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Some global results for nonlinear Sturm–Liouville problems with spectral parameter in the boundary condition

by ZIYATKHAN S. ALIYEV and GUNAY M. MAMEDOVA (Baku)

Abstract. We consider nonlinear Sturm–Liouville problems with spectral parameter in the boundary condition. We investigate the structure of the set of bifurcation points, and study the behavior of two families of continua of nontrivial solutions of this problem contained in the classes of functions having oscillation properties of the eigenfunctions of the corresponding linear problem, and bifurcating from the points and intervals of the line of trivial solutions.

1. Introduction. We consider the following nonlinear eigenvalue problem:

$$\begin{aligned} \ell(y)(x) &\equiv -y''(x)+q(x)y(x) \\ &= \lambda y(x)+h(x,y(x),y'(x),\lambda), \qquad 0<x<1, \end{aligned} \tag{1.1}$$

$$b_0 y(0)=d_0 y'(0), \tag{1.2}$$

$$(a_1\lambda+b_1)y(1)=(c_1\lambda+d_1)y'(1), \tag{1.3}$$

where λ is a real parameter, q is a real continuous function on $[0,1]$, b_0, d_0, a_1, b_1, c_1, d_1 are real numbers with $|b_0|+|d_0|>0$ and

$$\sigma_1=a_1d_1-b_1c_1>0. \tag{1.4}$$

We also assume that the nonlinear term h has the form $h=g+f$, where g and f are continuous functions on $[0,1]\times\mathbb{R}^3$ satisfying the conditions:

$$\left|\frac{f(x,u,s,\lambda)}{u}\right| \le M, \forall x\in[0,1],\ \forall u,s\in\mathbb{R},\ 0<|u|\le 1,\ |s|\le 1,\ \forall\lambda\in\mathbb{R}, \tag{1.5}$$

where M is a positive constant; and

$$g(x,u,s,\lambda)=o(|u|+|s|) \tag{1.6}$$

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near $(u,s)=(0,0)$, uniformly for $x\in[0,1]$ and $\lambda\in\Lambda$, for every bounded interval $\Lambda\subset\mathbb{R}$.

In the study of nonlinear eigenvalue problems, an important role is played, when it exists, by the linearization about zero of the problem under consideration, i.e., its Fréchet derivative at the origin (cf. [12]). In the context of linearizability, Rabinowitz [15] gives a nonlinear version of the classical results for linear Sturm–Liouville problems, namely the existence of two families of unbounded continua of nontrivial solutions of problem (1.1)–(1.3) in the case

$$f\equiv 0 \quad\text{and}\quad a_1=c_1=0,$$

and bifurcating from the points of the line of trivial solutions, corresponding to the eigenvalues of the linear problem, and contained in the classes of functions having the usual oscillation properties.

Because of the presence of the term f, problem (1.1)–(1.3) does not in general have a linearization about zero. For this reason, the set of bifurcation points for (1.1)–(1.3) with respect to the line of trivial solutions need not be discrete (cf. the example in [3, p. 381]. Therefore, to investigate bifurcation for (1.1)–(1.3), one has to consider bifurcation from intervals rather than from bifurcation points. We say that bifurcation occurs from an interval if this interval contains at least one bifurcation point [3].

Berestycki [3], Schmitt and Smith [17], Chiappinelli [6], Aliyev [1], Rynne [16] and Dai [10] obtained a number of global results for bifurcation of solutions of the nonlinear Sturm–Liouville problem (1.1)–(1.3) when $a_1=c_1=0$ (i.e., when the spectral parameter is not involved in the boundary conditions) and f has a sublinear growth in y and y'. In these papers the existence of two families of unbounded continua of solutions, corresponding to the usual oscillation properties and bifurcating from intervals of the line of trivial solutions is proved.

Problem (1.1)–(1.3) was previously considered in [2], where results similar to those of [14] and [3] were obtained. It should be noted that in [2] there is a gap, as the function $(a_1\lambda+b_1)/(c_1\lambda+d_1)$ in the boundary condition (1.3) is strictly increasing in each of the intervals $(-\infty,-d_1/c_1)$ and $(-d_1/c_1,\infty)$, but is not increasing for all $\lambda\in\mathbb{R}$. Therefore, in this case the method of [3] cannot be applied. Also, the behavior of continua of solutions having oscillation properties of the eigenfunctions of the corresponding linear problems and bifurcating from the bifurcation points and intervals has not been completely investigated. This is due to the fact that in the case $c_1\neq 0$ there is a positive integer N_0 such that the eigenfunctions corresponding to the N_0th and (N_0+1)th eigenvalues of the linear Sturm–Liouville problem obtained from (1.1)–(1.3) for $h\equiv 0$ have the same number of zeros.

In [5], using an extension of the Prüfer transformation, the authors obtain results similar to that of [9] for a nonlinear Sturm–Liouville problem with spectral parameter in the boundary condition.

In this paper we study the structure of bifurcation points, and completely investigate the behavior of two families of continua of solutions of problem (1.1)–(1.3) having the oscillation properties of the eigenfunctions of the corresponding linear problem, and bifurcating from the points and intervals of the line of trivial solutions.

2. Global bifurcation of solutions of the nonlinear problem (1.1)–(1.3) **for** $f \equiv 0$**.** Alongside problem (1.1)–(1.3) we consider the spectral problems

$$\begin{cases} -y''(x) + q(x)y(x) = \lambda y\,(x), \quad x \in (0,1), \\ b_0 y(0) = d_0 y'(0), \\ y(1) = 0, \end{cases} \tag{2.1}$$

$$\begin{cases} -y''(x) + q(x)y(x) = \lambda y(x), \quad x \in (0,1), \\ b_0 y(0) = d_0 y'(0), \\ (a_1\lambda + b_1)y(1) = (c_1\lambda + d_1)y'(1). \end{cases} \tag{2.2}$$

The eigenvalues of the Sturm–Liouville problem (2.1) are denoted by η_k, $k = 1, 2, \dots$, where $\eta_k \nearrow \infty$.

