ANNALES
POLONICI MATHEMATICI
115.2 (2015)

Hyper-order and order of meromorphic functions sharing functions

by Jianming Qi (Shanghai), Wenjun Yuan (Guangzhou) and Hongxun Yi (Jinan)

Abstract. In this paper we mainly estimate the hyper-order of an entire function which shares one function with its derivatives. Some examples are given to show that the conclusions are meaningful.

1. Introduction and main results. In Nevanlinna theory, the order and the hyper-order of a meromorphic function are two important concepts. They are defined respectively by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}, \qquad \sigma(f) = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

The respective definitions for an entire function f are

$$\rho(f) = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r}, \qquad \sigma(f) = \limsup_{r\to\infty} \frac{\log\log\log M(r,f)}{\log r}$$

(see [9]).

Consider a rational function R which behaves asymptotically as cr^β as $r\to\infty$, where $c\neq 0$ and β are constants. Define the degree of R at infinity as $\deg R = \deg_\infty R = \max\{0,\beta\}$. Let $f(z)$ and $g(z)$ be two nonconstant meromorphic functions in the complex plane $\mathbb{C}$, and let $\alpha(z)$ be a meromorphic function or a finite complex number. If $g(z)-\alpha(z)=0$ whenever $f(z)-\alpha(z)=0$, we write $f(z)=\alpha(z)\Rightarrow g(z)=\alpha(z)$. If $f(z)=\alpha(z)\Rightarrow g(z)=\alpha(z)$ and $g(z)=\alpha(z)\Rightarrow f(z)=\alpha(z)$, we write $f(z)=\alpha(z)\Leftrightarrow g(z)=\alpha(z)$ and say that $f(z)$ and $g(z)$ share $\alpha(z)$ IM (ignoring multiplicity).

In 2008, Li and Gao [4] proved the following

2010 *Mathematics Subject Classification*: Primary 30D35; Secondary 30D45.
Key words and phrases: complex differential equation, normal family, Nevanlinna theory.

DOI: 10.4064/ap115-2-6

THEOREM 1.1. *Let Q_1 and Q_2 be nonzero polynomials, and let P be a polynomial. If f is a nonconstant solution of the equation*

$$f^{(k)} - Q_1 = e^P(f - Q_2),$$

then $\sigma(f) = n$, where k is a positive integer and n denotes the degree of P.

The uniqueness problem for meromorphic functions sharing values with their derivatives is closely related to some kind of complex differential equations. Therefore, it is of interest to consider the growth properties of meromorphic functions under conditions involving sharing value.

In 2012, Lü and Xu [6] obtained the following result.

THEOREM 1.2. *Let f be a nonconstant entire function, and let $\alpha = Pe^Q$ $(\alpha \neq \alpha')$ where P $(\neq 0)$ and Q are polynomials. If*

$$f(z) = \alpha(z) \Rightarrow f'(z) = \alpha(z) \quad \textit{and} \quad f'(z) = \alpha(z) \Rightarrow f''(z) = \alpha(z),$$

then f is of finite order.

We propose four natural questions, related to Theorem 1.2.

PROBLEM 1.3. *From Theorem 1.2, we see that f, f', f'' share a function of finite order. What will happen if they share a function of infinite order?*

PROBLEM 1.4. *Can the polynomial P be replaced by a rational function R?*

PROBLEM 1.5. *Can f be a meromorphic function in Theorem 1.2?*

PROBLEM 1.6. *Can the order of f be estimated sharply?*

In this paper, we discuss the above problems and derive the following results.

MAIN THEOREM 1.7. *Let P be a polynomial and f, γ be entire functions. If*

$$f(z) = \alpha(z) \Rightarrow f'(z) = \alpha(z) \quad \textit{and} \quad f'(z) = \alpha(z) \Rightarrow f''(z) = \alpha(z),$$

where $\alpha = Pe^\gamma$ $(\alpha \neq \alpha')$, and if $\alpha - \alpha'$ has at most finitely many zeros, then $\sigma(f) \leq \sigma(\alpha) = \rho(\gamma)$.

REMARK. The following examples show that our conclusion $\sigma(f) \leq \rho(\gamma)$ is sharp.

