ANNALES
POLONICI MATHEMATICI
115.3 (2015)

Higher-order linear differential equations with solutions having a prescribed sequence of zeros and lying in the Dirichlet space

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Abstract. The aim of this paper is to consider the following three problems:

(1) for a given uniformly q-separated sequence satisfying certain conditions, find a coefficient function $A(z)$ analytic in the unit disc such that $f''' + A(z)f = 0$ possesses a solution having zeros precisely at the points of this sequence;

(2) find necessary and sufficient conditions for the differential equation

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1 f' + A_0 f = 0 \tag{$*$}$$

in the unit disc to be Blaschke-oscillatory;

(3) find sufficient conditions on the analytic coefficients of the differential equation $(*)$ for all analytic solutions to belong to the Dirichlet space $\mathcal{D}$.

Our results are a generalization of some earlier results due to J. Heittokangas and J. Gröhn.

1. Introduction and main results. Initiated by the work of S. Bank and I. Laine [3], a remarkable amount of research has been directed to the zero distribution of entire solutions of

$$f'' + A(z)f = 0, \tag{1.1}$$

where $A(z)$ is entire. See [18] for an extensive collection of these results, as well as for further references. The early results on oscillation theory in the case of the unit disc $\mathbb{D} = \{z : |z| < 1\}$ go back to the work of Nehari and his students Beesack and Schwarz in the 1940s and 1950s. In the 1960s and 1970s results on nonoscillation were obtained by Hadass, Kim, Lavie and London, to name but a few. After a quiet period, the unit disc oscillation theory begins to flourish again, starting from the 1990s. In particular, a sequence of papers due to Chuaqui, Duren, Osgood and Stowe continue the classical considerations in oscillation theory, while Belaidi, Cao and Yi are inspired by

2010 *Mathematics Subject Classification*: Primary 34M10; Secondary 30J10.
Key words and phrases: Blaschke-oscillatory, Blaschke product, uniformly q-separated sequence, Dirichlet space, prescribed zero sequence, differential equation.

DOI: 10.4064/ap115-3-6

the complex plane case, and consider oscillation of solutions in terms of the exponent of convergence. Oscillation results in terms of Blaschke sequences are considered by Heittokangas [7–9, 13, 14].

1.1. A prescribed sequence of c-points. We denote the Nevanlinna class by N, it consists of all functions f meromorphic in $\mathbb{D}$ and having bounded characteristic $T(r,f)$. Further, a meromorphic function f in $\mathbb{D}$ is called *nonadmissible* if

$$\limsup_{r\to 1^-} \frac{T(r,f)}{-\log(1-r)} < \infty.$$

In addition, the *order* of a meromorphic function $f(z)$ in $\mathbb{D}$ is defined by

$$\sigma(f) = \limsup_{r\to 1^-} \frac{\log^+ T(r,f)}{-\log(1-r)}.$$

Let H^∞ denote the space of all bounded analytic functions in $\mathbb{D}$. For $p \geq 0$, the space H_p^∞ consists of functions g analytic in $\mathbb{D}$ satisfying

$$\|g\|_p^\infty = \sup_{z\in\mathbb{D}} (1-|z|^2)^p |g(z)| < \infty.$$

The union $\bigcup_{p\geq 0} H_p^\infty$ is known as the *Korenblum space* $\mathcal{A}^{-\infty}$ [17].

A sequence $\{z_n\}$ in $\mathbb{D}$ satisfying

$$1-|z_{n+1}| \leq K(1-|z_n|), \qquad n\in\mathbb{N},$$

for some constant $K\in(0,1)$ is called an *exponential sequence* in $\mathbb{D}$ [5, p. 156]. We see that every exponential sequence $\{z_n\}$ in $\mathbb{D}$ satisfies

$$\sum_{n=1}^{\infty} (1-|z_n|)^\alpha < \infty \tag{1.2}$$

for any $\alpha > 0$. A sequence $\{z_n\}$ satisfying (1.2) for some $\alpha\in(0,1]$ will be called an *α-Blaschke sequence*. In particular, $\{z_n\}$ is a *Blaschke sequence* when $\alpha = 1$, and the product

$$B(z) = \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1-\overline{z_n}z},$$

known as the *Blaschke product*, represents an analytic function in $\mathbb{D}$ and has zeros precisely at the points $\{z_n\}$.

A sequence $\{z_n\}$ of points in $\mathbb{D}$ is called *uniformly separated* if

$$\inf_k \prod_{n\neq k} \left| \frac{z_n - z_k}{1-\overline{z_n}z_k} \right| > 0,$$

and *uniformly q-separated* [7] if there is a constant $q \geq 0$ such that

$$\inf_{k\in\mathbb{N}}\left\{\left(\frac{1}{1-|z_k|}\right)^q\prod_{n\neq k}\left|\frac{z_n-z_k}{1-\overline{z_n}z_k}\right|\right\}>0. \tag{1.3}$$

A uniformly 0-separated sequence is known as a uniformly separated sequence [5, p. 148]. Every exponential sequence in $\mathbb{D}$ is uniformly separated [5, Theorem 9.2].

The following result can be found in [7, Theorem 6].

THEOREM 1.1. *Let $\{z_n\}$ be a uniformly q-separated sequence of nonzero points in $\mathbb{D}$.*

(a) *Suppose that $\{z_n\}$ is an α-Blaschke sequence. Then there exists a function $A \in H^\infty_{2(1+\alpha+2q)}$ satisfying*

$$\limsup_{|z|\to 1^-}(1-|z|^2)^2|A(z)| \geq 1 \tag{1.4}$$

such that (1.1) *possesses a solution whose zero sequence is $\{z_n\}$.*

(b) *Suppose that $\{z_n\}$ is a finite union of uniformly separated* (*or exponential*) *sequences in $\mathbb{D}$. Then there exists $A \in H^\infty_{2(1+2q)}$ satisfying* (1.4) *such that* (1.1) *possesses a solution whose zero sequence is $\{z_n\}$.*

We are unaware whether an analog of Theorem 1.1 in the case of higher order linear differential equations can be found in the existing literature. Hence we state a result in the case $f''' + A(z)f = 0$.

THEOREM 1.2. *Let $\{z_n\}$ be a uniformly q-separated sequence of nonzero points in $\mathbb{D}$.*

(a) *Suppose that $\{z_n\}$ is an α-Blaschke sequence. Then there exists a function $A \in H^\infty_{3+9\alpha+24q}$ such that*

$$f''' + A(z)f = 0 \tag{1.5}$$

possesses a solution whose zero sequence is $\{z_n\}$ and the multiplicity of each zero is two.