For $c_1 \neq 0$ let N_0 be the integer such that $\eta_{N_0-1} < -d_1/c_1 \leq \eta_{N_0}$, where $\eta_0 = -\infty$.

In [4] it is established that the eigenvalues of problem (2.2) are real, simple, and form an infinite increasing sequence $\mu_k \nearrow \infty$; moreover, the eigenfunctions of this problem have the following oscillation properties:

(A) if $c_1 = 0$, then the eigenfunction $y_k(x)$, $k \in \mathbb{N}$, corresponding to the eigenvalue μ_k, has exactly $k-1$ simple zeros, all lying in $(0,1)$;

(B) if $c_1 \neq 0$, then the eigenfunction $y_k(x)$, $k \in \mathbb{N}$, corresponding to the eigenvalue μ_k, has exactly $k-1$ simple zeros for $k \leq N_0$, and exactly $k-2$ simple zeros for $k > N_0$, all lying in $(0,1)$.

In the Hilbert space $H = L_2(0,1) \oplus \mathbb{C}$ with the inner product

$$(\hat{y}, \hat{u}) = (\{y, m\}, \{u, s\}) = (y, u)_{L_2} + \sigma_1^{-1} m\bar{s},$$

define the operator

$$L\hat{y} = L\{y, m\} = \{\ell(y), d_1 y'(1) - b_1 y(1)\}$$

with the domain

$$D(L) = \bigl\{\{y, m\} \in H : y, y' \in AC(0,1),\ \ell(y) \in L_2(0,1),\\ b_0 y(0) = d_0 y'(0),\ m = a_1 y(1) - c_1 y'(1)\bigr\},$$

where $(y,u)_{L_2} = \int_0^1 y\,(x)\,\overline{u(x)}\,dx$ and $AC(0,1)$ is the class of absolutely continuous functions in $(0,1)$. Obviously, the operator L is well defined in H. Thus problem (2.2) takes the form

$$L\hat{y} = \lambda \hat{y}, \qquad \hat{y} \in D(L), \tag{2.3}$$

i.e., the eigenvalues of (2.2) and of the operator L coincide together with their multiplicities, and between the root functions, there is a correspondence

$$y_k \leftrightarrow \{y_k, m_k\}, \qquad m_k = a_1 y_k(1) - c_1 y_k'(1).$$

Define $F : \mathbb{R} \times D(L) \to H$ and $G : \mathbb{R} \times D(L) \to H$ as follows:

$$G(\lambda, \hat{y}) = G(\lambda, \{y, m\}) = \{g(x, y, y', \lambda), 0\},$$
$$F(\lambda, \hat{y}) = F(\lambda, \{y, m\}) = \{f(x, y, y', \lambda), 0\},$$

where $m = a_1 y(1) - c_1 y'(1)$. Then problem (1.1)–(1.3) reduces to the nonlinear problem

$$L\hat{y} = \lambda \hat{y} + F(\lambda, \hat{y}) + G(\lambda, \hat{y}), \tag{2.4}$$

i.e., between the solutions of these problems, there is a correspondence

$$(\lambda, \hat{y}) \leftrightarrow (\lambda, y).$$

The set

$$\hat{E} = \{\hat{y} = \{y, m\} \in C^1[0,1] \oplus \mathbb{C} : b_0 y(0) = d_0 y'(0),\ m = a_1 y(1) - c_1 y'(1)\}$$

is a Banach space with the norm $\|\hat{y}\|_1 = \|\{y, m\}\|_1 = |y|_1 + |m|$, where $|y|_1 = \max_{x\in[0,1]} |y(x)| + \max_{x\in[0,1]} |y'(x)|$.

It is known [18] that if $\lambda = 0$ is not an eigenvalue of the linear problem (2.2), then L^{-1} exists and $L^{-1} : \hat{E} \to \hat{E}$. Denote

$$\hat{L} = L^{-1}, \qquad \hat{G}(\lambda, \hat{y}) = \hat{L}G(\lambda, \hat{y}), \qquad \hat{F}(\lambda, \hat{y}) = \hat{L}F(\lambda, \hat{y}).$$

Then problem (2.4) can be written in the equivalent form

$$\hat{y} = \lambda \hat{L}\hat{y} + \hat{G}(\lambda, \hat{y}) + \hat{F}(\lambda, \hat{y}). \tag{2.5}$$

From [18] it follows that $\hat{L} : \hat{E} \to \hat{E}$ is compact. Hence $\hat{G} : \mathbb{R} \times \hat{E} \to \hat{E}$ and $\hat{F} : \mathbb{R} \times \hat{E} \to \hat{E}$ are completely continuous.

Let $\hat{S}_k^+$, $k \in \mathbb{N}$, denote the set of vectors $\hat{y} = \{y, m\} \in \hat{E}$ such that y has exactly $k-1$ simple zeros in $(0,1)$ and is positive near $x = 0$, and set $\hat{S}_k^- = -\hat{S}_k^+$ and $\hat{S}_k = \hat{S}_k^- \cup \hat{S}_k^+$. From now on ν will denote $+$ or $-$. The sets S_k^ν are disjoint and open in $\hat{E}$. Moreover, if $\hat{y} = \{y, m\} \in \hat{S}_k$, then y has at least one double zero in $[0,1]$ (see [9]).

Let

$$\hat{T}_k^\nu = \begin{cases} \hat{S}_k^\nu & \text{if } k \le N_0, \\ \hat{S}_{k-1}^\nu & \text{if } k > N_0, \end{cases} \qquad \hat{T}_k = T_k^+ \cup \hat{T}_k^-,\ k \in \mathbb{N}.$$

We denote by $\hat{\Im}$ the closure in $\mathbb{R} \times \hat{E}$ of the set of nontrivial solutions of problem (2.5).

