ANNALES
POLONICI MATHEMATICI
115.2 (2015)

Chebyshev and Robin constants on algebraic curves

by Jesse Hart and Sione Ma'u (Auckland)

Abstract. We define directional Robin constants associated to a compact subset of an algebraic curve. We show that these constants satisfy an upper envelope formula given by polynomials. We use this formula to relate the directional Robin constants of the set to its directional Chebyshev constants. These constants can be used to characterize algebraic curves on which the Siciak–Zaharjuta extremal function is harmonic.

1. Introduction. In [1], *directional Chebyshev constants* associated to a compact subset of a complex algebraic curve were defined and studied. The aim of the present paper is to relate these constants to pluripotential theory.

Pluripotential theory has been studied in some depth on complex algebraic varieties by Sadullaev [11] and Zeriahi [15]. As in classical pluripotential theory in $\mathbb{C}^N$, the *Siciak–Zaharjuta extremal function* (or the *pluricomplex Green function with pole at infinity*) associated to a compact set K plays a central role. We will denote this function by V_K and usually refer to it simply as the *extremal function*.

In classical potential theory in $\mathbb{C}$, and for $K \subset \mathbb{C}$ compact, the logarithmic growth of V_K is described by the *Robin constant of* K (denoted ρ_K), and in $\mathbb{C}^N$, $N > 1$, this generalizes to the notion of *Robin function*. In this paper we define, on an algebraic curve, an analogous notion of *directional Robin constant*. Under some additional assumptions, we can construct d directional Robin constants associated to a compact subset of an algebraic curve of degree d ($d \in \mathbb{N}$). These constants describe the logarithmic growth of the extremal function along the different directions the curve takes to infinity. Our main theorem directly relates these directional Robin constants to the directional Chebyshev constants defined in [1].

The present paper relies on classical results about the extremal function for regular compact sets (as given e.g. in [6]); results on curves and varieties

2010 *Mathematics Subject Classification*: Primary 32U35; Secondary 32C25.
Key words and phrases: extremal function, pluricomplex Green function, Chebyshev constant, Robin constant, algebraic curve.

DOI: 10.4064/ap115-2-1

follow from the classical theory by approximation. To make our approximation arguments work we need a recent result of Coman, Guedj, and Zeriahi [5] on extending a psh function of logarithmic growth from a variety in $\mathbb{C}^N$ to the whole space.

The extremal function on an algebraic variety is only *weakly plurisubharmonic*, as a stronger notion of plurisubharmonicity may fail at singular points. This is not a major issue for our main results, as the singular points on a curve can be handled fairly easily in our proofs. Working on a curve also allows us to exploit classical potential theory in the plane on occasion.

Section 2 recalls basic facts about the extremal function on an algebraic variety A, and introduces the notions of A-regularity and A-maximality. Some approximation lemmas for later use are also given. In Section 3 we define directional Robin constants on an algebraic curve of degree d that satisfies a certain condition $(*)$. This section in particular makes essential use of the one-variable nature of algebraic curves. In Section 4 we prove an ‘upper envelope’ polynomial formula for directional Robin constants. This follows by approximation from a polynomial formula for the Robin function in $\mathbb{C}^N$. In Section 5 we prove our main theorem:

Theorem 5.7. *Let K be a compact subset of an algebraic curve A that satisfies $(*)$, and let λ be a direction of A. Then*

$$e^{-\rho_{A,K}(\lambda)} = \tau(K,\lambda).$$

Here $\rho_{A,K}(\lambda)$ is the directional Robin constant and $\tau(K,\lambda)$ the directional Chebyshev constant for the compact set K and the direction λ. The proof is a straightforward application of the polynomial formula derived in Section 4.

Finally, in Section 6, we relate the directional Robin and Chebyshev constants on an algebraic curve to so-called *extremal curves* associated to the extremal function of a nonpluripolar compact set in $\mathbb{C}^N$.

2. Preliminaries. We will use the following notation and terminology. Suppose $u : \Omega \to [-\infty,\infty)$ is a function on some metric space Ω and $A \subset \Omega$ is a subset. Then $u^{*A} : A \to [-\infty,\infty)$ is defined by

$$u^{*A}(z) := \limsup_{t\to z,\, t\in A} u(t).$$

If $u(z) = u^{*A}(z)$ for all $z \in A$ then we say that u is *upper semicontinuous* (*usc*) *on* A. The function u^{*A} is called the *upper regularization of* u *on* A. Taking A to be the whole space we recover the usual notions of upper semicontinuity and upper regularization, and write $u^* = u^{*\Omega}$.

In our context Ω will be a domain in $\mathbb{C}^N$ (usually all of $\mathbb{C}^N$), and A will be an analytic variety in Ω of pure dimension $m \le N$ (usually $m = 1$). We write $\operatorname{reg}(A)$ to denote the set of *regular points of* A (at which A is locally a complex m-dimensional manifold), and then $\operatorname{sing}(A) := A \setminus \operatorname{reg}(A)$ is the set of singular points.

Let $\Omega \subset \mathbb{C}^N$ be an open set, and let A be an analytic variety in Ω of pure dimension m. Following Sadullaev [11], let $\mathcal{P}(A)$ denote the collection of *weakly plurisubharmonic* (*weakly psh*) functions on A: here $u \in \mathcal{P}(A)$ if $u : A \to [-\infty, \infty)$ is usc on A and psh on $\operatorname{reg}(A)$. A set $T \subset A$ is *pluripolar in* A if for every point $z \in T$ there is an open neighborhood U of z in $\mathbb{C}^N$ and a function $u \in \mathcal{P}(A \cap U)$ such that $U \cap T \subseteq \{z \in U : u(z) = -\infty\}$.

We have the following properties of families of functions in $\mathcal{P}(A)$ (cf. [11, 1.1–1.2]):

Proposition 2.1.

(1) *If* $\{u_j\} \subset \mathcal{P}(A)$ *is a decreasing sequence of functions, then* $u(z) := \lim_j u_j(z)$ *also belongs to* $\mathcal{P}(A)$.

(2) *If* $\{u_\alpha\} \subset \mathcal{P}(A)$ *is a locally uniformly bounded family of functions and* $u(z) = \sup_\alpha u_\alpha(z)$, *then the usc regularization of* u *on* A,

$$u^{*_A}(z) = \limsup_{w \to z,\, w \in A} u(w),$$

also belongs to $\mathcal{P}(A)$, *and the set* $\{z \in A : u(z) < u^{*_A}(z)\}$ *is pluripolar in* A. ∎

Suppose A is an analytic variety in $\mathbb{C}^N$ of pure dimension $m \le N$. We write $\mathcal{L}(A)$ for the collection of weakly psh functions of *logarithmic growth*, i.e., $u \in \mathcal{L}(A)$ if $u \in \mathcal{P}(A)$ and

$$u(z) \le \log(1 + |z|) + c, \qquad \forall z \in A, \tag{2.1}$$

for some constant c depending on u.

Let $K \subset A$ be a compact set. We denote by $\mathcal{L}(A, K)$ the class of functions given by

$$\mathcal{L}(A, K) = \{u \in \mathcal{L}(A) : u(z) \le 0 \text{ if } z \in K\}, \tag{2.2}$$

and define $V_{A,K} : A \to (-\infty, \infty]$ by $V_{A,K}(z) := \sup\{u(z) : z \in \mathcal{L}(A, K)\}$. With this notation the classical Siciak–Zaharjuta extremal function in $\mathbb{C}^N$ is $V_{\mathbb{C}^N,K} =: V_K$. We will call the $V_{A,K}$ the *extremal function of* K *on* A. Sadullaev [11] has shown the following.

Theorem 2.2. *Let* $K \subset A$ *be compact, where* A *is an irreducible algebraic variety in* $\mathbb{C}^N$. *If* K *is non-pluripolar in* A *then* $(V_{A,K})^{*_A} \in \mathcal{L}(A)$, *and* $V_K(z) = V_{A,K}(z)$ *for all* $z \in A$. ∎

While $V_{A,K} \equiv V_K$ on A, we will usually write $V_{A,K}$ if we consider the domain to be A, and V_K if we consider the domain to be $\mathbb{C}^N$.

The function $V_{A,K}$ satisfies the well-known formula of Siciak and Zaharjuta [15]:

Theorem 2.3. *For a compact set $K \subset A$, we have*

$$V_{A,K}(z) = \sup\Big\{\frac{1}{\deg p}\log|p(z)| : p \text{ is a polynomial with } \|p\|_K \le 1\Big\}. \ \blacksquare$$

We will verify a limiting version of this theorem later on.

Definition 2.4. A compact set $K \subset A$ is said to be *A-regular* if $V_{A,K}$ is continuous on $\operatorname{reg}(A)$. Recall also that a compact set $K \subset \mathbb{C}^N$ is *regular* if V_K is continuous on $\mathbb{C}^N$.

