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Fixed points and solutions of boundary value problems at resonance

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Abstract. We consider a simple boundary value problem at resonance for an ordinary differential equation. We employ a shift argument and construct a regular fixed point operator. In contrast to current applications of coincidence degree, standard fixed point theorems are applied to give sufficient conditions for the existence of solutions. We provide three applications of fixed point theory. They are delicate and an application of the contraction mapping principle is notably missing. We give a partial explanation as to why the contraction mapping principle is not a viable tool for boundary value problems at resonance.

1. Introduction. Assume $L > 0$ and assume $g : [0, L] \to \mathbb{R} \times \mathbb{R}$ is continuous. We shall consider boundary value problems for ordinary differential equations of the form

$$y''(t) = g(t, y(t)), \quad 0 \le t \le L, \tag{1.1}$$

$$y'(0) = 0, \quad y'(L) = 0. \tag{1.2}$$

The boundary value problem (1.1), (1.2) is said to be *at resonance* because the homogeneous problem

$$y''(t) = 0, \quad 0 < t < L, \quad y'(0) = 0, \quad y'(L) = 0,$$

has nontrivial constant solutions.

Boundary value problems at resonance have been investigated for many years, and Mawhin's [Ma1, Ma2] coincidence degree theory has been a very useful tool to obtain sufficient conditions for existence of solutions; we cite, for example, [BGS, DLG1, DLG2, FW, GS, Gu2, KT, Ka1, Ko1, Ko2]. More recently, there has been an interest in considering conditions that imply the existence of a solution in a cone, and a broad collection of methods have been developed and successfully employed. For example, a coincidence theorem of Schauder type has been developed and employed [S], the Lyapunov–Schmidt

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procedure has been employed [MY], topological degree [D, Gu1, NP, RS] has been employed, and a Leggett–Williams type theorem for coincidences has been developed [OZ] and employed [Fr, IZ, OZ]. A fixed point index theorem developed by Cremins [Cr] has been useful and has been employed by Bai and Fang [BF] and Kaufmann [Ka1, Ka2], for example. Webb and Zima [WZ] successfully employ fixed point index theory. Han [H] modifies the problem at resonance and considers an equivalent boundary value problem not at resonance in order to apply the Krasnosel'skiĭ–Guo fixed point theorem.

This work is initially motivated by Han [H] who produced an interesting application of the Krasnosel'skiĭ–Guo fixed point theorem. It is interesting in light of the observation that if y is a solution of (1.1), (1.2), then $\int_0^L g(s, y(s))\,ds = 0$. Using a shift argument [IPT], Han constructs an equivalent boundary value problem, for which $y \in C[0, L]$ is a solution if, and only if, y is a fixed point of an appropriate fixed point operator. In particular, if y is such a fixed point, then $\int_0^L g(s, y(s))\,ds = 0$. Infante, Pietramala and Tojo themselves [IPT] provided a thorough study of boundary value problems related to the Neumann boundary conditions (1.2) using the shift argument.

Motivated by [H] and [IPT], we too shall apply the shift argument. We present three applications, one using the Krasnosel'skiĭ–Guo fixed point theorem, one using the Schauder fixed point theorem and one using the Leray–Schauder nonlinear alternative. Our purpose is to obtain analogues of standard results for regular boundary value problems. We then consider a standard question in the study of boundary value problems for ordinary differential equations, and consider the questions of existence and uniqueness of solutions and existence of solutions for $L > 0$, in the case L is small. We show that standard applications of the contraction mapping principle fail and we provide partial explanation for the failure. Han's construction is specifically used to give a partial explanation.

We close the paper with a simple example to illustrate the results.