(b) *Suppose that $\{z_n\}$ is a finite union of uniformly separated* (*or exponential*) *sequences in $\mathbb{D}$. Then there exists $A \in H^\infty_{3+24q}$ such that* (1.5) *possesses a solution whose zero sequence is $\{z_n\}$ and the multiplicity of each zero is two.*

Part (b) has the following immediate consequence:

COROLLARY 1.3. *If $\{z_n\}$ is a uniformly separated* (*or exponential*) *sequence of nonzero points in $\mathbb{D}$, then there exists $A \in H^\infty_3$ such that* (1.5) *possesses a solution whose zero sequence is $\{z_n\}$ and the multiplicity of each zero is two.*

In [7], the authors found that a solution that maps a prescribed Blaschke sequence to some fixed nonzero value is very easy to find; see the following theorem.

THEOREM 1.4. *Let $\{z_n\}$ be a Blaschke sequence of nonzero points in $\mathbb{D}$. Then there exists $A \in H_2^\infty$ such that for each $\zeta \in \mathbb{C} \setminus \{0\}$,* (1.1) *possesses a solution $f_\zeta \in H^\infty$ taking the value ζ precisely at the points z_n.*

The above result can be considered as a solution to the problem of a prescribed sequence of c-points, where $c \in \mathbb{C} \setminus \{0\}$. Does a similar result hold if (1.1) is replaced by a higher order equation in Theorem 1.4? The answer is affirmative and given in the following theorem.

THEOREM 1.5. *Let $\{z_n\}$ be a Blaschke sequence of nonzero points in $\mathbb{D}$. Then there exists $A \in H_k^\infty$ such that for each $\zeta \in \mathbb{C} \setminus \{0\}$,*

$$f^{(k)} + A(z)f = 0 \quad (k \geq 2) \tag{1.6}$$

possesses a solution $f_\zeta \in H^\infty$ taking the value ζ precisely at the points z_n.

1.2. Blaschke-oscillatory equations. Before we state some further results, it is convenient to recall some classical terminology. Equation (1.1), where $A(z)$ is analytic in the unit disc, is called

- *disconjugate* (resp. *nonoscillatory*) if each nontrivial solution has at most one zero (resp. finitely many zeros) in $\mathbb{D}$;
- *oscillatory* if there is at least one solution with infinitely many zeros in $\mathbb{D}$;
- *Blaschke-oscillatory* if the zero sequence $\{z_n\}$ of each nontrivial solution satisfies the Blaschke condition (1.2) with $\alpha = 1$.

It is clear that a disconjugate equation is a nonoscillatory equation, which in turn is a Blaschke-oscillatory equation.

Nehari [22] proved that if $A(z)$ is analytic in $\mathbb{D}$ satisfying

$$|A(z)| \leq \frac{1}{(1-|z|^2)^2} \tag{1.7}$$

for all $z \in \mathbb{D}$, then (1.1) is disconjugate.

In 1955 Schwarz [23] showed that Nehari's condition (1.7) can be relaxed in the following sense: suppose that $A(z)$ is analytic in $\mathbb{D}$ and there exists a constant $R \in (0,1)$ such that (1.7) holds for all z with $R \leq |z| < 1$. Then (1.1) is nonoscillatory.

Using the result of Nehari, D. London [21] showed that if $A(z)$ is analytic in $\mathbb{D}$ and

$$\iint_{\mathbb{D}} |A(z)|\, d\sigma(z) \leq \pi, \tag{1.8}$$

then (1.1) is disconjugate. Here and in what follows, $d\sigma(z) = r dr\, d\theta$ stands for the Euclidean area measure.

London [21] showed further that (1.8) can be relaxed in the following sense: if $A(z)$ is analytic in $\mathbb{D}$ and

$$\iint_{\mathbb{D}} |A(z)|\, d\sigma(z) < \infty,$$

then (1.1) is nonoscillatory.

Heittokangas [8] extended the classical results due to Nehari and London as follows. Let $A(z)$ be analytic in $\mathbb{D}$ satisfying

$$\iint_{\mathbb{D}} |A(z)|^{1/2}\, d\sigma(z) < \infty \quad \text{or} \quad \iint_{\mathbb{D}} |A(z)|(1-|z|)\, d\sigma(z) < \infty.$$

Then (1.1) is Blaschke-oscillatory. As a corollary, Heittokangas [8] showed that if $A \in H^p$, where $1/4 \le p \le \infty$, then (1.1) is Blaschke-oscillatory. In particular, if $1 \le p \le \infty$, then (1.1) is nonoscillatory. Here H^p $(0 < p \le \infty)$ denotes the classical Hardy space [5] of all functions f analytic in $\mathbb{D}$ satisfying

$$\sup_{0 \le r < 1} M_p(r, f) < \infty,$$

where

$$M_p(r,f) = \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta \right)^{1/p}, \qquad 0 < p < \infty,$$
$$M_\infty(r,f) = M(r,f) = \max_{0 \le \theta \le 2\pi} |f(re^{i\theta})|.$$

Heittokangas [12] pointed out that the above results can be generalized to higher order linear differential equations

$$f^{(k)} + A_{k-1}(z) f^{(k-1)} + \cdots + A_0(z) f = 0 \qquad (k \ge 2), \tag{1.9}$$

where $A_j(z)$ $(j = 0, \dots, k-1)$ are analytic in $\mathbb{D}$.

Following the second order case above, we call (1.9) *Blaschke-oscillatory* if the zero sequence of any nontrivial solution of (1.9) satisfies the Blaschke condition. The following proposition is from [12].

Proposition 1.6. *Let $A_j(z)$ be analytic in $\mathbb{D}$ satisfying*

$$\iint_{\mathbb{D}} |A_j(z)|^{1/(k-j)}\, d\sigma(z) < \infty, \qquad j = 0, \dots, k-1, \tag{1.10}$$

or

$$\iint_{\mathbb{D}} |A_j(z)|(1-|z|)^{k-j-1}\, d\sigma(z) < \infty, \qquad j = 0, \dots, k-1. \tag{1.11}$$

Then (1.9) *is Blaschke-oscillatory.*

Using the Hardy–Littlewood theorem, we have the following consequence.

Corollary 1.7. *Let* $A_j \in H^{p_j}$, $j = 0, \dots, k-1$, *where* $1/(2(k-j)) \le p_j \le \infty$. *Then* (1.9) *is Blaschke-oscillatory.*

Assuming that (1.1) is Blaschke-oscillatory, what can we say about the properties of the coefficient function $A(z)$? Heittokangas [9] showed that if $A(z)$ is analytic in $\mathbb{D}$ and (1.1) is Blaschke-oscillatory, then

$$\iint_{\mathbb{D}} |A(z)|^{\alpha}\, d\sigma(z) < \infty$$

for every $\alpha \in (0, 1/2)$. Here we give another result.

Theorem 1.8. *If* $A(z)$ *is an analytic function in* $\mathbb{D}$ *such that* (1.1) *is Blaschke-oscillatory, then*

$$\iint_{\mathbb{D}} |A(z)|(1-|z|)^{\alpha}\, d\sigma(z) < \infty$$

for every $\alpha > 3$.