We list some results concerning A-regularity. Proofs are omitted. They follow standard arguments based on the fact that psh functions in $\mathcal{L}(A)$ can be locally smoothed at regular points, and (for property (4)) the fact that a finite set is (pluri-)polar.

Proposition 2.5. *Suppose $K \subset A$ is compact, where A is an irreducible algebraic variety. Then*

1. (1) *$V_{A,K}$ is continuous at $z \in \operatorname{reg}(A)$ if and only if $(V_{A,K})^{*A}(z) = V_{A,K}(z)$* (1).
2. (2) *$V_{A,K}$ is continuous on $\operatorname{reg}(A)$ if and only if $(V_{A,K})^{*A}(z) = 0$ for all $z \in K$.*
3. (3) *If K_1, K_2 are A-regular compact sets, then $K_1 \cup K_2$ is also A-regular.*
4. (4) *Let $K \subset A$ be a compact set and $\zeta \in A$. Set $L = K \cup \{\zeta\}$. Then for all $z \in A \setminus \{\zeta\}$ we have $V_K(z) = V_L(z)$.* $\blacksquare$

The last property will be useful for handling singular points on an algebraic curve. It also provides easy examples of sets that are not A-regular: if $K \subset A$ and $\zeta \in \operatorname{reg}(A) \setminus K$, then $K \cup \{\zeta\}$ is not A-regular whenever $V_K(\zeta) > 0$.

Definition 2.6. Let $A \subset \mathbb{C}^N$ be an algebraic variety. Given an open subset Ω of $\operatorname{reg}(A)$, let us define a function $u \in \mathcal{P}(A)$ to be *A-maximal on Ω* if, given a relatively compact domain $D \subset \Omega$ (i.e., D is open in $\operatorname{reg}(A)$) and $v \in \mathcal{P}(A)$, we have

$$v(z) \le u(z) \text{ for all } z \in \partial D \ \Rightarrow\ v(z) \le u(z) \text{ for all } z \in D.$$

We remark that A-maximality of a psh function at regular points may be given locally in terms of the complex Monge–Ampère operator in local coordinates. When A is an algebraic curve ($m = 1$), this says that the (generalized) Laplacian in local coordinates of an A-maximal function u is zero, i.e., u is harmonic. The following is proved in [11].

Theorem 2.7. *If A is an algebraic curve and $K \subset A$ is a compact subset such that V_K is locally bounded on A, then V_K is A-maximal on $\operatorname{reg}(A) \setminus K$. Hence V_K is harmonic on $\operatorname{reg}(A) \setminus K$.* $\blacksquare$

(1) Equivalently, $V_{A,K}$ is lower semicontinuous at z, i.e., $\liminf_{t\to z,\, t\in A} V_{A,K}(t) = V_{A,K}(z)$.

We can compute some extremal functions explicitly. Suppose A is an algebraic curve with the property that

$$A \subset \{z = (z_1, \dots, z_N) \in \mathbb{C}^N : |z_1|^2 \geq C(1 + |z_2|^2 + \dots + |z_N|^2)\} \tag{2.3}$$

for some constant $C > 0$. Define $\pi : A \to \mathbb{C}$ by $\pi(z) = \pi(z_1, \dots, z_N) := z_1$. An easy argument using the maximum principle for harmonic functions shows that for $K := \pi^{-1}(\{t \in \mathbb{C} : |t - a| \leq r\})$ (where $a \in \mathbb{C}$), we have

$$V_K(z) = \log^+ \frac{|z_1 - a|}{r}. \tag{2.4}$$

REMARK 2.8. The following example shows that if K is A-regular, then $V_{A,K}$ may still be discontinuous at a singular point of A. Consider the curve

$$A = \{z = (z_1, z_2) \in \mathbb{C}^2 : z_2^2 = cz_1^2 - z_1^3\},$$

where $c > 0$ is a fixed constant; then A has a singular point at $(0,0)$. There is a parametrization of A given by

$$z_1(t) = c - t^2, \quad z_2(t) = t(c - t^2), \quad t \in \mathbb{C},$$

and the origin is given by the parameters $t = \pm\sqrt{c}$. Consider the following set parametrized by a small disk:

$$K = \{(z_1(t), z_2(t)) : |t - \sqrt{c}| < \epsilon\}.$$

We claim that away from $(0,0)$ we have $V_{A,K}(z) = 3\log^+(|t - \sqrt{c}|/\epsilon)$. One can check that the right-hand side defines a function in $\mathcal{L}(A, K)$ that is continuous on $\mathrm{reg}(A)$, identically zero on K and harmonic on $\mathrm{reg}(A) \setminus K$. A standard argument using the maximum principle shows that it must be the extremal function. However it is not continuous at $(0,0)$ for $\epsilon > 0$ sufficiently small (precisely, $\epsilon < 2\sqrt{c}$), since

$$\lim_{\substack{z(t)\to(0,0)\\ t\to-\sqrt{c}}} \log^+ \frac{|t - \sqrt{c}|}{\epsilon} = \log \frac{2\sqrt{c}}{\epsilon} \neq 0 = \lim_{\substack{z(t)\to(0,0)\\ t\to\sqrt{c}}} \log^+ \frac{|t - \sqrt{c}|}{\epsilon}.$$

We close this section by listing some approximation lemmas that we will need. We use the following notation: if $\delta > 0$ and $K \subset \mathbb{C}^N$ then we write

$$K^\delta := \{z \in \mathbb{C}^N : \exists w \in K \text{ such that } |z - w| \leq \delta\}. \tag{2.5}$$

The *Hausdorff distance* between compact sets A and B, which we will denote simply by $\mathrm{dist}(A, B)$, is the smallest $\delta \geq 0$ for which $A \subseteq B^\delta$ and $B \subseteq A^\delta$.

LEMMA 2.9 ([6, Corollary 5.1.5]). *If $K \subset \mathbb{C}^N$ is compact then K^δ is regular for each $\delta > 0$, and* $\lim_{\delta\to0} V_{K^\delta} = V_K$. ■

We also want a similar result on an algebraic curve $A \subset \mathbb{C}^N$, and here it is convenient to use classical potential theory in the plane. Suppose the boundary of a compact body $D \subset A$ is a union of smooth arcs, and all singular points of A are in the interior of D. Then using the standard methods in [10] or [13] in solving the Dirichlet problem, one can construct a harmonic function $A \setminus D$ of logarithmic growth that goes to zero at every point of ∂D. It is easy to see that this function coincides with $V_{A,D}$, and therefore D is an A-regular set. The lemma below now follows by approximating a compact set K from above by compact bodies bounded by smooth arcs. We may assume that $\operatorname{sing}(A) \subset K$ by Proposition 2.5(4).

LEMMA 2.10. *Let $K \subset A$ be compact, where $A \subset \mathbb{C}^N$ is an algebraic curve. Then there is a sequence $K_1 \supset K_2 \supset \cdots$ of A-regular sets with $\bigcap_j K_j \supseteq K$ and $\lim_{j\to\infty} V_{K_j}(z) = V_K(z)$ for all $z \in \operatorname{reg}(A)$.* ∎

3. Directional Robin constants. Let $A \subset \mathbb{C}^N$ be an irreducible algebraic curve of degree d. Recall that a *linear asymptote* of A is a line L in $\mathbb{C}^N$ which may be characterized by the property that

$$\lim_{\substack{|z|\to\infty \\ z\in L}} |z - z_A| = 0;$$

here z_A is the the closest point to L that lies on $H \cap A$, where H is the orthogonal hyperplane to L through z.

Following [1], we will assume that A satisfies the following condition:

(∗) A has d distinct non-parallel linear asymptotes $L_1, \ldots, L_d$ and for each j, L_j may be parametrized by $t \mapsto c_j + t\lambda_j$ $(t \in \mathbb{C})$, where $c_j = (c_{j1}, \ldots, c_{jN})$, $\lambda_j = (1, \lambda_{j2}, \ldots, \lambda_{jN})$, and

$$\lambda_{jm} \neq \lambda_{km} \quad \text{if } j \neq k \quad \text{for all } m = 2, \ldots, N. \tag{3.1}$$

If A has d distinct non-parallel linear asymptotes, then almost any rotation of coordinates will place us in this situation. In particular, no asymptote is parallel to any hyperplane of the form $z_1 = c$ $(c \in \mathbb{C})$, which we will refer to as a *vertical hyperplane*. In other words, there are no vertical asymptotes, and this also means that A satisfies (2.3). We call $\{\lambda_j\}_{j=1}^d$ the set of *directions of A*.

REMARK 3.1. None of the proofs in this paper require (3.1), but it is essential for the arguments in [1]. We use (3.1) implicitly in the next section when we make use of results in that paper.

