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Cohen–Kuznetsov liftings of quasimodular forms

by

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1. Introduction. Quasimodular forms generalize classical modular forms, and they were introduced by Kaneko and Zagier in [3]. For a discrete subgroup Γ of $\mathrm{SL}(2,\mathbb{R})$ commensurable with $\mathrm{SL}(2,\mathbb{Z})$ and for nonnegative integers m and λ, a quasimodular form ϕ of weight λ and depth at most m for Γ corresponds to holomorphic functions $\phi_0, \phi_1, \ldots, \phi_m$ on the Poincaré upper half-plane $\mathcal{H}$ satisfying

$$\frac{1}{(cz+d)^\lambda}\phi\left(\frac{az+b}{cz+d}\right) = \phi_0(z) + \phi_1(z)\left(\frac{c}{cz+d}\right) + \cdots + \phi_m(z)\left(\frac{c}{cz+d}\right)^m$$

for all $z \in \mathcal{H}$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. The functions ϕ_k are also quasimodular forms and are determined uniquely by ϕ. Thus ϕ determines the corresponding polynomial

$$\Phi(z,X) = \sum_{r=0}^{m} \phi_r(z) X^r$$

of degree at most m in X. Such a polynomial is called *quasimodular*, and to study various aspects of quasimodular forms it is often convenient to work with quasimodular polynomials.

Jacobi-like forms for Γ are formal power series which generalize Jacobi forms, and they were studied by Cohen, Manin and Zagier [2], [6]. It is known that there is a one-to-one correspondence between Jacobi-like forms and certain sequences of modular forms. In particular, for a modular form f, there is a Jacobi-like form $\widetilde{f}(z,X)$ corresponding to the sequence whose only nonzero term is f, which is known as the *Cohen–Kuznetsov lifting* of f.

Although the coefficient functions of a Jacobi-like form are not modular forms in general, they are in fact quasimodular forms. There is a surjective map from the space of Jacobi-like forms to the space of quasimodular poly-

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nomials, and it was proved in [1] that this surjective map has a right inverse. This result shows the existence of liftings of quasimodular polynomials to Jacobi-like forms. For a quasimodular polynomial, the proof of the existence of its lifting in the above-mentioned paper was carried out by induction on the degree of the given quasimodular polynomial, and therefore the proof does not provide a formula for this lifting. The goal of the present paper is to obtain an explicit formula for such a lifting, which can be regarded as the Cohen–Kuznetsov lifting of the given quasimodular polynomial or of the corresponding quasimodular form. Indeed, when the degree of a quasimodular polynomial or the depth of the corresponding quasimodular form is zero, the quasimodular polynomial or form can be identified with a modular form, and the lifting coincides with the usual Cohen–Kuznetsov lifting of that modular form.

2. Jacobi-like forms and quasimodular forms. In this section we review certain properties of Jacobi-like forms studied by Cohen, Manin and Zagier (see [2] and [6] for details) and their connections with modular forms and quasimodular forms. We also describe quasimodular polynomials, which correspond to quasimodular forms.

Let $\mathcal{H}$ be the Poincaré upper half-plane, and let $\mathcal{F}$ be the ring of holomorphic functions on $\mathcal{H}$ that are bounded by powers of

$$\frac{|z|^2+1}{\operatorname{Im}(z)}. \tag{2.1}$$

We denote by $\mathcal{F}[[X]]$ the complex algebra of formal power series in X with coefficients in $\mathcal{F}$. If δ is a nonnegative integer, we set

$$\mathcal{F}[[X]]_\delta = X^\delta \mathcal{F}[[X]], \tag{2.2}$$

so that an element $\Phi(z,X) \in \mathcal{F}[[X]]_\delta$ can be written in the form

$$\Phi(z,X) = \sum_{k=0}^{\infty} \phi_k(z) X^{k+\delta} \tag{2.3}$$

with $\phi_k \in \mathcal{F}$ for each $k \geq 0$. Given such $\Phi(z,X) \in \mathcal{F}[[X]]_\delta$ and a nonnegative integer λ, we consider two other formal power series

$$(\mathfrak{S}_{\lambda,\delta}\Phi)(z,X), (\mathfrak{T}_{\lambda,\delta}\Phi)(z,X) \in \mathcal{F}[[X]]_\delta$$

defined by

$$(\mathfrak{S}_{\lambda,\delta}\Phi)(z,X) = \sum_{k=0}^{\infty} \phi_k^{\mathfrak{S}}(z) X^{k+\delta}, \tag{2.4}$$

$$(\mathfrak{T}_{\lambda,\delta}\Phi)(z,X) = \sum_{k=0}^{\infty} \phi_k^{\mathfrak{T}}(z) X^{k+\delta}, \tag{2.5}$$

where

$$\phi_k^{\mathfrak{S}} = \sum_{r=0}^{k} \frac{1}{r!(2k+2\delta+\lambda-r-1)!} \phi_{k-r}^{(r)}, \tag{2.6}$$

$$\phi_k^{\mathfrak{T}} = (2k+2\delta+\lambda-1) \sum_{r=0}^{k} (-1)^r \frac{(2k+2\delta+\lambda-r-2)!}{r!} \phi_{k-r}^{(r)}$$

for each $k \geq 0$.

We now turn to the usual action of the group $\mathrm{SL}(2,\mathbb{R})$ on $\mathcal{H}$ by linear fractional transformations, so that

$$\gamma z = \frac{az+b}{cz+d}$$

for all $z \in \mathcal{H}$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2,\mathbb{R})$. For the same z and γ, by setting

$$\mathfrak{J}(\gamma, z) = cz + d, \qquad \mathfrak{K}(\gamma, z) = \frac{c}{cz+d},$$

we obtain the maps $\mathfrak{J}, \mathfrak{K} : \mathrm{SL}(2,\mathbb{R}) \times \mathcal{H} \to \mathbb{C}$ which satisfy

$$\mathfrak{J}(\gamma\gamma', z) = \mathfrak{J}(\gamma, \gamma' z)\mathfrak{J}(\gamma', z), \quad \mathfrak{K}(\gamma, \gamma' z) = \mathfrak{J}(\gamma', z)^2 (\mathfrak{K}(\gamma\gamma', z) - \mathfrak{K}(\gamma', z))$$

for all $z \in \mathcal{H}$ and $\gamma, \gamma' \in \mathrm{SL}(2,\mathbb{R})$.

