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On the lattice of polynomials with integer coefficients: the covering radius in $L_p(0,1)$

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Abstract. The paper deals with the approximation by polynomials with integer coefficients in $L_p(0,1)$, $1 \le p \le \infty$. Let $\boldsymbol{P}_{n,r}$ be the space of polynomials of degree $\le n$ which are divisible by the polynomial $x^r(1-x)^r$, $r \ge 0$, and let $\boldsymbol{P}^{\mathbb{Z}}_{n,r} \subset \boldsymbol{P}_{n,r}$ be the set of polynomials with integer coefficients. Let $\mu(\boldsymbol{P}^{\mathbb{Z}}_{n,r}; L_p)$ be the maximal distance of elements of $\boldsymbol{P}_{n,r}$ from $\boldsymbol{P}^{\mathbb{Z}}_{n,r}$ in $L_p(0,1)$. We give rather precise quantitative estimates of $\mu(\boldsymbol{P}^{\mathbb{Z}}_{n,r}; L_2)$ for $n \gtrsim 6r$. Then we obtain similar, somewhat less precise, estimates of $\mu(\boldsymbol{P}^{\mathbb{Z}}_{n,r}; L_p)$ for $p \ne 2$. It follows that $\mu(\boldsymbol{P}^{\mathbb{Z}}_{n,r}; L_p) \asymp n^{-2r-2/p}$ as $n \to \infty$. The results partially improve those of Trigub [Izv. Akad. Nauk SSSR Ser. Mat. 26 (1962)].

1. Introduction. Notation and results. In the present paper we consider the following question: how well can a polynomial of degree $\le n$ be approximated in $L_p(0,1)$ by integer polynomials of degree $\le n$? By an *integer polynomial* we mean a polynomial with integer coefficients. For the first time this question (in the more general case of $L_p(a,b)$, $b-a<4$) appeared in the papers by Aparicio [A] and Gel'fond [G].

Let X be a real normed space. By a *lattice* in X we mean a non-zero finite-dimensional discrete additive subgroup of X. Every lattice $\varLambda$ in X may be represented in the form

$$\varLambda = \{k_1x_1 + \cdots + k_nx_n : k_1,\ldots,k_n \in \mathbb{Z}\},$$

where $n = \dim \operatorname{span} \varLambda$ and $x_1,\ldots,x_n$ is a system of linearly independent vectors; any such system is then called a *basis* of $\varLambda$.

Let $\varLambda$ be a lattice in X. We denote by $\mu(\varLambda; X)$ the *covering radius* of $\varLambda$:

$$\mu(\varLambda; X) := \max\{d(x,\varLambda) : x \in \operatorname{span} \varLambda\},$$

where $d(x,\varLambda)$ is the distance of x from $\varLambda$. In other words, when we approximate vectors in $\operatorname{span}\varLambda$ by elements of $\varLambda$, then $\mu(\varLambda; X)$ is the maximal error.

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Throughout the paper, m, n, r are non-negative integers.

Let $\boldsymbol{P}$ be the space of polynomials with real coefficients and let $\boldsymbol{P}_n$ be the subspace of polynomials of degree $\le n$. If $\boldsymbol{L}$ is a finite-dimensional linear subspace of $\boldsymbol{P}$, then we denote by $\boldsymbol{L}^{\mathbb{Z}}$ the lattice in $\boldsymbol{L}$ consisting of integer polynomials.

We denote by $\boldsymbol{M}_r$, $r \ge 0$, the subspace of $\boldsymbol{P}$ consisting of polynomials divisible by the polynomial $x^r(1-x)^r$. Thus $\boldsymbol{M}_0 := \boldsymbol{P}$,

$$\boldsymbol{M}_1 := \{P \in \boldsymbol{P} : P(0) = P(1) = 0\}$$

and, for $r \ge 2$,

$$\boldsymbol{M}_r := \{P \in \boldsymbol{P} : P^{(k)}(0) = P^{(k)}(1) = 0 \text{ for } k = 0, 1, \ldots, r-1\}.$$

For $n, r \ge 0$ we denote $\boldsymbol{P}_{n,r} := \boldsymbol{P}_n \cap \boldsymbol{M}_r$. We assume here that $n \ge 2r$; otherwise $\boldsymbol{P}_n \cap \boldsymbol{M}_r = \{0\}$.

Let $[a, b]$ be an interval with $b - a < 4$. If $p \in [1, \infty)$, then

($*$) every function $f \in L_p(a, b)$ can be approximated in $L_p(a, b)$ by integer polynomials.

This was proved by Aparicio [A] for $p = 2$, and by Gel'fond [G] for an arbitrary $p < \infty$. The case $p = \infty$ is more complicated: a continuous function f can be uniformly approximated on $[a, b]$ by integer polynomials if and only if f satisfies certain additional conditions; see [HZ].

Since polynomials are dense in $L_p(a, b)$, to prove ($*$) it is enough to show that every polynomial can be approximated in $L_p(a, b)$ by integer polynomials. This, in turn, is a consequence of the fact that $\mu(\boldsymbol{P}_n^{\mathbb{Z}}; L_p(a, b)) \to 0$ as $n \to \infty$.

The proofs of ($*$) given in [A] and [G] were based on estimates which may be written in the form $\mu(\boldsymbol{P}_n^{\mathbb{Z}}; L_p(a, b)) = O(n^{-1/kp})$ as $n \to \infty$, where k is a positive integer which depends only on the interval $[a, b]$. The estimates obtained by Trigub [Tr1, Sec. 4] show that k may be replaced by 1.

We shall restrict ourselves to the special case $[a, b] = [0, 1]$. The space $L_p(0, 1)$, $1 \le p \le \infty$, will be denoted by L_p. We denote by $\|\cdot\|_p$ the usual norm in L_p, and $d_p(f, A)$ denotes the corresponding distance of a function $f \in L_p$ from a subset $A \subset L_p$.

It is a standard fact that

($**$) a continuous function f on $[0, 1]$ can be uniformly approximated by integer polynomials if and only if $f(0), f(1) \in \mathbb{Z}$

(see e.g. Ferguson [F2]). Naturally, it is enough to prove that every polynomial P with $P(0), P(1) \in \mathbb{Z}$ can be uniformly approximated on $[0, 1]$ by integer polynomials. This, in turn, is a consequence of the fact that $\mu(\boldsymbol{P}_{n,1}^{\mathbb{Z}}; L_\infty) \to 0$ as $n \to \infty$. The proof of ($**$) given by Kantorovich [K] used the fact that the polynomials $x^k(1-x)^{n-k}$, where $1 \le k \le n-1$, form a basis of the lattice $\boldsymbol{P}^{\mathbb{Z}}_{n,1}$, and was based on an estimate which may be written in the form

$$\mu(\boldsymbol{P}^{\mathbb{Z}}_{n,1}; L_\infty) \le \frac{1}{2} \max_{0\le x\le 1} \sum_{k=1}^{n-1} x^k(1-x)^{n-k} < \frac{1}{2n}.$$

The same argument shows that

$$\mu(\boldsymbol{P}^{\mathbb{Z}}_{n,r}; L_\infty) \le \frac{1}{2} \max_{0\le x\le 1} \sum_{k=r}^{n-r} x^k(1-x)^{n-k} < \frac{1}{2}\binom{n}{r}^{-1}, \qquad r = 2, 3, \ldots.$$

It is also not hard to see that

$$\mu(\boldsymbol{P}^{\mathbb{Z}}_{n,0}; L_\infty) = \frac{1}{2}, \qquad n = 0, 1, 2, \ldots. \tag{1.1}$$

The estimates obtained in [Tr1, Sec. 2] yield $\mu(\boldsymbol{P}^{\mathbb{Z}}_{n,1}; L_\infty) = O(n^{-2})$. Lipnicki [Li], applying a similar method, proved that $\mu(\boldsymbol{P}^{\mathbb{Z}}_{n,r}; L_\infty) \le c^r r^{2r} n^{-2r}$ for $r \ge 1$ and $n \ge 6r$, where c is a numerical constant. An analysis of the proof shows that

$$\mu(\boldsymbol{P}^{\mathbb{Z}}_{n,r}; L_\infty) \le \frac{r^{2r}}{n^{2r}}(1 + O(n^{-1})) \qquad \text{as } n \to \infty. \tag{1.2}$$

Trigub [Tr1, Sec. 4] made a remark which implies that if $p < \infty$, then $\mu(\boldsymbol{P}^{\mathbb{Z}}_{n}; L_p) = O(n^{-2/p})$, and that this estimate cannot be improved. It seems that the proof was never published.

More information on the subject is given in the survey article [Tr2]. Historical and bibliographical information on approximation by polynomials with integer coefficients can be found in Ferguson [F1].

The aim of this paper is to give quantitative estimates of $\mu(\boldsymbol{P}^{\mathbb{Z}}_{n,r}; L_p)$, $1 \le p \le \infty$. Before formulating the results we will introduce some more notation.