LEMMA 3.2. *Let $\epsilon > 0$. Then there exists $R = R(\epsilon) > 0$ and a ball $B = B(R) = \{z\colon |z| < R\} \subset \mathbb{C}^N$ such that:*

1. *$A \setminus \overline{B} \subseteq \operatorname{reg}(A)$;*
2. *$A \setminus \overline{B} = D_1 \cup \cdots \cup D_d$, where $D_1, \ldots, D_d$ are domains in A that are pairwise disjoint; and*
3. *for each $j = 1, \ldots, d$, $\operatorname{dist}(D_j, L_j) < \epsilon$.*

Proof. The singular points of A are a finite set. The set $\bigcup_{j<k}(L_j \cap L_k)$ is also finite. As non-parallel lines diverge, the distance between them grows

linearly. Hence when $j \neq k$ there is $r_0 > 0$ and $c = c(j,k) > 0$ with $L_j \cap L_k \subset B(r_0)$ and $\operatorname{dist}(L_j \setminus B(r), L_k \setminus B(r)) \geq cr$ when $r > r_0$.

Given $\epsilon > 0$, we can choose $R_0 > 0$ sufficiently large that $B(R_0)$ contains all singular points of A, $\operatorname{dist}(L_j \setminus B(R_0), L_k \setminus B(R_0)) > 3\epsilon$, and, by the fact that the L_j's are asymptotes of A, $A \setminus B(R_0) \subset \bigcup_{j=1}^d (L_j)^\epsilon$.

For each j, let $D_j := (A \setminus B(R_0)) \cap (L_j)^\epsilon$; then by the previous paragraph $\operatorname{dist}(D_j, D_k) > \epsilon$ if $j \neq k$. In particular, the D_j's are disjoint. The lemma follows by choosing any $R > R_0$. ∎

Lemma 3.3. *For each $j = 1, \ldots, d$:*

(1) *The projection $\pi : D_j \to \mathbb{C}$ given by $z = (z_1, \ldots, z_N) \mapsto z_1 = \pi(z)$ is one-to-one.*
(2) *The limit $\rho_{A,K}(\lambda_j) := \lim_{|z|\to\infty,\, z \in D_j} (V_K(z) - \log|z_1|)$ exists for any compact set K that is non-pluripolar in A.*

Proof. Since A is of degree d, for each $c \in \mathbb{C}$ the intersection $A \cap \{z_1 = c\}$ has precisely d points counting multiplicity. Choose $R > 0$ as in the previous lemma. If $|c| > R$ then $(A \cap \{z_1 = c\}) \subset (A \setminus \overline{B}) = \bigcup_{j=1}^d D_j$, and hence

$$A \cap \{z_1 = c\} = \bigcup_{j=1}^{d} D_j \cap \{z_1 = c\}.$$

For each j, $L_j \cap \{z_1 = c\}$ is non-empty, since L_j is not a vertical line. As L_j is an asymptote of D_j, it follows easily that $D_j \cap \{z_1 = c\}$ is also non-empty. The intersection $D_j \cap \{z_1 = c\}$ has precisely one point, since the intersection $A \cap \{z_1 = c\}$ has d points. Hence $\pi : D_j \to \mathbb{C}$ given by $\pi(z) = \pi(z_1, \ldots, z_N) = z_1$ is one-to-one.

Let $\zeta_j : \pi(D_j) \to D_j$ be the local inverse, $\pi \circ \zeta_j(z) = z$, and on a small disk about the origin in $\mathbb{C}$ define

$$h(s) := V_K(\zeta_j(1/s)) + \log|s|. \tag{3.2}$$

By Theorems 2.2 and 2.7, V_K is harmonic off K. Since $V_K \in \mathcal{L}(A)$, it is easy to see that h is harmonic away from $s = 0$ and bounded in a neighborhood of $s = 0$. So h extends harmonically, hence smoothly, across $s = 0$. In particular,

$$\begin{aligned} h(0) = \lim_{s\to 0} h(s) &= \lim_{s\to 0} (V_K(\zeta_j(1/s)) + \log|s|) = \lim_{|z_1|\to\infty} (V_K(\zeta_j(z_1)) - \log|z_1|) \\ &= \lim_{\substack{|z|\to\infty \\ z \in D_j}} (V_K(z) - \log|z_1|). \end{aligned}$$

Finally, set $\rho_{A,K}(\lambda_j) := h(0)$. ∎

Definition 3.4. We call the number $\rho_{A,K}(\lambda_j)$ the *Robin constant for K in the direction λ_j*.

Since, by construction, $z/z_1 \to \lambda_j$ if and only if $|z_1| \to \infty$ and $z \in D_j$, we have

$$\rho_{A,K}(\lambda_j) = \lim_{\substack{|z_1|\to\infty \\ z/z_1\to\lambda_j \\ z\in A}} (V_K(z) - \log|z_1|). \tag{3.3}$$

Lemma 3.5. *Suppose $K_1 \supset K_2 \supset \cdots$ is a sequence of compact subsets of A with $K = \bigcap_n K_n$ and $\lim_{n\to\infty} V_{K_n} = V_K$. Let λ be a direction of A. Then $\lim_{n\to\infty} \rho_{A,K_n}(\lambda) = \rho_{A,K}(\lambda)$.*

Proof. In a neighborhood of the origin in $\mathbb{C}$ one can construct harmonic functions h_n and h using V_{K_n} and V_K, as in (3.2). It is easy to see (e.g. by Harnack's theorem) that $h_n(0) \nearrow h(0)$ as $n \to \infty$, and this implies the conclusion. ∎

4. A polynomial formula for directional Robin constants. Let A be an irreducible algebraic curve that satisfies the condition $(*)$ in the previous section, and let λ be one of the directions of A. The condition that λ is a direction of A may be rephrased in terms of projective space: embed $\mathbb{C}^N$ into $\mathbb{CP}^N$ via the usual map

$$z = (z_1, \ldots, z_N) \hookrightarrow [1 : z_1 : \cdots : z_N] = [1 : z] = Z, \tag{4.1}$$

where $Z = [Z_0 : Z_1 : \cdots : Z_N]$ denotes homogeneous coordinates and $H_\infty := \{Z_0 = 0\}$ is the hyperplane at infinity. Let us continue to denote by A the closure of A in $\mathbb{CP}^N$; then $\lambda = (1, \lambda_2, \ldots, \lambda_N)$ is a direction of A if and only if

$$[0 : \lambda] = [0 : 1 : \lambda_2 : \cdots : \lambda_N] \in A \cap H_\infty.$$

Given $\epsilon > 0$, choose $R > 0$ as in the previous section such that $A \setminus \overline{B} = D_1 \cup \cdots \cup D_d$, where as in Lemma 3.2 the D_j's are disjoint and $D_j \subset (L_j)^\epsilon$.

We recall some standard notation.

Notation 4.1.

(1) Given a multi-index $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{Z}_+^N$, write $z^\alpha = z_1^{\alpha_1} \cdots z_N^{\alpha_N}$ and $|\alpha| = \alpha_1 + \cdots + \alpha_N$.
(2) For a polynomial $p(z) = \sum_{|\alpha|\le m} a_\alpha z^\alpha$, write $\widehat{p}(z) = \sum_{|\alpha|=m} a_\alpha z^\alpha$ for its leading homogeneous part.
(3) Write $\|p\|_K = \sup_K |p(z)|$ for the sup norm of p on the compact set K.

The aim of this section is to prove the following formula.

Theorem 4.2. *Let $A \subset \mathbb{C}^N$ be an irreducible algebraic curve that satisfies $(*)$, and let $K \subset A$ be a compact subset that is non-pluripolar on A. Let*

$\lambda = (1, \lambda_2, \ldots, \lambda_N)$ *be a direction of* A. *Then*

$$\rho_{A,K}(\lambda) = \sup\left\{\tfrac{1}{\deg p} \log |\widehat{p}(\lambda)| : p \text{ is a polynomial, } \|p\|_K \leq 1\right\}. \tag{4.2}$$

Proof. Let P be a polynomial with $\|P\|_K \leq 1$. Then on A we have $\frac{1}{\deg P} \log |P| \in \mathcal{L}(A, K)$ so that $\frac{1}{\deg P} \log |P| \leq V_{A,K}$. Fix a direction λ of A. We have

$$\begin{aligned} \tfrac{1}{\deg P} \log |\widehat{P}(\lambda)| &= \lim_{\substack{t\to 0\\ z\to\lambda}} \left(\tfrac{1}{\deg P} \log |P(z/t)| + \log |t|\right) \\ &\leq \lim_{\substack{(t,z)\to(0,\lambda)\\ [t:z]\in A}} (V_{A,K}(z/t) + \log |t|) = \rho_{A,K}(\lambda). \end{aligned}$$

Since P was arbitrary, we see that

$$\sup\left\{\tfrac{1}{\deg P} |\widehat{P}(\lambda)| : P \text{ a polynomial in } \mathbb{C}^N \text{ with } \|P\|_K \leq 1\right\} \leq \rho_{A,K}(\lambda). \tag{4.3}$$

We will end the proof for now and complete it at the end of the section. ∎

We now review some basic results in classical pluripotential theory concerning the *Lelong class* $\mathcal{L}$ of global psh functions, given by $\mathcal{L} \equiv \mathcal{L}(\mathbb{C}^N)$ as in (2.1).