Now suppose that $f \equiv 0$ (in effect, we suppose that the nonlinearity $h \equiv 0$ itself satisfies (1.6)). In this case problem (1.1)–(1.3) is equivalent to

$$\hat{y} = \lambda \hat{L} \hat{y} + \hat{G}(\lambda, \hat{y}). \tag{2.6}$$

The linearization of (2.6) at $\hat{y} = \hat{0}$ is the spectral problem

$$\hat{y} = \lambda \hat{L} \hat{y}, \tag{2.7}$$

where $\hat{0} = \{0,\, 0\} \in \hat{E}$. Obviously, problem (2.7) is equivalent to the spectral problem (2.2) (and also to problem (2.3)).

We denote by $r(\hat{L})$ the set of characteristic values of $\hat{L}$.

From properties (A) and (B) it follows that the eigenvector $\hat{y}_k$, $k \in \mathbb{N}$, of the operator L, corresponding to the eigenvalue λ_k, is in the set $\hat{T}_k$.

We will denote by $\hat{y}_k^+$ the unique eigenfunction of (2.7) associated to λ_k with

$$\hat{y}_k^+ \in \hat{T}_k^+ \quad \text{and} \quad \|\hat{y}_k^+\|_1 = 1.$$

Theorem 2.1. *Let $f \equiv 0$. Then for each $k \in \mathbb{N}$ and each ν, there exists a continuum $\hat{Y}_k^\nu$ of solutions of problem* (2.6) *in* $(\mathbb{R} \times \hat{T}_k^\nu) \cup \{(\lambda_k, \hat{0})\}$*, which contains $(\lambda_k, \hat{0})$ and is unbounded in $\mathbb{R} \times \hat{E}$.*

Proof. Note that if (λ, y) is a solution of (1.1)–(1.3) for $f \equiv 0$, and if y has a double zero, then the growth estimate on g near the double zero and linearity of ℓ imply that $y \equiv 0$ on $[0,\, 1]$. Therefore, in particular, any solution $(\lambda, \hat{y})$ of problem (2.6) with $\hat{y} \in \partial S_k^\nu$ has $\hat{y} = \hat{0}$.

From (1.6) and the definition of $\hat{G}$, it follows that $\hat{G}(\lambda, \hat{y}) = o(\|\hat{y}\|_1)$ near $\hat{0} \in \hat{E}$ uniformly for λ in each bounded interval of $\mathbb{R}$. The eigenvalues of L are the characteristic values of $\hat{L}$ and are simple. Therefore the hypotheses of Theorem 1.3 in [15] are satisfied, and there exists a continuum $\hat{Y}_{\lambda_k} \equiv \hat{Y}_k$ of solutions of (2.6) in $\hat{\Im}$ which contains $(\lambda_k, \hat{0})$ and is either unbounded in $\mathbb{R} \times \hat{E}$, or contains $(\lambda_s, \hat{0})$, where $\lambda_k \neq \lambda_s \in r(\hat{L})$.

If $k \in \mathbb{N}$ and $k \neq N_0, N_0 + 1$, then the proof is similar to that of [15, Theorem 2.3].

Suppose now that $k = N_0$ or $N_0 + 1$. Note that $\hat{T}_{N_0} = \hat{T}_{N_0+1}$. Lemma 1.24 from [15] implies that if $(\lambda, \hat{y}) \in \hat{Y}_k$ and is near $(\lambda_k, \hat{0})$, then

$$\hat{y} = \alpha \hat{y}_k^+ + \hat{\vartheta} \quad \text{with } \hat{\vartheta} = o(|\alpha|), \quad \lambda = \lambda_k + o(1) \quad \text{as } \alpha \to 0. \tag{2.8}$$

Since $\hat{T}_k$ is open and $\hat{y}_k^+ \in \hat{T}_k$, we have

$$(\lambda, \hat{y}) \in \mathbb{R} \times \hat{T}_k \quad \text{and} \quad (\hat{Y}_k \setminus \{(\lambda_k, \hat{0})\}) \cap \hat{\mathrm{B}}_z(\lambda_k) \subset \mathbb{R} \times \hat{T}_k \tag{2.9}$$

for all $z > 0$ small, where $\hat{\mathrm{B}}_z(\lambda_k)$ denotes the open ball in $\mathbb{R} \times \hat{E}$ of radius z centered at $(\lambda_k, \hat{0})$. By the above remark, if $\hat{y} \in \partial \hat{T}_k \cap \hat{\Im}$, then $\hat{y} = \hat{0}$.

Hence $(\hat{Y}_k \setminus \{(\lambda_k, \hat{0})\}) \cap \partial\hat{T}_k = \emptyset$. Thus $\hat{Y}_k \subset (\mathbb{R} \times \hat{T}_k) \cup \{(\lambda_k, \hat{0})\}$. Although $\hat{T}_{N_0} = \hat{T}_{N_0+1}$, we will prove that alternative (ii) of Theorem 1.3 of [15] is not possible.