2. A fixed point operator. Let β be real and consider the equivalent boundary value problem

$$y''(t) + \beta^2 y(t) = g(t, y(t)) + \beta^2 y(t) = h(t, y(t)), \qquad 0 \le t \le L, \tag{2.1}$$

together with the boundary conditions (1.2). Throughout this article, we make the assumption that

$$\beta \in \left(0, \frac{\pi}{2L}\right). \tag{2.2}$$

A direct computation gives a Green's function $G(\beta, L; t, s)$ for the boundary value problem (2.1), (1.2):

$$G(\beta, L; t, s) = \frac{1}{\beta \sin(\beta L)} \begin{cases} \cos(\beta t)\cos(\beta(s-L)), & 0 \le t \le s \le L, \\ \cos(\beta s)\cos(\beta(t-L)), & 0 \le s \le t \le L, \end{cases} \tag{2.3}$$

which has the following properties:

(i) it satisfies the boundary conditions

$$G_t(\beta, L; 0, s) = 0, \qquad G_t(\beta, L; L, s) = 0;$$

(ii) it is continuous across the line $t = s$,

$$G(\beta, L; s^+, s) - G(\beta, L; s^-, s) = 0;$$

(iii) its t-partial derivative has a jump discontinuity across the line $t = s$,

$$G_t(\beta, L; s^+, s) - G_t(\beta, L; s^-, s) = 1.$$

THEOREM 2.1. *Assume $g : [0, L] \times \mathbb{R} \to \mathbb{R}$ is continuous. Then y is a solution of* (2.1), (1.2) *if, and only if, $y \in C[0, 1]$ and*

$$y(t) = \int_0^L G(\beta, L; t, s) h(s, y(s))\, ds, \qquad 0 \le t \le L.$$

Since it is clear that y is a solution of (1.1), (1.2) if, and only if, y is a solution of (2.1), (1.2), Theorem 2.1 can be restated in the following way.

THEOREM 2.2. *Assume $g : [0, L] \times \mathbb{R} \to \mathbb{R}$ is continuous. Then y is a solution of* (1.1), (1.2) *if, and only if, $y \in C[0, 1]$ and*

$$y(t) = \int_0^L G(\beta, L; t, s) h(s, y(s))\, ds, \qquad 0 \le t \le L.$$

Theorems 2.1 and 2.2 motivate the following definition of a fixed point operator $K : C[0, 1] \to C[0, 1]$:

$$Ky(t) = \int_0^L G(\beta, L; t, s) h(s, y(s))\, ds, \qquad 0 \le t \le L. \tag{2.4}$$

Thus, obtaining a fixed point of K in $C[0, L]$ is equivalent to obtaining a solution of (1.1), (1.2). We point out that if $h : [0, L] \times \mathbb{R} \to \mathbb{R}$ is continuous, then since G is uniformly continuous on $[0, L] \times [0, L]$, standard arguments show that $K : C[0, L] \to C[0, L]$, defined by (2.4), is a completely continuous operator.

LEMMA 2.3. *The Green function given by* (2.3) *satisfies*

$$\frac{\cos^2(\beta L)}{\beta \sin(\beta L)} \le G(\beta, L; s, t) \le \frac{1}{\beta \sin(\beta L)}, \qquad (t, s) \in [0, L] \times [0, L]. \tag{2.5}$$

Proof. From (2.3) we deduce that

$$G(\beta, L; t, s) \leq \frac{1}{\beta \sin(\beta L)}, \qquad (t, s) \in [0, L] \times [0, L].$$

Furthermore,

$$G(\beta, L; t, s) \geq \frac{1}{\beta \sin(\beta L)} \begin{cases} \cos(\beta L)\cos(\beta L), & 0 \leq t \leq s \leq L, \\ \cos(\beta L)\cos(\beta L), & 0 \leq s \leq t \leq L. \end{cases}$$

Hence,

$$G(\beta, L; t, s) \geq \frac{\cos^2(\beta L)}{\beta \sin(\beta L)}, \qquad (t, s) \in [0, L] \times [0, L]. \ \blacksquare$$

3. Applications of fixed point theorems. In this section we present three applications: of the Krasnosel'skiĭ–Guo [GL] fixed point theorem, of the Schauder [J] fixed point theorem, and of the Leray–Schauder nonlinear alternative [LS, DG]. We also give a partial explanation as to the notable lack of application of the contraction mapping principle.