Let $R \subset \mathbb{C}^N$ be a regular compact set. This means that R is non-pluripolar and that its Siciak–Zaharjuta extremal function

$$V_R(z) := \sup\{u(z) : u \in \mathcal{L},\ u \leq 0 \text{ on } R\} \tag{4.4}$$

is a continuous function in the class $\mathcal{L}$.

The *Robin function* $\rho_R : \mathbb{C}^N \setminus \{0\} \to [-\infty, \infty)$ of R is defined by

$$\rho_R(z) := \limsup_{|\lambda|\to\infty} (V_R(\lambda z) - \log |\lambda|).$$

It is easy to verify that ρ_R is logarithmically homogeneous, i.e.,

$$\rho_R(\lambda z) = \rho_R(z) + \log |\lambda|, \qquad \lambda \in \mathbb{C}.$$

The following proposition is a consequence of results of Bedford and Taylor [2]. (See also [3, Corollaries 4.4 & 4.6].)

Proposition 4.3. *Let* $R \subset \mathbb{C}^N$ *be a regular compact set. Then* ρ_R *is continuous on* $\mathbb{C}^N \setminus \{0\}$ *and*

$$\rho_R(z) = \lim_{|\lambda|\to\infty} (V_R(\lambda z) - \log |\lambda|), \tag{4.5}$$

i.e., the limit exists. Moreover, the limit is uniform on the sphere $\{|z| = 1\}$, *i.e.,* $|V_R(\lambda z) - \log |\lambda| - \rho_R(z)| \leq \epsilon(\lambda)$, *where the quantity* $\epsilon(\lambda)$ *is independent of* z *and* $\epsilon(\lambda) \to 0$ *as* $|\lambda| \to \infty$. ∎

Let (t, z) denote coordinates in $\mathbb{C}^{N+1}$ where $t \in \mathbb{C}$ and $z \in \mathbb{C}^N$. Following Siciak, define the function $h_R : \mathbb{C}^{N+1} \to \mathbb{R}$ by

$$(4.6)\qquad h_R(t,z)=\begin{cases}|t|e^{V_R(z/t)} & \text{if } t\neq 0,\\ e^{\rho_R(z)} & \text{if } t=0,\ z\neq 0,\\ 0 & \text{if } (t,z)=(0,0).\end{cases}$$

It is easy to see that h_R is homogeneous, i.e., $h_R(\lambda t,\lambda z)=|\lambda|h_R(t,z)$ for all $\lambda\in\mathbb{C}$. As a consequence of Proposition 4.3, one can also verify that h_R is a non-negative, continuous psh function on $\mathbb{C}^{N+1}$ that satisfies $h_R^{-1}(0)=\{(0,0)\}$. The following is proved in [12] (see also [6, Theorem 5.1.6]).

THEOREM 4.4. *Let $R\subset\mathbb{C}^N$ be a regular compact set. With h_R defined as in* (4.6), *we have*

$$h_R(t,z)=\sup\{|Q(t,z)|^{1/\deg Q} : Q \textit{ is a homogeneous polynomial with } |Q|^{1/\deg Q}\le h_R\}. \blacksquare$$

COROLLARY 4.5. *Let $R\subset\mathbb{C}^N$ be a regular compact set. Then*

$$(4.7)\qquad \rho_R(z)=\sup\Big\{\frac{1}{\deg p}\log|\widehat{p}(z)| : p \textit{ is a polynomial with } \|p\|_R\le 1\Big\}.$$

Sketch of proof. The result is well-known, so we will only sketch a proof (2). Using continuity of h_R at points of the form $(0,z)$ with $z\neq 0$, one shows that

$$e^{\rho_R(z)}=\sup\{|Q(0,z)|^{1/\deg Q} : Q \text{ is a homogeneous polynomial with } |Q|^{1/\deg Q}\le h_R\}.$$

The desired formula (4.7) is essentially the logarithm of the above equation. To see this, suppose $p(z):=Q(1,z)$. If $\deg p=\deg Q$ then it is easy to verify that $\widehat{p}(z)=Q(0,z)$ and $\|p\|_R\le 1$. $\blacksquare$

Recall that an algebraic variety $W\subset\mathbb{C}^N$ is said to be *homogeneous* if $z\in W$ implies $\lambda z\in W$ for all $\lambda\in\mathbb{C}$. Equivalently, there are a finite number of homogeneous polynomials $p_1,\dots,p_m$ such that

$$W=\{z\in\mathbb{C}^N:\ p_1(z)=\dots=p_m(z)=0\}.$$

PROPOSITION 4.6. *Let $W\subset\mathbb{C}^N$ be a homogeneous algebraic variety and $u:W\to[-\infty,\infty)$ a psh function on W that is continuous on $W\setminus\{0\}$ and logarithmically homogeneous, i.e., $u(\lambda z)=u(z)+\log|\lambda|$ for all $\lambda\neq 0$. Let $Z:=\{z\in W : u(z)\le 0\}$. Then $u^+(z):=\max\{u(z),0\}=V_{W,Z}(z)$ for all $z\in W$.*

Proof. Clearly $V_{W,Z}=u^+$ on Z, so take $z\in W\setminus Z$. Then $\varphi(\lambda):=V_{W,Z}(\lambda z)-u^+(\lambda z)$ defines a bounded subharmonic function on the open set $\Omega=\{\lambda\in\mathbb{C}:\ \lambda z\notin Z\}$ with $\lim_{\lambda\to\zeta}\varphi(\zeta)=0$ for all $\zeta\in\partial\Omega$. Hence by the maximum principle, $\varphi\le 0$ on Ω; in particular $\varphi(1)\le 0$, so $V_{W,Z}(z)\le u^+(z)$.

(2) The result is a simple consequence of [14, Theorem 2].

On the other hand, $V_{W,Z}(z) \geq u^+(z)$ is immediate since $u^+ \in \mathcal{L}(W, Z)$. The result follows. ∎

Let $W \subset \mathbb{C}^N$ be an algebraic variety. Recall that a compact set $K \subset W$ is *W-regular* if $V_{W,K}$ is continuous.

LEMMA 4.7. *Let $W \subset \mathbb{C}^N$ be an algebraic variety and let K_1, K_2 be compact sets with $K_1 \subset K_2 \subset W$. If K_1 is W-regular then $\|V_{K_1} - V_{K_2}\|_W \leq \|V_{K_1}\|_{K_2}$.*

Proof. Define $u\colon W \to (-\infty, \infty]$ by $u(z) := V_{W,K_1}(z) - \|V_{K_1}\|_{K_2}$. Since V_{W,K_1} is continuous, $V_{W,K_1} \in \mathcal{L}(W)$ and hence $u \in \mathcal{L}(W, K_2)$, so that $u \leq V_{W,K_2}$. On the other hand, $V_{K_2} \leq V_{K_1}$ on W since $\mathcal{L}(W, K_2) \subset \mathcal{L}(W, K_1)$. The result follows. ∎

COROLLARY 4.8. *Let $K \subset W$ be a W-regular compact set. Given $\delta > 0$ define $K_W^\delta := K^\delta \cap W$ (where $K^\delta \subset \mathbb{C}^N$ is as defined in (2.5)). Then $\|V_K - V_{K_W^\delta}\|_W \to 0$ as $\delta \to 0$.* ∎

To compare an extremal function in $\mathbb{C}^N$ with an extremal function on an algebraic variety $W \subset \mathbb{C}^N$, we will use an extension result proved in [5]. For this we need a stronger notion than weak plurisubharmonicity.

DEFINITION 4.9. A function $u : W \to [-\infty, \infty)$ is *plurisubharmonic* (*psh*) *at* $z \in W$ if there exists a neighborhood U of z in $\mathbb{C}^N$ and a psh function $\tilde{u} : U \to [-\infty, \infty)$ such that $\tilde{u}|_{U \cap W} = u$. If u is psh at each $z \in W$ then u is said to be *psh on* W.

REMARK 4.10. Obviously if u is psh on W then it is weakly psh on W. Observe also that a weakly psh function is psh at each point of $\mathrm{reg}(A)$: if W is of pure dimension m then one can make a local (holomorphic) change of coordinates at a regular point a so that a is the origin and W is the hyperplane given by $z_{m+1} = \cdots = z_N = 0$, and we may define $\tilde{u}(z) := u(z_1, \ldots, z_m, 0, \ldots, 0)$.