Notation. We write $\boldsymbol{E}$ (resp. $\boldsymbol{F}$) for the subspace of $\boldsymbol{P}$ consisting of polynomials P such that $P(x) = P(1-x)$ (resp. $P(x) = -P(1-x)$) for $x \in [0,1]$. Every polynomial P can be written in the form $E + F$, where $E \in \boldsymbol{E}$ and $F \in \boldsymbol{F}$ are given by

$$E(x) = \frac{P(x) + P(1-x)}{2}, \qquad F(x) = \frac{P(x) - P(1-x)}{2}.$$

Thus P is the direct sum of $\boldsymbol{E}$ and $\boldsymbol{F}$ (it is clear that $\boldsymbol{E} \cap \boldsymbol{F} = \{0\}$). Notice that $\boldsymbol{E}$ and $\boldsymbol{F}$ are orthogonal subspaces of $L_2(0,1)$.

Let U_r, V_r, S_r and T_r, where $r \ge 0$, be the polynomials given by

$$U_r(x) = x^r(1-x)^r, \qquad V_r(x) = (2x-1)x^r(1-x)^r,$$
$$S_r(x) = x^{r+1}(1-x)^r, \quad T_r(x) = x^r(1-x)^{r+1}.$$

Notice that $U_r \in \boldsymbol{E}$ and $V_r \in \boldsymbol{F}$.

Let $0 \le r \le m$. We denote

$$\boldsymbol{E}_{m,r} := \boldsymbol{E} \cap \boldsymbol{P}_{2m,r} = \operatorname{span}\{U_r, \dots, U_m\}, \tag{1.3}$$

$$\boldsymbol{F}_{m,r} := \boldsymbol{F} \cap \boldsymbol{P}_{2m+1,r} = \operatorname{span}\{V_r, \dots, V_m\}. \tag{1.4}$$

It is not hard to see that $U_r, \dots, U_m$ is a basis of the lattice $\boldsymbol{E}^{\mathbb{Z}}_{m,r}$. Similarly, $V_r, \dots, V_m$ is a basis of $\boldsymbol{F}^{\mathbb{Z}}_{m,r}$. Next, $S_r, T_r, \dots, S_m, T_m$ is a basis of the lattice $\boldsymbol{P}^{\mathbb{Z}}_{2m+1,r}$, while $S_r, T_r, \dots, S_{m-1}, T_{m-1}, U_m$ is a basis of $\boldsymbol{P}^{\mathbb{Z}}_{2m,r}$ ($\boldsymbol{P}^{\mathbb{Z}}_{2m,m} \equiv \boldsymbol{E}^{\mathbb{Z}}_{m,m}$ is the 1-dimensional lattice generated by U_m). By definition we have

$$U_r = S_r + T_r, \qquad V_r = S_r - T_r, \qquad S_r = \frac{U_r + V_r}{2}, \qquad T_r = \frac{U_r - V_r}{2}.$$

Hence it follows that

$$\boldsymbol{E}^{\mathbb{Z}}_{m,r} + \boldsymbol{F}^{\mathbb{Z}}_{m,r} \subsetneq \boldsymbol{P}^{\mathbb{Z}}_{2m+1,r} \subsetneq \tfrac{1}{2}(\boldsymbol{E}^{\mathbb{Z}}_{m,r} + \boldsymbol{F}^{\mathbb{Z}}_{m,r}), \quad 0 \le r \le m, \tag{1.5}$$

$$\boldsymbol{E}^{\mathbb{Z}}_{m,r} + \boldsymbol{F}^{\mathbb{Z}}_{m-1,r} \subsetneq \boldsymbol{P}^{\mathbb{Z}}_{2m,r} \subsetneq \tfrac{1}{2}(\boldsymbol{E}^{\mathbb{Z}}_{m,r} + \boldsymbol{F}^{\mathbb{Z}}_{m-1,r}), \quad 0 \le r \le m-1. \tag{1.6}$$

We will denote

$$C_r := \sqrt{2(2r)!(2r+1)!}, \qquad r = 0, 1, 2, \dots, \tag{1.7}$$

$$K_r = C_s^2, \qquad r \text{ even}, \ r = 2s, \tag{1.8}$$

$$K_r = C_s C_{s+1}, \qquad r \text{ odd}, \ r = 2s+1. \tag{1.9}$$

Next, we will write

$$a_{m,r} := C_r^2 \frac{(2m-2r)!}{(2m+2r+2)!}, \qquad 0 \le r \le m, \tag{1.10}$$

$$b_{m,r} := C_r^2 \frac{(2m-2r+1)!}{(2m+2r+3)!}, \qquad 0 \le r \le m, \tag{1.11}$$

$$c_{n,r} := \frac{n+1}{2} C_r^2 \frac{(n-2r-1)!}{(n+2r+2)!}, \qquad r \ge 0, \ n \ge 2r+1. \tag{1.12}$$

Thus

$$c_{n,r} = \frac{a_{m,r} + b_{m,r}}{4}, \qquad n = 2m+1, \ 0 \le r \le m, \tag{1.13}$$

$$c_{n,r} = \frac{a_{m,r} + b_{m-1,r}}{4}, \qquad n = 2m, \ 0 \le r \le m-1. \tag{1.14}$$

The results. The most precise estimates of $\mu(\boldsymbol{P}^{\mathbb{Z}}_{n,r}; L_p)$ are obtained for $p = 2$:

THEOREM 1.1. *Let $r \ge 0$ and $n \ge 6r + 7$. Then*

$$\frac{c_{n,r}^{1/2}}{2} \le \mu(\boldsymbol{P}^{\mathbb{Z}}_{n,r}; L_2) < 1.014\, c_{n,r}^{1/2}.$$

THEOREM 1.2. *Let $r \ge 0$. Then*

$$(1.15)\quad \frac{\sqrt{2}}{4}\cdot\frac{C_r}{n^{2r+1}}(1+O(n^{-1})) \le \mu(\boldsymbol{P}^{\mathbb{Z}}_{n,r};L_2) \le \frac{\sqrt{2}}{2}\cdot\frac{C_r}{n^{2r+1}}(1+O(n^{-1}))$$

as $n\to\infty$.

A more precise analysis shows that

$$\mu(\boldsymbol{P}^{\mathbb{Z}}_{n,r};L_2) = \frac{1}{2}\cdot\frac{C_r}{n^{2r+1}}(1+O(n^{-1}))$$

as $n\to\infty$. The proof will be given in a separate paper.

The proofs of Theorems 1.1 and 1.2 are given in Section 3. The problem is reduced to the corresponding estimates for the lattices $\boldsymbol{E}_{m,r}$ and $\boldsymbol{F}_{m,r}$. These, in turn, are consequences of certain inequalities connected with the behaviour of the quantities $d_2(U_r,\boldsymbol{E}_{m,r+1})$ and $d_2(V_r,\boldsymbol{F}_{m,r+1})$.

For $p\ne 2$ the estimates obtained are less precise:

THEOREM 1.3. *Let $r\ge 0$ and $1\le p<2$. Then*

$$(1.16)\quad 2^{-3/2}\frac{C_r}{n^{2r+2/p}}(1+O(n^{-1})) \le \mu(\boldsymbol{P}^{\mathbb{Z}}_{n,r};L_p) \le \frac{2^{2r+2}K_r}{n^{2r+2/p}}(1+O(n^{-1}))$$

as $n\to\infty$.

THEOREM 1.4. *Let $r\ge 0$ and $2<p\le\infty$. Then*

$$(1.17)\quad \mu(\boldsymbol{P}^{\mathbb{Z}}_{n,r};L_p) \le 2^{-1/2}\frac{C_r}{n^{2r+2/p}}(1+O(n^{-1})) \qquad \textit{as } n\to\infty.$$

PROPOSITION 1.5. *Let $r\ge 0$ and $n\ge 2r$, $n\ge 1$. Then*

$$(1.18)\quad \mu(\boldsymbol{P}^{\mathbb{Z}}_{n,r};L_p) \ge 6^{-1/2}\frac{(2r)!}{2^{2r+1}}\cdot\frac{1}{n^{2r+2/p}}, \qquad 2<p<\infty,$$

$$(1.19)\quad \mu(\boldsymbol{P}^{\mathbb{Z}}_{n,r};L_\infty) \ge \frac{(2r)!}{2^{2r+1}}\cdot\frac{1}{n^{2r}}.$$

The proofs are given in Section 4. Theorems 1.3 and 1.4 are consequences of the corresponding results for $p=2$ and the Markov–Nikol'skiĭ inequalities between L_p norms in $\boldsymbol{P}_n$. Proposition 1.5 is an easy consequence of standard facts.

From Theorem 1.4 it follows in particular that

$$\mu(\boldsymbol{P}^{\mathbb{Z}}_{n,r};L_\infty) \le 2^{-1/2}\frac{C_r}{n^{2r}}(1+O(n^{-1})) \qquad \text{as } n\to\infty.$$

For $r\ge 6$ this estimate is better than (1.2).