Given a function $f \in \mathcal{F}$, a formal power series $\Phi(z, X) \in \mathcal{F}[[X]]$, a nonnegative integer λ, and an element $\gamma \in \mathrm{SL}(2,\mathbb{R})$, we set

$$(f|_\lambda \gamma)(z) = \mathfrak{J}(\gamma, z)^{-\lambda} f(z),$$

$$(\Phi|_\lambda^J \gamma)(z, X) = \mathfrak{J}(\gamma, z)^{-\lambda} e^{-\mathfrak{K}(\gamma,z)X} \Phi(\gamma z, \mathfrak{J}(\gamma, z)^{-2} X), \tag{2.7}$$

$$(\Phi|_\lambda^M \gamma)(z, X) = \mathfrak{J}(\gamma, z)^{-\lambda} \Phi(\gamma z, \mathfrak{J}(\gamma, z)^{-2} X) \tag{2.8}$$

for $z \in \mathcal{H}$.

PROPOSITION 2.1. *The maps* $\mathfrak{S}_{\lambda,\delta}, \mathfrak{T}_{\lambda,\delta} : \mathcal{F}[[X]]_\delta \to \mathcal{F}[[X]]_\delta$ *given by* (2.4) *and* (2.5) *are complex linear isomorphisms with*

$$(\mathfrak{T}_{\lambda,\delta})^{-1} = \mathfrak{S}_{\lambda,\delta}. \tag{2.9}$$

Proof. This is a slightly modified version of a result that follows from the equivalence of (4) and (5) in Proposition 2 of [2] and can be proved in a straightforward manner. ∎

If α and ν are integers with $\nu > 0$, we note that a function $f \in \mathcal{F}$ satisfies

$$\frac{d^\nu}{dz^\nu}(f|_\alpha \gamma)(z) = \sum_{r=0}^{\nu} (-1)^{\nu-r} \frac{\nu!}{r!} \binom{\alpha+\nu-1}{\nu-r} \frac{\mathfrak{K}(\gamma, z)^{\nu-r}}{\mathfrak{J}(\gamma, z)^{\alpha+2r}} f^{(r)}(\gamma z) \tag{2.10}$$

for $z \in \mathcal{H}$ and $\gamma \in \mathrm{SL}(2,\mathbb{R})$ (see [2, (1.9)]). The next proposition shows the $\mathrm{SL}(2,\mathbb{R})$-equivariance of the maps $\mathfrak{S}_{\lambda,\delta}$ and $\mathfrak{T}_{\lambda,\delta}$.

Proposition 2.2. *The isomorphisms* $\mathfrak{S}_{\lambda,\delta}, \mathfrak{T}_{\lambda,\delta} : \mathcal{F}[[X]]_\delta \to \mathcal{F}[[X]]_\delta$ *satisfy*

$$(\mathfrak{S}_{\lambda,\delta}\Phi)|^J_\lambda \gamma = \mathfrak{S}_{\lambda,\delta}(\Phi|^M_\lambda \gamma), \tag{2.11}$$
$$(\mathfrak{T}_{\lambda,\delta}\Phi)|^M_\lambda \gamma = \mathfrak{T}_{\lambda,\delta}(\Phi|^J_\lambda \gamma)$$

for each $\gamma \in \mathrm{SL}(2,\mathbb{R})$ *and* $\Phi(z,X) \in \mathcal{F}[[X]]_\delta$.

Proof. Since $\mathfrak{T}_{\lambda,\delta}$ is the inverse of $\mathfrak{S}_{\lambda,\delta}$ by (2.9), it suffices to prove the relation (2.11). Given $\gamma \in \mathrm{SL}(2,\mathbb{R})$ and $\Phi(z,X) \in \mathcal{F}[[X]]_\delta$ as in (2.3), using (2.4) and (2.7), we have

$$\begin{aligned}
&((\mathfrak{S}_{\lambda,\delta}\Phi)|^J_\lambda \gamma)(z,X)\\
&\quad = \mathfrak{J}(\gamma,z)^{-\lambda}\bigg(\sum_{r=0}^\infty \frac{(-1)^r}{r!}\mathfrak{K}(\gamma,z)^r X^r\bigg)\bigg(\sum_{\ell=0}^\infty \phi^{\mathfrak{S}}_\ell(\gamma z)\mathfrak{J}(\gamma,z)^{-2\ell-2\delta}X^{\ell+\delta}\bigg)\\
&\quad = \sum_{r=0}^\infty\sum_{\ell=0}^\infty \frac{(-1)^r}{r!}\mathfrak{J}(\gamma,z)^{-\lambda-2\ell-2\delta}\mathfrak{K}(\gamma,z)^r\phi^{\mathfrak{S}}_\ell(\gamma z)X^{\ell+r+\delta}\\
&\quad = \sum_{k=0}^\infty\sum_{r=0}^k \frac{(-1)^r}{r!}\mathfrak{J}(\gamma,z)^{-\lambda-2k+2r-2\delta}\mathfrak{K}(\gamma,z)^r\phi^{\mathfrak{S}}_{k-r}(\gamma z)X^{k+\delta}.
\end{aligned}$$

