ANNALES
POLONICI MATHEMATICI
115.3 (2015)

Rigidity of Einstein manifolds and generalized quasi-Einstein manifolds

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Abstract. We discuss the rigidity of Einstein manifolds and generalized quasi-Einstein manifolds. We improve a pinching condition used in a theorem on the rigidity of compact Einstein manifolds. Under an additional condition, we confirm a conjecture on the rigidity of compact Einstein manifolds. In addition, we prove that every closed generalized quasi-Einstein manifold is an Einstein manifold provided $\mu = -1/(n-2)$, $\lambda \leq 0$ and $\beta \leq 0$.

1. Introduction. Recently, many authors have shown their interest in rigid properties of manifolds with various curvature conditions. One of the most important results is the $\frac{1}{4}$-pinching sphere theorem. Its formulation in [BS] states that a compact Riemannian manifold is diffeomorphic to a spherical space form provided it has positive sectional curvature and the ratio of the minimum and the maximum of the sectional curvatures is always strictly greater than a quarter. This result can also be found in [CD].

In [XG], Xu and Gu studied the rigidity of compact Einstein manifolds with positive scalar curvature. Let $K(\pi)$ be the sectional curvature of M for the 2-plane $\pi \subset T_xM$, and set $K_{\max}(x) = \max_{\pi\subset T_xM} K(\pi)$, $K_{\min}(x) = \min_{\pi\subset T_xM} K(\pi)$. Theorem 1.1 in [XG] states that if M is an n-dimensional compact Einstein manifold with $n \geq 4$ and $R_0 > \left[1 - \frac{6}{5(n-1)}\right] K_{\max}$, then M is isometric to a spherical space form of constant curvature c, where R_0 is the normalized scalar curvature of M and $R_0 = c$. In addition, Xu and Gu [XG] proposed the following conjecture.

Conjecture A. *Let M be an n-dimensional compact Einstein manifold with $n \geq 4$. If $R_0 > \frac{3}{5} K_{\max}$, then M is isometric to a spherical space form.*

In general, Conjecture A is very difficult to prove. If $n = 4$, from [XG, Theorem 1.1] we conclude that Conjecture A is true. Denote by R

2010 *Mathematics Subject Classification*: Primary 53C25; Secondary 53C21.
Key words and phrases: rigidity, Einstein manifolds, generalized quasi-Einstein manifold, spherical space form.

DOI: 10.4064/ap115-3-3

the scalar curvature of M. From [CR] and [S], we know that $\mathbb{S}^4$ and $\mathbb{CP}^2$ are the only compact simply-connected four-dimensional manifolds with positive bi-orthogonal curvature that can have (weakly) $\frac{1}{4}$-pinched bi-orthogonal curvature, or non-negative isotropic curvature, or satisfy $K^{\perp} \geq R/24 > 0$. This shows that Conjecture A is true in dimension four under a pinching condition weaker than $R_0 > \frac{3}{5}K_{\max}$. In fact, it follows from [MM] that if $w^{\pm} \leq 0$, then M^4 has non-negative isotropic curvature, where $w^{\pm}$ is the largest eigenvalue of the Weyl tensor $W^{\pm}$. Therefore, Conjecture A is true under weaker pinching conditions in dimension four. However, at present a solution for general dimension is not known.

Inspired by [XG] and the $\frac{1}{4}$-pinching sphere theorem in [BS], we will discuss the rigidity of compact Einstein manifolds with non-negative sectional curvature. We prove that the condition $R_0 > \left[1 - \frac{6}{5(n-1)}\right]K_{\max}$ in [XG, Theorem 1.1] can be relaxed to $R_0 > \left[1 - \frac{4}{3(n-1)}\right]K_{\max}$. Furthermore, we show that Conjecture A is true under an additional pinching condition.

In [C], Catino introduced the notion of generalized quasi-Einstein manifold. Let (M, g) be an n-dimensional Riemannian manifold with $n \geq 3$. If there exist three smooth functions f, μ and λ on (M, g) such that

$$\mathrm{Ric} + \nabla^2 f - \mu df \otimes df = \lambda g, \tag{1.1}$$

then (M, g) is called a *generalized quasi-Einstein manifold*.

In [BR] and [HW], Barros–Ribeiro and Huang–Wei studied the rigidity of some closed generalized quasi-Einstein manifolds under the condition that $\mu = 1/m > 0$. In [JW], Jauregui and Wylie proved that a Riemannian metric is conformal to an Einstein metric if and only if it admits a generalized quasi-Einstein structure with $\mu = -1/(n-2)$. Therefore, generalized quasi-Einstein manifolds with $\mu = -1/(n-2)$ are important in Riemannian geometry. In this paper, we prove that some generalized quasi-Einstein manifolds with $\mu = -1/(n-2)$ are exactly Einstein manifolds.