As an immediate consequence of Theorems 1.2–1.4 and Proposition 1.5 we obtain the following result:

THEOREM 1.6. *Let $r\ge 0$ and $1\le p\le\infty$. Then*

$$\mu(\boldsymbol{P}^{\mathbb{Z}}_{n,r};L_p) \asymp \frac{1}{n^{2r+2/p}} \qquad \textit{as } n\to\infty.$$

In particular, for $r = 0$ we get $\mu(\boldsymbol{P}_n^{\mathbb{Z}}; L_p) \asymp n^{-2/p}$, the above-mentioned result announced by Trigub.

2. Estimates for $d_2(U_r, \boldsymbol{E}_{m,r+1})$ and $d_2(V_r, \boldsymbol{F}_{m,r+1})$. We denote by $\Gamma(Q_1, \ldots, Q_n)$ the Gram determinant of the polynomials $Q_1, \ldots, Q_n$ in $L_2(0,1)$.

LEMMA 2.1. *Let $0 \le r \le m$. Then*

$$\Gamma(U_r, \ldots, U_m) = (m+r)! \prod_{i=2r}^{2m} i! \prod_{i=0}^{m-r} (2i)! / \prod_{i=m+r}^{2m} (2i+1)!. \tag{2.1}$$

Proof. We have $\Gamma(U_r, \ldots, U_m) = \det[(U_i|U_j)]_{i,j=r}^m$, where

$$(U_i|U_j) = \int_0^1 U_i(x)U_j(x)\,dx = \int_0^1 x^{i+j}(1-x)^{i+j}\,dx = \frac{(i+j)!(i+j)!}{(2i+2j+1)!}.$$

So, it remains to show that

$$\det\left[\frac{(i+j)!(i+j)!}{(2i+2j+1)!}\right]_{i,j=1}^r = (m+r)! \prod_{i=2r}^{2m} i! \prod_{i=0}^{m-r} (2i)! / \prod_{i=m+r}^{2m} (2i+1)!,$$

which is a standard exercise. ∎

LEMMA 2.2. *Let $m \ge 0$. Then $\|U_m\|_2 = a_{m,m}^{1/2}$ and*

$$d_2(U_r, \boldsymbol{E}_{m,r+1}) = a_{m,r}^{1/2}, \quad 0 \le r \le m-1.$$

Proof. Setting $r = m$ in (2.1) we get

$$\|U_m\|^2 = \frac{(2m)!(2m)!}{(4m+1)!} = \frac{2(2m)!(2m+1)!}{(4m+2)!} \overset{(1.7)}{=} \frac{C_m^2}{(4m+2)!} \overset{(1.10)}{=} a_{m,m}.$$

Now, let $r \le m-1$. Replacing r by $r+1$ in (1.3) we may write

$$d_2(U_r, \boldsymbol{E}_{m,r+1}) = d_2(U_r, \operatorname{span}\{U_{r+1}, \ldots, U_m\}). \tag{2.2}$$

Next we have

$$[d_2(U_r, \operatorname{span}\{U_{r+1}, \ldots, U_m\})]^2 = \frac{\Gamma(U_r, U_{r+1}, \ldots, U_m)}{\Gamma(U_{r+1}, \ldots, U_m)}. \tag{2.3}$$

Replacing r by $r+1$ in (2.1) we get

$$\Gamma(U_{r+1}, \ldots, U_m) = \prod_{i=2r+2}^{m+r+1} i! \prod_{i=m+r+1}^{2m} i! \prod_{i=0}^{m-r-1} (2i)! / \prod_{i=m+r+1}^{2m} (2i+1)!. \tag{2.4}$$

Finally, from (2.2)–(2.4) and (2.1) we obtain

$$[d_2(U_r, \boldsymbol{E}_{m,r+1})]^2 = \frac{(2r)!(2r+1)!(2m-2r)!}{(m+r+1)(2m+2r+1)!}$$
$$= \frac{2(2r)!(2r+1)!(2m-2r)!}{(2m+2r+2)!} \overset{(1.7)}{=} C_r^2 \frac{(2m-2r)!}{(2m+2r+2)!} \overset{(1.10)}{=} a_{m,r}. \blacksquare$$

LEMMA 2.3. *Let* $0 \le r \le m$. *Then*

$$(2.5) \qquad \Gamma(V_r, \ldots, V_m) = 2^{m-r+1} \prod_{i=2r}^{2m+1} i! \prod_{i=0}^{m-r} (2i+1)! / \prod_{i=m+r}^{2m} (2i+3)!.$$

Proof. We have $\Gamma(V_r, \ldots, V_m) = \det[(V_i|V_j)]_{i,j=r}^m$, where

$$(V_i|V_j) = \int_0^1 V_i(x)V_j(x)\,dx = \int_0^1 (2x-1)^2 x^{i+j}(1-x)^{i+j}\,dx$$
$$= \frac{2(i+j)!(i+j+1)!}{(2i+2j+3)!}.$$

So, it remains to show that

$$\det\left[\frac{(i+j)!(i+j+1)!}{(2i+2j+3)!}\right]_{i,j=1}^r = \prod_{i=2r}^{2m+1} i! \prod_{i=0}^{m-r} (2i+1)! / \prod_{i=m+r}^{2m} (2i+3)!,$$

which is a standard exercise. ∎

LEMMA 2.4. *Let* $m \ge 0$. *Then* $\|V_m\|_2 = b_{m,m}^{1/2}$ *and*

$$d_2(V_r, \boldsymbol{F}_{m,r+1}) = b_{m,r}^{1/2}, \qquad 0 \le r \le m-1.$$

The proof is analogous to that of Lemma 2.2. It is enough to replace (2.1) by (2.5), and (1.3) by (1.4).

LEMMA 2.5. *Let* $1 \le k \le m$. *Then*

$$(2.6) \quad \frac{a_{m,k}}{a_{m,k-1}} = \frac{(2k-1)(2k)^2(2k+1)}{(2m-2k+1)(2m-2k+2)(2m+2k+1)(2m+2k+2)}.$$

Hence

$$(2.7) \qquad \frac{a_{m,k}}{a_{m,k-1}} < \frac{k^4}{[(m+1/2)^2 - k^2]^2}.$$

Consequently, if $k/m \le \vartheta < 1$, *then*

$$\frac{a_{m,k}}{a_{m,k-1}} < \left(\frac{\vartheta^2}{1-\vartheta^2}\right)^2.$$

Proof. According to the definition of $a_{m,k}$ (see (1.10) and (1.7)), we may write

$$\frac{a_{m,k}}{a_{m,k-1}} = \frac{2(2k)!(2k+1)!}{2(2k-2)!(2k-1)!} \cdot \frac{(2m-2k)!}{(2m-2k+2)!} \cdot \frac{(2m+2k)!}{(2m+2k+2)!}$$
$$= \frac{(2k-1)(2k)^2(2k+1)}{(2m-2k+1)(2m-2k+2)(2m+2k+1)(2m+2k+2)}.$$

This proves (2.6). Next, it is clear that $(2k-1)(2k)^2(2k+1) < (2k)^4$, and it is not hard to see that

$$(2m-2k+1)(2m-2k+2)(2m+2k+1)(2m+2k+2) > [(2m+1)^2-(2k)^2]^2.$$

Hence, by (2.6),

$$\frac{a_{m,k}}{a_{m,k-1}} < \frac{(2k)^4}{[(2m+1)^2-(2k)^2]^2} = \frac{k^4}{[(m+1/2)^2-k^2]^2}.$$

This proves (2.7). Finally, if $k/m \le \vartheta < 1$, then, by (2.7),

$$\frac{a_{m,k}}{a_{m,k-1}} < \frac{k^4}{(m^2-k^2)^2} = \left[\frac{(k/m)^2}{1-(k/m)^2}\right]^2 \le \left(\frac{\vartheta^2}{1-\vartheta^2}\right)^2,$$

because the function $x^2/(1-x^2)$ is increasing on $(0,1)$. ∎

Lemma 2.6. *Let* $1 \le k \le m$. *Then*

$$\frac{b_{m,k}}{b_{m,k-1}} = \frac{(2k-1)(2k)^2(2k+1)}{(2m-2k+2)(2m-2k+3)(2m+2k+2)(2m+2k+3)}.$$

Hence

$$\frac{b_{m,k}}{b_{m,k-1}} < \frac{k^4}{[(m+1)^2-k^2]^2}. \tag{2.8}$$

Consequently, if $k/m \le \vartheta < 1$, *then*

$$\frac{b_{m,k}}{b_{m,k-1}} < \left(\frac{\vartheta^2}{1-\vartheta^2}\right)^2.$$

The proof is analogous to that of Lemma 2.5.