EXAMPLE 1.8. Let $f(z) = Ae^z$, where A is a nonzero constant. Let $\alpha(z) = e^{e^{-z}+z}$. Noting that $f = f' = f''$, we have

$$f(z) = \alpha(z) \Rightarrow f'(z) = \alpha(z) \quad \text{and} \quad f'(z) = \alpha(z) \Rightarrow f''(z) = \alpha(z).$$

Obviously, $\alpha(z) - \alpha'(z) = e^{e^{-z}}$ has no zeros. Thus it satisfies the assumptions of Theorem 1.7 and $\sigma(f) = 0 < \sigma(\alpha) = 1$.

Example 1.9. Let $f(z) = 2e^z$ and $\alpha(z) = (4z^2 - z + 2)e^{z^2}$. Noting that $f = f' = f''$, we see that

$$f(z) = \alpha(z) \Rightarrow f'(z) = \alpha(z) \quad \text{and} \quad f'(z) = \alpha(z) \Rightarrow f''(z) = \alpha(z).$$

Thus f satisfies the assumptions of Theorem 1.7 and $\sigma(f) = 0$.

Example 1.10. Let $f(z) = 4z^2 - 8z + 8$ and $\alpha(z) = 2z^2$. Note that $f \neq f' \neq f''$, and $f(z) - \alpha(z) = 2(z-2)^2$, $f'(z) - \alpha(z) = -2(z-2)^2$ and $f''(z) - \alpha(z) = 2(2-z)(2+z)$. It is easy to see $f(z) = \alpha(z) \Rightarrow f'(z) = \alpha(z)$ and $f'(z) = \alpha(z) \Rightarrow f''(z) = \alpha(z)$ and $\sigma(f) = \sigma(\alpha)$. Thus f satisfies the assumptions of Theorem 1.7 and $\sigma(f) = \sigma(\alpha) = 0$.

Example 1.11. Let $f(z) = z^4 Ae^z + z^4 + 8z^3 + 24z^2 + 48z + 48$ and $\alpha(z) = z^4 + 8z^3 + 24z^2 + 48z + 48$, where $A = e^4$ is a constant. Differentiating f twice yields $f'(z) = (z^4 + 4z^3)Ae^z + 4z^3 + 24z^2 + 48z + 48$ and $f''(z) = (z^4 + 8z^3 + 12z^2)Ae^z + 12z^2 + 48z + 48$. Then $f(z) = \alpha(z) \Rightarrow f'(z) = \alpha(z)$ and $f'(z) = \alpha(z) \Rightarrow f''(z) = \alpha(z)$. Thus $\sigma(f) \leq \sigma(\alpha)$, but $f \neq f'$.

Remark. The condition that $\alpha - \alpha'$ has at most finitely many zeros is essential in our proof of Theorem 1.7. But we do not know whether it is necessary or not. If γ is a polynomial, then this condition obviously holds.

Remark. In Theorem 1.7, if the order of γ is zero, for example γ is a polynomial, then $\sigma(f) = 0$, which is an important property for a meromorphic function f.

Main Theorem 1.12. *Let f be a meromorphic function with at most finitely many poles, and let $\alpha = Re^Q$ $(\alpha \neq \alpha')$, where R $(\neq 0)$ is a rational function and Q is a nonconstant polynomial. If*

$$f(z) = \alpha(z) \Rightarrow f'(z) = \alpha(z) \quad \textit{and} \quad f'(z) = \alpha(z) \Rightarrow f''(z) = \alpha(z),$$

then $\rho(f) \leq 2 \deg Q$.

Main Theorem 1.13. *Let f be a nonconstant entire function, and let $\alpha = Re^Q$ $(\alpha \neq \alpha')$, where R $(\neq 0)$ is a rational function and Q is a nonconstant polynomial. If*

$$f(z) = \alpha(z) \Rightarrow f'(z) = \alpha(z) \quad \textit{and} \quad f'(z) = \alpha(z) \Rightarrow f''(z) = \alpha(z),$$

then $\rho(f) \leq \deg Q$.

Example 1.14. Let $f(z) = 2e^z$, $\alpha(z) = ze^{z+1}$, so $\rho(f) = 1$, $\deg Q = 1$. Note that

$$f(z) = \alpha(z) \Rightarrow f'(z) = \alpha(z) \quad \text{and} \quad f'(z) = \alpha(z) \Rightarrow f''(z) = \alpha(z).$$

Thus $\rho(f) = 1 \leq 1$.