We now proceed to find necessary conditions for (1.9) to be Blaschke-oscillatory. When studying the oscillatory behavior of solutions of (1.9), we may suppose that $A_{k-1} \equiv 0$. For if ϕ denotes a primitive function of $A_{k-1}(z)$, then the standard substitution $g = fe^{-(1/k)\phi}$ has no effect on the zeros, and it transforms (1.9) to an equation where the coefficient of the $(k-1)$th derivative vanishes. It is proved in [12] that if $A_0, \dots, A_{k-2}$ are analytic functions in $\mathbb{D}$ such that

$$f^{(k)} + A_{k-2} f^{(k-2)} + \dots + A_1 f' + A_0 f = 0 \tag{1.12}$$

is Blaschke-oscillatory, then

$$\iint_{D(0,r)} |A_j(z)|^{1/(k-j)}\, d\sigma(z) = O\left(\log^2 \frac{e}{1-r}\right), \qquad j = 0, \dots, k-2,$$

where $D(0,r) = \{z \in \mathbb{D} : |z| < r\}$.

The next result is a generalization of Theorem 1.8.

Theorem 1.9. *If* $A_0, \dots, A_{k-2}$ *are analytic functions in* $\mathbb{D}$ *such that* (1.12) *is Blaschke-oscillatory, then*

$$\iint_{\mathbb{D}} |A_j(z)|(1-|z|)^{\alpha}\, d\sigma(z) < \infty, \qquad j = 0, \dots, k-2,$$

for every $\alpha > 2(k-j) - 1$.

1.3. Linear differential equations with solutions in the Dirichlet space $\mathcal{D}$. Heittokangas [10] gave a condition on the analytic coefficient $A(z)$ of (1.1) which implies that all solutions f of (1.1) are in $\bigcap_{0<p<\infty} Q_p$. For

$0 < p < \infty$, Q_p is the space of all analytic functions in $\mathbb{D}$ for which

$$\|f\|_{Q_p} = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 g(a,z)^p \, d\sigma(z) < \infty,$$

where $g(z,a) = \log|\frac{1-\bar{a}z}{z-a}|$ is the Green's function in $\mathbb{D}$.

THEOREM 1.10. *Let $A(z) = \sum_{n=0}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$, be the analytic coefficient of* (1.1) *in $\mathbb{D}$ with $|a_n| \le 1$ for all n. Then all solutions f of* (1.1) *belong to $\bigcap_{0<p<\infty} Q_p$.*

An analytic function f in $\mathbb{D}$ is said to belong to $\mathcal{D}$, the *Dirichlet space*, if

$$\int_{\mathbb{D}} |f'(z)|^2 \, d\sigma(z) < \infty.$$

We have the strict inclusion $\mathcal{D} \subset \bigcap_{0<p<\infty} Q_p$ (see [1, Corollary 4, Theorem 1]).

In 2011, Hao Li and Hasi Wulan [19] improved Theorem 1.10 as follows.

THEOREM 1.11. *Let $A(z) = \sum_{n=0}^{\infty} a_n z^n$, $a_n \in \mathbb{C}$, be the analytic coefficient of* (1.1) *in $\mathbb{D}$ with $|a_n| \le 1$ for all n. Then all solutions of* (1.1) *belong to the Dirichlet space $\mathcal{D}$.*

We take this opportunity to give a similar result to Theorem 1.11 for (1.9), which also improves Theorem 1.8 of [20].

THEOREM 1.12. *Let $A_j(z) = \sum_{n=0}^{\infty} a_{j,n} z^n$, $a_{j,n} \in \mathbb{C}$, be analytic coefficients of* (1.9) *in $\mathbb{D}$ with $|a_{j,n}| \le (n+2)^{k-2-j}$, $k \ge 2$, $j = 0,\dots,k-1$, for all n. Then all solutions of* (1.9) *belong to the Dirichlet space $\mathcal{D}$.*

2. Auxiliary lemmas. The following lemma associates uniformly q-separated sequences with interpolation in H_p^∞ spaces, and reduces to Carleson's result on H^∞ interpolation in the case when $q = 0$.

LEMMA 2.1 ([7]). *Let $\{z_n\}$ be an infinite sequence in $\mathbb{D}$, and let $s \ge 0$. Suppose that σ_n is any sequence of points in $\mathbb{C}$ (not necessarily distinct) satisfying*

$$\|\sigma_n\|_s = \sup_{n\in\mathbb{N}} (1-|z_n|^2)^s |\sigma_n| < \infty.$$

(a) *If $\{z_n\}$ is a uniformly q-separated α-Blaschke sequence in $\mathbb{D}$, then there exists $G \in H_{\alpha+q+s}^\infty$ such that $G(z_n) = \sigma_n$ for all $n \in \mathbb{N}$.*
(b) *If $\{z_n\}$ is a uniformly q-separated sequence which is a finite union of separated sequences, then there exists $G \in H_{q+s}^\infty$ such that $G(z_n) = \sigma_n$ for all $n \in \mathbb{N}$.*

LEMMA 2.2 ([7]). *Let B be a Blaschke product with zeros $z_n \neq 0$, and let $k \in \mathbb{N}$.*

(a) *Suppose that $\{z_n\}$ is a uniformly q-separated α-Blaschke sequence. Then there exists a set $E \subset [0,1)$ satisfying*

$$\int_E \frac{dr}{1-r} < \infty \tag{2.1}$$

such that for all $z \in \mathbb{D}$ satisfying $|z| \notin E$, we have

$$\left|\frac{B^{(k)}(z)}{B(z)}\right| = O\left(\left(\frac{1}{1-|z|}\right)^{(1+\alpha)k}\left(1+q\log\frac{e}{1-|z|}\right)^{k}\right).$$

(b) *Suppose that $\{z_n\}$ is a finite union of uniformly separated sequences. Let $\delta > 0$ denote the infimum in (1.3), and let $p \geq 0$. Then for all $z \notin \bigcup_n \triangle(z_n, (\delta/2)(1-|z_n|)^p)$, we have*

$$\left|\frac{B^{(k)}(z)}{B(z)}\right| = O\left(\left(\frac{1}{1-|z|}\right)^{(1+p)k}\right).$$

Here $\triangle(\omega,\tau) = \left\{z \in \mathbb{D} : \left|\frac{\omega - z}{1-\overline{\omega}z}\right| < \tau\right\}$ is known as the *pseudo-hyperbolic disc* of radius $\tau \in (0,1)$ centered at $\omega \in \mathbb{D}$. It is a true Euclidean disc with Euclidean radius $R = \frac{\tau(1-|\omega|^2)}{1-\tau^2|\omega|^2}$ and Euclidean center $\gamma = \frac{(1-\tau^2)\omega}{1-\tau^2|\omega|^2}$ (see [6]).