First we state for convenience:

THEOREM 3.1 (Krasnosel'skiĭ–Guo fixed point theorem). *Let X be a Banach space and let $D \subset X$ be a cone in X. Assume Ω_1 and Ω_2 are open balls of X with $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. Let $K : D \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right) \mapsto D$ be a completely continuous operator such that either*

(1) $\|KU\| \leq \|U\|$, $U \in D \subset \partial\Omega_1$, *and* $\|KU\| \geq \|U\|$, $U \in D \subset \partial\Omega_2$, *or*
(2) $\|KU\| \geq \|U\|$, $U \in D \subset \partial\Omega_1$, *and* $\|KU\| \leq \|U\|$, $U \in D \subset \partial\Omega_2$.

Then K has a fixed point in $D \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Let $X = C[0, L]$ with $\|y\| = \max_{0 \leq t \leq L} |y(t)|$. Define the cone $D \subset X$ by

$$D = \{y \in C[0, L] : y(t) \geq \cos^2(\beta L)\|y\|, \, t \in [0, L]\}.$$

THEOREM 3.2. *Assume $h : [0, L] \times \mathbb{R} \to \mathbb{R}$ is continuous and assume $h(t, y) \geq 0$ for $(t, y) \in [0, L] \times [0, \infty)$. Let K be the operator defined by* (2.4). *Then*

$$K : D \to D.$$

Proof. First note that the Green function is uniformly continuous on $[0, L] \times [0, L]$. So the continuity of h implies $Ky \in C[0, L]$ if $y \in C[0, L]$. Moreover, the condition $h \geq 0$ if $y \geq 0$ and (2.5) imply that if $y \geq 0$, then $Ky \geq 0$. So, $y \in D$ implies $Ky \geq 0$.

Now, let $y \in D$. Then

$$|Ky(t)| \leq \int_0^L |G(\beta, L; t, s)| \, |h(y(s))| \, ds,$$

which implies

$$\|Ky\| \le \frac{1}{\beta\sin(\beta L)}\int_0^L |h(y(s))|\,ds. \tag{3.1}$$

Employ (3.1) to obtain,

$$\begin{aligned}Ky(t) &= \int_0^L G(\beta, L; t, s)h(y(s))\,ds\\ &\ge \frac{\cos^2(\beta L)}{\beta\sin(\beta L)}\int_0^L h(y(s))\,ds \ge \cos^2(\beta L)\|Ky\|\end{aligned}$$

and

$$Ky(t) \ge \cos^2(\beta L)\|Ky\|. \tag{3.2}$$

Thus, $KD \subset D$. ∎

To apply Theorem 3.1, suppose in addition to assumption (2.2) that

$$g(t,y) \ge -\beta^2 y, \qquad (t,y) \in [0,L]\times[0,\infty). \tag{3.3}$$

THEOREM 3.3. *Assume $g : [0,L]\times\mathbb{R} \to \mathbb{R}$ is continuous and assume* (2.2) *and* (3.3) *hold. Assume g satisfies the asymptotic properties*

$$\limsup_{y\to 0^+}\max_{t\in[0,1]}\frac{g(t,y)}{y} = -\beta^2, \tag{3.4}$$

$$\liminf_{y\to\infty}\min_{t\in[0,1]}\frac{|g(t,y)|}{y} = \infty. \tag{3.5}$$

Then there is at least one positive solution of (1.1), (1.2).

Proof. For $0 < r < \infty$, let

$$\Omega_r = \{y \in C[0,L] : \|y\| < r\}.$$

We shall determine values of $0 < r < R$ such that the hypotheses of Theorem 3.1 are satisfied with $\Omega_r = \Omega_1$ and $\Omega_R = \Omega_2$.

First, note by (3.4) there exists $r > 0$ sufficiently small such that

$$\|Ky\| \le \|y\| \quad \text{for all } y \in D\cap\partial\Omega_r. \tag{3.6}$$

To see this, let $m > 0$ and assume

$$\frac{mL}{\beta\sin(\beta L)} \le 1.$$

By (3.4), there exists $r > 0$ such that if $y \in D\cap\partial\Omega_r$, then

$$g(t,y(t)) + \beta^2 y(t) \le my(t) \quad \text{or} \quad h(t,y(t)) \le my(t).$$

Then

$$Ky(t) = \int_0^L G(\beta, L; s, t)h(y(s))\,ds \le m\|y\| \int_0^L G(\beta, L; s, t)\,ds$$
$$\le \|y\| \frac{mL}{\beta \sin(\beta L)} \le \|y\|.$$

Thus, $\|Ky\| \le \|y\|$ if $y \in D \cap \partial\Omega_r$, and (3.6) is true.