Remark. Example 5 illustrates both Theorem 1.12 and Theorem 1.13.

2. Some lemmas. In order to prove our theorems, we need the following lemmas.

Using the famous Pang–Zalcman lemma [8, Lemma 2] and the result of F. Lü, J. F. Xu and A. Chen [7, Lemma 2.1, p. 595], it is easy to obtain the following lemma. It plays an important role in the proofs of Theorems 1.7 and 1.12.

LEMMA 2.1. *Let $\{f_n\}$ be a family of meromorphic (resp. analytic) functions in the unit disc $\triangle$. If $a_n \to a$, $|a| < 1$, and $f_n^\sharp(a_n) \to \infty$, and if there exists $A \geq 1$ such that $|f_n'(z)| \leq A$ whenever $f_n(z) = 0$, then there exist*

(i) *a subsequence of f_n (still denoted $\{f_n\}$),*
(ii) *points $z_n \to z_0$, $|z_0| < 1$,*
(iii) *positive numbers $\rho_n \to 0$,*

such that $\rho_n^{-1} f_n(z_n + \rho_n \xi) = g_n(\xi) \to g(\xi)$ locally uniformly, where g is a nonconstant meromorphic (resp. entire) function on $\mathbb{C}$ such that $\rho(g) \leq 2$ (resp. $\rho(g) \leq 1$), $g^\sharp(\xi) \leq g^\sharp(0) = A + 1$ and

$$\rho_n \leq M/f_n^\sharp(a_n),$$

where M is a constant independent of n.

Here, as usual, $g^\sharp(\xi) = \frac{|g'(\xi)|}{1+|g(\xi)|^2}$ is the spherical derivative.

LEMMA 2.2 ([5]). *Let f be a meromorphic function with $\sigma(f) > 0$. Then, for any $\epsilon > 0$, there exists a sequence $z_n \to \infty$ such that $f^\sharp(z_n) > e^{|z_n|^{\sigma(f)-\epsilon}}$ if n is large enough.*

LEMMA 2.3 ([10]). *Let $f(z)$ be a meromorphic function in the complex plane with $\rho(f) > 2$. Then for each $0 < \mu < (\rho(f) - 2)/2$, there exist points $a_n \to \infty$ $(n \to \infty)$ such that*

$$\lim_{n\to\infty} \frac{f^\sharp(a_n)}{|a_n|^\mu} = \infty.$$

LEMMA 2.4 ([3]). *Let $f(z)$ be an entire function with $\rho(f) > 1$. Then for each $0 < \mu < \rho(f) - 1$, there exist points $a_n \to \infty$ $(n \to \infty)$ such that*

$$\lim_{n\to\infty} \frac{f^\sharp(a_n)}{|a_n|^\mu} = \infty.$$

LEMMA 2.5 ([1]). *Let g be a nonconstant entire function with $\rho(g) \leq 1$, let $k \geq 2$ be an integer, and let a be a nonzero finite value. If $g(z) = 0 \Rightarrow g'(z) = a$ and $g'(z) = a \Rightarrow g^{(k)}(z) = 0$, then $g(z) = a(z - z_0)$, where z_0 is a constant.*

LEMMA 2.6 ([9]). *Suppose $f(z)$ and $g(z)$ are nonconstant meromorphic functions in the complex plane. Then*

$$\rho(fg) \leq \max\{\rho(f), \rho(g)\}, \qquad \rho(f + g) \leq \max\{\rho(f), \rho(g)\}.$$

3. Proof of Theorem 1.7. In the proof, we use some ideas of [2, 7, 5].

Since $\alpha = Pe^{\gamma}$, we have $\sigma(\alpha) = \rho(\gamma)$. So, we just need to obtain $\sigma(f) \leq \rho(\gamma)$.

On the contrary, assume that $\sigma(f) = d > c = \rho(\gamma)$. Set $H = f - \alpha$. Then

(I) $H(z) = 0 \Rightarrow H'(z) = \alpha(z) - \alpha'(z)$,
(II) $H'(z) = \alpha - \alpha' \Rightarrow H''(z) = \alpha - \alpha''$.