Now by a construction as in [11], we decompose $\hat{Y}_{N_0}$ into two subcontinua $\hat{Y}^+_{N_0}$ and $\hat{Y}^-_{N_0}$ which contain $(\lambda_{N_0}, \hat{0})$. We also decompose $\hat{Y}_{N_0+1}$ into two subcontinua $\hat{Y}^+_{N_0+1}$ and $\hat{Y}^-_{N_0+1}$ which contain $(\lambda_{N_0+1}, \hat{0})$. If $(\lambda, \hat{y})$ is in $\hat{Y}^\nu_{N_0} \setminus \{(\lambda_{N_0}, \hat{0})\}$ (resp. in $\hat{Y}^\nu_{N_0+1} \setminus \{(\lambda_{N_0+1}, \hat{0})\}$) and is near $(\lambda_{N_0}, \hat{0})$ (resp. $(\lambda_{N_0+1}, \hat{0})$), then (2.8) holds for $k = N_0$ (resp. $k = N_0 + 1$). Taking into account that $\alpha\hat{y}^+_k \in \hat{T}^\nu_{N_0}$ for $k = N_0$ (resp. $k = N_0 + 1$) and $\alpha \in \mathbb{R}^\nu \setminus \{0\}$, from (2.9) we obtain

$$(\hat{Y}^\nu_{N_0} \setminus \{(\lambda_{N_0}, \hat{0})\}) \cap \hat{\mathrm{B}}_z(\lambda_{N_0}) \subset \mathbb{R} \times \hat{T}^\nu_{N_0},$$
$$(\text{resp. } (Y^\nu_{N_0+1} \setminus \{(\lambda_{N_0+1}, \hat{0})\}) \cap \hat{\mathrm{B}}_z(\lambda_{N_0+1}) \subset \mathbb{R} \times \hat{T}^\nu_{N_0}),$$

for all $z > 0$ small, where $\mathbb{R}^\nu = \{\gamma \in \mathbb{R} : 0 \leq \nu\gamma \leq \infty\}$. Since

$$(\hat{Y}^\nu_{N_0} \setminus \{(\lambda_{N_0}, \hat{0})\}) \cap (\mathbb{R} \times \partial\hat{T}^\nu_{N_0}) = \emptyset$$
$$(\text{resp. } (\hat{Y}^\nu_{N_0+1} \setminus \{(\lambda_{N_0+1}, \hat{0})\}) \cap (\mathbb{R} \times \partial\hat{T}^\nu_{N_0}) = \emptyset),$$

$\hat{Y}^\nu_{N_0} \setminus \{(\lambda_{N_0}, \hat{0})\}$ (resp. $\hat{Y}^\nu_{N_0+1} \setminus \{(\lambda_{N_0+1}, \hat{0})\}$) cannot leave $\mathbb{R} \times \hat{T}^\nu_{N_0}$ outside of a neighborhood of $(\lambda_{N_0}, \hat{0})$ (resp. $(\lambda_{N_0+1}, \hat{0})$). Moreover, $\hat{Y}^-_{N_0}$ (resp. $\hat{Y}^-_{N_0+1}$) does not intersect $\hat{Y}^+_{N_0}$ (resp. $\hat{Y}^+_{N_0+1}$) outside of a neighborhood of $(\lambda_{N_0}, \hat{0})$ (resp. $(\lambda_{N_0+1}, \hat{0})$). By the remark to Theorem 2 in [11], either (i) $\hat{Y}^-_{N_0}$ (resp. $\hat{Y}^-_{N_0+1}$) is unbounded in $\mathbb{R} \times \hat{E}$, or (ii) $\hat{Y}^-_{N_0}$ (resp. $\hat{Y}^-_{N_0+1}$) meets $\hat{Y}^+_{N_0}$ (resp. $Y^+_{N_0+1}$) outside of a neighborhood of $(\lambda_{N_0}, \hat{0})$ (resp. $(\lambda_{N_0+1}, \hat{0})$) (this also shows that a similar result holds for $\hat{Y}^+_{N_0}$ (resp. $\hat{Y}^+_{N_0+1}$)). Consequently, all these sets are unbounded in $\mathbb{R} \times \hat{E}$. ∎

The set $E = C^1[0, 1] \cap \mathrm{B.C_0}$. is a Banach space with the norm $|\cdot|_1$, where $\mathrm{B.C_0}$. is the set of the boundary conditions (1.2).

Let

$$\Im = \{(\lambda, y) \in \mathbb{R} \times E : (\lambda, \hat{y}) \in \hat{\Im}\},$$
$$T^\nu_k = \{y \in E : \hat{y} = \{y, m\} \in \hat{T}^\nu_k\}, \quad T_k = T^+_k \cup T^-_k,\ k \in \mathbb{N}.$$

Since between solutions of problem (2.6) and (1.1)–(1.3) for $f \equiv 0$ there exists an isomorphism $(\lambda, \hat{y}) \leftrightarrow (\lambda, y)$, Theorem 2.1 yields the following result.

Theorem 2.2. *Let $f \equiv 0$. Then for each $k \in \mathbb{N}$ and each ν, there exists a continuum Y^ν_k of solutions of problem (1.1)–(1.3) in $(\mathbb{R} \times T^\nu_k) \cup \{(\lambda_k, 0)\}$ which contains $(\lambda_k, 0)$ and is unbounded in $\mathbb{R} \times E$.*

3. Global bifurcation of solutions of (1.1)–(1.3)**.** Problem (1.1)–(1.3) need not have any linearization at the origin, but still can be related to some linear problems. The general idea is to approximate this equation by linearizable ones, for which we apply Theorem 2.1. Then we pass to the limit using a priori bounds which are obtained with the aid of the maximum-minimum properties of the eigenvalues of some linear problems.

We say that $(\lambda, \hat{0})$ is a *bifurcation point* of (2.3) *with respect to the set* $\mathbb{R}\times\hat{S}_k^\nu$, $k\in\mathbb{N}$, if in every small neighborhood of this point there is a solution to this problem which is contained in $\mathbb{R}\times\hat{S}_k^\nu$.