The next lemma allows us to avoid exceptional sets E with $\int_E \frac{dr}{1-r} < \infty$.

LEMMA 2.3 ([2]). *Let $g(r)$ and $h(r)$ be increasing real valued functions on $[0,1)$ such that $g(r) \leq h(r)$ possibly outside an exceptional set $E \subset [0,1)$ with $\int_E \frac{dr}{1-r} < \infty$. Then there exists a constant $b \in (0,1)$ such that if $s(r) = 1 - b(1-r)$, then $g(r) \leq h(s(r))$ for all $r \in [0,1)$.*

LEMMA 2.4 ([6]). *Any two points z and ω in the same pseudohyperbolic disc $\triangle(\alpha, r)$ satisfy*

$$\frac{1}{C} \leq \frac{1-|z|}{1-|\omega|} \leq C$$

with $C = 8\left(\frac{1+r^2}{1-r^2}\right)^2$ depending only on r.

LEMMA 2.5 ([5, Hardy–Littlewood]). *If $0 < p < q \leq \infty$, $f \in H^p$, $\lambda \geq p$, and $\alpha = 1/p - 1/q$, then*

$$\int_0^1 (1-r)^{\lambda\alpha - 1}\{M_q(r,f)\}^{\lambda}\, dr < \infty.$$

LEMMA 2.6 ([4]). *Let f be a meromorphic function in $\mathbb{D}$ of finite order σ. Let $\varepsilon > 0$ be a constant, and k and j be integers satisfying $k > j \geq 0$. Assume that $f^{(j)} \not\equiv 0$. Then:*

(i) *There exists a set* $E_1 \subset [0,1)$ *with* $\int_{E_1} \frac{dr}{1-r} < \infty$ *such that for all* $z \in \mathbb{D}$ *satisfying* $|z| \notin E_1$, *we have*

$$\left|\frac{f^{(k)}(z)}{f^{(j)}(z)}\right| \le \left(\frac{1}{1-|z|}\right)^{(k-j)(\sigma+2+\varepsilon)}. \tag{2.2}$$

(ii) *There exists a set* $E_2 \subset [0,2\pi)$ *of linear measure zero such that if* $\theta \in [0,2\pi) \setminus E_2$, *then there is a constant* $R = R(\theta) \in (0,1)$ *such that* (2.2) *holds for all* z *satisfying* $\arg z = \theta$ *and* $R \le |z| < 1$.

LEMMA 2.7 ([11]). *Suppose that* $S_\beta = \sum_{n=1}^{\infty}(1-|z_n|)^\beta < \infty$ $(\beta \in (0,1])$ *for a sequence* $\{z_n\}$ *of nonzero points in* $\mathbb{D}$. *For each* $r \in [0,1)$, *let* $n(r)$ *be the number of points* z_n *lying in* $\{z : |z| \le r\}$. *Then*

$$n(r) \le \frac{S_\beta}{(1-r)^\beta}.$$

LEMMA 2.8 ([15, 16]). *Let* $f_1, \dots, f_k$ *be linearly independent solutions of* (1.12), *where* $A_0, \dots, A_{k-2}$ *are analytic in* $\mathbb{D}$. *Let*

$$\omega_1 = f_1/f_k,\ \dots,\ \omega_{k-1} = f_{k-1}/f_k, \tag{2.3}$$

and let X_j *be the determinant*

$$X_j = \begin{vmatrix} \omega_1' & \omega_2' & \cdots & \omega_{k-1}' \\ \vdots & \vdots & & \vdots \\ \omega_1^{(j-1)} & \omega_2^{(j-1)} & \cdots & \omega_{k-1}^{(j-1)} \\ \omega_1^{(j+1)} & \omega_2^{(j+1)} & \cdots & \omega_{k-1}^{(j+1)} \\ \vdots & \vdots & & \vdots \\ \omega_1^{(k)} & \omega_2^{(k)} & \cdots & \omega_{k-1}^{(k)} \end{vmatrix}, \qquad j = 1, \dots, k. \tag{2.4}$$

Then

$$A_j = \sum_{i=0}^{k-j} (-1)^{2k-i} \delta_{ki} \binom{k-i}{k-i-j} \frac{X_{k-i}}{X_k} \frac{(\sqrt[k]{X_k})^{(k-i-j)}}{\sqrt[k]{X_k}}, \qquad j = 0, \dots, k-2, \tag{2.5}$$

where $\delta_{kk} = 0$ *and* $\delta_{ki} = 1$ *otherwise.*

It is easily checked that $(-1)^{k-1} X_k = W/f_k^k$, where W is the Wronskian of $f_1, \dots, f_k$ (see e.g. [19, p. 12]). Since W is a nonzero constant, we may set $C = -1/\sqrt[k]{W}$ to obtain the equality $\sqrt[k]{X_k} = 1/(Cf_k)$. This shows that $\sqrt[k]{X_k}$ is a well-defined meromorphic function in $\mathbb{D}$.

LEMMA 2.9 ([12]). *Suppose that* $A_0, \dots, A_{k-1}$ *are analytic in* $\mathbb{D}$, *and let* $\{f_1, \dots, f_k\}$ *be any solution base of* (1.9). *Then* $f_n/f_m \in N$ *for any* $n, m \in \{1, \dots, k\}$ *if and only if* (1.9) *is Blaschke-oscillatory.*

In order to obtain the next lemma, one may follow the proof of [18, Lemma 7.7] step by step, and use Lemma 2.6 each time when logarithmic derivatives are involved. These calculations culminate in dealing with identities (7.15) in [18]. We omit the details.

LEMMA 2.10. *Let $\varepsilon > 0$. Let $f_1, \dots, f_k$ be linearly independent meromorphic solutions of a linear differential equation of type* (1.9) *with coefficients $A_0(z), \dots, A_{k-1}(z)$ meromorphic in $\mathbb{D}$. Let $\sigma \geq 0$, and suppose that $\sigma(f_j) \leq \sigma$ for all $j = 1, \dots, k$. Then there exists a set $E \subset [0,1)$ with $\int_E \frac{dr}{1-r} < \infty$ such that*

$$|A_j(z)| \leq \left(\frac{1}{1-|z|}\right)^{(k-j)(\sigma+2)+\varepsilon}, \qquad j = 0, \dots, k-1,$$

for all $z \in \mathbb{D}$ with $|z| \notin E$.