Set

$$\beta = \alpha - \alpha' = (P - P' - P\gamma')e^{\gamma}, \qquad \varphi = \alpha - \alpha'' = (P'' + 2P'\gamma' + P\gamma'' + P\gamma'^2)e^{\gamma}.$$

Set $F = H/\beta$. Obviously, $\sigma(F) = \sigma(f) = d$. By Lemma 2.2, for $0 < \epsilon < (d - c)/2$, there exists a sequence $w_n \to \infty$ as $n \to \infty$ such that

$$F^{\sharp}(w_n) > e^{|w_n|^{\sigma(F) - \epsilon}} = e^{|w_n|^{d - \epsilon}}.$$

As $\beta = \alpha - \alpha'$ has finitely many zeros, there exists a positive number r such that F has no poles in $D = \{z : |z| > r\}$.

In view of $w_n \to \infty$ as $n \to \infty$, we may assume $|w_n| \geq r + 1$ for all n. Define $D_1 = \{z : |z| < 1\}$ and

$$F_n(z) = F(w_n + z) = \frac{H(w_n + z)}{\beta(w_n + z)};$$

then every F_n is analytic in D_1. Now, fix $z \in D_1$. If $F_n(z) = 0$, then $H(w_n + z) = 0$. It is clear from (I) that $H'(w_n + z) = \beta(w_n + z)$. Hence (for n large enough)

$$|F_n'(z)| = \left| \frac{H'(w_n + z)}{\beta(w_n + z)} - \frac{H(w_n + z)}{\beta(w_n + z)} \frac{\beta'(w_n + z)}{\beta(w_n + z)} \right| = 1.$$

Also $F_n^{\sharp}(0) \to \infty$ as $n \to \infty$. It follows from Marty's criterion that $(F_n)_n$ is not normal at $z = 0$.

Therefore, we can apply Lemma 2.1. Choosing an appropriate subsequence of $(F_n)_n$ if necessary, we may assume that there exist sequences $(z_n)_n$ and $(\rho_n)_n$ with $|z_n| < r < 1$ and $\rho_n \to 0$ such that

$$g_n(\zeta) := \rho_n^{-1} F_n(z_n + \rho_n \zeta) = \rho_n^{-1} \frac{H(w_n + z_n + \rho_n \zeta)}{\beta(w_n + z_n + \rho_n \zeta)} \to g(\zeta) \tag{3.1}$$

locally uniformly in $\mathbb{C}$, where g is a nonconstant entire function of order at most 1. Moreover, $g^{\sharp}(\xi) \leq g^{\sharp}(0) = 2$ for all $\xi \in \mathbb{C}$ and

$$\rho_n \leq \frac{M}{F_n^{\sharp}(0)} = \frac{M}{F^{\sharp}(w_n)} \leq M e^{-|w_n|^{d - \epsilon}} \tag{3.2}$$

for a positive number M.

From (3.1), we have

$$(3.3)\quad \begin{aligned} g_n'(\zeta) &= \frac{H'(w_n+z_n+\rho_n\zeta)}{\beta(w_n+z_n+\rho_n\zeta)} - \frac{H(w_n+z_n+\rho_n\zeta)\beta'(w_n+z_n+\rho_n\zeta)}{\beta(w_n+z_n+\rho_n\zeta)\beta(w_n+z_n+\rho_n\zeta)} \\ &= \frac{H'(w_n+z_n+\rho_n\zeta)}{\beta(w_n+z_n+\rho_n\zeta)} - \rho_n g_n(\zeta)\frac{\beta'(w_n+z_n+\rho_n\zeta)}{\beta(w_n+z_n+\rho_n\zeta)} \to g'(\zeta). \end{aligned}$$

Since $\beta = \alpha - \alpha' = (P - P' - P\gamma')e^{\gamma}$ and $\varphi = \alpha - \alpha'' = (P'' + 2P'\gamma' + P\gamma'' + P\gamma'^2)e^{\gamma}$, we have

$$\frac{\beta'}{\beta} = \frac{P' + P\gamma' - P'' - 2P'\gamma' - P\gamma'' - P\gamma'^2}{P - P' - P\gamma'}$$

and $\rho(\gamma'') = \rho(\gamma') = \rho(\gamma'^2) = \rho(\gamma) = c$. In view of the definition of order, we have