Thus we may write

$$((\mathfrak{S}_{\lambda,\delta}\Phi)|^J_\lambda \gamma)(z,X) = \sum_{k=0}^\infty \xi^{\mathfrak{S}}_k(z)X^{k+\delta}$$

with

$$\begin{aligned}
\xi^{\mathfrak{S}}_k(z) &= \sum_{r=0}^k \frac{(-1)^r}{r!}\mathfrak{J}(\gamma,z)^{-\lambda-2k+2r-2\delta}\mathfrak{K}(\gamma,z)^r\phi^{\mathfrak{S}}_{k-r}(\gamma z) \qquad (2.12)\\
&= \sum_{r=0}^k\sum_{\ell=0}^{k-r} \frac{(-1)^r\mathfrak{J}(\gamma,z)^{-\lambda-2k+2r-2\delta}\mathfrak{K}(\gamma,z)^r\phi^{(\ell)}_{k-r-\ell}(\gamma z)}{r!\ell!(2k-2r+2\delta+\lambda-\ell-1)!},
\end{aligned}$$

where we used (2.6). On the other hand, from (2.8) we obtain

$$\begin{aligned}
(\Phi|^M_\lambda \gamma)(z,X) &= \mathfrak{J}(\gamma,z)^{-\lambda}\sum_{k=0}^\infty \phi_k(\gamma z)\mathfrak{J}(\gamma,z)^{-2k-2\delta}X^{k+\delta}\\
&= \sum_{k=0}^\infty (\phi_k|_{\lambda+2k+2\delta}\gamma)(z)X^{k+\delta};
\end{aligned}$$

hence we see that

$$(\mathfrak{S}_{\lambda,\delta}(\Phi|^M_\lambda \gamma))(z,X) = \sum_{k=0}^\infty \eta^{\mathfrak{S}}_k(z)X^{k+\delta},$$

where

$$\eta_k^{\mathfrak{S}}(z) = \sum_{r=0}^{k} \frac{1}{r!(\lambda + 2k + 2\delta - r - 1)!} (\phi_{k-r}|_{\lambda+2k-2r+2\delta}\gamma)^{(r)}(z).$$

However, using (2.10), we have

$$\begin{aligned}(\phi_{k-r}|_{\lambda+2k-2r+2\delta}\gamma)^{(r)}(z) \\ = \sum_{\ell=0}^{r}(-1)^{r-\ell}\frac{r!}{\ell!}\binom{\lambda+2k-r+2\delta-1}{r-\ell}\frac{\mathfrak{K}(\gamma,z)^{r-\ell}}{\mathfrak{J}(\gamma,z)^{\lambda+2k-2r+2\delta+2\ell}}\phi_{k-r}^{(\ell)}(\gamma z).\end{aligned}$$

Thus it follows that

$$\begin{aligned}\eta_k^{\mathfrak{S}}(z) &= \sum_{r=0}^{k}\sum_{\ell=0}^{r}\frac{(-1)^{r-\ell}}{\ell!(r-\ell)!(2k-2r+2\delta+\lambda+\ell-1)!} \\ &\qquad \times \frac{\mathfrak{K}(\gamma,z)^{r-\ell}}{\mathfrak{J}(\gamma,z)^{\lambda+2k-2r+2\delta+2\ell}}\phi_{k-r}^{(\ell)}(\gamma z) \\ &= \sum_{\ell=0}^{k}\sum_{r=\ell}^{k}\frac{(-1)^{r-\ell}}{\ell!(r-\ell)!(\lambda+2k-2r+2\delta+\ell-1)!} \\ &\qquad \times \frac{\mathfrak{K}(\gamma,z)^{r-\ell}}{\mathfrak{J}(\gamma,z)^{\lambda+2k-2r+2\delta+2\ell}}\phi_{k-r}^{(\ell)}(\gamma z).\end{aligned}$$

Changing the index r to $r+\ell$, we get

$$\begin{aligned}\eta_k^{\mathfrak{S}}(z) &= \sum_{\ell=0}^{k}\sum_{r=0}^{k-\ell}\frac{(-1)^{r}}{\ell!r!(\lambda+2k-2r+2\delta-\ell-1)!} \\ &\qquad \times \frac{\mathfrak{K}(\gamma,z)^{r}}{\mathfrak{J}(\gamma,z)^{\lambda+2k-2r+2\delta}}\phi_{k-\ell-r}^{(\ell)}(\gamma z).\end{aligned}$$

Comparing this with (2.12), we see that

$$\xi_k^{\mathfrak{S}} = \eta_k^{\mathfrak{S}}$$

for each $k \ge 0$, and therefore (2.11) follows. ∎

We now consider a discrete subgroup Γ of $\mathrm{SL}(2,\mathbb{R})$ commensurable with $\mathrm{SL}(2,\mathbb{Z})$. Then a *modular form of weight* λ *for* Γ is a holomorphic function $f \in \mathcal{F}$ satisfying

$$f|_\lambda\gamma = f$$

for all $\gamma \in \Gamma$. We denote by $M_\lambda(\Gamma)$ the space of such modular forms.

REMARK 2.3. For the growth condition at the cusps we note that the functions belonging to $\mathcal{F}$ are bounded by powers of the quotient in (2.1). This condition was suggested by Cohen, Manin and Zagier [2].

Definition 2.4.

(i) A formal power series $\Phi(z, X) \in \mathcal{F}[[X]]$ is a *Jacobi-like form of weight* λ *for* Γ if it satisfies

$$(\Phi|_\lambda^J \gamma)(z, X) = \Phi(z, X)$$

for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$.

(ii) A formal power series $\Phi(z, X) \in \mathcal{F}[[X]]$ is a *modular series of weight* λ *for* Γ if it satisfies

$$(\Phi|_\lambda^M \gamma)(z, X) = \Phi(z, X)$$

for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$.

We denote by $\mathcal{J}_\lambda(\Gamma)$ and $\mathcal{M}_\lambda(\Gamma)$ the spaces of Jacobi-like forms and modular series, respectively, of weight λ for Γ. We see easily that $\Phi(z, X)$ in (2.3) belongs to $\mathcal{M}_\lambda(\Gamma)$ if and only if

$$\phi_k \in M_{2k+2\delta+\lambda}(\Gamma)$$

for each $k \geq 0$. Given a nonnegative integer δ, let $\mathcal{J}_\lambda(\Gamma)_\delta$ and $\mathcal{M}_\lambda(\Gamma)_\delta$ denote the subspaces of $\mathcal{J}_\lambda(\Gamma)$ and $\mathcal{M}_\lambda(\Gamma)$, respectively, defined by

$$\mathcal{J}_\lambda(\Gamma)_\delta = \mathcal{J}_\lambda(\Gamma) \cap \mathcal{F}[[X]]_\delta, \quad \mathcal{M}_\lambda(\Gamma)_\delta = \mathcal{M}_\lambda(\Gamma) \cap \mathcal{F}[[X]]_\delta,$$

where $\mathcal{F}[[X]]_\delta$ is as in (2.2). Then by Proposition 2.2 the automorphisms $\mathfrak{S}_{\lambda,\delta}$ and $\mathfrak{T}_{\lambda,\delta}$ of $\mathcal{F}[[X]]_\delta$ induce the isomorphisms

$$\mathfrak{S}_{\lambda,\delta} : \mathcal{M}_\lambda(\Gamma)_\delta \to \mathcal{J}_\lambda(\Gamma)_\delta, \quad \mathfrak{T}_{\lambda,\delta} : \mathcal{J}_\lambda(\Gamma)_\delta \to \mathcal{M}_\lambda(\Gamma)_\delta \tag{2.13}$$

with

$$\mathfrak{S}_{\lambda,\delta} = \mathfrak{T}_{\lambda,\delta}^{-1}. \tag{2.14}$$

We note that this result provides a slight variation of the correspondence between Jacobi-like forms and sequences of modular forms established by Cohen, Manin and Zagier in [2, Proposition 2].