PROPOSITION 4.11 (cf. [5, Proposition 3.1]). *Let W be an algebraic variety in $\mathbb{C}^N$, which we extend projectively to $\overline{W} \subset \mathbb{CP}^N = \mathbb{C}^N \cup H_\infty$. Suppose for all $a \in \overline{W} \cap H_\infty$ that the germ of $\overline{W}$ at a is irreducible. Suppose $u \in \mathcal{L}(W)$ is psh at each point of W. Then there exists $v \in \mathcal{L}(\mathbb{C}^N)$ such that $v|_W \equiv u$.* ∎

In this proposition $\mathbb{C}^N$ is extended to $\mathbb{CP}^N$ via the usual embedding $\mathbb{C}^N \hookrightarrow \mathbb{CP}^N$ given by (4.1).

We only need a special case of the previous result. Let $A \subset \mathbb{C}^N$ be an algebraic curve of degree d with d distinct directions $\lambda_1, \ldots, \lambda_d$. Then extending A projectively, we obtain $A \cap H_\infty = \{\lambda_j\}_{j=1}^d$. These are all regular points of A since by Bézout's theorem they must intersect H_∞ with multiplicity one. Hence the germ of A at each of these points is irreducible.

Corollary 4.12. *Suppose A is an algebraic curve of degree d with d distinct directions. Then for every $u \in \mathcal{L}(A)$ that is psh on A there exists $v \in \mathcal{L}(\mathbb{C}^N)$ such that $v|_A \equiv u$.* ∎

Suppose $K \subset A$ is an A-regular compact set that covers the singular points of A. Precisely, if $a \in A$ is a singular point then there is a neighborhood U of a in $\mathbb{C}^N$ such that $(U \cap A) \subseteq K$. Equivalently,

$$\overline{A \setminus K} \subseteq \operatorname{reg}(A) \quad \text{(where we take the usual closure in } \mathbb{C}^N). \tag{4.8}$$

This guarantees that V_K is continuous on A, and hence weakly psh. It is easy to see that V_K is in fact psh on A: it extends locally at each regular point (see the previous remark), and extends locally at each singular point using the zero function.

Lemma 4.13. *Suppose $K \subset A$ is compact and A-regular, and $\overline{A \setminus K} \subset \operatorname{reg}(A)$. Then*

$$\|V_K - V_{K^\epsilon}\|_A \to 0 \quad \text{as } \epsilon \to 0. \tag{4.9}$$

Proof. Let $\eta > 0$. Using Corollary 4.8, choose $\delta \in (0, \eta)$ with $\|V_K - V_{K_A^\delta}\|_A < \eta$, where $K_A^\delta = K^\delta \cap A$. By Lemma 2.10, we can find an A-regular set $L^\delta \supset K_A^\delta$ for which $\|V_K - V_{L^\delta}\|_A < 2\eta$.

Now, L^δ satisfies (4.8), so by the previous paragraph, V_{L^δ} is psh on A. Let u_δ be the extension to $\mathcal{L}(\mathbb{C}^N)$ of V_{A,L^δ} given by Corollary 4.12. For all $z \in K$, the set $\Omega_\delta := \{z \in \mathbb{C}^N : u_\delta(z) < \delta\}$ is an open neighborhood of K in $\mathbb{C}^N$, since u_δ is usc. Hence we can find $\epsilon_0 \in (0, \delta)$ such that for all $\epsilon \in (0, \epsilon_0)$, $K^\epsilon \subset \Omega_\delta$; it follows that $u_\delta(z) - \delta \in \mathcal{L}(\mathbb{C}^N, K^\epsilon)$ and thus $u_\delta(z) - \delta \le V_{K^\epsilon}(z)$ for all $z \in \mathbb{C}^N$. When $z \in A$ this means that

$$V_{A,L^\delta}(z) - \delta \le V_{K^\epsilon}(z) \le V_K(z),$$

so that $0 \le V_K(z) - V_{K^\epsilon}(z) \le V_K(z) - V_{A,L^\delta}(z) + \delta$. Hence

$$\|V_K - V_{K^\epsilon}\|_A \le \|V_K - V_{A,L^\delta}\|_A + \delta \le 3\eta \quad \text{if } \epsilon \in (0, \epsilon_0).$$

As $\eta > 0$ is arbitrary, the right-hand side can be made arbitarily small with an appropriately chosen ϵ. ∎

Remark 4.14. Classical pluripotential theory already gives the pointwise convergence $V_{K^\delta}(z) \nearrow V_K(z)$ for all $z \in \mathbb{C}^N$, but it is not uniform since it includes convergence to $+\infty$ for points $z \in \mathbb{C}^N \setminus A$.

Define $A_h \subset \mathbb{C}^{N+1}$ as the closure in $\mathbb{C}^{N+1}$ of the set $\{(t, tz) \in \mathbb{C}^{N+1} : z \in A\}$. It is easy to see that A_h is a homogeneous variety, and

$$(t, w) \in A_h \setminus \{(0, 0)\} \quad \text{if and only if} \quad [t : w] \in A$$

in homogeneous coordinates (cf. (4.1)). In addition, A_h is irreducible if and only if A is irreducible. If λ is a direction of A and $K \subset A$ is a compact

A-regular set, then by (3.3),

$$\rho_K(\lambda) = \lim_{\substack{(t,z)\to(0,\lambda) \\ (t,z)\in A_h}} (V_K(z/t) + \log|t|). \tag{4.10}$$

Lemma 4.15. *Let $K \subset A$ be an A-regular compact set that satisfies* (4.8)*, where $A \subset \mathbb{C}^N$ is an irreducible algebraic variety. Given $\delta > 0$, define the function $h_{K^\delta} : \mathbb{C}^{N+1} \to [0,\infty)$ as in* (4.6)*, replacing R by K^δ. Define $h_K : A_h \to [0,\infty)$ similarly by*

$$h_K(t,z) = \begin{cases} |t|e^{V_K(z/t)} & \text{if } t \neq 0 \text{ and } (t,z) \in A_h, \\ e^{\rho_{A,K}(z)} & \text{if } t = 0, z \neq 0 \text{ and } (0,z) \in A_h, \\ 0 & \text{if } (t,z) = (0,0). \end{cases}$$

Then $h_{K^\delta} \nearrow h_K$ on A_h as $\delta \searrow 0$.

Proof. The conclusion is obvious at $(0,0)$, and when $(t,z) \in A_h$ with $t \neq 0$ it follows from the convergence $V_{K^\delta} \nearrow V_K$.

When $t = 0$, $z \neq 0$, we need to show that $\rho_{K^\delta}(z) \nearrow \rho_{A,K}(z)$ as $\delta \searrow 0$. Fix $z \neq 0$ with $(0,z) \in A_h$ and let $\epsilon > 0$. Then by Lemma 3.3 and Proposition 4.3 we can choose λ sufficiently large and $\tilde{z}$ sufficiently close to z such that $\lambda\tilde{z} \in A$ and

$$|V_{K^\delta}(\lambda\tilde{z}) - \log|\lambda| - \rho_{K^\delta}(z)| < \epsilon, \qquad |V_K(\lambda\tilde{z}) - \log|\lambda| - \rho_{A,K}(z)| < \epsilon.$$

By Lemma 4.13,

$$|\rho_{A,K}(z) - \rho_{K^\delta,A}(z)| \le |V_K(\lambda\tilde{z}) - V_{K^\delta}(\lambda\tilde{z})| + 2\epsilon \le \|V_K - V_{K^\delta}\|_A + 2\epsilon.$$

Since ϵ was arbitrary, $|\rho_K(z) - \rho_{K^\delta}(z)| \le \|V_K - V_{K^\delta}\|_A$, and by Lemma 4.13 again, this goes to zero as $\delta \to 0$. Since ρ_{K^δ} is monotone in δ, $\rho_{K^\delta} \nearrow \rho_K$. ∎

Corollary 4.16. *Under the hypotheses of the previous lemma, the convergence $h_{K^\delta} \nearrow h_K$ as $\delta \searrow 0$ is uniform on some open neighborhood in A_h of the set $\{(t,z) \in A_h : \ t = 0, z_1 = 1\}$.*

Proof. If $z \neq 0$ then $(0,z) \in A_h$ if and only if $[0:z] \in A \cap H_\infty$, so

$$S := \{(t,z) = (t, z_1, \ldots, z_N) \in A_h : t = 0,\ z_1 = 1\}$$

is a finite set. Hence it is contained in a bounded open set D (in A_h). Since K is A-regular, the continuity of $V_{A,K}$ together with equation (4.10) shows that h_K is continuous on $A_h \setminus \{(0,0)\}$. Hence the convergence $h_{K^\delta} \nearrow h_K$ is uniform on $\overline{D}$ by Dini's theorem. ∎