Now we exhibit $R > 0$ sufficiently large such that

$$(3.7) \qquad \|Ky\| \ge \|y\| \quad \text{for all } y \in D \cap \partial\Omega_R.$$

There exists $R_0 > 0$ such that if $y \ge R_0$, then

$$g(t, y) \ge \rho y$$

where $\rho > 0$ is such that

$$\frac{\cos^4(\beta L)}{\beta \sin(\beta L)}(\rho + \beta^2)L \ge 1.$$

Let $R > \max\{R_0/\cos^2(\beta L), 2r\} > r$. Then $y \in D \cap \Omega_R$ implies

$$y(t) \ge \cos^2(\beta L)\|y\| > R_0$$

and

$$h(t, y(t)) = g(t, y(t)) + \beta^2 y(t) \ge \rho y(t) + \beta^2 y(t) = (\rho + \beta^2)y(t).$$

By Lemma 2.3, we have

$$Ky(t) = \int_0^L G(\beta, L; s, t)h(s, y(s))\,ds \ge (\rho + \beta^2) \int_0^L G(\beta, L; s, t)y(s)\,ds$$
$$\ge (\rho + \beta^2)\frac{\cos^2(\beta L)}{\beta \sin(\beta L)} \cos^2(\beta L)L\|y\| \ge \|y\|.$$

Thus, $\|Ky\| \ge \|y\|$ if $y \in D \cap \partial\Omega_R$, and (3.7) is true. ■

Remark. Many related results are readily obtained now that the basic framework is established to apply the Krasnosel'skiĭ–Guo fixed point theorem, Theorem 3.1. One can modify the asymptotic conditions (3.4), (3.5) with superlinear growth at the origin and sublinear growth at infinity, or produce growth properties to imply the existence of multiple positive solutions, or introduce nonlinear eigenvalue problems and obtain eigenvalue intervals on which positive solutions exist; see [EHW, EW, HW] for the seminal papers along these lines. As this is not the intent of this particular article, we do not develop these related results here.

We now consider an application of the Schauder fixed point theorem [J], stated here for convenience:

THEOREM 3.4 (Schauder fixed point theorem). *If V is a closed convex subset of a Banach space X, if $K : V \to V$ is continuous on V, and if $\overline{KV}$ is a compact subset of X, then K has a fixed point in V.*

Consider the Banach space $X = C[0, L]$, equipped with the maximum norm

$$\|y\|_0 = \max_{0\le t\le L} |y(t)|.$$

From this point on in the paper, we do not consider positive solutions and so condition (3.3) is not assumed throughout the remainder of the article.

THEOREM 3.5. *Assume that $g : [0, L] \times \mathbb{R} \to \mathbb{R}$ is continuous. Assume there exists $\beta \in \left(0, \frac{\pi}{2L}\right)$ such that $h(t, y) = g(t, y) + \beta^2 y$ is bounded. Then there exists a solution of (1.1), (1.2).*

Proof. Since h is bounded, let $Q = \sup\{|h(t, y)| : (t, y) \in [0, L] \times \mathbb{R}\}$. Let $V \subset X$ be defined by

$$V = \left\{ y \in X : |y(t)| \le \frac{QL}{\beta \sin(\beta L)},\ 0 \le t \le L \right\}.$$

Then, clearly, that V is a closed bounded convex subset of X. For any $y \in V$,

$$(3.8)\qquad \begin{aligned} |Ky(t)| &\le \int_0^L |G(\beta, L; t, s)|\,|h(s, y(s))|\,ds \le Q \int_0^L |G(\beta, L; t, s)|\,ds \\ &\le \frac{QL}{\beta \sin(\beta L)}. \end{aligned}$$

So K maps V into V. ∎

REMARK. It is a standard application of the Schauder fixed point theorem to show that a bounded nonlinear term is a common sufficient condition for existence of solutions. However, in this case, the application is more interesting. Consider the boundary value problem (1.1), (1.2), and assume $g : [0, L] \times \mathbb{R} \to (0, \infty)$. Then $\int_0^L g(t, y(t))\,dt > 0$ for any $y \in X$. In particular, there is no fixed point for the operator K if g is moreover assumed to be continuous and bounded.