We now fix a nonnegative integer m and denote by $\mathcal{F}_m[X]$ the complex vector space of polynomials in X over $\mathcal{F}$ of degree at most m. If λ is a nonnegative integer and

$$\Psi(z, X) = \sum_{r=0}^{m} \psi_r(z) X^r \in \mathcal{F}_m[X], \tag{2.15}$$

we set

$$(\Psi|_\lambda^X \gamma)(z, X) = \sum_{r=0}^{m} (\psi_r|_{\lambda+2r}\gamma)(z) X^r, \tag{2.16}$$

$$(\Psi\|_\lambda \gamma)(z, X) = \mathfrak{J}(\gamma, z)^{-\lambda} \Psi(\gamma z, \mathfrak{J}(\gamma, z)^2 (X - \mathfrak{K}(\gamma, z))) \tag{2.17}$$

for all $z \in \mathcal{H}$ and $\gamma \in \mathrm{SL}(2, \mathbb{R})$.

If a polynomial $\varPsi(z,X) \in \mathcal{F}_m[X]$ is as in (2.15) and if $\lambda > 2m$, we introduce two additional polynomials

$$(\varXi_\lambda^m \varPsi)(z,X), (\varLambda_\lambda^m \varPsi)(z,X) \in \mathcal{F}_m[X]$$

defined by

$$(\varXi_\lambda^m \varPsi)(z,X) = \sum_{r=0}^{m} \psi_r^{m,\varXi}(z) X^r, \qquad (\varLambda_\lambda^m \varPsi)(z,X) = \sum_{r=0}^{m} \psi_r^{m,\varLambda}(z) X^r, \tag{2.18}$$

where

$$\begin{aligned} \psi_r^{m,\varXi} &= \frac{1}{r!} \sum_{j=0}^{m-r} \frac{1}{j!(\lambda - 2r - j - 1)!} \psi_{m-r-j}^{(j)}, \\ \psi_r^{m,\varLambda} &= (\lambda + 2r - 2m - 1) \\ &\quad \times \sum_{j=0}^{r} \frac{(-1)^j}{j!} (m - r + j)!(2r + \lambda - 2m - j - 2)! \psi_{m-r+j}^{(j)}, \end{aligned} \tag{2.19}$$

for each $r \in \{0, 1, \dots, m\}$. These formulas determine isomorphisms

$$\varXi_\lambda^m, \varLambda_\lambda^m : \mathcal{F}_m[X] \to \mathcal{F}_m[X] \tag{2.20}$$

with

$$(\varLambda_\lambda^m)^{-1} = \varXi_\lambda^m,$$

and they are known to satisfy

$$((\varXi_\lambda^m \varPsi) \|_\lambda \gamma)(z,X) = \varXi_\lambda^m (\varPsi|_{\lambda-2m}^X \gamma)(z,X), \tag{2.21}$$

$$((\varLambda_\lambda^m \varPsi)|_{\lambda-2m}^X \gamma)(z,X) = \varLambda_\lambda^m (\varPsi \|_\lambda \gamma)(z,X) \tag{2.22}$$

for all $\varPsi(z,X) \in \mathcal{F}_m[X]$ and $\gamma \in \mathrm{SL}(2,\mathbb{R})$ (see [4]).

Definition 2.5. Let $\varGamma$ be a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ commensurable with $\mathrm{SL}(2,\mathbb{Z})$ as before.

(i) A *modular polynomial for $\varGamma$ of weight λ and degree at most m* is an element $F(z,X) \in \mathcal{F}_m[X]$ satisfying

$$F|_\lambda^X \gamma = F$$

for all $\gamma \in \varGamma$.

(ii) An element $\varPsi(z,X) \in \mathcal{F}_m[X]$ is a *quasimodular polynomial for $\varGamma$ of weight λ and degree at most m* if it satisfies

$$\varPsi \|_\lambda \gamma = \varPsi$$

for all $\gamma \in \varGamma$.

We denote by $\mathrm{MP}_\lambda^m(\varGamma)$ and $\mathrm{QP}_\lambda^m(\varGamma)$ the spaces of, respectively, modular polynomials and quasimodular polynomials for $\varGamma$ of weight λ and degree at most m. From (2.21) and (2.22) we see that the maps Ξ_λ^m and Λ_λ^m induce the isomorphisms

$$\Xi_\lambda^m : \mathrm{MP}_{\lambda-2m}^m(\Gamma) \to \mathrm{QP}_\lambda^m(\Gamma), \qquad \Lambda_\lambda^m : \mathrm{QP}_\lambda^m(\Gamma) \to \mathrm{MP}_{\lambda-2m}^m(\Gamma) \tag{2.23}$$

for each integer $\lambda > 2m$.

Definition 2.6. An element $\psi \in \mathcal{F}$ is a *quasimodular form for Γ of weight λ and depth at most m* if there are functions $\psi_0, \dots, \psi_m \in \mathcal{F}$ satisfying

$$(\psi|_\lambda \gamma)(z) = \sum_{r=0}^{m} \psi_r(z) \mathfrak{K}(\gamma, z)^r \tag{2.24}$$

for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$. We denote by $\mathrm{QM}_\lambda^m(\Gamma)$ the space of such quasimodular forms.