To study the bifurcation of solutions of problem (1.1)–(1.3) with respect to $\mathbb{R}\times\hat{S}_k^\nu$, $k\in\mathbb{N}$, we introduce the approximate equation

$$
\begin{cases}
\ell(y) = \lambda y + f(x, |y|^\varepsilon y, y', \lambda) + g(x, y, y', \lambda), \quad x\in(0,1),\\
b_0 y(0) = d_0 y'(0),\\
(a_1\lambda + b_1)y(1) = (c_1\lambda + d_1)y'(1),
\end{cases}
\tag{3.1}
$$

where $\varepsilon\in(0,1]$; this form of approximation is similar to that used in [1, 3, 16].

Note that by (1.5) for $\varepsilon\in(0,1]$, the function $f(x,|u|^\varepsilon u, s,\lambda)$ satisfies the condition

$$
f(x, |u|^\varepsilon u, s, \lambda) = o(|u| + |s|)
\tag{3.2}
$$

near $(0,0)$ uniformly for $x\in[0,1]$ and λ in a bounded real interval.

The following lemma will ensure that the set of bifurcation points of problem (1.1)–(1.3) is nonempty.

Lemma 3.1. *For each $k\in\mathbb{N}$ and each ν, and for sufficiently small $\tau>0$, there exists a solution $(\zeta_{\tau,k}, \hat{w}_{\tau,k})$ of* (2.4) *such that $\hat{w}_{\tau,k}\in\hat{T}_k^\nu$ and $\|\hat{w}_{\tau,k}\|_1 = \tau$.*

Proof. Problem (3.1) can be written in the equivalent form

$$
L\hat{y} = \lambda\hat{y} + F_\varepsilon(\lambda,\hat{y}) + G(\lambda,\hat{y}),
\tag{3.3}
$$

where

$$
F_\varepsilon(\lambda,\hat{y}) = F_\varepsilon(\lambda,\{y,m\}) = \{f(x, |y(x)|^\varepsilon y(x), y'(x), \lambda), 0\}.
$$

Problem (3.3) is equivalent to

$$
\hat{y} = \lambda\hat{L}\hat{y} + \hat{F}_\varepsilon(\lambda, y) + \hat{G}(\lambda, y),
\tag{3.4}
$$

where $\hat{F}_\varepsilon(\lambda,\hat{y}) = \hat{L}F_\varepsilon(\lambda,\hat{y})$.

By (3.2) we have $\hat{F}_\varepsilon(\lambda,\hat{y}) = o(\|\hat{y}\|_1)$ in a small neighborhood of $\hat{0}\in\hat{E}$ uniformly for λ in each bounded interval of $\mathbb{R}$. Then by Theorem 2.1, for each $k\in\mathbb{N}$ and each ν, there exists an unbounded continuum $\hat{C}_{k,\varepsilon}^\nu$ of solutions of

(3.4) (as well as of (3.3)) such that

$$(\lambda_k, \hat{0}) \in \hat{C}^{\nu}_{k,\varepsilon} \subset (\mathbb{R} \times \hat{T}^{\nu}_k) \cup \{(\lambda_k, \hat{0})\}.$$

Then for any $\varepsilon \in (0, 1]$, there exists a solution $(\zeta_{\tau,k,\varepsilon}, \hat{w}_{\tau,k,\varepsilon}) \in \mathbb{R} \times \hat{E}$ of (3.3) such that $\hat{w}_{\tau,k,\varepsilon} \in \partial\hat{B}_\tau \cap \hat{T}^{\nu}_k$, where $\partial\hat{B}_\tau$ is the boundary of the open ball $\hat{B}_\tau \subset \hat{E}$ of radius τ centered at $\hat{0}$. By (3.1) we know that $(\zeta_{\tau,k,\varepsilon}, w_{\tau,k,\varepsilon})$ is a solution of the nonlinear problem

$$\text{(3.5)} \qquad \begin{cases} \ell(y) + h_\varepsilon y = \lambda y + g(x, y, y', \lambda), \quad x \in (0, 1), \\ b_0 y(0) = d_0 y'(0), \\ (a_1\lambda + b) y(1) = (c_1\lambda + d) y'(1), \end{cases}$$

where

$$\text{(3.6)} \qquad h_\varepsilon(x) = \begin{cases} f(x, |w_{\tau,k,\varepsilon}(x)|^\varepsilon w_{\tau,k,\varepsilon}(x), w'_{\tau,k,\varepsilon}(x), \xi_{\tau,k,\varepsilon})/w_{\tau,k,\varepsilon}(x) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } w_{\tau,k,\varepsilon}(x) \neq 0, \\ 0 \quad \text{if } w_{\tau,k,\varepsilon}(x) = 0. \end{cases}$$

Define $L_{h_\varepsilon} : H \to H$ as follows:

$$\text{(3.7)} \qquad L_{h_\varepsilon}\hat{y} = L_{h_\varepsilon}\{y, m\} = \{h_\varepsilon(x) y(x), 0\}.$$

Then problem (3.5) can be rewritten in the following equivalent form:

$$\text{(3.8)} \qquad L\hat{y} + L_{h_\varepsilon}\hat{y} = \lambda\hat{y} + G(\lambda, \hat{y}).$$

Based on the maximum-minimum properties of the eigenvalues [7] (see also [8, Ch. 4, §4]), the kth eigenvalue $\lambda_{k\varepsilon}$ of the linear problem

$$\text{(3.9)} \qquad L\hat{y} + L_{h_\varepsilon}\hat{y} = \lambda\hat{y}$$

is determined from the relation

$$\text{(3.10)} \qquad \lambda_{k\varepsilon} = \max_{V^{(k-1)}} \min_{\hat{y} \in \hat{E}} \{R_{h_\varepsilon}[\hat{y}] : (\hat{y}, \hat{\varphi}) = 0,\ \hat{\varphi} \in V^{(k-1)}\},$$

where $R_{h_\varepsilon}[\hat{y}]$ is the Rayleigh ratio

$$\text{(3.11)} \qquad R_{h\varepsilon}[\hat{y}] = \frac{((L + L_{h_\varepsilon})\hat{y}, \hat{y})}{(\hat{y}, \hat{y})}$$

and $V^{(k-1)}$ is any set of $k - 1$ linearly independent vectors in $\hat{E}$.