$$(3.4)\quad \begin{aligned} \left|\frac{\beta'}{\beta}\Big|_{z=w_n+z_n+\rho_n\zeta}\right| &\le |w_n|^q M(|w_n+z_n+\rho_n\zeta|, \gamma') \\ &\le |w_n|^q M(2|w_n|, \gamma') \le |w_n|^q e^{A|w_n|^{c+\epsilon}}, \end{aligned}$$

where A is a positive constant and q is an integer. As $0 < \epsilon < (d-c)/2$, we have $d - \epsilon > c + \epsilon$. Combining (3.2) and (3.4) yields

$$(3.5)\quad \begin{aligned} \left|\frac{H(w_n+z_n+\rho_n\zeta)\beta'(w_n+z_n+\rho_n\zeta)}{\beta(w_n+z_n+\rho_n\zeta)\beta(w_n+z_n+\rho_n\zeta)}\right| &= \left|\rho_n g_n(\zeta)\frac{\beta'(w_n+z_n+\rho_n\zeta)}{\beta(w_n+z_n+\rho_n\zeta)}\right| \\ &\le M|g_n(\zeta)|\,|w_n|^q e^{A|w_n|^{c+\epsilon} - |w_n|^{d-\epsilon}} \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

From (3.3) and (3.5), we deduce that

$$(3.6)\quad \frac{H'(w_n+z_n+\rho_n\zeta)}{\beta(w_n+z_n+\rho_n\zeta)} \to g'(\zeta).$$

In a similar way, we obtain

$$(3.7)\quad \rho_n\frac{H''(w_n+z_n+\rho_n\zeta)}{\beta(w_n+z_n+\rho_n\zeta)} \to g''(\zeta).$$

We claim that

(1) $g(\zeta) = 0 \Rightarrow g'(\zeta) = 1$,
(2) $g'(\zeta) = 1 \Rightarrow g''(\zeta) = 0$.

Suppose that $g(\zeta_0) = 0$. Then by Hurwitz's theorem there exist $\zeta_n \to \zeta_0$ such that (for n sufficiently large)

$$g_n(\zeta_n) = \rho_n^{-1}\frac{H(w_n+z_n+\rho_n\zeta_n)}{\beta(w_n+z_n+\rho_n\zeta_n)} = 0.$$

Thus $H(w_n+z_n+\rho_n\zeta_n) = 0$, and by (I) we have

$$(3.8)\quad H'(w_n+z_n+\rho_n\zeta_n) = \beta(w_n+z_n+\rho_n\zeta_n).$$

From (3.6), we derive

$$g'(\zeta_0) = \lim_{n\to\infty} \frac{H'(w_n + z_n + \rho_n\zeta_n)}{\beta(w_n + z_n + \rho_n\zeta_n)} = 1,$$

which implies that $g(\zeta) = 0 \Rightarrow g'(\zeta) = 1$.

To prove (2), suppose that $g'(\eta_0) = 1$. We know that $g' \not\equiv 1$, since otherwise $g^\sharp(0) \le 1 < 2$, a contradiction. Hence by (3.6) and Hurwitz's theorem, there exist $\eta_n \to \eta_0$ such that (for n sufficiently large)

$$H'(w_n + z_n + \rho_n\eta_n) = \beta(w_n + z_n + \rho_n\eta_n).$$

It is obvious from (II) that $H''(w_n + z_n + \rho_n\eta_n) = \varphi(w_n + z_n + \rho_n\eta_n)$. By (3.7), similarly to (3.4) and (3.5), we obtain

$$\begin{aligned}(3.9)\quad g''(\eta_0) &= \lim_{n\to\infty} \rho_n \frac{H''(w_n + z_n + \rho_n\eta_n)}{\beta(w_n + z_n + \rho_n\eta_n)} = \lim_{n\to\infty} \rho_n \frac{\varphi(w_n + z_n + \rho_n\eta_n)}{\beta(w_n + z_n + \rho_n\eta_n)} \\ &= \lim_{n\to\infty} \rho_n \frac{(P'' + 2P'\gamma' + P\gamma'' + P\gamma'^2)(w_n + z_n + \rho_n\eta_n)}{(P - P' - P\gamma')(w_n + z_n + \rho_n\eta_n)} = 0,\end{aligned}$$

which yields (2).

From Lemma 2.5, it is now easy to deduce that $g(\zeta) = \zeta - b_0$, where b_0 is a constant; then $g^\sharp(0) \le 1 < 2$, which is also a contradiction. This completes the proof of Theorem 1.7.