Lemma 2.7. *For each integer* $s \ge 10$ *one has*

$$u_s := \frac{s!(s+1)!(2s)!}{(4s+2)!} \cdot \frac{(6s+2)!}{(3s)!(3s+1)!} > \frac{18}{5}.$$

Proof. One can verify directly that $u_{10} > 18/5$, so it remains to show that the sequence (u_s) is increasing. After standard simplifications, we obtain

$$\frac{u_{s+1}}{u_s} = \frac{4}{3} \cdot \frac{(s+2)(2s+1)(6s+3)(6s+5)(6s+7)}{(3s+1)(3s+2)(4s+3)(4s+5)(4s+6)}$$
$$= \frac{1728s^5+8640s^4+15936s^3+13680s^2+5508s+840}{1728s^5+7776s^4+13236s^3+10578s^2+3942s+540} > 1. \ ∎$$

Lemma 2.8. *For all integers $r \geq 5$ and $m \geq 3r$ one has*

$$a_{m,m} < \frac{5}{18} a_{m,r}. \tag{2.9}$$

Proof. According to the definition of $a_{m,m}$ and $a_{m,r}$ (see (1.10) and (1.7)), inequality (2.9) may be written in the form

$$\frac{2(2m)!(2m+1)!}{(4m+2)!} < \frac{5}{18} \cdot \frac{2(2r)!(2r+1)!(2m-2r)!}{(2m+2r+2)!}.$$

Set $n = 2m$ and $s = 2r$. It is enough to show that

$$v_n := \frac{s!(s+1)!(n-s)!}{(n+s+2)!} \cdot \frac{(2n+2)!}{n!(n+1)!} > \frac{18}{5}, \qquad n \geq 3s,\ s \geq 10.$$

Lemma 2.7 says that $v_{3s} > 18/5$ for $s \geq 10$, so it remains to prove that the sequence $(v_n)_{n \geq 3s}$ is increasing. After standard simplifications, we obtain

$$\frac{v_{n+1}}{v_n} = 2 \cdot \frac{(2n+3)(n-s+1)}{(n+1)(n+s+3)} > 4 \cdot \frac{n-s+1}{n+s+3}.$$

It is clear that the right-hand side is greater than 1, at least for $n \geq 3s$. ∎

Lemma 2.9. *Let $m \geq 1$ and $p := \lfloor m/3 \rfloor$. Then*

$$\sum_{k=p+1}^{m} a_{m,k} < \frac{290}{429} a_{m,p} < 0.676\, a_{m,p}. \tag{2.10}$$

Proof. For $m = 1, \ldots, 14$ inequality (2.10) can be verified directly (the numbers $a_{m,k}$, defined by (1.10) and (1.7), can be easily computed; the coefficient $290/429$ is attained for $m = 3$ and $p = 1$). Assume in what follows that $m \geq 15$.

We deduce from Lemma 2.5 that $a_{m,k}/a_{m,k-1}$ is an increasing function of k (the numerator of the right-hand side of (2.6) is increasing; the denominator is decreasing). Lemma 2.5 also implies that $a_{m,k+1}/a_{m,k} < 1$ if $(k+1)/m < \sqrt{2}/2$. Setting $k = m$ in (2.6) we get

$$\frac{a_{m,m}}{a_{m,m-1}} = \frac{2m-1}{4m+1} m^2 \geq \frac{29}{61} m^2, \tag{2.11}$$

because $m \geq 15$. Hence it follows that the sequence $(a_{m,k})_{k=1}^{m}$ initially decreases, attains its minimum at some point k_0 such that

$$\frac{k_0+1}{m} > \frac{\sqrt{2}}{2}, \tag{2.12}$$

and finally increases. Let $q := \lfloor m/2 \rfloor$. We may write

$$\sum_{k=p+1}^{m} a_{m,k} = \sum_{k=p+1}^{q} a_{m,k} + \sum_{k=q+1}^{k_0} a_{m,k} + \sum_{k=k_0+1}^{m} a_{m,k}. \tag{2.13}$$

We shall separately estimate each of the three components on the right-hand side.

It follows from Lemma 2.5 that $a_{m,k}/a_{m,k-1} < 1/9$ if $k/m \le 1/2$. Hence

$$a_{m,k} < \frac{a_{m,p}}{9^{k-p}}, \qquad k = p+1, \dots, q. \tag{2.14}$$

In particular, for $k = q$ we have

$$a_{m,q} < \frac{a_{m,p}}{9^{q-p}}. \tag{2.15}$$

From (2.14) we get

$$\sum_{k=p+1}^{q} a_{m,k} < \sum_{k=p+1}^{q} \frac{a_{m,p}}{9^{k-p}} < \frac{1}{8} a_{m,p}. \tag{2.16}$$

Now we shall estimate the second component in (2.13). We may write

$$\sum_{k=q+1}^{k_0} a_{m,k} < (k_0 - q) a_{m,q+1} < \frac{m}{2} \cdot a_{m,q} \overset{(2.15)}{<} \frac{m}{2} \cdot \frac{a_{m,p}}{9^{q-p}} \le \frac{m}{2 \cdot 9^{(m-3)/6}} a_{m,p},$$

because $q - p = \lfloor m/2 \rfloor - \lfloor m/3 \rfloor \ge (m-3)/6$. As $m \ge 15$, we have

$$\frac{m}{2 \cdot 9^{(m-3)/6}} \le \frac{5}{54},$$

whence

$$\sum_{k=q+1}^{k_0} a_{m,k} < \frac{5}{54} a_{m,p}. \tag{2.17}$$

To estimate the third component in (2.13), we may write

$$\sum_{k=k_0+1}^{m} a_{m,k} = \sum_{k=k_0+1}^{m-1} a_{m,k} + a_{m,m} < (m - k_0 - 1) a_{m,m-1} + a_{m,m}. \tag{2.18}$$

From (2.12) we get $m - k_0 - 1 < (1 - \sqrt{2}/2)m$, and (2.11) yields

$$a_{m,m-1} \le \frac{61}{29m^2} a_{m,m}.$$

Hence

$$(m - k_0 - 1) a_{m,m-1} < \frac{61}{29} \left(1 - \frac{\sqrt{2}}{2}\right) \frac{1}{m} a_{m,m} < \frac{0.62}{m} a_{m,m} \le 0.05 a_{m,m},$$

because $m \ge 15$. Thus

$$(m - k_0 - 1) a_{m,m-1} + a_{m,m} < 1.05 a_{m,m}. \tag{2.19}$$

From Lemma 2.8 it follows that $a_{m,m} < \frac{5}{18} a_{m,p}$. So, by (2.18) and (2.19), we get

$$\sum_{k=k_0+1}^{m} a_{m,k} < 1.05 \cdot \frac{5}{18} a_{m,p} < 0.3 a_{m,p}. \tag{2.20}$$

Finally, from (2.13), (2.16), (2.17) and (2.20) we obtain

$$\sum_{k=p+1}^{m} a_{m,k} < \frac{1}{8} a_{m,p} + \frac{5}{54} a_{m,p} + 0.3 a_{m,p} < 0.52 a_{m,p} < \frac{290}{429} a_{m,p}. \ ■$$

Lemma 2.10. *Let $m \geq 1$ and $p := \lfloor m/3 \rfloor$. Then*

$$\sum_{k=p+1}^{m} b_{m,k} < \frac{2}{13} b_{m,p} < 0.154 b_{m,p}.$$

The proof is similar to the preceding one; analogues of Lemmas 2.7 and 2.8 are needed. The coefficient $2/13$ is attained for $m = 3$ and $p = 1$.

Lemma 2.11. *Let $r \geq 0$ and $m \geq 3r + 3$. Then*

$$\sum_{k=r+1}^{m} a_{m,k} < 0.027 a_{m,r}.$$

Proof. Let $p := \lfloor m/3 \rfloor$. We may write

$$\sum_{k=r+1}^{m} a_{m,k} = \sum_{k=r+1}^{p} a_{m,k} + \sum_{k=p+1}^{m} a_{m,k}. \tag{2.21}$$

From Lemma 2.5 it follows that $a_{m,k}/a_{m,k-1} < 1/64$ if $k/m \leq 1/3$. Hence

$$a_{m,k} < \frac{a_{m,r}}{64^{k-r}}, \qquad k = r+1, \ldots, p, \tag{2.22}$$

and therefore

$$\sum_{k=r+1}^{p} a_{m,k} < \sum_{k=r+1}^{p} \frac{a_{m,r}}{64^{k-r}} < \frac{1}{63} a_{m,r}. \tag{2.23}$$

Setting $k = p$ in (2.22) we get

$$a_{m,p} < \frac{a_{m,r}}{64^{p-r}} \leq \frac{1}{64} a_{m,r},$$

because

$$p - r = \left\lfloor \frac{m}{3} \right\rfloor - r \geq \left\lfloor \frac{3r+3}{3} \right\rfloor - r = 1.$$

Hence, Lemma 2.9 yields

$$\sum_{k=p+1}^{m} a_{m,k} < 0.676 a_{m,p} < \frac{0.676}{64} a_{m,r} < 0.011 a_{m,r}. \tag{2.24}$$

From (2.21), (2.23) and (2.24), we finally obtain

$$\sum_{k=r+1}^{m} a_{m,k} < \frac{1}{63} a_{m,r} + 0.011 a_{m,r} < 0.027 a_{m,r}. \blacksquare$$

LEMMA 2.12. *Let $r \geq 0$ and $m \geq 3r+6$. Then*

$$a_{m,r}^{-1} \sum_{k=r+1}^{m} a_{m,k} < 1.027 \frac{(r+1)^4}{[(m+1/2)^2 - (r+1)^2]^2}.$$