2. Rigidity of compact Einstein manifolds

THEOREM 2.1. *Let M be an n-dimensional compact Einstein manifold with non-negative sectional curvature and $n \geq 4$. Denote by $R_0 := c$ the normalized scalar curvature of M. If $R_0 > \left[1 - \frac{4}{3(n-1)}\right]K_{\max}$, then M is isometric to a spherical space form of constant curvature c.*

Proof. Since M is an n-dimensional compact Einstein manifold, by [XG] we have $\mathrm{Ric}(e_i, e_i) = (n-1)R_0$ for an orthonormal frame $\{e_1, \ldots, e_n\}$ and any $i \in \{1, \ldots, n\}$. According to the definition of $\mathrm{Ric}(e_i, e_i)$, we have

$$\mathrm{Ric}(e_i, e_i) \leq K_{\min} + (n-2)K_{\max}.$$

Therefore,

$$K_{\min} \geq \operatorname{Ric}(e_i, e_i) - (n-2)K_{\max}. \tag{2.1}$$

Suppose that $\{w_1, w_2, w_3, w_4\}$ is an orthonormal four-frame. Since M has non-negative sectional curvature, by Berger's inequality (see [B1], [S] and [XG]) we have

$$R_{1234} \leq \tfrac{2}{3}(K_{\max} - K_{\min}) \leq \tfrac{2}{3}K_{\max}. \tag{2.2}$$

According to (2.1), (2.2) and the definition of $\operatorname{Ric}(e_i, e_i)$, we obtain

$$\begin{aligned} &R_{1313} + R_{2323} + R_{1414} + R_{2424} - 2R_{1234} \\ &= (R_{1313} + R_{1414}) + (R_{2323} + R_{2424}) - 2R_{1234} \\ &\geq (\operatorname{Ric}(e_1, e_1) - (n-3)K_{\max}) + (\operatorname{Ric}(e_2, e_2) - (n-3)K_{\max}) - \tfrac{4}{3}K_{\max} \\ &= 2(n-1)R_0 - 2(n-3)K_{\max} - \tfrac{4}{3}K_{\max} \\ &= 2(n-1)\bigg[R_0 - \bigg(1 - \frac{4}{3(n-1)}\bigg)K_{\max}\bigg]. \end{aligned} \tag{2.3}$$

Since $R_0 > \big[1 - \frac{4}{3(n-1)}\big]K_{\max}$, by (2.3) we conclude that the isotropic curvature of M is positive. Therefore, M is isometric to a spherical space form of constant curvature c. ∎

THEOREM 2.2. *Let M be an n-dimensional compact Einstein manifold with $n \geq 4$. If $R_0 > \frac{3}{5}K_{\max}$ and*

$$K_{\min} \geq \tfrac{3}{4}\big[\big(2n - \tfrac{79}{15}\big)K_{\max} - (2n-3)R_0\big], \tag{2.4}$$

then M is isometric to a spherical space form.

Proof. Suppose $\{w_1, w_2, w_3, w_4\}$ is an orthonormal four-frame. From Berger's inequality we have

$$R_{1234} \leq \tfrac{2}{3}(K_{\max} - K_{\min}).$$

Similar to the proof of Theorem 1.1, we deduce

$$\begin{aligned} &R_{1313} + R_{2323} + R_{1414} + R_{2424} - 2R_{1234} \\ &\qquad \geq 2\operatorname{Ric}(e_i, e_i) - 2(n-3)K_{\max} - \tfrac{4}{3}(K_{\max} - K_{\min}) \\ &\qquad = 2(n-1)R_0 - 2(n-3)K_{\max} - \tfrac{4}{3}K_{\max} + \tfrac{4}{3}K_{\min}. \end{aligned} \tag{2.5}$$

By (2.4) and (2.5), we obtain

$$\begin{aligned} &R_{1313} + R_{2323} + R_{1414} + R_{2424} - 2R_{1234} \\ &\qquad \geq 2(n-1)R_0 - 2(n-3)K_{\max} \\ &\qquad\quad - \tfrac{4}{3}K_{\max} + \big(2n - \tfrac{79}{15}\big)K_{\max} - (2n-3)R_0 \\ &\qquad = R_0 - \tfrac{3}{5}K_{\max}. \end{aligned} \tag{2.6}$$

Since $R_0 > \frac{3}{5}K_{\max}$, we may use (2.6) to conclude that M has positive isotropic curvature. Therefore, we invoke Brendle's theorem [B2] to conclude that M is isometric to a spherical space form. ∎

3. Rigidity of generalized quasi-Einstein manifolds

THEOREM 3.1. *Suppose that (M,g) is a closed generalized quasi-Einstein manifold with $\mu = -1/(n-2)$. Then there exists a constant β such that*

$$\Delta f - |\nabla f|^2 + (n-2)\lambda + \beta e^{\frac{2}{2-n}f} = 0. \tag{3.1}$$

Furthermore, if $\lambda, \beta \le 0$, then (M,g) is an Einstein manifold.