From (3.11) we have

$$\text{(3.12)} \qquad R_{h\varepsilon}[\hat{y}] = \frac{(L\hat{y}, \hat{y})}{(\hat{y}, \hat{y})} + \frac{(L_{h_\varepsilon}\hat{y}, \hat{y})}{(\hat{y}, \hat{y})}.$$

It is obvious that

$$\text{(3.13)} \qquad \lambda_k = \max_{V^{(k-1)}} \min_{\hat{y} \in \hat{E}} \left\{ R[\hat{y}] = \frac{(L\hat{y}, \hat{y})}{(\hat{y}, \hat{y})} : (\hat{y}, \hat{\varphi}) = 0,\ \hat{\varphi} \in V^{(k-1)} \right\}.$$

By (1.5), from (3.6) we find that $|h_\varepsilon(x)| \leq M$ for all $x \in [0,1]$. Then taking into account (3.7), (3.11)–(3.13), from (3.10) we have

$$\lambda_{k\varepsilon} \in [\lambda_k - M, \lambda_k + M]. \tag{3.14}$$

By Theorem 2.1, $(\lambda_{k\varepsilon}, \hat{0})$ is a bifurcation point of problem (3.8) with respect to $\mathbb{R} \times \hat{T}_k^\nu$, and this bifurcation point corresponds to a continuous branch of nontrivial solutions of (3.8). Consequently, for each $\tau > 0$ there exists $\rho_{\tau\varepsilon} > 0$ such that

$$\zeta_{\tau,k,\varepsilon} \in (\lambda_{k\varepsilon} - \rho_{\tau\varepsilon}, \lambda_{k\varepsilon} + \rho_{\tau\varepsilon}) \subset [\lambda_k - M - \rho_0, \lambda_k + M + \rho_0], \tag{3.15}$$

where $\rho_0 = \sup_{\tau,\varepsilon} \rho_{\tau\varepsilon} > 0$.

Since $\{\hat{w}_{\tau,k,\varepsilon} : 0 < \varepsilon \leq 1\}$ is bounded in $\hat{E}$ and $\{\zeta_{\tau,k,\varepsilon} : 0 < \varepsilon \leq 1\}$ is bounded in $\mathbb{R}$, and since the operators $\hat{F}_\varepsilon, \hat{G} : \mathbb{R} \times \hat{E} \to \hat{E}$ are completely continuous, (3.4) implies that the set $\{\hat{w}_{\tau,k,\varepsilon} \in \hat{E} : 0 < \varepsilon \leq 1\}$ is compact in $\hat{E}$.

Let $\{\varepsilon_n\}_{n=1}^\infty \subset (0,1)$ with $\varepsilon_n \downarrow 0$ as $n \to \infty$ be such that $(\zeta_{\tau,k,\varepsilon_n}, \hat{w}_{\tau,k,\varepsilon_n}) \to (\xi_{\tau,k}, \hat{w}_{\tau,k})$ in $\mathbb{R} \times \hat{E}$ (hence $(\zeta_{\tau,k,\varepsilon_n}, w_{\tau,k,\varepsilon_n}) \to (\xi_{\tau,k}, w_{\tau,k})$ in $\mathbb{R} \times E$). Letting $n \to \infty$ in (3.4), we find that $(\xi_{\tau,k}, \hat{w}_{\tau,k})$ is a solution of the nonlinear problem (2.5). Consequently, it is a solution of (2.4). Moreover, $\hat{w}_{\tau,k} \in \hat{T}_k^\nu \cup \partial \hat{T}_k^\nu$. Since $\|\hat{w}_{\tau,k}\|_1 = \tau$, using the reasoning of [3, p. 379], we obtain $\hat{w}_{\tau,k} \in \hat{T}_k^\nu$. The proof of Lemma 3.1 is complete.

Remark 3.1. By Lemma 3.1, we have $\hat{w}_{\tau,k} \in \hat{S}_k^\nu$ if $k < N_0$, $\hat{w}_{\tau,k} \in \hat{S}_{k-1}^\nu$ if $k > N_0 + 1$ and $\hat{w}_{\tau,N_0}, \hat{w}_{\tau,N_0+1} \in \hat{S}_{N_0}^\nu$.

Corollary 3.1. *The set of bifurcation points of* (2.4) *with respect to* $\mathbb{R} \times \hat{S}_k^\nu$ *is nonempty.*

The next lemma will provide uniform a priori bounds for the solutions of (3.3) near the trivial solutions, and will ensure that the bifurcation points of problem (2.4) with respect to $\mathbb{R} \times \hat{S}_k^\nu$ are contained in intervals of radius M centered at $(\lambda_k, \hat{0})$.

Lemma 3.2. *Let* ε_n, $0 \leq \varepsilon_n \leq 1$, *be a sequence converging to* 0. *Let* $(\xi_n, \hat{w}_n) \in \mathbb{R} \times \hat{S}_k^\nu$ *be a solution of problem* (3.3) *corresponding to* $\varepsilon = \varepsilon_n$, *and suppose* $\{(\zeta_n, \hat{w}_n)\}_{n=1}^\infty$ *converges to* $(\zeta, \hat{0})$ *in* $\mathbb{R} \times \hat{E}$. *Then* $\zeta \in J_k$, *where*

$$J_k = \begin{cases} [\lambda_k - M, \lambda_k + M] & \textit{if } k < N_0, \\ [\lambda_{N_0} - M, \lambda_{N_0} + M] \textit{ or } [\lambda_{N_0+1} - M, \lambda_{N_0+1} + M] & \textit{if } k = N_0, \\ [\lambda_{k+1} - M, \lambda_{k+1} + M] & \textit{if } k > N_0. \end{cases}$$

Proof. Suppose that $\xi \notin J_k$. We define $\rho = \operatorname{dist}\{\zeta, J_k\}$. Since $\xi_n \to \xi$ as $n \to \infty$, there is an $n_\rho \in \mathbb{N}$ such that for all $n > n_\rho$ we have $|\xi_n - \xi| < \rho/2$. Hence, $\operatorname{dist}\{\zeta_n, J_k\} > \rho/2$ at $n > n_\rho$.