Proof. Replacing r by $r+1$ in Lemma 2.11 we get

$$\sum_{k=r+2}^{m} a_{m,k} < 0.027 a_{m,r+1}.$$

Hence

$$a_{m,r}^{-1} \sum_{k=r+1}^{m} a_{m,k} = a_{m,r}^{-1} \Big(a_{m,r+1} + \sum_{k=r+2}^{m} a_{m,k} \Big) < 1.027 \frac{a_{m,r+1}}{a_{m,r}}.$$

Next, replacing k by $r+1$ in (2.7), we obtain

$$\frac{a_{m,r+1}}{a_{m,r}} < \frac{(r+1)^4}{[(m+1/2)^2 - (r+1)^2]^2}. \blacksquare$$

LEMMA 2.13. *Let $r \geq 0$ and $m \geq 3r+3$. Then*

$$\sum_{k=r+1}^{m} b_{m,k} < 0.019 b_{m,r}.$$

Proof. Let $p := \lfloor m/3 \rfloor$. We may write

$$\sum_{k=r+1}^{m} b_{m,k} = \sum_{k=r+1}^{p} b_{m,k} + \sum_{k=p+1}^{m} b_{m,k}. \tag{2.25}$$

By repeating the corresponding part of the proof of Lemma 2.11, with $a_{m,k}$ replaced by $b_{m,k}$, we obtain

$$\sum_{k=r+1}^{p} b_{m,k} < \frac{1}{63} b_{m,r}, \tag{2.26}$$

$$b_{m,p} < \frac{1}{64} b_{m,r}. \tag{2.27}$$

By Lemma 2.10, we have

$$\sum_{k=p+1}^{m} b_{m,k} < 0.154 b_{m,p} \overset{(2.27)}{<} \frac{0.154}{64} b_{m,r} < 0.003 b_{m,r}. \tag{2.28}$$

From (2.25), (2.26) and (2.28), we finally obtain

$$\sum_{k=r+1}^{m} b_{m,k} < \frac{1}{63} b_{m,r} + 0.003 b_{m,r} < 0.019 b_{m,r}. \ ■$$

LEMMA 2.14. *Let $r \ge 0$ and $m \ge 3r + 6$. Then*

$$b_{m,r}^{-1} \sum_{k=r+1}^{m} b_{m,k} < 1.019 \frac{(r+1)^4}{[(m+1)^2 - (r+1)^2]^2}.$$

The proof is similar to that of Lemma 2.12; Lemma 2.11 should be replaced by Lemma 2.13, and (2.7) by (2.8).

3. The covering radius in the L_2 norm

LEMMA 3.1. *Let $x_0, x_1, \dots, x_k$ be a sequence of linearly independent vectors in $L_2(0,1)$ and let Λ be the lattice generated by $x_0, x_1, \dots, x_k$. Let $h_0 := \|x_0\|_2$ and let*

$$h_i := d_2(x_i, \operatorname{span}\{x_0, x_1, \dots, x_{i-1}\}), \qquad i = 1, \dots, k.$$

Then

$$\frac{h_k}{2} \le \mu(\Lambda; L_2) \le \frac{1}{2}\Big(\sum_{i=0}^{k} h_i^2\Big)^{1/2} = \frac{h_k}{2}\Big(1 + h_k^{-2} \sum_{i=0}^{k-1} h_i^2\Big)^{1/2}. \tag{3.1}$$

Proof. Let $\tilde{\Lambda}$ be the lattice generated by $x_0, x_1, \dots, x_{k-1}$. Let

$$M := \operatorname{span}\{x_0, x_1, \dots, x_k\}, \qquad \tilde{M} := \operatorname{span}\{x_0, x_1, \dots, x_{k-1}\}.$$

To prove the first inequality in (3.1) it is enough to observe that $\Lambda = \tilde{\Lambda} + \mathbb{Z}x_k \subset \tilde{M} + \mathbb{Z}x_k$, whence

$$\mu(\Lambda; L_2) \ge d_2\big(\tfrac{1}{2}x_k, \Lambda\big) \ge d_2\big(\tfrac{1}{2}x_k, \tilde{M} + \mathbb{Z}x_k\big) = d_2(\tfrac{1}{2}x_k, \tilde{M}) = \tfrac{1}{2}h_k.$$

Let $u_0, u_1, \dots, u_k$ be the orthogonalization of $x_0, x_1, \dots, x_k$. Then $\|u_i\| = h_i$ for $i = 0, 1, \dots, k$. Let

$$P := \{t_0 u_0 + t_1 u_1 + \dots + t_k u_k : -1/2 < t_0, t_1, \dots, t_k \le 1/2\}.$$

It is easily seen that the parallelepipeds $P + x$, $x \in \Lambda$, form a disjoint covering of M. Let $\varrho := (\sum_{i=0}^{k} h_i^2)^{1/2}$ and let B be the closed unit ball in M. It is clear that $P \subset \frac{1}{2}\varrho B$. So, the balls $\frac{1}{2}\varrho B + x$, $x \in \Lambda$, cover M, which means that $\mu(\Lambda; L_2) \le \frac{1}{2}\varrho$. ■

Let us denote

$$\alpha_{m,r} := a_{m,r}^{-1} \sum_{j=r+1}^{m} a_{m,j}, \qquad r \ge 0,\ m \ge r+1.$$

Lemma 2.11 says that if $m \ge 3r + 3$, then

$$\alpha_{m,r} < 0.027. \tag{3.2}$$

PROPOSITION 3.2. *Let $r \geq 0$ and $m \geq 3r+3$. Then*

$$\frac{a_{m,r}^{1/2}}{2} \leq \mu(\boldsymbol{E}_{m,r}^{\mathbb{Z}}; L_2) \leq \frac{a_{m,r}^{1/2}}{2}(1+\alpha_{m,r})^{1/2} < 1.014\frac{a_{m,r}^{1/2}}{2}. \tag{3.3}$$

Proof. Let $k := m-r$ and $x_i := U_{m-i}$ for $i = 0, 1, \ldots, k$. Let $h_0, h_1, \ldots, h_k$ and Λ be defined as in Lemma 3.1. Then

$$\Lambda = \mathbb{Z}x_k + \mathbb{Z}x_{k-1} + \cdots + \mathbb{Z}x_0 = \mathbb{Z}U_r + \mathbb{Z}U_{r+1} + \cdots + \mathbb{Z}U_m = \boldsymbol{E}_{m,r}^{\mathbb{Z}}$$

because $U_r, \ldots, U_m$ is a basis of $\boldsymbol{E}_{m,r}^{\mathbb{Z}}$. We will prove that

$$h_i = a_{m,m-i}^{1/2}, \quad i = 0, 1, \ldots, k. \tag{3.4}$$

Suppose first that $i \geq 1$. Replacing r by $m-i+1$ in (1.3) we get $\operatorname{span}\{U_{m-i+1}, \ldots, U_m\} = \boldsymbol{E}_{m,m-i+1}$. Hence

$$\begin{aligned} h_i &= d_2(x_i, \operatorname{span}\{x_0, x_1, \ldots, x_{i-1}\}) \\ &= d_2(U_{m-i}, \operatorname{span}\{U_m, U_{m-1}, \ldots, U_{m-i+1}\}) \\ &= d_2(U_{m-i}, \boldsymbol{E}_{m,m-i+1}) = a_{m,m-i}^{1/2}, \end{aligned}$$

according to Lemma 2.2. For $i = 0$ the proof is even simpler: $h_0 = \|x_0\| = \|U_m\|_2 = a_{m,m}^{1/2}$.

From (3.4) it follows that $h_k = a_{m,m-k}^{1/2} = a_{m,r}^{1/2}$ and

$$h_k^{-2}\sum_{i=0}^{k-1} h_i^2 = a_{m,r}^{-1}\sum_{i=0}^{k-1} a_{m,m-i} = a_{m,r}^{-1}\sum_{j=r+1}^{m} a_{m,j} = \alpha_{m,r}.$$

It is now enough to apply Lemma 3.1. The last inequality in (3.3) follows from (3.2). ∎

If $m \geq 3r+6$, then, according to Lemma 2.12, inequality (3.2) may be strengthened to

$$\alpha_{m,r} < 1.027\frac{(r+1)^4}{[(m+1/2)^2-(r+1)^2]^2}.$$

This, in turn, allows one to replace (3.3) by

$$\frac{a_{m,r}^{1/2}}{2} \leq \mu(\boldsymbol{E}_{m,r}^{\mathbb{Z}}; L_2) < \frac{a_{m,r}^{1/2}}{2}\left[1 + 0.52\frac{(r+1)^4}{[(m+1/2)^2-(r+1)^2]^2}\right]. \tag{3.5}$$

PROPOSITION 3.3. *Let $r \geq 0$. Then*

$$\mu(\boldsymbol{E}_{m,r}^{\mathbb{Z}}; L_2) = \frac{a_{m,r}^{1/2}}{2}(1+O(m^{-4})) = \frac{C_r}{2(2m)^{2r+1}}(1+O(m^{-1}))$$

as $m \to \infty$.

