } 2 \nmid xy, \end{cases} \tag{2.32}$$

$$(\beta_1, \beta_2) = (x, y)$$

and let Λ' be defined as in (2.26). By (1.2), (2.26) and (2.32), we have $\alpha_1 \equiv 1 \pmod 4$, $|\Lambda'| = c^z$ and $\mathrm{ord}_2 \Lambda' \geq z$. By Lemma 2.2, we get

$$z \leq 208(\log a)(\log b)\left(\max\left\{10, 0.04 + \log\left(\frac{x}{\log b} + \frac{y}{\log a}\right)\right\}\right)^2. \tag{2.33}$$

If $10 \geq 0.04 + \log(x/\log b + y/\log a)$, then $e^{9.96} > \max\{x/\log b, y/\log a\}$ and

$$\max\{x, y\} < 21200 \log m. \tag{2.34}$$

Further, since $\max\{2a^x, 2b^y\} > c^z$ by (1.2), we see from (2.34) that (1.3) holds.

If $10 < 0.04 + \log(x/\log b + y/\log a)$, then from (2.23) and (2.33) we get

$$\begin{aligned} z &\leq 208(\log a)(\log b)\left(0.04 + \log\left(\frac{x}{\log b} + \frac{y}{\log a}\right)\right)^2 \\ &< 208(\log a)(\log b)\left(0.04 + \log\frac{2z\log c}{(\log a)(\log b)}\right)^2. \end{aligned} \tag{2.35}$$

Let $t = 2z(\log c)/((\log a)(\log b))$. Using the same method as above, we can also deduce from (2.35) that $t < 310000(\log c)^2$ and

$$z < 155000(\log a)(\log b)(\log c). \tag{2.36}$$

Therefore, by (2.23) and (2.36), we obtain (1.3) immediately. Thus, the theorem is proved.

3. Proof of Theorem 1.2. Let u, v be coprime positive integers with $u > v$. For any positive integer n, let $L_n(u,v) = u^n + \lambda v^n$, where $\lambda \in \{\pm 1\}$. For $n > 1$, a prime p is called a *primitive divisor* of $L_n(u,v)$ if $p \mid L_n(u,v)$ and $p \nmid L_1(u,v)\cdots L_{n-1}(u,v)$.

LEMMA 3.1 ([15]). *If $n > 1$ and $v > 1$, then $L_n(u,v)$ has a primitive divisor, except for $(n,u,v,\lambda) = (2, 2^r+1, 2^r-1, -1)$, where r is a positive integer with $r > 1$.*

LEMMA 3.2 ([7, Theorem 3.7.4]). *If r is a positive integer satisfying (1.4), then $r(f,g) \mid r$.*

For any fixed triple $(A,B,C) \in P(a,b,c)$, (1.2) can be rewritten as

$$A^X + \lambda B^Y = C^Z, \quad X, Y, Z \in \mathbb{N},\ \lambda \in \{\pm 1\}, \tag{3.1}$$

where (X,Y,Z) is the corresponding permutation of (x,y,z).

LEMMA 3.3. *Let $(X,Y,Z) = (X_1,Y_1,Z_1)$ and (X_2,Y_2,Z_2) be two solutions of (3.1) with $Z_1 \leq Z_2$. If $C > 2$, then $X_1Y_2 \neq X_2Y_1$ and*

$$A^{|X_1Y_2 - X_2Y_1|} \equiv (-\lambda)^{Y_1+Y_2} \pmod{C^{Z_1}}. \tag{3.2}$$

Proof. Let $d = \gcd(X_1, Y_1)$. Then

$$X_1 = dr, \quad Y_1 = ds, \quad r, s \in \mathbb{N}, \gcd(r, s) = 1, \tag{3.3}$$

and

$$(A^r)^d + \lambda(B^s)^d = C^{Z_1}. \tag{3.4}$$

If $X_1Y_2 = X_2Y_1$, then from (3.3) we get

$$X_2 = rk, \quad Y_2 = sk, \quad k \in \mathbb{N}, \tag{3.5}$$

and

$$(A^r)^k + \lambda(B^s)^k = C^{Z_2}. \tag{3.6}$$

Since $(X_1, Y_1, Z_1) \neq (X_2, Y_2, Z_2)$ and $Z_1 \leq Z_2$, we see from (3.4) and (3.6) that $k > d \geq 1$.

On the other hand, let $u = A^r, v = B^s$ and $L_n(u, v) = u^n + \lambda v^n$. For any positive integer n, we find from (3.4) and (3.6) that $L_k(u, v)$ has no primitive divisors. But, since $C > 2$, by Lemma 3.1, it is impossible. So $X_1Y_2 \neq X_2Y_1$ and $|X_1Y_2 - X_2Y_1|$ is a positive integer. Further, by (3.1),

$$\begin{aligned} A^{X_1Y_2} &\equiv (-\lambda)^{Y_2} B^{Y_1Y_2} \pmod{C^{Z_1}}, \\ A^{X_2Y_1} &\equiv (-\lambda)^{Y_1} B^{Y_1Y_2} \pmod{C^{Z_2}}. \end{aligned} \tag{3.7}$$

Since $Z_1 \leq Z_2$, from (3.7), we obtain (3.2) immediately. The lemma is proved.