Note that $(\zeta_n, \hat{w}_n) \in \mathbb{R} \times \hat{S}_k^\nu$ is a solution of the nonlinear problem (3.8) corresponding to $\varepsilon = \varepsilon_n$. Since $(\lambda_{k,\varepsilon_n}, \hat{0})$ is a bifurcation point of (3.8) with respect to $\mathbb{R} \times \hat{T}_k^\nu$, for each $n > n_\rho$, there is $\rho_n > 0$ such that $\rho_n < \rho/2$ and $\zeta_n \in (\lambda_{k,\varepsilon_n} - \rho_n, \lambda_{k,\varepsilon_n} + \rho_n)$ if $k < N_0$; either $\zeta_n \in (\lambda_{N_0,\varepsilon_n} - \rho_n, \lambda_{N_0,\varepsilon_n} + \rho_n)$ or $\zeta_n \in (\lambda_{N_0+1,\varepsilon_n} - \rho_n, \lambda_{N_0+1,\varepsilon_n} + \rho_n)$ if $k = N_0$ (by Remark 3.1); and $\zeta_n \in (\lambda_{k+1,\varepsilon_n} - \rho_n, \lambda_{k+1,\varepsilon_n} + \rho_n)$ if $k > N_0$. By (3.14), we have $\operatorname{dist}\{\zeta_n, J_k\} < \rho/2$, a contradiction.

Corollary 3.2. *If $(\lambda, \hat{0})$ is a bifurcation point of (2.4) with respect to $\mathbb{R} \times \hat{S}_k^\nu$, then $\lambda \in J_k$.*

For each $k \in \mathbb{N}$, we define $\widetilde{\hat{D}}_k^\nu \subset \hat{\Im}$ to be the union of all the components $\hat{D}_{k,\lambda}^\nu$ of $\hat{\Im}$ which bifurcate from the bifurcation points $(\lambda, \hat{0})$ of (2.4) with respect to $\mathbb{R} \times \hat{S}_k^\nu$. By Lemma 3.2 and Corollary 3.2, the set $\widetilde{\hat{D}}_k^\nu$ is nonempty.

Let $\hat{D}_k^\nu = \widetilde{\hat{D}}_k^\nu \cup (I_k \times \{\hat{0}\})$, where

$$I_k = \begin{cases} J_k & \text{if } k \neq N_0, \\ [\lambda_{N_0} - M, \lambda_{N_0+1} + M] & \text{if } k = N_0. \end{cases}$$

The set $\hat{D}_{N_0}^\nu \cup (I_{N_0} \times \{\hat{0}\})$ is connected in $\mathbb{R} \times \hat{E}$, but

$$\hat{D}_{N_0}^\nu \cup \big(([\lambda_{N_0} - M, \lambda_{N_0} + M] \cup [\lambda_{N_0+1} - M, \lambda_{N_0+1} + M]) \times \{\hat{0}\}\big)$$

may not be connected.

Theorem 3.3. *For each $k \in \mathbb{N}$ and each ν, the set $\hat{D}_k^\nu$ is unbounded in $\mathbb{R} \times \hat{E}$, and lies in $(\mathbb{R} \times \hat{S}_k^\nu) \cup (I_k \times \{\hat{0}\})$.*

Proof. Apply Lemmas 3.1, 3.2 and Corollaries 3.1, 3.2, and an argument similar to that of [13, Theorem 2.1] (see also [14, Theorem 3.1]).

For each $k \in \mathbb{N}$, we define $\widetilde{D}_k^\nu \subset \Im$ to be the union of all the components $D_{k,\lambda}^\nu$ of $\Im$ which bifurcate from the bifurcation points $(\lambda, 0)$ of (1.1)–(1.3) with respect to $\mathbb{R} \times S_k^\nu$. Let $D_k^\nu = \widetilde{D}_k^\nu \cup (I_k \times \{\hat{0}\})$.

Since between solutions of problem (2.4) and (1.1)–(1.3) there exists an isomorphism $(\lambda, \hat{y}) \leftrightarrow (\lambda, y)$, Theorem 3.3 yields the following result.

Theorem 3.4. *For each $k \in \mathbb{N}$ and each ν, the set D_k^ν is unbounded in $\mathbb{R} \times E$, and lies in $(\mathbb{R} \times S_k^\nu) \cup (I_k \times \{0\})$.*

Now suppose that

$$g \equiv 0 \tag{3.16}$$

and that the nonlinearity f satisfies the condition (1.5) for any (x, u, s, λ) in $[0, 1] \times \mathbb{R}^3$. In this case problem (1.1)–(1.3) is equivalent to

$$L\hat{y} = \lambda \hat{y} + F(\lambda, \hat{y}). \tag{3.17}$$

Following the arguments used in the proof of Lemma 3.1, we find that if $(\lambda, \hat{y})$ is a solution of problem (3.17) such that $\hat{y} \in \hat{S}_k^\nu$, then $\lambda \in I_k$. Consequently, by Theorems 3.3 and 3.4 we have the following

THEOREM 3.5. *Suppose that* (3.16) *holds. Then for each* $k \in \mathbb{N}$ *and each* ν,

$$\hat{D}_k^\nu \subset I_k \times (\hat{S}_k^\nu \cup \{\hat{0}\}) \quad \textit{and} \quad D_k^\nu \subset I_k \times (S_k^\nu \cup \{0\}).$$

REMARK 3.2. When problem (1.1)–(1.3) is not linearizable, the structure of the set of bifurcation points within $I_k \times \{0\}$ is not clear.