Proof. The first equality follows directly from (3.5). To obtain the second one, it is enough to observe that

$$a_{m,r} \overset{(1.10)}{=} C_r^2 \frac{(2m-2r)!}{(2m+2r+2)!} = \frac{C_r^2}{(2m)^{4r+2}}(1+O(m^{-1})). \blacksquare$$

Let us denote

$$\beta_{m,r} := b_{m,r}^{-1} \sum_{j=r+1}^{m} b_{m,j}, \qquad r \ge 0,\, m \ge r+1.$$

Lemma 2.13 says that if $m \ge 3r+3$, then

$$\beta_{m,r} < 0.019. \tag{3.6}$$

PROPOSITION 3.4. *Let $r \ge 0$ and $m \ge 3r+3$. Then*

$$\frac{b_{m,r}^{1/2}}{2} \le \mu(\boldsymbol{F}^{\mathbb{Z}}_{m,r}; L_2) \le \frac{b_{m,r}^{1/2}}{2}(1+\beta_{m,r})^{1/2} < 1.010\frac{b_{m,r}^{1/2}}{2}. \tag{3.7}$$

Proof. The proof is analogous to that of Proposition 3.2. It is enough to replace U_{m-i}, $\boldsymbol{E}^{\mathbb{Z}}_{m,r}$ and $a_{m,m-i}$ by V_{m-i}, $\boldsymbol{F}^{\mathbb{Z}}_{m,r}$ and $b_{m,m-i}$, respectively; (3.2) should be replaced by (3.6). ■

If $m \ge 3r+6$, then, according to Lemma 2.14, inequality (3.6) may be strengthened to

$$\beta_{m,r} < 1.019\frac{(r+1)^4}{[(m+1)^2-(r+1)^2]^2}.$$

This, in turn, allows one to replace (3.7) by

$$\frac{b_{m,r}^{1/2}}{2} \le \mu(\boldsymbol{F}^{\mathbb{Z}}_{m,r}; L_2) < \frac{b_{m,r}^{1/2}}{2}\left[1+0.51\frac{(r+1)^4}{[(m+1)^2-(r+1)^2]^2}\right]. \tag{3.8}$$

PROPOSITION 3.5. *Let $r \ge 0$. Then*

$$\mu(\boldsymbol{F}^{\mathbb{Z}}_{m,r}; L_2) = \frac{b_{m,r}^{1/2}}{2}(1+O(m^{-4})) = \frac{C_r}{2(2m)^{2r+1}}(1+O(m^{-1}))$$

as $m \to \infty$.

Proof. The first equality follows directly from (3.8). To obtain the second one, it is enough to observe that

$$b_{m,r} \overset{(1.11)}{=} C_r^2 \frac{(2m-2r+1)!}{(2m+2r+3)!} = \frac{C_r^2}{(2m)^{4r+2}}(1+O(m^{-1})). \blacksquare$$

Let us denote

$$\boldsymbol{Q}^{\mathbb{Z}}_{n,r} := \boldsymbol{E}^{\mathbb{Z}}_{m,r} + \boldsymbol{F}^{\mathbb{Z}}_{m,r}, \qquad n = 2m+1,\, 0 \le r \le m,$$
$$\boldsymbol{Q}^{\mathbb{Z}}_{n,r} := \boldsymbol{E}^{\mathbb{Z}}_{m,r} + \boldsymbol{F}^{\mathbb{Z}}_{m-1,r}, \qquad n = 2m,\, 0 \le r \le m-1.$$

Then (1.5) and (1.6) may be written jointly as

$$\boldsymbol{Q}^{\mathbb{Z}}_{n,r} \subsetneq \boldsymbol{P}^{\mathbb{Z}}_{n,r} \subsetneq \tfrac{1}{2}\boldsymbol{Q}^{\mathbb{Z}}_{n,r}, \qquad r \ge 0,\, n \ge 2r+1,$$

which implies that

$$\tfrac{1}{2}\mu(\boldsymbol{Q}^{\mathbb{Z}}_{n,r};L_2) \le \mu(\boldsymbol{P}^{\mathbb{Z}}_{n,r};L_2) \le \mu(\boldsymbol{Q}^{\mathbb{Z}}_{n,r};L_2). \tag{3.9}$$

As $\boldsymbol{E}$ and $\boldsymbol{F}$ are mutually orthogonal, it is clear that

$$\mu(\boldsymbol{Q}^{\mathbb{Z}}_{n,r};L_2)^2 = \mu(\boldsymbol{E}^{\mathbb{Z}}_{m,r};L_2)^2 + \mu(\boldsymbol{F}^{\mathbb{Z}}_{m,r};L_2)^2, \qquad n = 2m+1, \tag{3.10}$$

$$\mu(\boldsymbol{Q}^{\mathbb{Z}}_{n,r};L_2)^2 = \mu(\boldsymbol{E}^{\mathbb{Z}}_{m,r};L_2)^2 + \mu(\boldsymbol{F}^{\mathbb{Z}}_{m-1,r};L_2)^2, \quad n = 2m. \tag{3.11}$$

Proof of Theorem 1.1. In view of (3.9) it is enough to prove that

$$c_{n,r}^{1/2} \le \mu(\boldsymbol{Q}^{\mathbb{Z}}_{n,r};L_2) < 1.014 c_{n,r}^{1/2}. \tag{3.12}$$

Suppose first that n is odd, $n = 2m+1$, $m \ge 3r+3$. From Propositions 3.2 and 3.4 we obtain respectively

$$\frac{a_{m,r}}{4} \le \mu(\boldsymbol{E}^{\mathbb{Z}}_{m,r};L_2)^2 \le (1+\alpha_{m,r})\frac{a_{m,r}}{4} \overset{(3.2)}{<} 1.027\frac{a_{m,r}}{4}, \tag{3.13}$$

$$\frac{b_{m,r}}{4} \le \mu(\boldsymbol{F}^{\mathbb{Z}}_{m,r};L_2)^2 \le (1+\beta_{m,r})\frac{b_{m,r}}{4} \overset{(3.6)}{<} 1.019\frac{b_{m,r}}{4}. \tag{3.14}$$

Consequently

$$\frac{a_{m,r}}{4} + \frac{b_{m,r}}{4} \le \mu(\boldsymbol{E}^{\mathbb{Z}}_{m,r};L_2)^2 + \mu(\boldsymbol{F}^{\mathbb{Z}}_{m,r};L_2)^2 < 1.027\biggl(\frac{a_{m,r}}{4} + \frac{b_{m,r}}{4}\biggr),$$

which means that

$$c_{n,r} \le \mu(\boldsymbol{Q}^{\mathbb{Z}}_{n,r};L_2)^2 < 1.027 c_{n,r} \tag{3.15}$$

according to (1.13) and (3.10). This yields (3.12).

If n is even, $n = 2m$, $m \ge 3r+4$, then the proof is analogous: replacing m by $m-1$ in (3.14), we obtain

$$\frac{a_{m,r}}{4} + \frac{b_{m-1,r}}{4} \le \mu(\boldsymbol{E}^{\mathbb{Z}}_{m,r};L_2)^2 + \mu(\boldsymbol{F}^{\mathbb{Z}}_{m-1,r};L_2)^2 < 1.027\biggl(\frac{a_{m,r}}{4} + \frac{b_{m-1,r}}{4}\biggr),$$

which means that (3.15) is true also in this case (here we use (1.14) and (3.11)). ∎

Proof of Theorem 1.2. From (3.10), (3.11) and Propositions 3.3 and 3.5 it follows directly that

$$\mu(\boldsymbol{Q}^{\mathbb{Z}}_{n,r};L_2) = \frac{\sqrt{2}}{2}\cdot\frac{C_r}{n^{2r+1}}(1+O(n^{-1}))$$

as $n \to \infty$. Hence, by (3.9), we obtain (1.15). ∎

4. The covering radius in the L_p norm. Let k,s be non-negative integers, $0 \le s \le k$. If $s \le k-1$, then we denote by $U_{k,s}$ the shortest (in the L_2 norm) polynomial in the hyperplane $\boldsymbol{E}_{k,s+1} + U_s$. It is clear that $\|U_{k,s}\|_2 = d_2(U_s, \boldsymbol{E}_{k,s+1})$. If $s = k$, then we define $U_{k,k} := U_k$. From Lemma

2.2 and (1.10) it follows that

$$\|U_{k,s}\|_2 = C_s \left[\frac{(2k-2s)!}{(2k+2s+2)!} \right]^{1/2}. \tag{4.1}$$

It is easily seen that if $0 \le s_1 \le k_1$ and $0 \le s_2 \le k_2$, then

$$U_{k_1,s_1} \cdot U_{k_2,s_2} \in \boldsymbol{E}_{k_1+k_2,s_1+s_2+1} + U_{s_1+s_2}.$$

Let $0 \le r \le m-1$. We denote

$$g_{m,r} := K_r \left[\frac{(m-r-1)!(m-r+1)!}{(m+r+1)!(m+r+3)!} \right]^{1/2}. \tag{4.2}$$

LEMMA 4.1. *Let $r \ge 0$ and $m \ge r+1$. There is a polynomial $Q_{m,r}$ in $\boldsymbol{E}_{m,r+1} + U_r$ with $\|Q_{m,r}\|_1 \le g_{m,r}$.*

Proof. We will consider four cases.