4. Proof of Theorem 1.12. We mimic the previous proof, so we will omit the identical calculations.

Set $H = f - \alpha$. Then

(I) $H(z) = 0 \Rightarrow H'(z) = \alpha(z) - \alpha'(z)$,
(II) $H'(z) = \alpha - \alpha' \Rightarrow H''(z) = \alpha - \alpha''$.

Set

$$\begin{aligned}\beta &= \alpha - \alpha' = (R - R' - RQ')e^Q = R_1 e^Q, \\ \varphi &= \alpha - \alpha'' = (R - R'' - 2R'Q' - RQ'' - RQ'^2)e^Q = R_2 e^Q,\end{aligned}$$

where R_1 ($\neq 0$) and R_2 are rational functions. Set $F = H/\beta$.

Suppose that $\rho(F) > 2\deg Q$. By Lemma 2.3, for every $0 < \mu < (\rho(F) - 2)/2$, there exist $w_n \to \infty$ such that

$$\lim_{n\to\infty} \frac{F^\sharp(w_n)}{|w_n|^\mu} = \infty. \tag{4.1}$$

Since $\beta = \alpha - \alpha'$ has at most finitely many zeros and f has finitely many poles, there exists $r > 0$ such that F has no poles in $D = \{z : |z| > r\}$.

As $w_n \to \infty$, we may assume that $|w_n| \geq r+1$ for all n. Define $D_1 = \{z : |z| < 1\}$ and

$$F_n(z) = F(w_n + z) = \frac{H(w_n+z)}{\beta(w_n+z)};$$

then every F_n is analytic in D_1. Now, fix $z \in D_1$. If $F_n(z) = 0$, then $H(w_n+z) = 0$. It is clear from (I) that $H'(w_n+z) = \beta(w_n+z)$. Hence (for n large enough) $|F_n'(z)| = 1$. Also $F_n^\sharp(0) \to \infty$ as $n \to \infty$. It follows from Marty's criterion that $(F_n)_n$ is not normal at $z = 0$.

Therefore, we can apply Lemma 2.1. Choosing an appropriate subsequence of $(F_n)_n$ if necessary, we may assume that there exist sequences $(z_n)_n$ and $(\rho_n)_n$ with $|z_n| < r < 1$ and $\rho_n \to 0$ such that

$$g_n(\zeta) := \rho_n^{-1} F_n(z_n + \rho_n \zeta) \to g(\zeta) \tag{4.2}$$

locally uniformly in $\mathbb{C}$, where g is a nonconstant entire function of order at most 1. Moreover, $g^\sharp(\xi) \leq g^\sharp(0) = 2$ for all $\xi \in \mathbb{C}$ and

$$\rho_n \leq \frac{M}{F_n^\sharp(0)} = \frac{M}{F^\sharp(w_n)} \leq M|w_n|^{-\mu-\epsilon} \tag{4.3}$$

for a positive number M.

From (4.2) we have, as in (3.3),

$$g_n'(\zeta) \to g'(\zeta). \tag{4.4}$$

Since $\beta = \alpha - \alpha' = R_1 e^Q$ and $\varphi = \alpha - \alpha'' = R_2 e^Q$, we have

$$\begin{aligned} \left| \frac{\beta'}{\beta}\Big|_{z=w_n+z_n+\rho_n\zeta} \right| &= \left| \frac{R_1' + R_1 Q'}{R_1}\Big|_{z=w_n+z_n+\rho_n\zeta} \right| \\ &= \left| \frac{R' - R'' - 2R'Q' - RQ'' + RQ' - RQ'^2}{R - R' - RQ'}\Big|_{z=w_n+z_n+\rho_n\zeta} \right| = O(|w_n|^{l_1}). \end{aligned} \tag{4.5}$$