REMARK. Also note that if $h(t, 0) = 0$, then the trivial solution is a solution of (1.1), (1.2), and the Schauder fixed point theorem provides no new information. So, for all practical purposes, assume in Theorem 3.5 additionally that $h(t, 0) \ne 0$.

For our third application, we use the Leray–Schauder nonlinear alternative, again stated here for convenience:

THEOREM 3.6 (Leray–Schauder nonlinear alternative). *Let V be a closed and bounded subset of a Banach space X, let U be an open subset of V and $0 \in U$. Suppose $K : \overline{U} \to V$ is a continuous and compact operator. Then either*

(1) *K has fixed point in $\overline{U}$, or*
(2) *there exists a point $u \in \partial U$ such that $u = \lambda K u$ for some $\lambda \in (0,1)$ where ∂U denotes the boundary of U in V.*

THEOREM 3.7. *Assume that $g : [0,L] \times \mathbb{R} \to \mathbb{R}$ is continuous. Assume there exists $\beta \in \left(0, \frac{\pi}{2L}\right)$, $\sigma \in C[0,L] \times R^+$ and a nondecreasing function $\Psi : R^+ \to R^+$ such that if $h(t,y) = g(t,y) + \beta^2 y$, then*

$$|h(t,y)| \le \sigma(t)\Psi(|y|), \qquad (t,y) \in [0,L] \times \mathbb{R}.$$

Moreover, assume there exists $M > 0$ such that

$$\frac{M\beta \sin(\beta L)}{L\|\sigma\|\Psi(M)} > 1. \tag{3.9}$$

Then the boundary value problem (1.1), (1.2) has a solution.

Proof. Again consider the Banach space $X = C[0,L]$ equipped with the maximum norm $\|y\|_0 = \max_{0\le t\le L} |y(t)|$. Assume that $h(t,y) = g(t,y) + \beta^2 y$ is continuous on $[0,L] \times \mathbb{R}$ and define the mapping $K : X \to X$ by

$$Ky(t) = \int_0^L G(\beta, L; t, s) h(s, y(s))\, ds.$$

Now, suppose for $y \in X$ and for some $\lambda \in (0,1)$ we have

$$y(t) = \lambda K y(t). \tag{3.10}$$

By the definition of K we can rewrite (3.10) to obtain

$$|y(t)| \le \int_0^L |G(\beta, L; t, s)|\, |h(s, y(s))|\, ds \le \|\sigma\|\Psi(\|y\|) \frac{L}{\beta \sin(\beta L)}$$

and conclude that

$$\|y\| \le \|\sigma\|\Psi(\|y\|) \frac{L}{\beta \sin(\beta L)}.$$

Hence,

$$\frac{\|y\|\beta \sin(\beta L)}{L\|\sigma\|\Psi(\|y\|)} \le 1.$$

Thus, if y satisfies (3.10), then $\|y\| \ne M$, where M is given by (3.9).

To apply the Leray–Schauder nonlinear alternative and complete the proof, define

$$U = \{y \in \beta : \|y\| < M\}.$$

Then the operator $K : \overline{U} \to \beta$ is continuous and compact and there is no point $y \in \partial U$ and $\lambda \in (0,1)$ such that $y = \lambda K y$. Thus, K has a fixed point in $\overline{U}$. ∎

Corollary 3.8. *Assume that $g : [0,L] \times \mathbb{R} \to \mathbb{R}$ is continuous. Assume there exists $\beta \in \left(0, \frac{\pi}{2L}\right)$, $\sigma \in C[0,L] \times \mathbb{R}^+$ and a nondecreasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that if $h(t,y) = g(t,y) + \beta^2 y$, then*

$$|h(t,y)| \le \sigma(t)\Psi(|y|), \qquad (t,y) \in [0,L] \times \mathbb{R}.$$

If

$$\limsup_{y \to \infty} \frac{y}{\Psi(y)} = \infty,$$

then there exists a solution of (1.1), (1.2).