If $\psi \in \mathrm{QM}_\lambda^m(\Gamma)$ satisfies (2.24), it can be shown that the functions ψ_r are uniquely determined; hence we can consider the map

$$\mathcal{Q}_\lambda^m : \mathrm{QM}_\lambda^m(\Gamma) \to \mathcal{F}_m[X]$$

defined by

$$(\mathcal{Q}_\lambda^m \psi)(z, X) = \sum_{r=0}^{m} \psi_r(z) X^r. \tag{2.25}$$

In fact, it is also known that this map determines an isomorphism

$$\mathcal{Q}_\lambda^m : \mathrm{QM}_\lambda^m(\Gamma) \to \mathrm{QP}_\lambda^m(\Gamma), \tag{2.26}$$

whose inverse is given by

$$(\mathcal{Q}_\lambda^m)^{-1} \Psi(z, X) = \psi_0 \tag{2.27}$$

for $\Psi(z, X) \in \mathrm{QP}_\lambda^m(\Gamma)$ as in (2.15) (see [1]).

3. Liftings of quasimodular forms. In this section we obtain an explicit formula for a lifting of a quasimodular polynomial to a Jacobi-like form whose existence was proved in [1]. Since quasimodular forms correspond to quasimodular polynomials, the same formula also determines a lifting of a quasimodular form to a Jacobi-like form, which generalizes the Cohen–Kuznetsov lifting of a modular form.

Let $\mathcal{F}[[X]]_\delta$ and $\mathcal{F}_m[X]$ with $\delta, m \geq 0$ be as in Section 2, and consider two surjective complex linear maps

$$\Pi_m^\delta, \widehat{\Pi}_m^\delta : \mathcal{F}[[X]]_\delta \to \mathcal{F}_m[X] \tag{3.1}$$

defined by

$$(\Pi^\delta_m \Phi)(z,X) = \sum_{r=0}^m \frac{1}{r!}\phi_{m-r}(z)X^r, \tag{3.2}$$

$$(\widehat{\Pi}^\delta_m \Phi)(z,X) = \sum_{r=0}^m \phi_r(z)X^r \tag{3.3}$$

for a formal power series of the form

$$\Phi(z,X) = \sum_{k=0}^\infty \phi_k(z)X^{k+\delta} \in \mathcal{F}[[X]]_\delta. \tag{3.4}$$

Proposition 3.1. *For each nonnegative integer* λ *the diagram*

$$\begin{array}{ccc} \mathcal{F}[[X]]_\delta & \xrightarrow{\mathfrak{T}_{\lambda,\delta}} & \mathcal{F}[[X]]_\delta \\ \Pi^\delta_m \big\downarrow & & \big\downarrow \widehat{\Pi}^\delta_m \\ \mathcal{F}_m[X] & \xrightarrow{\Lambda^m_{\lambda+2\delta+2m}} & \mathcal{F}_m[X] \end{array}$$

commutes, where $\mathfrak{T}_{\lambda,\delta}$ *and* $\Lambda^m_{\lambda+2\delta+2m}$ *are the isomorphisms in Proposition* 2.1 *and* (2.20).

Proof. Given a formal power series $\Phi(z,x) \in \mathcal{F}[[X]]_\delta$ as in (3.4), from (2.5) and (3.3) we obtain

$$((\widehat{\Pi}^\delta_m \circ \mathfrak{T}_{\lambda,\delta})\Phi)(z,X) = \sum_{r=0}^m \phi^{\mathfrak{T}}_r(z)X^r,$$

where

$$\phi^{\mathfrak{T}}_r = (2r+\lambda+2\delta-1)\sum_{j=0}^r (-1)^j \frac{(2r+\lambda+2\delta-j-2)!}{j!}\phi^{(j)}_{r-j}$$

for $0 \le r \le m$. On the other hand, if we set

$$\widehat{\phi}_k = \frac{1}{k!}\phi_{m-k}$$

for $0 \le k \le m$, from (2.18) and (3.2) we see that

$$((\Lambda^m_{\lambda+2\delta+2m} \circ \Pi^\delta_m)\Phi)(z,X) = \sum_{r=0}^m \phi^{m,\Lambda}_r(z)X^r,$$

where

$$\phi^{m,\Lambda}_r = (\lambda+2\delta+2r-1)\sum_{j=0}^r \frac{(-1)^j}{j!}(m+r+j)!(2r+\lambda+2\delta-j-2)!\widehat{\phi}^{(j)}_{m-r+j}$$

with

$$\widehat{\phi}^{(j)}_{m-r+j} = \frac{1}{(m-r+j)!}\phi^{(j)}_{r-j}$$

for $0 \leq j \leq r \leq m$. Thus we have

$$\phi_r^{\mathfrak{T}} = \phi_r^{m,\Lambda}$$

for $0 \leq r \leq m$, which implies that

$$\widehat{\Pi}_m^\delta \circ \mathfrak{T}_{\lambda,\delta} = \Lambda_{\lambda+2\delta+2m}^m \circ \Pi_m^\delta; \tag{3.5}$$

hence the proposition follows. ∎

The surjective maps Π_m^δ and $\widehat{\Pi}_m^\delta$ in (3.1) are in fact equivariant with respect to the $\mathrm{SL}(2,\mathbb{R})$-actions in (2.7), (2.8), (2.16) and (2.17) in such a way that

$$\Pi_m^\delta(\Phi|_\lambda^J\gamma) = \Pi_m^\delta(\Phi)\|_{\lambda+2m+2\delta}\gamma, \qquad \widehat{\Pi}_m^\delta(\Phi|_\lambda^M\gamma) = \Pi_m^\delta(\Phi)|_{\lambda+2\delta}^X\gamma$$

for all $\Phi(z,X) \in \mathcal{F}[[X]]$ and $\gamma \in \mathrm{SL}(2,\mathbb{R})$ (cf. [1]). Thus, if Γ is a discrete subgroup of $\mathrm{SL}(2,\mathbb{R})$ considered in Section 2, they induce the complex linear maps

$$\Pi_m^\delta : \mathcal{J}_\lambda(\Gamma)_\delta \to \mathrm{QP}_{\lambda+2m+2\delta}^m(\Gamma), \qquad \widehat{\Pi}_m^\delta : \mathcal{M}_\lambda(\Gamma)_\delta \to \mathrm{MP}_{\lambda+2\delta}^m(\Gamma). \tag{3.6}$$

Hence we obtain the commutative diagram

$$\begin{array}{ccc} \mathcal{J}_\lambda(\Gamma)_\delta & \xrightarrow{\mathfrak{T}_{\lambda,\delta}} & \mathcal{M}_\lambda(\Gamma)_\delta \\ {\scriptstyle \Pi_m^\delta}\big\downarrow & & \big\downarrow{\scriptstyle \widehat{\Pi}_m^\delta} \\ \mathrm{QP}_{\lambda+2\delta+2m}^m(\Gamma) & \xrightarrow{\Lambda_{\lambda+2\delta+2m}^m} & \mathrm{MP}_{\lambda+2\delta}^m(\Gamma) \end{array}$$

for each nonnegative integer λ.