By (4.1) and (4.3), we deduce that

$$\lim_{n\to\infty} w_n^{l_1} \rho_n = 0, \tag{4.6}$$

where

$$\begin{aligned} l_1 &= \deg \frac{R' - R'' - 2R'Q' - RQ'' + RQ' - RQ'^2}{R - R' - RQ'} \\ &= \deg \frac{\frac{R'}{R} - \frac{R''}{R} - 2\frac{R'}{R}Q' - Q'' + Q' - Q'^2}{1 - \frac{R'}{R} - Q'} = \deg Q' \end{aligned}$$

is a fixed constant. Combining (4.1), (4.3), (4.5) and (4.6) yields, as in (3.5),

$$\left| \frac{H(w_n+z_n+\rho_n\zeta)\beta'(w_n+z_n+\rho_n\zeta)}{\beta(w_n+z_n+\rho_n\zeta)\beta(w_n+z_n+\rho_n\zeta)} \right| \leq M|g_n(\zeta)|\,|w_n|^{l_1-\mu-\epsilon} \to 0 \quad \text{as } n \to \infty. \tag{4.7}$$

From (4.4) and (4.7), we deduce that

$$\frac{H'(w_n + z_n + \rho_n\zeta)}{\beta(w_n + z_n + \rho_n\zeta)} \to g'(\zeta).$$

In a similar way,

$$\rho_n \frac{H''(w_n + z_n + \rho_n\zeta)}{\beta(w_n + z_n + \rho_n\zeta)} \to g''(\zeta).$$

We claim that

(1) $g(\zeta) = 0 \Rightarrow g'(\zeta) = 1$,
(2) $g'(\zeta) = 1 \Rightarrow g''(\zeta) = 0$.

The proof of (1) is exactly the same as the proof of Theorem 1.7. To prove (2), just replace (3.9) in the previous proof by

$$\begin{aligned} g''(\eta_0) &= \lim_{n\to\infty} \rho_n \frac{H''(w_n + z_n + \rho_n\eta_n)}{\beta(w_n + z_n + \rho_n\eta_n)} \\ &= \lim_{n\to\infty} \rho_n \frac{\varphi(w_n + z_n + \rho_n\eta_n)}{\beta(w_n + z_n + \rho_n\eta_n)} \\ &= \lim_{n\to\infty} \rho_n \frac{(R - R'' - 2R'Q' - RQ'' - RQ'^2)(w_n + z_n + \rho_n\eta_n)}{(R - R' - RQ')(w_n + z_n + \rho_n\eta_n)} \\ &= \lim_{n\to\infty} \rho_n (O(|w_n|^{l_2})), \end{aligned}$$

where

$$\begin{aligned} l_2 &= \deg \frac{R' - R'' - 2R'Q' - RQ'' - RQ'^2}{R - R' - RQ'} \\ &= \deg \frac{\frac{R'}{R} - \frac{R''}{R} - 2\frac{R'}{R}Q' - Q'' - Q'^2}{1 - \frac{R'}{R} - Q'} = \deg Q' \end{aligned}$$

is also a fixed constant.

By (4.1) and (4.3), we deduce that

$$\lim_{n\to\infty} w_n^{l_2}\rho_n = 0, \tag{4.8}$$

which yields (2).

From Lemma 2.5, it is easy to deduce that $g(\zeta) = \zeta - b_0$, where b_0 is a constant; then $g^\sharp(0) \le 1 < 2$, which is also a contradiction.

So $\rho(F) \le 2\deg Q$.

Next we will prove $\rho(f) \le \rho(F)$. We distinguish three cases.

Case 1. If $\rho(\alpha) < \rho(f)$, then since $\rho(\alpha - \alpha') \le \rho(\alpha)$, by Lemma 2.6 we have $\rho(F(\alpha - \alpha')) \le \max\{\rho(F), \rho(\alpha)\}$. Due to $F = \frac{f-\alpha}{\alpha-\alpha'}$ we have $f = \alpha + F(\alpha - \alpha')$. Thus, by Lemma 2.6, $\rho(f) \le \max\{\rho(\alpha), \rho(F(\alpha - \alpha'))\} \le \max\{\rho(\alpha), \rho(F)\}$, and $\rho(\alpha) < \rho(f)$ yields $\rho(f) \le \rho(F) \le 2\deg Q$.

Case 2. If $\rho(\alpha) = \rho(f)$, then since $\rho(\alpha) = \deg Q$, we have $\rho(f) = \rho(\alpha) = \deg Q$.