3. Proof of Theorem 1.2

(a) Let B be the Blaschke product associated with $\{z_n\}$. Write $f = B^2 e^g$, where g is some analytic function to be constructed later. Then f is a solution of (1.5) if and only if

$$(6B'B'' + 2BB''') + 6(B'^2 + BB'')g' + 6BB'(g'' + g'^2) + ((g')^3 + 3g'g'' + g''' + A)B^2 = 0.$$

At the point z_n, we find that

$$g'(z_n) = -B''(z_n)/B'(z_n).$$

Since $B(z_n) = 0$, we have $B'(z_n) \neq 0$. For convenience, set

$$\sigma_n = -B''(z_n)/B'(z_n). \tag{3.1}$$

Since $B \in H^\infty$, we have $B'' \in H_2^\infty$. Moreover, since $\{z_n\}$ is uniformly q-separated in $\mathbb{D}$, there exists a constant $\delta \in (0,1)$ such that

$$\frac{1}{|B'(z_n)|} = (1-|z_n|^2)\left(\prod_{j\neq n}\left|\frac{z_j - z_n}{1-\overline{z_j}z_n}\right|\right)^{-1} \leq \frac{1}{\delta}(1-|z_n|^2)^{1-q}. \tag{3.2}$$

We conclude that $\|\sigma_n\|_{1+q} < \infty$. By Lemma 2.1(a), there exists $G \in H^\infty_{1+\alpha+2q}$ such that $G(z_n) = \sigma_n$ for all $n \in \mathbb{N}$. Define

$$g(z) = \int^z [G(\zeta) + B(\zeta)F(\zeta)]\, d\zeta,$$

where the integral represents any fixed primitive function of $G(\zeta)+B(\zeta)F(\zeta)$, and $F(\zeta)$ is an analytic function to be constructed later. Define

$$A(z) = -\frac{(6B'B'' + 2BB''') + 6(B'^2 + BB'')g' + 6BB'(g'' + g'^2)}{B^2} - [(g')^3 + 3g'g'' + g'''].$$

For convenience, set

$$(6B'B'' + 2BB''') + 3(2B'^2 + 2BB'')g' + 6BB'(g'' + g'^2) = H(z).$$

If $A(z)$ is analytic in $\mathbb{D}$, then $H(z)$ has at least a double zero at every z_n. Hence $H(z_n) = 0$ and $H'(z_n) = 0$ for every n. The requirement $H'(z_n) = 0$ leads to the interpolation property

$$\begin{aligned} F(z_n) = -\frac{1}{12B'(z_n)^3}\{&6B''(z_n)^2 + 8B'(z_n)B'''(z_n) + 18B'(z_n)B''(z_n)G(z_n) \\ &+ 12B'(z_n)^2G'(z_n) + 6B'(z_n)^2G(z_n)^2\} \\ := s_n, \quad & n \in \mathbb{N}. \end{aligned}$$

Since $B \in H^\infty$ and $G \in H^\infty_{1+\alpha+2q}$, we have $B''' \in H^\infty_3$, $G' \in H^\infty_{2+\alpha+2q}$, $G^2 \in H^\infty_{2(1+\alpha+2q)}$. Moreover, taking into account (3.2), we conclude that $\|s_n\|_{1+2\alpha+7q} < \infty$. Again by Lemma 2.1(a), there exists $F \in H^\infty_{1+3\alpha+8q}$ such that $F(z_n) = s_n$. Hence $A(z)$ defined by (3.3) is analytic in $\mathbb{D}$. A routine calculation shows that $f = B^2e^g$ is a solution of (1.5), where $A(z)$ is defined by (3.3). Moreover, f has zeros precisely at these points, and of multiplicity two. It remains to estimate the growth of the coefficient function $A(z)$. Using (3.3), we obtain

$$\begin{aligned} |A(z)| \le 6\left|\frac{B'B''}{B^2}\right| + 2\left|\frac{B'''}{B}\right| + 6\left(\left|\frac{B'}{B}\right|^2 + \left|\frac{B''}{B}\right|\right)\left|g'\right| \\ + 6\left|\frac{B'}{B}\right|[|g''| + |g'|^2] + |g'|^3 + 3|g'g''| + |g'''|. \end{aligned}$$

Next, using Lemma 2.2(a), and the fact that $g' \in H^\infty_{1+3\alpha+8q}$ and $g^{(k)} \in H^\infty_{k+3\alpha+8q}$, it follows that

$$M(r, A) = O\left(\left(\frac{1}{1-r}\right)^{3+9\alpha+24q}\right) \tag{3.3}$$

outside of a possible exceptional set $E \subset [0, 1)$ of r-values satisfying (2.1). Finally, Lemma 2.3 applied to (3.4) yields $A \in H^\infty_{3+9\alpha+24q}$.

(b) Define $\{\sigma_n\}$ as in (3.1). Evidently $\|\sigma_n\|_{1+q} < \infty$. By Lemma 2.1(b), there exists $G \in H^\infty_{1+2q}$ such that $G(z_n) = \sigma_n$ for all $n \in \mathbb{N}$. Again, define

$$g(z) = \int^z [G(\zeta) + B(\zeta)F(\zeta)]\, d\zeta,$$

where the integral represents any fixed primitive function of $G(\zeta)+B(\zeta)F(\zeta)$, and $F(\zeta)$ is an analytic function with $F \in H^\infty_{1+8q}$. Define an analytic function $A(z)$ by (3.3). Let $\delta > 0$ be the infimum in (1.3). Then the pseudo-hyperbolic discs $\mathbb{D}_n = \triangle(z_n, \frac{\delta}{2}(1 - |z_n|)^q)$ are pairwise disjoint. Using Lemma 2.2(b) and the fact that $g' \in H^\infty_{1+8q}$ and $g^{(k)} \in H^\infty_{k+8q}$, we see that there exists a finite constant $C_1 > 0$ such that

$$(1-|z|^2)^{3+24q}|A(z)| \le C_1 \tag{3.4}$$

for all $z \notin \bigcup_n \mathbb{D}_n$.

Now assume that $z \in \bigcup_n \mathbb{D}_n$. Then $z \in \mathbb{D}_k$ for some $k \in \mathbb{N}$. By the maximum modulus principle there exists $\zeta \in \partial\mathbb{D}_k$ such that $|A(z)| \le |A(\zeta)|$ for all $z \in \mathbb{D}_k$. Recalling that $\mathbb{D}_k$ is in fact a Euclidean disc, there exists $\omega \in \partial\mathbb{D}_k$ of greatest modulus. In particular,

$$|\zeta| \le |\omega| = \frac{|z_k| + \frac{\delta}{2}(1-|z_k|)^q}{1+\frac{\delta}{2}(1-|z_k|)^q|z_k|}.$$

Since (3.5) holds at ζ, we have

$$\begin{aligned}|A(z)| \le |A(\zeta)| &\le \frac{C_1}{(1-|\zeta|^2)^{3+24q}} \le \frac{C_1}{(1-|\omega|)^{3+24q}} \\ &\le \left(\frac{2+\delta}{2-\delta}\right)^{3+24q} \frac{C_1}{(1-|z_k|)^{3+24q}}.\end{aligned}$$

As z and z_k both belong to $\overline{\triangle}(z_k, \delta/2)$, Lemma 2.4 yields

$$\frac{1-|z|}{1-|z_k|} \le 8\left(\frac{4+\delta^2}{4-\delta^2}\right)^2,$$

where the upper bound is independent of k. Hence

$$(1-|z|^2)^{3+24q}|A(z)| \le \left(\frac{1-|z|^2}{1-|z_k|^2}\right)^{3+24q} |A(z)|(1-|z_k|^2)^{3+24q} < C_2 < \infty,$$

where C_2 is independent of k. Combining this with (3.5), we conclude that $A \in H^\infty_{3+24q}$. This completes the proof of Theorem 1.2.