We now consider the natural embedding

$$\mathcal{E}_\delta^m : \mathcal{F}_m[X] \to \mathcal{F}[[X]]_\delta$$

defined by

$$(\mathcal{E}_\delta^m\Psi)(z,X) = \sum_{k=0}^{\infty} \widetilde{\psi}_k(z)X^{k+\delta} \tag{3.7}$$

for

$$\Psi(z,X) = \sum_{r=0}^{m} \psi_r(z)X^r \in \mathcal{F}_m[X], \tag{3.8}$$

where

$$\widetilde{\psi}_k = \begin{cases} \psi_k & \text{for } 0 \leq k \leq m, \\ 0 & \text{for } k > m. \end{cases}$$

Then we easily see that it induces an embedding

$$\mathcal{E}_\delta^m : \mathrm{MP}_\lambda^m(\Gamma) \to \mathcal{M}_{\lambda-2\delta}(\Gamma)_\delta \tag{3.9}$$

of modular polynomials into modular series satisfying

$$(\widehat{\Pi}_m^\delta \circ \mathcal{E}_\delta^m)\Psi(z,X) = \Psi(z,X) \tag{3.10}$$

for all $\Psi(z,X) \in \mathrm{MP}^m_\lambda(\Gamma)$. Given $\lambda > 2m$, we now define the linear map

$$\mathcal{L}^m_{\delta,\lambda} : \mathcal{F}_m[X] \to \mathcal{F}[[X]]_\delta$$

by setting

$$(\mathcal{L}^m_{\delta,\lambda}\Psi)(z,X) = \sum_{k=0}^{\infty} \psi^*_k(z) X^{k+\delta}$$

for $\Psi(z,X) \in \mathcal{F}_m[X]$ as in (3.8), where

$$\psi^*_k = \sum_{r=\max(k-m,0)}^{k} \sum_{j=0}^{k-r} (\lambda + 2k - 2r - 2m - 1) \times \frac{(-1)^j (m-k+r+j)!(2k-2r+\lambda-2m-j-2)!}{j!r!(2k+\lambda-2m-r-1)!} \psi^{(j+r)}_{m-k+r+j} \tag{3.11}$$

for each $k \geq 0$.

THEOREM 3.2. *The map* $\mathcal{L}^m_{\delta,\lambda}$ *induces a lifting*

$$\mathcal{L}^m_{\delta,\lambda} : \mathrm{QP}^m_\lambda(\Gamma) \to \mathcal{J}_{\lambda-2\delta-2m}(\Gamma)_\delta \tag{3.12}$$

of quasimodular polynomials to Jacobi-like forms such that

$$((\Pi^\delta_m \circ \mathcal{L}^m_{\delta,\lambda})\Psi)(z,X) = \Psi(z,X)$$

for all $\Psi(z,X) \in \mathrm{QP}^m_\lambda(\Gamma)$.

Proof. From (2.13), (2.23) and (3.9) we obtain the following sequence of maps:

$$\mathrm{QP}^m_\lambda(\Gamma) \xrightarrow{\Lambda^m_\lambda} \mathrm{MP}^m_{\lambda-2m}(\Gamma) \xrightarrow{\mathcal{E}^m_\delta} \mathcal{M}_{\lambda-2\delta-2m}(\Gamma)_\delta \xrightarrow{\mathfrak{S}_{\lambda-2\delta-2m,\delta}} \mathcal{J}_{\lambda-2\delta-2m}(\Gamma)_\delta.$$

We shall first show that the composite of these maps coincides with $\mathcal{L}^m_{\delta,\lambda}$. Given a quasimodular polynomial

$$\Psi(z,X) = \sum_{r=0}^{m} \psi_r(z) X^r \in \mathrm{QP}^m_\lambda(\Gamma),$$

using (2.18) and (3.7), we have

$$((\mathcal{E}^m_\delta \circ \Lambda^m_\lambda)\Psi)(z,X) = \sum_{k=0}^{\infty} \eta_k(z) X^{k+\delta} \in \mathcal{M}_{\lambda-2\delta-2m}(\Gamma)_\delta$$

with

$$\eta_k = \begin{cases} \psi^{m,\Lambda}_k & \text{for } 0 \leq k \leq m, \\ 0 & \text{for } k > m, \end{cases}$$

where $\psi^{m,\Lambda}_k$ is as in (2.19). From this and (2.4) we see that

$$\big((\mathfrak{S}_{\lambda-2\delta-2m,\delta} \circ \mathcal{E}^m_\delta \circ \Lambda^m_\lambda)\Psi\big)(z,X) = \sum_{k=0}^{\infty} \psi^*_k(z) X^{k+\delta} \in \mathcal{J}_{\lambda-2\delta-2m}(\Gamma)_\delta,$$

where

$$\psi_k^* = \sum_{r=0}^{k} \frac{1}{r!(2k+\lambda-2m-r-1)!}\eta_{k-r}^{(r)}$$
$$= \sum_{r=\max(k-m,0)}^{k} \frac{1}{r!(2k+\lambda-2m-r-1)!}(\psi_{k-r}^{m,\Lambda})^{(r)}$$

for each $k \geq 0$. Noting that

$$\psi_{k-r}^{m,\Lambda} = (\lambda+2k-2r-2m-1)$$
$$\times \sum_{j=0}^{k-r} \frac{(-1)^j}{j!}(m-k+r+j)!(2k-2r+\lambda-2m-j-2)!\psi_{m-k+r+j}^{(j)}$$

by (2.19), we obtain

$$\mathcal{L}_{\delta,\lambda}^m = \mathfrak{S}_{\lambda-2\delta-2m,\delta} \circ \mathcal{E}_\delta^m \circ \Lambda_\lambda^m.$$