Remark. Before proceeding to a discussion on the failure of an application of the contraction mapping principle, we point out that in [IPT], the authors consider the shift (1.1), and they consider the analogous shift

$$y''(t) - \beta^2 y(t) = g(t, y(t)) - \beta^2 y(t) = \hat{h}(t, y(t)), \qquad 0 \le t \le L. \tag{3.11}$$

The Green function for the boundary value problem (3.11), (1.2) is readily constructed and has the form

$$G(\beta, L; t, s) = \frac{1}{\beta \sinh(\beta L)} \begin{cases} \cosh(\beta t)\cosh(\beta(s - L)), & 0 \le t \le s \le L, \\ \cosh(\beta s)\cosh(\beta(t - L)), & 0 \le s \le t \le L. \end{cases}$$

Thus, analogues of Theorems 3.3, 3.5 and 3.7 can be stated and proved.

We close the section with a partial discussion as to why the contraction mapping principle fails to be a useful tool in the study of problems at resonance. Standard questions to consider when studying boundary value problems for ordinary differential equations involve uniqueness of solutions implying uniqueness or existence of solutions [J]; for example, if initial value problems are uniquely solvable and solutions extend to $[0,L]$, then under suitable hypotheses, one expects a boundary value problem to have a unique solution if the interval length, L, is small enough. This is clearly not the case for problems at resonance such as (1.1), (1.2). If (1.1), (1.2) has a solution $y(t)$, then $\int_0^L g(s, y(s))\, ds = 0$. So, for example, if $g = 1$, then (1.1), (1.2) has no solution for any $L > 0$.

To apply the contraction mapping principle for L sufficiently small, one generally begins by assuming that g is Lipschitz, i.e. there exists $\alpha > 0$ such that

$$|g(t, y_1) - g(t, y_2)| \le \alpha |y_1 - y_2|, \qquad 0 < t < L,\ y_1, y_2 \in \mathbb{R}.$$

Assume (2.2), let $G(L, \beta; t, s)$ be given by (2.3) and let the operator K be given by (3.10). Then, if $y_1, y_2 \in C[0,L]$, we have

$$\begin{aligned}|Ky_1(t) - Ky_2(t)| &\le \int_0^L |G(\beta, L; t, s)|\,|h(s, y_1(s)) - h(s, y_2(s))|\,ds \\ &\le \int_0^L |G(\beta, L; t, s)|\,ds\,(\alpha + \beta^2)\|y_1 - y_2\| \\ &\le \frac{L}{\beta \sin(\beta L)}(\alpha + \beta^2)\|y_1 - y_2\|.\end{aligned}$$

A possible contraction coefficient has the form

$$\frac{L}{\beta \sin(\beta L)}(\alpha + \beta^2)$$

and we consider the smaller coefficient with $\alpha = 0$. Then

$$\frac{L}{\beta \sin(\beta L)}\beta^2 = \frac{\beta L}{\sin(\beta L)} > 1$$

if $L > 0$ and $\beta \in \left(0, \frac{\pi}{2L}\right)$. So, the standard approach does not give rise to a contraction coefficient for any $\alpha > 0$.

Note that if one considers the analogous shift (3.11), with boundary conditions (1.2), then the corresponding possible contraction coefficient is

$$\frac{L \cosh^2(\beta L)}{\beta \sinh(\beta L)}\beta^2 \ge \frac{\beta L \cosh(\beta L)}{\sinh(\beta L)} > 1$$

if $L > 0$. Thus, the partial discussion for the failure of the contraction mapping principle applies if one considers (3.11).

4. An example. We provide one simple example to illustrate an application of Theorem 3.5. Consider the boundary value problem

$$y''(t) = g(t) - \frac{y^3(t)}{1 + y^2(t)}, \qquad 0 \le t \le 1, \tag{4.1}$$

with the boundary conditions (1.2), where $g \in C[0,1]$. So, $L = 1$. Let $\beta = 1 < \pi/2$. Then

$$h(t, y(t)) = g(t) - \frac{y^3(t)}{1 + y^2(t)} + y(t) = g(t) + \frac{y(t)}{1 + y^2(t)}.$$

The function h is bounded and continuous on $[0,1] \times \mathbb{R}$ and so, by Theorem 3.5, there exists a solution of the boundary value problem (4.1), (1.2).

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