Proof. Since (M,g) is a generalized quasi-Einstein manifolds with $\mu = -1/(n-2)$, by (1.1) we have

$$\mathrm{Ric} + \nabla^2 f - \frac{1}{2-n} df \otimes df = \lambda g. \tag{3.2}$$

We denote by R the scalar curvature of (M,g). So, the twice contracted Bianchi identity is given by $\nabla R = 2\,\mathrm{div}\,\mathrm{Ric}$. Since $\mathrm{div}(df \otimes df) = \Delta f \nabla f + \frac{1}{2}\nabla|\nabla f|^2$ and $\mathrm{div}\,\nabla^2 f = \mathrm{Ric}(\nabla f) + \nabla\Delta f$, similar to [BR] we may use (3.2) to infer that

$$\nabla R + 2\,\mathrm{Ric}(\nabla f) + 2\nabla\Delta f - \frac{2}{2-n}\Delta f \nabla f - \frac{1}{2-n}\nabla|\nabla f|^2 = 2\nabla\lambda. \tag{3.3}$$

Taking the trace on both sides of (3.2), we obtain

$$R + \Delta f - \frac{1}{2-n}|\nabla f|^2 = n\lambda. \tag{3.4}$$

According to (3.2), we get

$$\begin{aligned} \mathrm{Ric}(\nabla f) &= \lambda g(\nabla f) - \nabla^2 f(\nabla f) + \frac{1}{2-n} df \otimes df(\nabla f) \\ &= \lambda \nabla f - \frac{1}{2}\nabla|\nabla f|^2 + \frac{1}{2-n}|\nabla f|^2 \nabla f. \end{aligned} \tag{3.5}$$

Substituting (3.4) and (3.5) into (3.3), we arrive at

$$\begin{aligned} \nabla R = {} & \frac{4(1-n)}{2-n}\lambda\nabla f - \frac{1-n}{2-n}\nabla|\nabla f|^2 \\ & + \frac{2(1-n)}{(2-n)^2}|\nabla f|^2\nabla f + \frac{2}{2-n}R\nabla f + 2(n-1)\nabla\lambda. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \nabla R - \frac{2}{2-n}R\nabla f = {} & 2(n-1)\Big(\nabla\lambda - \frac{2}{2-n}\lambda\nabla f\Big) \\ & - \frac{1-n}{2-n}\Big(\nabla|\nabla f|^2 - \frac{2}{2-n}|\nabla f|^2\nabla f\Big). \end{aligned} \tag{3.6}$$

From (3.6), we deduce

$$\nabla(Re^{\frac{2}{n-2}f}) = 2(n-1)\nabla(\lambda e^{\frac{2}{n-2}f}) - \frac{1-n}{2-n}\nabla(|\nabla f|^2 e^{\frac{2}{n-2}f}).$$

Thus, we conclude that there exists a constant β such that

$$Re^{\frac{2}{n-2}f} - 2(n-1)(\lambda e^{\frac{2}{n-2}f}) + \frac{1-n}{2-n}(|\nabla f|^2 e^{\frac{2}{n-2}f}) = \beta. \tag{3.7}$$

According to (3.7), we have

$$R - 2(n-1)\lambda + \frac{1-n}{2-n}|\nabla f|^2 - \beta e^{\frac{2}{2-n}f} = 0. \tag{3.8}$$

Inserting (3.4) into (3.8), we get (3.1).

Since (M, g) is a closed manifold, we have $\int_M \Delta f = 0$. Therefore, from (3.1) and our assumptions on λ and β we deduce that $\int_M |\nabla f|^2 \le 0$. This forces f to be constant. We then use (3.1) once more to infer that λ is also constant. From this it follows that (M, g) is an Einstein manifold. ■

Acknowledgments. This research is supported by Scientific Research Fund of Hunan Provincial Education Department (grant no. 14A020), and by Hunan Provincial Natural Science Foundation of China (grant no. 14JJ2120). This research is supported partly by the Program of Developing Key Scientific Disciplines in Hunan Province (grant no. [2011]76). The authors would like to thank the referees for their valuable suggestions.

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Received 12.12.2014
and in final form 11.8.2015 (3573)