Proposition 4.17. *Suppose $K \subset A$ is an A-regular set satisfying* (4.8)*. Then*

$$\rho_{A,K}(\lambda) \le \sup\Big\{\frac{1}{\deg P}|\widehat{P}(\lambda)| : \ P \text{ is a polynomial in } \mathbb{C}^N \text{ with } \|P\|_K \le 1\Big\}. \tag{4.11}$$

Proof. Given $\delta > 0$, define K^δ as in (2.5). Then K^δ is regular, so by Theorem 4.4,

$$h_{K^\delta}(t,z) = \sup\{|Q(t,z)|^{1/\deg Q} : Q \text{ is a homogeneous polynomial in } \mathbb{C}^{N+1} \text{ with } |Q|^{1/\deg Q} \le h_{K^\delta}\}. \tag{4.12}$$

Let Q be a homogeneous polynomial in $\mathbb{C}^{N+1}$ for which $|Q|^{1/\deg Q} \le h_{K^\delta}$. Define the polynomial P on $\mathbb{C}^N$ by $P(z) := Q(1,z)$. If $z \in K$ then

$$|P(z)| = |Q(1,z)| \le (h_{K^\delta}(1,z))^{\deg Q} \le (h_K(1,z))^{\deg Q} = e^{V_K(z)\deg Q} = 1.$$

Hence $\|P\|_K \le 1$. Also, we have $\deg P \le \deg Q$, and $Q(0,z) = 0$ if $\deg Q > \deg P$; otherwise, if $\deg Q = \deg P$ then $Q(0,z) = \widehat{P}(z)$. Equation (4.12) implies, for all $z \in \mathbb{C}^N$ and $\delta > 0$, that

$$h_{K^\delta}(0,z) \le \sup\{|\widehat{P}(z)|^{1/\deg P} : P \text{ is a polynomial in } \mathbb{C}^N \text{ with } \|P\|_K \le 1\}.$$

Now take $z = \lambda$ where $\lambda = (1, \lambda_2, \ldots, \lambda_N)$ is a direction of A. Then $(0,\lambda) \in A_h$, so applying Corollary 4.16 we get

$$\begin{aligned} e^{\rho_{A,K}(\lambda)} &= h_K(0,\lambda) = \lim_{\delta\to 0} h_{K^\delta}(0,\lambda) \\ &\le \sup\{|\widehat{P}(\lambda)|^{1/\deg P} : P \text{ is a polynomial in } \mathbb{C}^N \text{ with } \|P\|_K \le 1\}. \end{aligned}$$

Equation (4.11) follows upon taking logarithms. ∎

The above proposition together with (4.3) yields Theorem 4.2 for A-regular sets that cover singular points. The general case will follow by approximation.

End of the proof of Theorem 4.2. Let $L := K \cup \operatorname{sing}(A)$. Then L is a compact set such that $V_{A,K}(z) = V_{A,L}(z)$ for all $z \notin L$ (using Proposition 2.5(4)), and hence $\rho_{A,L}(\lambda) = \rho_{A,K}(\lambda)$. Next, by Lemma 2.10 we can find a sequence $L_1 \supset L_2 \supset \cdots$ of A-regular sets with $\bigcap_n L_n = L$. Then for each n, we have

$$\begin{aligned} \rho_{A,L_n}(\lambda) &\le \sup\Big\{\tfrac{1}{\deg p}\log|\widehat{p}(\lambda)| : P \text{ is a polynomial in } \mathbb{C}^N \text{ with } \|P\|_{L_n} \le 1\Big\} \\ &\le \sup\Big\{\tfrac{1}{\deg p}\log|\widehat{p}(\lambda)| : P \text{ is a polynomial in } \mathbb{C}^N \text{ with } \|P\|_K \le 1\Big\} \\ &\le \rho_{A,K}(\lambda), \end{aligned}$$

where the first inequality uses (4.11) and the last inequality uses (4.3). Letting $n \to \infty$, we have $\rho_{A,L_n}(\lambda) \nearrow \rho_{A,L}(\lambda) = \rho_{A,K}(\lambda)$ on the left-hand side by Lemma 3.5. Hence the last inequality is an equality, proving the theorem. ∎

5. Directional Chebyshev constants. In this section we use Theorem 4.2 to relate directional Robin constants to directional Chebyshev constants. Throughout this section $A \subset \mathbb{C}^N$ is an irreducible algebraic curve that satisfies condition $(*)$ in Section 3.

Directional Chebyshev constants for a compact set $K \subset A$ were studied in [1]. We recall the basic notions and results. For a positive integer n we will denote by $D(n)$ an unspecified polynomial of degree $\leq n$.

Consider the factor ring of polynomials on A; polynomials p and q are considered to be equivalent if $p(z) = q(z)$ for all $z \in A$. For each equivalence class, one can construct a standard representative by a generalized division algorithm; this polynomial is called a *normal form*. The collection of normal forms is denoted by $\mathbb{C}[A]$.

Recall also the notation $\widehat{p}$ from the previous section (Notation 4.1).

PROPOSITION 5.1 (see [1, Section 4]). *Let $\lambda = (1, \lambda_2, \ldots, \lambda_N)$ be a direction of A. Then there is a unique polynomial $\mathbf{v}_\lambda \in \mathbb{C}[A]$ of minimal degree such that:*

(1) $\mathbf{v}_\lambda(\lambda) = 1$ *and* $\mathbf{v}_\lambda(\tilde{\lambda}) = 0$ *for any other direction* $\tilde{\lambda} \neq \lambda$.
(2) *For any polynomial p in $\mathbb{C}^N$,*

$$p(z)\mathbf{v}_\lambda(z) = \widehat{p}(\lambda) z_1^{\deg p} \mathbf{v}_\lambda(z) + D(\deg p + \deg \mathbf{v}_\lambda - 1).$$

If w is any other polynomial with the above properties, then $w(z) = z_1^a \mathbf{v}_\lambda(z)$ where $a = \deg w - \deg \mathbf{v}_\lambda$. ∎

DEFINITION 5.2. Given a direction λ of A, we call $\mathbf{v}_\lambda$ the *minimal directional polynomial for the direction* λ. We also define for a positive integer n the directional polynomial

$$\mathbf{v}_{\lambda,n}(z) := z_1^{n - \deg \mathbf{v}_\lambda} \mathbf{v}_\lambda(z)$$

of degree n.

Directional Chebyshev constants are defined in terms of directional polynomials:

DEFINITION 5.3. Suppose $K \subset A$ is compact and λ is a direction of A. Define

$$T_n(K, \lambda) := \inf\{\|p\|_K : p = \mathbf{v}_{\lambda,n}(z) + D(n-1)\}^{1/n},$$
$$\tau(K, \lambda) := \limsup_{n\to\infty} T_n(K, \lambda).$$

We call a polynomial of degree n which attains the infimum in the definition of $T_n(K, \lambda)$ a *Chebyshev polynomial of degree n in the direction* λ, and $T_n(K, \lambda)$ itself is the *directional Chebyshev constant of K of order n in the direction* λ. Finally $\tau(K, \lambda)$ is the *directional Chebyshev constant of K for the direction* λ.

It was proved in [1] that

$$\tau(K, \lambda) = \lim_{n\to\infty} T_n(K, \lambda), \tag{5.1}$$

i.e., the lim sup in Definition 5.3 may be replaced by the limit.

For a compact set K let

$$\Lambda_n(z,K) := \sup\{|\widehat{q}(z)| : \|q\|_K \le 1,\ \deg q \le n\}.$$

An easy consequence of Theorem 4.2 is the following:

LEMMA 5.4. *Let $K \subset A$ be compact. Then $e^{\rho_{A,K}(\lambda)} = \lim_{n\to\infty} \Lambda_n(\lambda,K)^{1/n}$ for any direction $\lambda = (1,\lambda_2,\ldots,\lambda_N)$ of A.* ∎

We relate $\Lambda_n(z,K)$ to directional Chebyshev constants:

LEMMA 5.5. *Suppose that $K \subset A$ is compact and non-pluripolar in A, and suppose that λ is a direction of A. Then*

$$\Lambda_n(\lambda,K) \le \|\mathbf{v}_\lambda\|_K T_{n+\deg \mathbf{v}_\lambda}(K,\lambda)^{-(n+\deg \mathbf{v}_\lambda)}$$

for a sufficiently large positive integer n.