4. Proof of Theorem 1.5. Let $B(z)$ be the Blaschke product associated with $\{z_n\}$. Write $f_\zeta(z) = 2\zeta(B(z)+2)^{-1} = 2\zeta(g(z))^{-1}$ for each fixed $\zeta \in \mathbb{C}\setminus\{0\}$, where $g(z) = B(z)+2$. Then a routine calculation shows that

$$\begin{aligned} f'_\zeta(z)g(z) &= -f_\zeta(z)g'(z), \\ f''_\zeta(z)g(z) + 2f'_\zeta(z)g'(z) &= -f_\zeta(z)g''(z), \\ &\vdots \\ f^{(k)}_\zeta(z)g(z) + \cdots + kf'_\zeta(z)g^{(k-1)}(z) &= -f_\zeta(z)g^{(k)}(z). \end{aligned}$$

By the classical Cramer rule,

$$f^{(k)}_\zeta(z) = (-f_\zeta(z))\frac{F_k(z)}{E_k(z)}, \tag{4.1}$$

where

$$(4.2)\qquad E_k(z)=\begin{vmatrix} 0 & 0 & \cdots & 0 & g(z) \\ 0 & 0 & \cdots & g(z) & 2g'(z) \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & g(z) & \cdots & C_{k-1}^{k-3}g^{(k-3)}(z) & (k-1)g^{(k-2)}(z) \\ g(z) & kg'(z) & \cdots & C_k^{k-2}g^{(k-2)}(z) & kg^{(k-1)}(z) \end{vmatrix}$$

$$= (-1)^{k(k-1)/2}g^k(z),$$

$$(4.3)\qquad F_k(z)=\begin{vmatrix} g'(z) & 0 & \cdots & 0 & g(z) \\ g''(z) & 0 & \cdots & g(z) & 2g'(z) \\ \vdots & \vdots & & \vdots & \vdots \\ g^{(k-1)}(z) & g(z) & \cdots & C_{k-1}^{k-3}g^{(k-3)}(z) & (k-1)g^{(k-2)}(z) \\ g^{(k)}(z) & kg'(z) & \cdots & C_k^{k-2}g^{(k-2)}(z) & kg^{(k-1)}(z) \end{vmatrix}$$

$$= (-1)^{k-1}\begin{vmatrix} 0 & \cdots & 0 & g(z) & g'(z) \\ 0 & \cdots & g(z) & 2g'(z) & g''(z) \\ \vdots & & \vdots & \vdots & \vdots \\ g(z) & \cdots & C_{k-1}^{k-3}g^{(k-3)}(z) & (k-1)g^{(k-2)}(z) & g^{(k-1)}(z) \\ kg'(z) & \cdots & C_k^{k-2}g^{(k-2)}(z) & kg^{(k-1)}(z) & g^{(k)}(z) \end{vmatrix}.$$

Set

$$(4.4)\qquad G_k(z)=\begin{vmatrix} 0 & \cdots & 0 & g(z) & g'(z) \\ 0 & \cdots & g(z) & 2g'(z) & g''(z) \\ \vdots & & \vdots & \vdots & \vdots \\ g(z) & \cdots & C_{k-1}^{k-3}g^{(k-3)}(z) & (k-1)g^{(k-2)}(z) & g^{(k-1)}(z) \\ kg'(z) & \cdots & C_k^{k-2}g^{(k-2)}(z) & kg^{(k-1)}(z) & g^{(k)}(z) \end{vmatrix}.$$

Then

$$G_k(z)=\sum_{j=1}^{k} C_k^j g^{(j)}(z)M_{kj}(z)=\sum_{j=1}^{N_k} D_{kj}[g],$$

where $M_{kj}(z)$ are the algebraic complements of $C_k^j g^{(j)}(z)$ for $j=1,\ldots,k$, N_k is some positive integer depending on k, the expressions $D_{kj}[g]$ are differential monomials in g of the form

$$D_{kj}[g]=C_{kj}(g)^{k_{j0}}(g')^{k_{j1}}\cdots(g^{(k)})^{k_{jk}}$$

with $k_{j0},\ldots,k_{jk}\in\mathbb{N}\cup\{0\}$ for $j=1,\ldots,N_k$, and C_{kj} are some constants. The sum $\nu_{kj}=k_{j1}+2k_{j2}+\cdots+k\cdot k_{jk}$ is the weight of $D_{kj}[g]$. The weight of $G_k = \sum_{j=1}^{N_k} D_{kj}[g]$ is defined by $\nu(G_k) = \max_{1\le j\le N_k} \nu_{kj}$. We assert that $\nu(G_k) = k$ for $k = 2, 3, \dots$. We use induction on k. For $k = 2$, $G_2 = \left|\begin{smallmatrix} g(z) & g'(z) \\ 2g'(z) & g''(z)\end{smallmatrix}\right| = g(z)g''(z) - 2g'^2(z)$, and obviously $\nu(G_2) = 2$. Assuming now that for $k = n$,

$$\nu(G_n) = \nu\Big(\sum_{j=1}^{n} C_n^j g^{(j)}(z) M_{nj}(z)\Big) = n, \tag{4.5}$$

consider $k = n + 1$. Since

$$\begin{aligned} &G_{n+1}(z) \\ &= \begin{vmatrix} 0 & 0 & \cdots & 0 & g & g' \\ 0 & 0 & \cdots & g & 2g' & g'' \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & g & \cdots & C_{n-1}^{n-3} g^{(n-3)} & (n-1)g^{(n-2)} & g^{(n-1)} \\ g & ng' & \cdots & C_n^{n-2} g^{(n-2)} & ng^{(n-1)} & g^{(n)} \\ (n+1)g' & C_{n+1}^2 g'' & \cdots & C_{n+1}^{n-1} g^{(n-1)} & (n+1)g^{(n)} & g^{(n+1)} \end{vmatrix} \\ &= (-1)^{n+2}(n+1)g'(z)G_n(z) + (-1)^{n+1} g(z) \sum_{j=1}^{n} C_{n+1}^{j+1} g^{(j+1)}(z) M_{nj}(z), \end{aligned} \tag{4.6}$$

the assertion $\nu(G_{n+1}) = n + 1$ follows immediately by (4.5) and (4.6).