In the following example the bifurcation points of problem (1.1)–(1.3) with respect to $\mathbb{R} \times S_k$ are discretely arranged in the interval $I_k \times \{0\}$, and two of them are bifurcation points of this problem with respect to $\mathbb{R} \times S_1^+$, and the other two with respect to $\mathbb{R} \times S_1^-$. Explicit representations of the sets D_1^+ and D_1^- are also given.

EXAMPLE 3.1. Consider the nonlinear eigenvalue problem

$$-y''(x) = \lambda y(x) + |y(x)|, \qquad 0 < x < \pi, \tag{3.18}$$

$$y(0) = 0, \qquad \lambda y(\pi) = (\lambda + 1) y'(\pi), \tag{3.19}$$

which is obtained from (1.1)–(1.3) for $q \equiv 0$, $h(x, y, y', \lambda) = |y|$ (i.e., $g \equiv 0$) and $d_0 = b_1 = 0$, $b_0 = 1$, $a_1 = c_1 = d_1 = 1$.

The linear problem

$$-y''(x) = \lambda y(x), \qquad 0 < x < \pi,$$
$$y(0) = 0, \qquad \lambda y(\pi) = (\lambda + 1) y'(\pi),$$

has only the eigenfunctions $y_k(x) = \sin \sqrt{\lambda_k}\, x$, $k \in \mathbb{N}$, with eigenvalues determined by the equation $\cot \sqrt{\lambda}\, \pi = \sqrt{\lambda}/(\lambda + 1)$, which has one negative root $\lambda_1 \in (-2.7, -2.5)$ and infinitely many positive roots λ_k, $k \geq 2$, where $\lambda_2 \in (0, 0.25)$ and $\lambda_k > 1$ for $k \geq 3$.

Since the eigenvalues of the problem

$$-y''(x) = \lambda y(x), \qquad 0 < x < \pi, \qquad y(0) = y(\pi) = 0,$$

are positive and $-d_1/c_1 = -1$, it follows that $N_0 = 1$. Consequently, $I_1 = [\lambda_1 - 1, \lambda_2 + 1]$ and $I_k = [\lambda_{k+1} - 1, \lambda_{k+1} + 1]$, $k = 2, 3, \ldots$.

Note that any eigenfunction $y \in S_1^+$ of problem (3.18)–(3.19) is also an eigenfunction of the linear problem

$$-y''(x) = (\lambda + 1) y(x), \qquad 0 < x < \pi,$$
$$y(0) = 0, \qquad \lambda y(\pi) = (\lambda + 1) y'(\pi),$$

which has only the eigenfunctions $v_k(x) = \sin \sqrt{\zeta_k + 1}\, x$, $k \in \mathbb{N}$, with eigenvalues ζ_k, $k \in \mathbb{N}$, found from the equation $\cot \sqrt{\zeta + 1}\, \pi = \zeta/(\zeta + 1)\sqrt{\zeta + 1}$. It is easy to verify that $\zeta_1 \in (-3.25, -3)$, $\zeta_2 \in (-0.75, 0)$ and $\zeta_k > 0.75$ for $k \geq 3$.

Therefore, the set D_1^+ admits the following representation:

$$D_1^+ = \{(\zeta_1, \alpha \operatorname{sh} \sqrt{|\zeta_1 + 1|}\, x) : \alpha \in (0, \infty)\}$$
$$\cup \{(\zeta_2, \beta \sin \sqrt{\zeta_2 + 1}\, x) : \beta \in (0, \infty)\} \cup (I_1 \times \{0\}) \subset (I_1 \times (S_1^+ \cup \{0\})).$$

If $y \in S_1^-$, then problem (3.18)–(3.19) reduces to

$$-y''(x) = (\lambda - 1)y(x), \quad 0 < x < \pi,$$
$$y(0) = 0, \quad \lambda y(\pi) = (\lambda + 1)y'(\pi),$$

which has only the eigenfunctions $w_k(x) = \sin \sqrt{\xi_k - 1}\, x$, $k \in \mathbb{N}$, with eigenvalues ξ_k, $k \in \mathbb{N}$, found from the equation $\cot \sqrt{\xi - 1}\, \pi = \xi/(\xi + 1)\sqrt{\xi - 1}$. It can be verified that $\xi_1 \in (-2.4, -2.2)$, $\xi_2 \in (0.75, 1)$ and $\xi_k > 1$ for $k \geq 3$.

Hence, the set D_1^- admits the following representation:

$$D_1^- = \{(\xi_1, \gamma \operatorname{sh} \sqrt{|\xi_1 - 1|}\, x) : \gamma \in (-\infty, 0)\}$$
$$\cup \{(\xi_2, \delta \operatorname{sh} \sqrt{|\xi_2 - 1|}\, x) : \delta \in (-\infty, 0)\} \cup (I_1 \times \{0\}) \subset (I_1 \times (S_1^- \cup \{0\})).$$

Remark 3.3. It would be interesting to have more information about the set of bifurcation points of problem (1.1)–(1.3) in $I_k \times \{0\}$, e.g.: under what conditions is this set finite? Or when does it contain an interval?, etc.

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Ziyatkhan S. Aliyev
Baku State University
and
IMM NAS of Azerbaijan
Baku, Azerbaijan
E-mail: z_aliyev@mail.ru

Gunay M. Mamedova
Baku State University
Baku, Azerbaijan
E-mail: m.g.m.400@mail.ru

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