Proof. Let $\{p_j\}$ be a sequence of polynomials satisfying $\|p_j\|_K \le 1$, $\deg p_j = n$, and $\lim_{j\to\infty} |\widehat{p}_j(\lambda)| = \Lambda_n(\lambda,K)$. Since K is not pluripolar in A, it follows that $\Lambda_n(\lambda,K)^{1/n} \to e^{\rho_K(\lambda)} \ne 0$ as $n\to\infty$. When n and j are sufficiently large, then $|\widehat{p}_j(\lambda)| \ne 0$ and

$$\frac{\mathbf{v}_\lambda(z)p_j(z)}{|\widehat{p}_j(\lambda)|} = \mathbf{v}_{\lambda,n+\deg \mathbf{v}_\lambda}(z) + D(n-1),$$

by property (2) of $\mathbf{v}_\lambda$ in Proposition 5.1, and this implies that

$$T_{n+\deg \mathbf{v}_\lambda}(K,\lambda)^{n+\deg \mathbf{v}_\lambda} \le \frac{\|\mathbf{v}_\lambda p_j\|_K}{|\widehat{p}_j(\lambda)|} \le \frac{\|\mathbf{v}_\lambda\|_K}{|\widehat{p}_j(\lambda)|}.$$

We note that this inequality holds for each member of the sequence $\{p_j\}_{j\in\mathbb{N}}$ and so we may take the limit on the right-hand side as $j\to\infty$. This gives

$$T_{n+\deg \mathbf{v}_\lambda}(K,\lambda)^{n+\deg \mathbf{v}_\lambda} \le \frac{\|\mathbf{v}_\lambda\|_K}{\Lambda_n(\lambda,K)}.$$

Hence $\Lambda_n(\lambda,K) \le \|\mathbf{v}_\lambda\|_K T_{n+\deg \mathbf{v}_\lambda}(K,\lambda)^{-(n+\deg \mathbf{v}_\lambda)}$. ∎

LEMMA 5.6. *Suppose that $K \subset A$ is compact and non-pluripolar, and suppose that λ is a direction of the curve A. Then $T_n(K,\lambda)^{-n} \le \Lambda_n(\lambda,K)$.*

Proof. Let p be a Chebyshev polynomial of degree n for K in the direction λ. We note that this means that $\widehat{p}(z) = \mathbf{v}_{\lambda,n}(z)$ and $\|p\|_K = T_n(K,\lambda)^n$. Let $q(z) = p(z)/\|p\|_K$. Since $\|q\|_K = 1$ it follows that

$$T_n(K,\lambda)^{-n} = \frac{1}{\|p\|_K} = \frac{|\widehat{p}(\lambda)|}{\|p\|_K} = |\widehat{q}(\lambda)| \le \Lambda_n(\lambda,K).$$

Hence $T_n(K,\lambda)^{-n} \le \Lambda_n(\lambda,K)$. ∎

We close this section with the main theorem of the paper.

THEOREM 5.7. *Let $K \subset A$ be compact and let λ be a direction of A. Then*

$$e^{-\rho_{A,K}(\lambda)} = \tau(K,\lambda).$$

Proof. Using Lemmas 5.5 and 5.6 we have

$$\Lambda_n(\lambda, K) \le \|\mathbf{v}_\lambda\|_K T_{n+\deg \mathbf{v}_\lambda}(K,\lambda)^{-(n+\deg \mathbf{v}_\lambda)} \le \|\mathbf{v}_\lambda\|_K \Lambda_{n+\deg \mathbf{v}_\lambda}(\lambda, K).$$

Noting that $\|\mathbf{v}_\lambda\|_K$ is just a constant, taking nth roots and the limit as $n \to \infty$ gives the result by Lemma 5.4. ∎

6. Extremal curves. Let $K \subset \mathbb{C}^N$ be a regular compact set. It is known that in certain special cases the extremal function V_K has extremal curves, i.e., holomorphic curves in $\mathbb{C}^N \setminus K$ on which the restriction of V_K is harmonic. When K is the closure of a bounded strictly lineally convex domain with smooth boundary, it was shown by Lempert ([7], [8]) that through each point of $\mathbb{C}^N \setminus K$ there is an extremal curve, and the collection of these curves gives a smooth foliation of $\mathbb{C}^N \setminus K$.

When $K \subset \mathbb{R}^N \subset \mathbb{C}^N$ is a real convex body, the existence of extremal curves through each point of $\mathbb{C}^N \setminus K$ was shown in [9] for K symmetric, and in [4] for an arbitrary convex body. In both cases the extremal curves are contained in complex ellipses (algebraic curves of degree 2) whose real points are inscribed in K. Proving the existence of extremal curves for V_K is a hard problem and more general existence results are not known. All of the results just described make essential use of the fact that K is assumed to be convex or lineally convex.

In this section, we relate extremal curves to the Robin and Chebyshev constants considered earlier.

LEMMA 6.1. *Let $K \subset \mathbb{C}^N$ be a regular compact set for which $A \cap K \neq \emptyset$, where $A \subset \mathbb{C}^N$ is an irreducible algebraic curve. Then $V_K \le V_{K\cap A}$, and if $A \cap K$ is A-regular then $e^{-\rho_{A,K\cap A}(\lambda)} \le e^{-\rho_K(\lambda)}$.*

Proof. Since $K \cap A \subset K$ it follows that $V_K \le V_{K\cap A}$.

If $A \cap K$ is A-regular then $V_{A,A\cap K} \in \mathcal{L}(A)$ and

$$\begin{aligned}\rho_{A,K}(\lambda) &= \lim_{\substack{(t,z)\to(0,\lambda)\\(t,z)\in A_h}} (V_{A,K\cap A}(z/t) + \log|t|)\\ &\ge \lim_{\substack{(t,z)\to(0,\lambda)\\(t,z)\in A_h}} (V_K(z/t) + \log|t|) = \rho_K(\lambda),\end{aligned}$$

where we use Proposition 4.3 for the last equality. The conclusion follows immediately. ∎

For convenience we will use the following notation below. Given a compact set $K \subset \mathbb{C}^N$ and a variety $A \subset \mathbb{C}^N$, write

$$V_{A,K} := V_{A,K\cap A} \quad \text{and} \quad \rho_{A,K} := \rho_{A,K\cap A}.$$

Let L_λ be the linear asymptote of A in the direction λ. Choose $\epsilon > 0$ and $R > 0$ such that $D_\lambda := (L_\lambda)^\epsilon \cap (A \setminus B(R))$ is a connected manifold whose projection to the z_1-axis, $\pi : D_\lambda \to \mathbb{C}$ given by $\pi(z) = \pi(z_1, \dots, z_N) = z_1$, is one-to-one (as in Lemmas 3.2–3.3).

LEMMA 6.2. *Suppose that $K \subset \mathbb{C}^N$ is a regular compact set and that $K \cap A$ is A-regular, where A is an irreducible algebraic curve. If $\rho_K(\lambda) = \rho_{A,K}(\lambda)$ then $V_K(z) = V_{K,A}(z)$ for all $z \in D_\lambda$.*

Proof. Let $\Omega := \{0\} \cup \{s \in \mathbb{C} : \exists z \in D_\lambda \text{ such that } \pi(z) = 1/s\}$. It is easy to see that Ω is an open neighborhood of the origin. Define $u : \Omega \setminus \{0\} \to [-\infty, \infty)$ by

$$u(s) := V_K(\zeta(1/s)) - V_{A,K}(\zeta(1/s)),$$

where $\zeta : \pi(D_\lambda) \to D_\lambda$ is the local inverse of the projection π (as in the proof of Lemma 3.3). Clearly $u \le 0$. Also, u is subharmonic on $\Omega \setminus \{0\}$ since V_K is psh, $s \mapsto \zeta(1/s)$ is holomorphic, and $V_{A,K}$ is harmonic on D_λ (Theorem 2.7). Since K is regular, $V_K(\zeta(1/s)) \to \rho_K(\lambda)$ (Proposition 4.3); similarly, $V_{A,K}(\zeta(1/s)) \to \rho_{A,K}(\lambda)$ by equation (3.3). Hence u extends continuously, and therefore subharmonically, to all of Ω with $u(0) = 0$. By the maximum principle for subharmonic functions, $u \equiv 0$ on Ω. The conclusion follows easily for all $z = \zeta(1/s) \in D_\lambda$. ■

PROPOSITION 6.3. *Let A be an irreducible algebraic curve and λ a direction of A. Suppose that $K \subset \mathbb{C}^N$ is a regular compact set and $A \cap K$ is A-regular. If $\rho_K(\lambda) = \rho_{A,K}(\lambda)$ then $V_K(z) = V_{A,K}(z)$ for all z in the connected component (in* $\operatorname{reg}(A)$*) of* $\operatorname{reg}(A) \setminus K$ *that contains the direction λ (i.e., the component of* $\operatorname{reg}(A)$ *that contains $[0 : \lambda]$ when we extend A projectively to $\mathbb{CP}^N$). Hence V_K is harmonic on this component.*