By (4.1)–(4.4), we obtain

$$f_\zeta^{(k)}(z) = (-f_\zeta(z))(-1)^{(k-1)(k+2)/2} \frac{G_k(z)}{g^k(z)}.$$

Define

$$A(z) = (-1)^{(k-1)(k+2)/2} G_k(z)/g^k(z).$$

Since $B \in H^\infty$, the Cauchy formula yields $B^{(i)} \in H_i^\infty$, $i = 1, \dots, k$. These imply immediately $g \in H^\infty$ and $g^{(k)} \in H_i^\infty$, $i = 1, \dots, k$. Therefore obviously $A \in H_k^\infty$ from the fact that the weight of $G_k(z)$ is k. For each fixed $\zeta \in \mathbb{C} \setminus \{0\}$, a simple computation reveals that (1.6) has an analytic solution $f_\zeta(z) = 2\zeta(B(z) + 2)^{-1}$ which takes the value ζ precisely at the points z_n. Evidently, $f_\zeta \in H^\infty$. Theorem 1.5 is proved.

5. Proofs of Proposition 1.6 and Corollary 1.7. We prove Proposition 1.6 here for the sake of completeness but the proof is essentially contained in [12].

Proof of Proposition 1.6. J. Heittokangas et al. [14] found sufficient conditions on $A_j(z)$ for all solutions of (1.9) to be in the Nevanlinna class N:

these conditions are (1.10) and (1.11). On the other hand, since any $f \in N$ can be factorized as $f = Bg$, where B is a finite or an infinite Blaschke product and $g \in N$ is nonvanishing and analytic in $\mathbb{D}$, Proposition 1.6 follows. ■

Proof of Corollary 1.7. We settle the case $p_j = 1/(2(k-j))$, which, by the nesting property of H^p-spaces, proves the whole statement. Lemma 2.5 gives (with $p_j = 1/(2(k-j))$, $q = \lambda = 1/(k-j)$ and $\alpha = k-j$)

$$\iint_{\mathbb{D}} |A_j(z)|^{1/(k-j)}\, d\sigma(z) \le 2\pi \int_0^1 M_{1/(k-j)}(r, A_j)^{1/(k-j)}\, dr < \infty.$$

The result follows by Proposition 1.6. ■

Remark 5.1. Using (1.11) instead of (1.10) in the proof of Corollary 1.7, we see that a corresponding result holds for $1/(k-j+1) \le p_j \le \infty$. However, $H^{1/(k-j+1)} \subset H^{1/(2(k-j))}$. Hence, by using (1.11), we get a weaker result than that in Corollary 1.7.

6. Proof of Theorem 1.8. Let $\{f_1, f_2\}$ be a fundamental system of solutions of (1.1), and define $f = f_1 - f_2$ and $G = f_1/f_2$. Since (1.1) is assumed to be Blaschke-oscillatory, the possible zeros of f and the zeros and poles of G are all Blaschke sequences. We also observe that the 1-points of G are the zeros of f, hence they form a Blaschke sequence as well. Now, by Nevanlinna's second fundamental theorem,

$$T(r,G) \le N(r,G) + N\left(r, \frac{1}{G}\right) + N\left(r, \frac{1}{G-1}\right) + S(r,G),$$

where the error term $S(r,G)$ satisfies

$$S(r,G) = O\left(\log \frac{1}{1-r}\right) + o(T(r,G)).$$

Applying Lemma 2.7 with $\beta = 1$, we conclude that G is nonadmissible and $\sigma(G) = 0$. It is well known that $2A(z) = S_G(z)$, where

$$S_G = \frac{G'''}{G'} - \frac{3}{2}\left(\frac{G''}{G'}\right)^2$$

is the Schwarzian derivative of G. Next, from Lemmas 2.6 and 2.3 we deduce that

$$\begin{aligned}|A(re^{i\theta})| &= O\left(\left|\frac{G'''(re^{i\theta})}{G'(re^{i\theta})}\right| + \left|\frac{G''(re^{i\theta})}{G'(re^{i\theta})}\right|^2\right) \\ &= O\left(\left(\frac{1}{1-r}\right)^{4+2\varepsilon}\right)\end{aligned}$$

and

$$\iint_{\mathbb{D}} |A(re^{i\theta})|(1-|z|)^{\alpha}\, d\sigma(z) = O\bigg(\int_0^1 \frac{1}{(1-r)^{4-\alpha+2\varepsilon}}\, dr\bigg).$$

It follows that $\iint_{\mathbb{D}} |A(z)|(1-|z|)^{\alpha}\, d\sigma(z) < \infty$, since $\alpha > 3$. This concludes the proof of Theorem 1.8.

7. Proof of Theorem 1.9. Let $f_1, \dots, f_k$ be linearly independent solutions of (1.12), and let $\omega_1, \dots, \omega_{k-1}$ be defined by (2.3). By Lemma 2.9, all ω_j $(j = 1, \dots, k-1)$ have bounded characteristic and $\sigma(\omega_j) = 0$ since (1.12) is Blaschke-oscillatory. It is stated in [16, p. 418] that the functions $1, \omega_1, \dots, \omega_{k-1}$ are linearly independent meromorphic solutions of the differential equation

$$\omega^{(k)} - \frac{X_{k-1}(z)}{X_k(z)}\omega^{(k-1)} + \cdots + (-1)^{k+1}\frac{X_1(z)}{X_k(z)}\omega' = 0,$$

where the functions X_j are defined by (2.4). From $\sigma(\omega_j) = 0$ and Lemma 2.10, we now conclude that

$$\left|\frac{X_j(z)}{X_k(z)}\right| \le \left(\frac{1}{1-|z|}\right)^{2(k-j)+\varepsilon/3}, \qquad j = 1, \dots, k-1,$$

that is,

$$\left|\frac{X_{k-i}(z)}{X_k(z)}\right| \le \left(\frac{1}{1-|z|}\right)^{2i+\varepsilon/3}, \qquad i = 1, \dots, k-1. \tag{7.1}$$

By (2.4) and $\sigma(\omega_j) = 0$ it is clear that $\sigma(X_k) = 0$. Since $\sqrt[k]{X_k}$ is a well-defined meromorphic function in $\mathbb{D}$, it follows that $\sigma(\sqrt[k]{X_k}) = 0$. By Lemma 2.10, there exists a set $E \subset [0, 1)$ with $\int_E \frac{dr}{1-r} < \infty$ such that, for all $z \in \mathbb{D}$ satisfying $|z| \notin E$, we have

$$\left|\frac{(\sqrt[k]{X_k})^{(k-i-j)}(z)}{\sqrt[k]{X_k}}\right| \le \left(\frac{1}{1-|z|}\right)^{2(k-i-j)+\varepsilon/3}, \tag{7.2}$$

where i and j are as in (2.5). By combining (2.5), (7.1), (7.2) and Lemma 2.3, we get

$$|A_j(z)| \le \left(\frac{1}{1-|z|}\right)^{2(k-j)+\varepsilon}, \qquad j = 0, \dots, k-2,$$

for all $z \in \mathbb{D}$ and

$$\iint_{D} |A_j(re^{i\theta})|(1-|z|)^{\alpha}\, d\sigma(z) = O\bigg(\int_0^1 \frac{1}{(1-r)^{2(k-j)-\alpha+\varepsilon}}\bigg).$$

The assertion follows from this since $\alpha > 2(k-j) - 1$.