On the other hand, from (3.5) we see that

$$\widehat{\Pi}_m^\delta \circ \mathfrak{T}_{\lambda-2\delta-2m,\delta} = \Lambda_\lambda^m \circ \Pi_m^\delta.$$

Using this, (2.14) and (3.10), we have

$$((\Pi_m^\delta \circ \mathcal{L}_{\delta,\lambda}^m)\Psi)(z,X)$$
$$= \big(((\Lambda_\lambda^m)^{-1} \circ \widehat{\Pi}_m^\delta \circ \mathfrak{T}_{\lambda-2\delta-2m,\delta} \circ \mathcal{L}_{\delta,\lambda}^m)\Psi\big)(z,X)$$
$$= \big(((\Lambda_\lambda^m)^{-1} \circ \widehat{\Pi}_m^\delta \circ \mathfrak{T}_{\lambda-2\delta-2m,\delta} \circ \mathfrak{S}_{\lambda-2\delta-2m,\delta} \circ \mathcal{E}_\delta^m \circ \Lambda_\lambda^m)\Psi\big)(z,X) = \Psi(z,X)$$

for all $\Psi(z,X) \in \mathrm{QP}_\lambda^m(\Gamma)$; hence the proof of Theorem 3.2 is complete. ∎

In order to describe the lifting in the previous theorem in terms of quasimodular forms, we consider the map

$$\pi_n^\delta : \mathcal{J}_\lambda(\Gamma)_\delta \to \mathcal{F}$$

for each nonnegative integer n defined by

$$\pi_n^\delta(\Phi) = \phi_n$$

if $\Phi(z,X) \in \mathcal{J}_\lambda(\Gamma)_\delta$ is given by (3.4). Then $\pi_n^\delta(\Phi)$ is the constant term in the quasimodular polynomial $(\Pi_n^\delta \Phi)(z,X) \in \mathrm{QP}_{\lambda+2\delta+2n}^n(\Gamma)$ and therefore is a quasimodular form belonging to $\mathrm{QM}_{\lambda+2\delta+2n}^n(\Gamma)$ by (2.27). Thus π_n^δ determines the map

$$\pi_n^\delta : \mathcal{J}_\lambda(\Gamma)_\delta \to \mathrm{QM}_{\lambda+2\delta+2n}^n(\Gamma).$$

Given a nonnegative integer m, we now introduce the map $\widetilde{\mathcal{L}}_{\delta,\lambda}^m$ defined by

$$\widetilde{\mathcal{L}}_{\delta,\lambda}^m = \mathcal{L}_{\delta,\lambda}^m \circ \mathcal{Q}_\lambda^m : \mathrm{QM}_\lambda^m(\Gamma) \to \mathcal{F}[[X]]_\delta,$$

where $\mathcal{Q}_\lambda^m$ is the isomorphism in (2.26).

COROLLARY 3.3. *The map* $\widetilde{\mathcal{L}}^m_{\delta,\lambda}$ *induces a lifting*

$$\widetilde{\mathcal{L}}^m_{\delta,\lambda} : \mathrm{QM}^m_\lambda(\Gamma) \to \mathcal{J}_{\lambda-2\delta-2m}(\Gamma)_\delta$$

of quasimodular forms to Jacobi-like forms such that

$$(\pi^\delta_m \circ \widetilde{\mathcal{L}}^m_{\delta,\lambda})(\psi) = \psi$$

for all $\psi \in \mathrm{QM}^m_\lambda(\Gamma)$.

Proof. This follows from Theorem 3.2 and the fact that the coefficient of X^m in the Jacobi-like form

$$(\widetilde{\mathcal{L}}^m_{\delta,\lambda}\psi)(z,X) = ((\mathcal{L}^m_{\delta,\lambda} \circ \mathcal{Q}^m_\lambda)\psi)(z,X) \in \mathcal{J}_{\lambda-2\delta-2m}(\Gamma)_\delta$$

coincides with the constant term in the quasimodular polynomial

$$(\mathcal{Q}^m_\lambda\psi)(z,X) \in \mathrm{QP}^m_\lambda(\Gamma),$$

which is equal to $\psi(z)$. ∎

EXAMPLE 3.4. (i) We consider the lifting (3.12) for $m = 0$. First, we note that $\mathrm{QP}^0_\lambda(\Gamma)$ can be identified with $M_\lambda(\Gamma)$. Thus we have $\Psi(z,X) = \psi_0(z)$ with $\psi_0 \in M_\lambda(\Gamma)$, and we see in formula (3.11) that $r = k$ and $j = 0$, so that

$$\psi^*_k = \frac{(\lambda-1)!}{k!(k+\lambda-1)!}\psi^{(k)}_0,$$

$$(3.13) \qquad (\mathcal{L}^0_{\delta,\lambda}\Psi)(z,X) = (\lambda-1)!\sum_{k=0}^\infty \frac{\psi^{(k)}_0(z)}{k!(k+\lambda-1)!}X^{k+\delta} \in \mathcal{J}_{\lambda-2\delta}(\Gamma)_\delta.$$

Thus the Jacobi-like form $(\mathcal{L}^0_{\delta,\lambda}\Psi)(z,X)/(\lambda-1)!$ is the well-known *Cohen–Kuznetsov lifting* of the modular form f_0 (see e.g. [2]).