Proof. At each point $z \in \operatorname{reg}(A)$, $V_{A,K}$ is harmonic (Theorem 2.7), and so the function $V_K - V_{A,K}$ is subharmonic in an open neighborhood of z in $\operatorname{reg}(A)$. Together with the fact that $V_K - V_{A,K}$ is continuous, one can use a standard argument (involving subharmonicity and the maximum principle) to show that the set $\{z \in \operatorname{reg}(A) \setminus K : V_K(z) - V_{A,K}(z) = 0\}$ is both open and closed in $\operatorname{reg}(A)$. Hence it must be a union of connected components; in particular, it contains any component of $\operatorname{reg}(A)$ that meets D_λ, i.e., the component containing the direction λ. ■

REMARK 6.4. Note that for any irreducible algebraic curve $W \subset \mathbb{C}^N$ and compact set $K \subset \mathbb{C}^N$ we have $V_{W,K} \ge V_K$ everywhere by definition (since $W \cap K \subseteq K$). Hence for any direction λ of W, we have $\rho_{W,K}(\lambda) \ge \rho_K(\lambda)$. If there exists some curve A for which $\rho_{A,K}(\lambda) = \rho_K(\lambda)$, then the minimum possible value is attained, so that $\rho_{A,K}(\lambda) = \inf_W \rho_{W,K}(\lambda)$. In addition, if A has property $(*)$ of Section 3, then by Theorem 5.7 this implies that

$$\tau_A(K \cap A, \lambda) = \sup_W \tau_W(K \cap W, \lambda). \tag{6.1}$$

where the sup is taken over all irreducible algebraic curves W that satisfy $(*)$. This gives a necessary property of extremal curves formulated in terms of Chebyshev constants.

Suppose W is an irreducible algebraic curve of degree d that satisfies property $(*)$, and $\{\lambda_1, \dots, \lambda_d\}$ are its asymptotic directions. Set

$$\tau_W(K \cap W) = \Big(\prod_{j=1}^{d} \tau_W(K \cap W, \lambda_j)\Big)^{1/d},$$

and call this quantity the *principal Chebyshev constant*.

The principal Chebyshev constant coincides with a notion of transfinite diameter studied in [1]. In that paper, a basis of monomials for the polynomials on W was constructed by Groebner basis techniques. Let us denote this basis by $\{z^{\alpha(j)}\}_{j=1}^{\infty}$, with the monomials enumerated according to some graded ordering (here $\alpha(j)$ is a multi-index for each j, and we have $\deg z^{\alpha(j)} \le \deg z^{\alpha(j+1)}$).

Write $V(\zeta_1, \dots, \zeta_\nu) = \det\big[z^{\alpha(j)}(\zeta_k)\big]_{j,k=1}^{\nu}$ for a finite set of points $\{\zeta_1, \dots, \zeta_\nu\}$, and $V_\nu = V_\nu(K \cap W) = \max\{|V(\zeta_1, \dots, \zeta_\nu)| : \zeta_j \in K \cap W, \forall j\}$. The transfinite diameter of $K \cap W$ on W is given by

$$d_W(K \cap W) = \lim_{s\to\infty} V_{m_s}^{1/l_s}$$

where m_s is the number of monomials of degree $\le s$ and where $l_s = \sum_{\nu=1}^{m_s} \nu(m_\nu - m_{\nu-1})$ is the sum of the degrees.

The main theorem of [1] states that

$$\tau_W(K \cap W) = d_W(K \cap W),$$

which is a direct analog of the classical result of Fekete–Szegö. Together with (6.1) and Proposition 6.3, this leads to the following.

Corollary 6.5. *Let $K \subset \mathbb{C}^N$ be a regular compact set. Suppose an irreducible algebraic curve A satisfies $(*)$, and also every connected component of $A \setminus K$ is an extremal curve for V_K. Then*

$$d_A(K \cap A) = \sup_W d_W(K \cap W), \tag{6.2}$$

where the supremum is taken over all irreducible algebraic curves W satisfying $(*)$.

We illustrate the above ideas on real convex bodies.

Example 6.6. The existence of extremal curves is known when K is a compact convex body in $\mathbb{R}^N \subset \mathbb{C}^N$. The main result of [4] says that extremal curves for V_K lie on complex ellipses whose real points are inscribed in K. It is also shown that such a complex ellipse, E, has a parametrization of the form

$$t \mapsto a + b/t + \bar{b}t, \qquad t \neq 0,$$

where $a \in \mathbb{R}^2$ and $b \in \mathbb{C}^2$. The points at infinity of the projective closure $\overline{E} \subset \mathbb{CP}^N$ may be computed by letting $t \to 0$ and $t \to \infty$. The real points of E, contained in K, form the real ellipse $E_{\mathbb{R}}$ given by

$$e^{i\theta} \mapsto a + 2\operatorname{Re}(b)\cos\theta + 2\operatorname{Im}(b)\sin\theta, \qquad \theta \in \mathbb{R}.$$

This shows that a is the center of the ellipse $E_{\mathbb{R}}$ (in $\mathbb{R}^2$), while the parameter b gives its eccentricity and orientation.

In the setting of Proposition 6.3 and Corollary 6.5, we have $\operatorname{reg}(E) = E$, and the real ellipse $E_{\mathbb{R}} = E \cap K$ divides E into two connected components, as can be seen immediately from the parametrization. One component contains the direction $[0 : b]$ and the other contains the direction $[0 : \bar{b}]$. These components form a pair of conjugate analytic disks on which V_K is harmonic. Hence equation (6.2) holds here with $A = E$.

We close the paper with a couple of open questions.

(1) Can the assumption of property $(*)$ in Corollary 6.5 be dropped? ([3])
(2) Let $d(K)$ denote the classical Fekete–Leja transfinite diameter of K in $\mathbb{C}^N$. If A is as in Corollary 6.5, and every connected component of $A \setminus K$ is an extremal curve, are the transfinite diameters $d_A(K \cap A)$ and $d(K)$ related?

Acknowledgements. The second author was partly supported by University of Auckland grant no. 3704154.

References

[1] W. Baleikorocau and S. Ma‘u, *Chebyshev constants, transfinite diameter, and computation on algebraic curves*, Comput. Methods Funct. Theory 15 (2015), 291–322.
[2] E. Bedford and B. A. Taylor, *Plurisubharmonic functions with logarithmic singularities*, Ann. Inst. Fourier (Grenoble) 38 (1988), no. 4, 133–171.
[3] T. Bloom, N. Levenberg, and S. Ma‘u, *Robin functions and extremal functions*, Ann. Polon. Math. 80 (2003), 55–84.
[4] D. Burns, N. Levenberg, and S. Ma‘u, *Pluripotential theory for convex bodies in* $\mathbb{R}^n$, Math. Z. 250 (2005), 91–111.
[5] D. Coman, V. Guedj, and A. Zeriahi, *Extension of plurisubharmonic functions with growth control*, J. Reine Angew. Math. 676 (2013), 33–49.
[6] M. Klimek, *Pluripotential Theory*, Oxford Univ. Press, Oxford, 1991.
[7] L. Lempert, *La métrique de Kobayashi et la représentation des domaines sur la boule*, Bull. Soc. Math. France 109 (1981), 427–424.

([3]) Note that transfinite diameter may be defined on a curve without reference to property $(*)$.

[8] L. Lempert, *Symmetries and other transformations of the complex Monge–Ampère equation*, Duke Math. J. 52 (1985), 869–885.

[9] M. Lundin, *The extremal PSH for the complement of convex, symmetric subsets of* $\mathbb{R}^N$, Michigan Math. J. 32 (1985), 197–201.

[10] T. Ransford, *Potential Theory in the Complex Plane*, Cambridge Univ. Press, Cambridge, 1995.

[11] A. Sadullaev, *Estimates of polynomials on analytic sets*, Izv. Akad. Nauk SSSR Ser. Mat. 46 (1982), 524–534 (in Russian).

[12] J. Siciak, *Extremal plurisubharmonic functions in* $\mathbb{C}^N$, Ann. Polon. Math. 39 (1981), 175–211.

[13] M. Tsuji, *Potential Theory in Modern Function Theory*, Maruzen, Tokyo, 1959.

[14] A. Zeriahi, *Capacité, constante de Čebyšev et polynômes orthogonaux associés à un compact de* $\mathbb{C}^n$, Bull. Sci. Math. (2) 109 (1985), 325–335.

[15] A. Zeriahi, *Pluricomplex Green functions and approximation of holomorphic functions*, in: Complex Analysis, Harmonic Analysis and Applications (Bordeaux, 1995), Pitman Res. Notes Math. Ser. 347, Longman, Harlow, 1996, 104–142.

Jesse Hart, Sione Ma‘u
Department of Mathematics
University of Auckland
Auckland, New Zealand
E-mail: jesse.hart@auckland.ac.nz
s.mau@auckland.ac.nz

Received 24.10.2014
and in final form 20.7.2015 (3527)