Case 3. If $\rho(\alpha) > \rho(f)$, then since

$$F = \frac{f-\alpha}{\alpha-\alpha'} = \frac{f-Re^Q}{R_1e^Q} = \frac{f}{R_1e^Q} - \frac{R}{R_1}$$

and R/R_1 is a rational function, because of $\rho(R/R_1) = 0$ and $\rho(\alpha) = \deg Q = \rho(R_1e^Q) > \rho(f)$, we obtain $\rho(f) < \deg Q = \rho(F)$.

Thus, we have proved that $\rho(f) \le \rho(F) \le 2\deg Q$. This completes the proof of Theorem 1.12.

5. Proof of Theorem 1.13. Similar to the proof of Theorem 1.12, we also set $H = f - \alpha$. Then

(I) $H(z) = 0 \Rightarrow H'(z) = \alpha(z) - \alpha'(z)$,
(II) $H'(z) = \alpha - \alpha' \Rightarrow H''(z) = \alpha - \alpha''$.

Set

$$\beta = \alpha - \alpha' = (R - R' - RQ')e^Q = R_1e^Q,$$
$$\varphi = \alpha - \alpha'' = (R - R'' - 2R'Q' - RQ'' - RQ'^2)e^Q = R_2e^Q,$$

where R_1 ($\neq 0$) and R_2 are rational functions. Set $F = H/\beta$.

If $\rho(F) > \deg Q$, by Lemma 2.4, for every $0 < \mu < \rho(f) - 1$, there exist $w_n \to \infty$ such that

$$\lim_{n\to\infty} \frac{F^\sharp(w_n)}{|w_n|^\mu} = \infty. \tag{5.1}$$

The remainder of the proof is very similar to the proof of Theorem 1.12, so we omit it.

Acknowledgements. The authors wish to thank the anonymous referee for his/her helpful comments and useful suggestions. This work was supported by Tianyuan Youth Fund of the NSF of China (grant no. 11326083), the NSF of China (grant no. 11171184, grant no. 11271090), Shanghai University Young Teacher Training Program (ZZSDJ12020), Innovation Program of Shanghai Municipal Education Commission (14YZ164), NSF of Guangdong Province (S2012010010121), and projects 13XKJC01 from the Leading Academic Discipline Project of Shanghai Dianji University.

References

[1] J. M. Chang, M. L. Fang and L. Zalcman, *Normal families of holomorphic functions*, Illinois J. Math. 48 (2004), 319–337.

[2] J. Grahl and C. Meng, *Entire functions sharing a polynomial with their derivatives and normal families*, Analysis (Munich) 28 (2008), 51–61.
[3] R. M. Gu, Z. R. Li and W. J. Yuan, *The growth of entire solutions of some algebraic differential equations*, Georgian Math. J. 18 (2011), 489–495.
[4] X. M. Li and C. C. Gao, *Entire functions sharing one polynomial with their derivatives*, Proc. Indian Acad. Sci. Math. Sci. 118 (2008), 13–26.
[5] F. Lü and J. M. Qi, *A note on the hyper-order of entire functions*, Bull. Korean Math. Soc. 50 (2013), 1209–1219.
[6] F. Lü and J. F. Xu, *Entire functions that share a function with their first and second derivatives*, Ann. Polon. Math. 104 (2012), 81–96.
[7] F. Lü, J. F. Xu and A. Chen, *Entire functions sharing polynomials with their first derivatives*, Arch. Math. (Basel) 92 (2009), 593–601.
[8] X. C. Pang and L. Zalcman, *Normal families and shared values*, Bull. London Math. Soc. 32 (2000), 325–331.
[9] H. X. Yi and C. C. Yang, *Uniqueness Theory of Meromorphic Functions*, Science Press, Beijing, 1995.
[10] W. J. Yuan, B. Xiao and J. J. Zhang, *The general result of Gol'dberg's theorem concerning the growth of meromorphic solutions of algebraic differential equations*, Comput. Math. Appl. 58 (2009), 1788–1791.

Jianming Qi
Department of Mathematics and Physics
Shanghai Dianji University
201306, Shanghai, China
E-mail: qijianmingsdju@163.com

Wenjun Yuan
School of Mathematics
and Information Science
Guangzhou University
510006, Guangzhou, China
E-mail: wjyuan1957@126.com

Hongxun Yi
School of Mathematics
Shandong University
250100, Jinan, China
E-mail: hxyi@sdu.edu.cn

Received 7.10.2014
and in final form 18.9.2015 (3398)