(ii) We now consider the case of $m = 1$. First, for $k = 0$ we have $r = j = 0$ in the sum in (3.12), and therefore

$$\psi^*_0 = (\lambda-3)\frac{(\lambda-4)!}{(\lambda-3)!}\psi_1 = \psi_1.$$

On the other hand, using (3.12) for $k \geq 1$, we obtain

$$\psi^*_k = \sum_{r=k-1}^{k}\sum_{j=0}^{k-r}(\lambda+2k-2r-3) \times \frac{(-1)^j(1-k+r+j)!(2k-2r+\lambda-j-4)!}{j!r!(2k+\lambda-r-3)!}\psi^{(j+r)}_{1-k+r+j}$$

$$= (\lambda-1)\frac{(\lambda-2)!}{(k-1)!(k+\lambda-2)!}\psi_0^{(k-1)} - (\lambda-1)\frac{(\lambda-3)!}{(k-1)!(k+\lambda-2)!}\psi_1^{(k)}$$
$$+ (\lambda-3)\frac{(\lambda-4)!}{k!(k+\lambda-3)!}\psi_1^{(k)}$$
$$= (\lambda-1)\frac{(\lambda-2)!}{(k-1)!(k+\lambda-2)!}\psi_0^{(k-1)} - \frac{(\lambda-2)!(k-1)}{k!(k+\lambda-2)!}\psi_1^{(k)}$$
$$= \frac{(\lambda-2)!}{k!(k+\lambda-2)!}\big((\lambda-1)k\psi_0^{(k-1)} - (k-1)\psi_1^{(k)}\big).$$

Thus it follows that

$$(3.14)\qquad (\mathcal{L}^1_{\delta,\lambda}\Psi)(z,X) = \psi_1(z)X^\delta + \sum_{k=1}^{\infty}\frac{(\lambda-2)!}{k!(k+\lambda-2)!}\big((\lambda-1)k\psi_0^{(k-1)}(z) - (k-1)\psi_1^{(k)}(z)\big)X^{k+\delta},$$

which belongs to $\mathcal{J}_{\lambda-2\delta-2}(\Gamma)_\delta$.

(iii) Let f be a modular form belonging to $M_w(\Gamma)$. Then it can be shown that f' is a quasimodular form belonging to $\mathrm{QM}^1_{w+2}(\Gamma)$ which satisfies

$$(f'|_{w+2}\gamma)(z) = f'(z) + wf(z)\mathfrak{K}(\gamma,z)$$

for all $z \in \mathcal{H}$ and $\gamma \in \Gamma$. Thus, if we set

$$\Psi(z,X) = (\mathcal{Q}^1_{w+2}(f'))(z,X) = \psi_0(z) + \psi_1(z)X$$

with $\mathcal{Q}^1_{w+2}$ as in (2.25), by using (3.14) for $\lambda = w+2$ we obtain

$$\psi_0 = f', \qquad \psi_1 = wf,$$
$$(\lambda-1)k\psi_0^{(k-1)} - (k-1)\psi_1^{(k)} = (w+1)kf^{(k)} - (k-1)wf^{(k)} = (k+w)f^{(k)}.$$

Thus

$$(\mathcal{L}^1_{\delta,w+2}\Psi)(z,X) = wf(z)X^\delta + w!\sum_{k=1}^{\infty}\frac{(k+w)f^{(k)}}{k!(k+w)!}X^{k+\delta}$$
$$= w!\sum_{k=0}^{\infty}\frac{f^{(k)}}{k!(k+w-1)!}X^{k+\delta},$$

which belongs to $\mathcal{J}_{w-2\delta}(\Gamma)_\delta$. Comparing this with (3.13), we see that the lifting of the quasimodular polynomial $(\mathcal{Q}^1_{w+2}(f'))(z,X)$ corresponding to the quasimodular form f' is equal to the lifting of the modular form f times the weight of f.

4. Concluding remarks. We see easily that the kernel of the complex linear map Π^δ_m in (3.6) is equal to $\mathcal{J}_\lambda(\Gamma)_{\delta+m+1}$. On the other hand, by Theorem 3.2 the same map is surjective. Thus we obtain a short exact sequence of the form

$$0 \to \mathcal{J}_\lambda(\Gamma)_{\delta+m+1} \to \mathcal{J}_\lambda(\Gamma)_\delta \xrightarrow{\Pi_m^\delta} \mathrm{QP}^m_{\lambda+2m+2\delta}(\Gamma) \to 0, \tag{4.1}$$

where the second arrow represents the inclusion map. Furthermore, from (3.12) we obtain the map

$$\mathcal{L}^m_{\delta,\lambda+2\delta+2m} : \mathrm{QP}^m_{\lambda+2\delta+2m}(\Gamma) \to \mathcal{J}_\lambda(\Gamma)_\delta$$

satisfying

$$(\Pi_m^\delta \circ \mathcal{L}^m_{\delta,\lambda+2\delta+2m})\Psi = \Psi$$

for all $\Psi(z, X) \in \mathrm{QP}^m_{\lambda+2\delta+2m}(\Gamma)$. Thus it follows that the short exact sequence (4.1) splits.

As we noted in Section 2, if $\psi \in \mathrm{QM}^m_\lambda(\Gamma)$ satisfies (2.24), the functions ψ_r are uniquely determined. Since it is known that $\psi_r \in \mathrm{QM}^{m-r}_{\lambda-2r}(\Gamma)$ for $0 \le r \le m$ (see e.g. [5]), we can consider the complex linear map

$$\mathfrak{S}_r : \mathrm{QM}^m_\lambda(\Gamma) \to \mathrm{QM}^{m-r}_{\lambda-2r}(\Gamma)$$

with $0 \le r \le m$ defined by

$$\mathfrak{S}_r(\psi) = \psi_r$$

for $\psi \in \mathrm{QM}^m_\lambda(\Gamma)$ as in (2.24). Then the formula for the lifting map

$$\widetilde{\mathcal{L}}^m_{\delta,\lambda} : \mathrm{QM}^m_\lambda(\Gamma) \to \mathcal{J}_{\lambda-2\delta-2m}(\Gamma)_\delta$$

in Corollary 3.3 can be written as

$$(\widetilde{\mathcal{L}}^m_{\delta,\lambda}\psi)(z, X) = \sum_{k=0}^\infty \psi_k^\#(z) X^{k+\delta}$$

for $\psi \in \mathrm{QM}^m_\lambda(\Gamma)$, where

$$\psi_k^\# = \sum_{r=\max(k-m,0)}^{k} \sum_{j=0}^{k-r} (\lambda + 2k - 2r - 2m - 1) \times \frac{(-1)^j (m-k+r+j)!(2k-2r+\lambda-2m-j-2)!}{j!r!(2k+\lambda-2m-r-1)!} (\mathfrak{S}_{m-k+r+j}\psi)^{(j+r)}$$

for each $k \ge 0$.

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