CASE 1: m even, $m = 2k$; r even, $r = 2s$. Let $Q_{m,r} := U_{k,s}^2$. Then $Q_{m,r} \in \boldsymbol{E}_{2k,2s+1} + U_{2s} = \boldsymbol{E}_{m,r+1} + U_r$ and

$$\|Q_{m,r}\|_1 = \|U_{k,s}\|_2^2 \overset{(4.1)}{=} C_s^2 \frac{(2k-2s)!}{(2k+2s+2)!} \overset{(1.8)}{=} K_r \frac{(m-r)!}{(m+r+2)!},$$

which, as easily checked, is less than $g_{m,r}$.

CASE 2: m odd, $m = 2k+1$; r even, $r = 2s$. Let $Q_{m,r} := U_{k,s} \cdot U_{k+1,s}$. Then $Q_{m,r} \in \boldsymbol{E}_{2k+1,2s+1} + U_{2s} = \boldsymbol{E}_{m,r+1} + U_r$ and, by the Schwarz inequality,

$$\begin{aligned} \|Q_{m,r}\|_1 &\le \|U_{k,s}\|_2 \cdot \|U_{k+1,s}\|_2 \\ &\overset{\text{(i)}}{=} C_s^2 \left[\frac{(2k-2s)!}{(2k+2s+2)!} \cdot \frac{(2k-2s+2)!}{(2k+2s+4)!} \right]^{1/2} \\ &\overset{(1.8)}{=} K_r \left[\frac{(m-r-1)!(m-r+1)!}{(m+r+1)!(m+r+3)!} \right]^{1/2} = g_{m,r}. \end{aligned} \tag{4.3}$$

To obtain equality (i) we use (4.1), and then (4.1) with k replaced by $k+1$.

CASE 3: m even, $m = 2k$; r odd, $r = 2s+1$. Let $Q_{m,r} := U_{k,s} \cdot U_{k,s+1}$. Then $Q_{m,r} \in \boldsymbol{E}_{2k,2s+2} + U_{2s+1} = \boldsymbol{E}_{m,r+1} + U_r$ and

$$\begin{aligned} \|Q_{m,r}\|_1 &\le \|U_{k,s}\|_2 \cdot \|U_{k,s+1}\|_2 \\ &\overset{\text{(ii)}}{=} C_s C_{s+1} \left[\frac{(2k-2s)!}{(2k+2s+2)!} \cdot \frac{(2k-2s-2)!}{(2k+2s+4)!} \right]^{1/2} \\ &\overset{(1.9)}{=} K_r \left[\frac{(m-r+1)!(m-r-1)!}{(m+r+1)!(m+r+3)!} \right]^{1/2} = g_{m,r}. \end{aligned} \tag{4.4}$$

To obtain equality (ii) we use (4.1), and then (4.1) with s replaced by $s+1$.

CASE 4: m odd, $m = 2k+1$; r odd, $r = 2s+1$. Let $Q_{m,r} := U_{k,s} \cdot U_{k+1,s+1}$. Then $Q_{m,r} \in \boldsymbol{E}_{2k+1,2s+2} + U_{2s+1} = \boldsymbol{E}_{m,r+1} + U_r$ and

$$\text{(4.5)} \qquad \begin{aligned} \|Q_{m,r}\|_1 &\le \|U_{k,s}\|_2 \cdot \|U_{k+1,s+1}\|_2 \\ &\overset{\text{(iii)}}{=} C_s C_{s+1} \left[\frac{(2k-2s)!}{(2k+2s+2)!} \cdot \frac{(2k-2s)!}{(2k+2s+6)!} \right]^{1/2} \\ &\overset{(1.9)}{=} K_r \left[\frac{(m-r)!(m-r)!}{(m+r)!(m+r+4)!} \right]^{1/2}, \end{aligned}$$

which is easily checked to be less than $g_{m,r}$. To obtain equality (iii) we use (4.1), and then (4.1) with k replaced by $k+1$ and s replaced by $s+1$. ∎

LEMMA 4.2. *Let $r \ge 0$ and $m \ge r+1$. There is a polynomial $R_{m,r}$ in $\boldsymbol{F}_{m,r+1} + V_r$ with $\|R_{m,r}\|_1 < g_{m,r}$.*

Proof. Let $Q_{m,r}$ be the polynomial from Lemma 4.1. Set $R_{m,r} = Q_{m,r} \cdot V_0$, where $V_0(x) = 2x - 1$. It is easily seen that $R_{m,r} \in \boldsymbol{F}_{m,r+1} + V_r$. It remains to observe that

$$\|R_{m,r}\|_1 = \|Q_{m,r} \cdot V_0\|_1 < \|Q_{m,r}\|_1 \cdot \|V_0\|_\infty = \|Q_{m,r}\|_1 \le g_{m,r}. \ ∎$$

LEMMA 4.3. *Let $r \ge 0$ and $m \ge 3r+6$. Then*

$$\text{(4.6)} \qquad \mu(\boldsymbol{E}^{\mathbb{Z}}_{m,r}; L_1) < 2^{-1} g_{m,r} + 2^{-1/2} a^{1/2}_{m,r+1},$$

$$\text{(4.7)} \qquad \mu(\boldsymbol{F}^{\mathbb{Z}}_{m,r}; L_1) < 2^{-1} g_{m,r} + 2^{-1/2} b^{1/2}_{m,r+1}.$$

Proof. The lattice $\boldsymbol{E}^{\mathbb{Z}}_{m,r}$ (resp. $\boldsymbol{E}^{\mathbb{Z}}_{m,r+1}$) is generated by $U_r, \ldots, U_m$ (resp. by $U_{r+1}, \ldots, U_m$); we may write

$$\boldsymbol{E}_{m,r} = \boldsymbol{E}_{m,r+1} + \mathbb{R} U_r, \qquad \boldsymbol{E}^{\mathbb{Z}}_{m,r} = \boldsymbol{E}^{\mathbb{Z}}_{m,r+1} + \mathbb{Z} U_r.$$

Let B be the closed unit ball in L_1. According to the definition of $\mu(\boldsymbol{E}^{\mathbb{Z}}_{m,r}; L_1)$ we have

$$\boldsymbol{E}_{m,r+1} \subset \boldsymbol{E}^{\mathbb{Z}}_{m,r+1} + \mu(\boldsymbol{E}^{\mathbb{Z}}_{m,r+1}; L_1) B,$$

and it is not hard to see that

$$\boldsymbol{E}_{m,r} \subset \boldsymbol{E}_{m,r+1} + \mathbb{Z} U_r + \tfrac{1}{2} d_1(U_r, \boldsymbol{E}_{m,r+1}) B.$$

Hence

$$\begin{aligned} \boldsymbol{E}_{m,r} &\subset \boldsymbol{E}^{\mathbb{Z}}_{m,r+1} + \mathbb{Z} U_r + \tfrac{1}{2} d_1(U_r, \boldsymbol{E}_{m,r+1}) B + \mu(\boldsymbol{E}^{\mathbb{Z}}_{m,r+1}; L_1) B \\ &= \boldsymbol{E}^{\mathbb{Z}}_{m,r+1} + \left[\tfrac{1}{2} d_1(U_r, \boldsymbol{E}_{m,r+1}) + \mu(\boldsymbol{E}^{\mathbb{Z}}_{m,r+1}; L_1)\right] B, \end{aligned}$$

which means that

$$\text{(4.8)} \qquad \mu(\boldsymbol{E}^{\mathbb{Z}}_{m,r}; L_1) \le \tfrac{1}{2} d_1(U_r, \boldsymbol{E}_{m,r+1}) + \mu(\boldsymbol{E}^{\mathbb{Z}}_{m,r+1}; L_1).$$

Lemma 4.1 says that

$$\text{(4.9)} \qquad d_1(U_r, \boldsymbol{E}_{m,r+1}) \le g_{m,r}.$$

As $\|\cdot\|_1 \le \|\cdot\|_2$, we have

$$\mu(\boldsymbol{E}^{\mathbb{Z}}_{m,r+1}; L_1) \le \mu(\boldsymbol{E}^{\mathbb{Z}}_{m,r+1}; L_2). \tag{4.10}$$

Finally, Proposition 3.2 (with r replaced by $r+1$) implies that

$$\mu(\boldsymbol{E}^{\mathbb{Z}}_{m,r+1}; L_2) < 2^{-1/2} a^{1/2}_{m,r+1}. \tag{4.11}$$

From (4.8)–(4.11) we obtain (4.6).