Proof of Theorem 1.2. We now assume that (1.2) has two solutions $(x, y, z) = (x_1, y_1, z_1)$ and (x_2, y_2, z_2) with $\min\{x, y, z\} > 1$. Let (A, B, C) in $P(a, b, c)$ satisfy (1.8) and (1.9). Since (1.2) is equivalent to (3.1), the latter has two solutions $(X, Y, Z) = (X_1, Y_1, Z_1)$ and (X_2, Y_2, Z_2) with $\min\{X, Y, Z\} > 1$. Since $C > 2$ and $\min\{Z_1, Z_2\} \geq 2$, by Lemma 3.3 we have $X_1Y_2 \neq X_2Y_1$ and

$$A^{|X_1Y_2 - X_2Y_1|} \equiv (-\lambda)^{Y_1+Y_2} \pmod{C^2}. \tag{3.8}$$

Further, since $(-\lambda)^{Y_1+Y_2} \in \{\pm 1\}$, applying Lemma 3.2 to (3.8) we get $r(A, C) \mid |X_1Y_2 - X_2Y_1|$ and

$$|X_1Y_2 - X_2Y_1| = r(A, C)n, \quad n \in \mathbb{N}. \tag{3.9}$$

Let $\delta = \delta(A, C)$. By (1.5) and (1.6), we have

$$A^{r(A,C)} - \delta = Ck, \quad k \in \mathbb{N}, \delta \in \{\pm 1\}. \tag{3.10}$$

Further let

$$d_0 = d(k, S(C)), \quad d_1 = \gcd(k, C), \quad d_2 = C/d_1. \tag{3.11}$$

By (1.7) and (3.11), we have $d_1 \mid d_0$ and $d_1 \leq d_0$. Since $d_0 \leq C^{1-\varepsilon_2}$ by (1.9), we get $d_1 \leq C^{1-\varepsilon_2}$ and

$$d_2 \geq C^{\varepsilon_2} \tag{3.12}$$

by (3.11).

From (3.8)–(3.10), we have

$$(3.13)\qquad ((A^{r(A,C)}-\delta)+\delta)^n-(-\lambda)^{Y_1+Y_2}\equiv (Ck+\delta)^n-(-\lambda)^{Y_1+Y_2}$$
$$\equiv (\delta^n-(-\lambda)^{Y_1+Y_2})+Ck\sum_{i=1}^{n}\binom{n}{i}(Ck)^{i-1}\delta^{n-i}\equiv 0 \pmod{C^2}.$$

Since $C>2$, we see from (3.13) that $\delta^n=(-\lambda)^{Y_1+Y_2}$ and

$$(3.14)\qquad k\sum_{i=1}^{n}\binom{n}{i}(Ck)^{i-1}\delta^{n-i}\equiv kn\delta^{n-1}\equiv 0 \pmod{C}.$$

Further, since $\gcd(k,C)=d_1$ and $d_2=C/d_1$ by (3.11), we infer from (3.14) that $d_2\mid n$ and

$$(3.15)\qquad n\geq d_2.$$

Therefore, by (3.9), (3.12) and (3.15),

$$(3.16)\qquad \max\{X_1Y_2,X_2Y_1\}>|X_1Y_2-X_2Y_1|\geq n\geq C^{\varepsilon_2}.$$

On the other hand, since (X,Y,Z) is a permutation of (x,y,z), by Theorem 1.1 we have

$$(3.17)\qquad \max\{X_1Y_2,X_2Y_1\}<155000^2(\log m)^6.$$

The combination of (1.8), (3.16) and (3.17) yields

$$(3.18)\qquad 155000^2(\log m)^6\geq C^{\varepsilon_2}\geq m^{\varepsilon_1\varepsilon_2}=m^{\varepsilon},$$

where $\varepsilon=\varepsilon_1\varepsilon_2$. Let

$$(3.19)\qquad f(t)=t-\frac{155000^2}{\varepsilon^6}(\log t)^6$$

and $t_0=\rho(2\log\rho)^6$, where $\rho=(155000/\varepsilon^3)^2$. Since $f(t_0)>0$ and $f'(t)\geq 0$ for $t\geq t_0$, $f(t)$ is an increasing function for $t\geq t_0$. From (3.18) and (3.19) we have $m^{\varepsilon}<t_0$, which contradicts (1.10). Thus, under our assumption, (1.2) has at most one solution (x,y,z) with $\min\{x,y,z\}>1$. The theorem is proved.

Acknowledgements. Research supported by the National Natural Science Foundation of China (No. 10971184).

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Received on 17.11.2014
and in revised form on 17.8.2015 (8002)