The proof of (4.7) is analogous; it is enough to replace $\boldsymbol{E}$ by $\boldsymbol{F}$, the polynomial U_r by V_r, Lemma 4.1 by Lemma 4.2, and Proposition 3.2 by Proposition 3.4. ∎

Let us denote

$$h_{n,r} := g_{m,r}, \qquad n = 2m+1,\ 0 \le r \le m-1,$$
$$h_{n,r} := \frac{g_{m,r} + g_{m-1,r}}{2}, \qquad n = 2m,\ 0 \le r \le m-2.$$

LEMMA 4.4. *Let $r \ge 0$ and $n \ge 6r+14$. Then*

$$\mu(\boldsymbol{P}^{\mathbb{Z}}_{n,r}; L_1) < h_{n,r} + 2c^{1/2}_{n,r+1}.$$

Proof. Suppose that n is even, $n = 2m$. Then $m \ge 3r+7$. From (1.6) it follows that $\mu(\boldsymbol{P}^{\mathbb{Z}}_{n,r}; L_1) \le \mu(\boldsymbol{E}^{\mathbb{Z}}_{m,r}; L_1) + \mu(\boldsymbol{F}^{\mathbb{Z}}_{m-1,r}; L_1)$. Next, by Lemma 4.3,

$$\mu(\boldsymbol{E}^{\mathbb{Z}}_{m,r}; L_1) + \mu(\boldsymbol{F}^{\mathbb{Z}}_{m-1,r}; L_1) < \frac{g_{m,r} + g_{m-1,r}}{2} + 2^{-1/2}(a^{1/2}_{m,r+1} + b^{1/2}_{m-1,r+1})$$
$$\le h_{n,r} + (a_{m,r+1} + b_{m-1,r+1})^{1/2} \overset{(1.14)}{=} h_{n,r} + 2c^{1/2}_{n,r+1}.$$

The proof for n odd is analogous. ∎

COROLLARY 4.5. *Let $r \ge 0$. Then*

$$\mu(\boldsymbol{P}^{\mathbb{Z}}_{n,r}; L_1) \le \frac{2^{2r+2}K_r}{n^{2r+2}}(1 + O(n^{-1})) \quad \text{as } n \to \infty. \tag{4.12}$$

Proof. It follows directly from (4.2) that

$$g_{m,r} = \frac{K_r}{m^{2r+2}}(1 + O(m^{-1})) \quad \text{as } m \to \infty.$$

Hence, by the definition of $h_{n,r}$, we have

$$h_{n,r} = \frac{2^{2r+2}K_r}{n^{2r+2}}(1 + O(n^{-1})) \quad \text{as } n \to \infty. \tag{4.13}$$

From (1.12), with r replaced by $r+1$, it follows that

$$c^{1/2}_{n,r+1} = O(n^{-2r-3}) = \frac{1}{n^{2r+2}} O(n^{-1}) \quad \text{as } n \to \infty. \tag{4.14}$$

Combining Lemma 4.4, (4.13) and (4.14) we obtain (4.12). ∎

Lemma 4.6. *Let* $0 \neq f \in L_\infty(0,1)$. *Then*

$$\frac{\|f\|_p}{\|f\|_1} \le \left(\frac{\|f\|_\infty}{\|f\|_2}\right)^{2-2/p}, \quad 1 \le p \le \infty, \tag{4.15}$$

$$\frac{\|f\|_p}{\|f\|_2} \le \left(\frac{\|f\|_\infty}{\|f\|_2}\right)^{1-2/p}, \quad 2 \le p \le \infty, \tag{4.16}$$

$$\frac{\|f\|_2}{\|f\|_p} \le \left(\frac{\|f\|_\infty}{\|f\|_2}\right)^{2/p-1}, \quad 1 \le p \le 2. \tag{4.17}$$

These inequalities follow easily from basic properties of L_p norms.

Lemma 4.7. *Let* $P \in \boldsymbol{P}_n$, *where* $n \ge 0$. *Then*

$$\|P\|_\infty \le (n+1)\|P\|_2. \tag{4.18}$$

This is an easy consequence of elementary properties of Legendre polynomials on $[0,1]$; see e.g. Labelle [La].

Lemma 4.8. *Let* $1 \le p \le q \le \infty$ *and let* $P \in \boldsymbol{P}_n$, *where* $n \ge 1$. *Then*

$$\|P\|_q \le [2(p+1)]^{1/p-1/q} n^{2/p-2/q} \|P\|_p. \tag{4.19}$$

This is a standard fact; see e.g. [T, Sec. 4.9.6].

Lemma 4.9. *Let* $P \in \boldsymbol{P}_n$, *where* $n \ge 0$. *Then*

$$\|P\|_p \le (n+1)^{2-2/p}\|P\|_1, \quad 1 \le p \le 2, \tag{4.20}$$

$$\|P\|_p \le (n+1)^{1-2/p}\|P\|_2, \quad 2 \le p \le \infty, \tag{4.21}$$

$$\|P\|_2 \le (n+1)^{2/p-1}\|P\|_p, \quad 1 \le p \le 2, \tag{4.22}$$

$$\|P\|_\infty \le 6^{1/2} n^{2/p}\|P\|_p, \quad 2 < p < \infty,\ n \ge 1. \tag{4.23}$$

Proof. Inequalities (4.20)–(4.22) follow from (4.15)–(4.17), respectively, and (4.18). Inequality (4.23) follows from (4.19) (for $q = \infty$). ∎

Corollary 4.10. *Let* $r \ge 0$ *and* $n \ge 2r$. *Then*

$$\mu(\boldsymbol{P}^{\mathbb{Z}}_{n,r}; L_p) \le (n+1)^{2-2/p}\mu(\boldsymbol{P}^{\mathbb{Z}}_{n,r}; L_1), \quad 1 \le p \le 2, \tag{4.24}$$

$$\mu(\boldsymbol{P}^{\mathbb{Z}}_{n,r}; L_p) \le (n+1)^{1-2/p}\mu(\boldsymbol{P}^{\mathbb{Z}}_{n,r}; L_2), \quad 2 \le p \le \infty, \tag{4.25}$$

$$\mu(\boldsymbol{P}^{\mathbb{Z}}_{n,r}; L_p) \ge (n+1)^{1-2/p}\mu(\boldsymbol{P}^{\mathbb{Z}}_{n,r}; L_2), \quad 1 \le p \le 2, \tag{4.26}$$

$$\mu(\boldsymbol{P}^{\mathbb{Z}}_{n,r}; L_p) \ge 6^{-1/2}\, n^{-2/p}\mu(\boldsymbol{P}^{\mathbb{Z}}_{n,r}; L_\infty), \quad 2 < p < \infty,\ n \ge 1. \tag{4.27}$$

Proof of Theorem 1.3. The first inequality in (1.16) follows from (4.26) and the first inequality in (1.15). The second inequality in (1.16) follows from (4.24) and (4.12). ∎

Proof of Theorem 1.4. Inequality (1.17) follows from (4.25) and the second inequality in (1.15). ∎

Proof of Proposition 1.5. Inequality (1.18) is an immediate consequence of (1.19) and (4.27). We will prove (1.19).

Let f be the linear functional on $(\boldsymbol{P}_n, \|\cdot\|_\infty)$ given by $f(P) = P^{(r)}(0)/r!$. We have $f(\boldsymbol{P}_n^{\mathbb{Z}}) = \mathbb{Z}$ and $f(U_r) = 1$. For any $P \in \boldsymbol{P}_n^{\mathbb{Z}}$ we may write

$$\tfrac{1}{2} \le \left|f\left(\tfrac{1}{2}U_r\right) - f(P)\right| \le \|f\| \cdot \left\|\tfrac{1}{2}U_r - P\right\|_\infty.$$

Hence

$$\mu(\boldsymbol{P}_{n,r}^{\mathbb{Z}}; L_\infty) \ge d_\infty\left(\tfrac{1}{2}U_r, \boldsymbol{P}_{n,r}^{\mathbb{Z}}\right) \ge \tfrac{1}{2}\|f\|^{-1}. \tag{4.28}$$

If $r = 0$, then, obviously, $\|f\| = 1$, whence $\mu(\boldsymbol{P}_{n,0}^{\mathbb{Z}}; L_\infty) \ge 1/2$ (in fact, one has equality here; see (1.1)). So, assume that $r \ge 1$.

If $P \in \boldsymbol{P}_n$, then, by the Markov inequality,

$$|P^{(r)}(0)| \le \|P^{(r)}\|_\infty \le 2^r \frac{n^2(n^2-1^2)\ldots(n^2-(r-1)^2)}{1\cdot 3\ldots(2r-1)}\|P\|_\infty.$$

Thus

$$\|f\| \le \frac{2^r}{r!} \cdot \frac{n^2(n^2-1^2)\ldots(n^2-(r-1)^2)}{1\cdot 3\ldots(2r-1)}.$$

We may write $n^2(n^2-1^2)\ldots(n^2-(r-1)^2) \le n^{2r}$ and $1\cdot 3\ldots(2r-1) = (2r)!/2^r r!$. Consequently, $\|f\| \le 2^{2r}n^{2r}/(2r)!$. Hence, (4.28) leads to (1.19). ∎

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