8. Proof of Theorem 1.12. Let $f(z)=\sum_{n=0}^{\infty} b_n z^n$ be a formal solution of (1.9). Then

$$
\begin{aligned}
(8.1)\qquad & f^{(k)}+A_{k-1}(z)f^{(k-1)}+\cdots+A_1(z)f'+A_0(z)f \\
&= \sum_{n=0}^{\infty}[(n+k)(n+k-1)\cdots(n+1)b_{n+k}+c_n]z^n=0,
\end{aligned}
$$

where

$$c_n=\sum_{i=0}^{k-1}\sum_{j=1}^{n+1}(n+k-j-i)(n+k-j-i-1)\cdots(n-j+2)b_{n+k-j-i}a_{k-i-1,j-1}.$$

Hence (8.1) holds if and only if

$$b_n=-\frac{c_{n-k}}{n(n-1)\cdots(n+1-k)} \tag{8.2}$$

for all $n=k,k+1,\ldots$. Choose a finite constant $M>0$ such that

$$|b_i|\le \frac{M}{(i+2)(i+1)^{1/2}} \tag{8.3}$$

for all $i=0,\ldots,n$ $(n>k)$. Then it follows from (8.2) that

$$
\begin{aligned}
(8.4)\qquad |b_{n+1}| &= \frac{|c_{n-k+1}|}{(n+1)n\cdots(n+2-k)} \\
&= \frac{\left|\sum_{i=0}^{k-1}\sum_{j=1}^{n-k+2}(n+1-j-i)\cdots(n-k-j+3)b_{n+1-j-i}a_{k-i-1,j-1}\right|}{(n+1)n\cdots(n+2-k)} \\
&\le \sum_{i=0}^{k-1} I_{k-1-i},
\end{aligned}
$$

where

$$
\begin{aligned}
&I_{k-1-i} \\
&= \frac{|\sum_{j=1}^{n-k+2}(n+1-j-i)(n-j-i)\cdots(n-k-j+3)b_{n+1-j-i}a_{k-i-1,j-1}|}{(n+1)n\cdots(n+2-k)}.
\end{aligned}
$$

It follows from $|a_{k-1,n}|\le 1/(n+2)$ for all n and (8.3) that

$$
\begin{aligned}
&(8.5) \\
&I_{k-1}=\frac{|n\cdots(n-k+2)b_n a_{k-1,0}+\cdots+(k-1)\cdots 2b_{k-1}a_{k-1,n-k+1}|}{(n+1)n\cdots(n+2-k)} \\
&\quad\le \frac{M}{(n+1)n\cdots(n+2-k)} \\
&\qquad\times\left[\frac{n\cdots(n-k+2)}{(n+2)(n+1)^{1/2}}\,\frac{1}{2}+\cdots+\frac{(k-1)\cdots 2}{(k+1)k^{1/2}}\,\frac{1}{n-k+3}\right]
\end{aligned}
$$

$$\begin{aligned}
&\le \frac{M}{2(n+1)n\cdots(n+2-k)}(n-k+2)(n-1)^{1/2}(n-2)\cdots(n-k+2)\\
&= \frac{M}{(n+3)(n+2)^{1/2}}\\
&\quad\times \frac{(n+3)(n+2)^{1/2}(n-k+2)(n-1)^{1/2}(n-2)\cdots(n-k+2)}{2(n+1)n\cdots(n+2-k)}\\
&\le \frac{2M}{3(n+3)(n+2)^{1/2}} \qquad (n > N_0 > k).
\end{aligned}$$

Here and in what follows, N_0 is a sufficiently large positive integer.

Similarly, it follows from $|a_{k-2,n}| \le 1$ for all n and (8.3) that

$$\begin{aligned}
(8.6)\qquad &I_{k-2}\\
&= \frac{|(n-1)\cdots(n-k+2)b_{n-1}a_{k-2,0} + \cdots + (k-2)\cdots 2b_{k-2}a_{k-2,n-k+1}|}{(n+1)n\cdots(n+2-k)}\\
&\le \frac{M}{(n+1)n\cdots(n+2-k)}\\
&\quad\times\left[\frac{(n-1)(n-2)\cdots(n-k+2)}{(n+1)n^{1/2}} + \cdots + \frac{(k-2)(k-3)\cdots 2}{k(k-1)^{1/2}}\right]\\
&\le \frac{M}{(n+1)n\cdots(n+2-k)}(n-k+2)(n-2)^{1/2}(n-3)\cdots(n-k+2)\\
&= \frac{M}{(n+3)(n+2)^{1/2}}\\
&\quad\times\frac{(n+3)(n+2)^{1/2}(n-k+2)(n-2)^{1/2}(n-3)\cdots(n-k+2)}{(n+1)n\cdots(n+2-k)}\\
&\le \frac{M}{3(k-1)}\,\frac{1}{(n+3)(n+2)^{1/2}} \qquad (n > N_0 > k).
\end{aligned}$$

For $i = 2, \ldots, k-3$, we have

$$\begin{aligned}
(8.7)\qquad I_{k-1-i} &= \frac{1}{(n+1)n\cdots(n+2-k)}\\
&\quad\times\big|(n-i)(n-i-1)\cdots(n-k+2)b_{n-i}a_{k-i-1,0}\\
&\qquad + \cdots + (k-1-i)(k-2-i)\cdots 2b_{k-1-i}a_{k-i-1,n-k+1}\big|\\
&\le \frac{M}{(n+1)n\cdots(n+2-k)}\\
&\quad\times\left[\frac{(n-i)(n-i-1)\cdots(n-k+2)}{(n+2-i)(n+1-i)^{1/2}}2^{i-1} + \cdots\right.\\
&\qquad\left. + \frac{(k-1-i)(k-2-i)\cdots 2}{(k+1-i)(k-i)^{1/2}}(n-k+3)^{i-1}\right]
\end{aligned}$$

$$\leq \frac{M}{(n+1)n\cdots(n+2-k)}\big[(n-k+2)(n-i-1)^{1/2}$$
$$\times (n-i-2)(n-i-3)\cdots(n-k+2)(n-k+3)^{i-1}\big]$$
$$= \frac{M}{(n+3)(n+2)^{1/2}}\,\frac{(n+3)(n+2)^{1/2}}{(n+1)n\cdots(n+2-k)} \times \big[(n-k+2)$$
$$\times (n-i-1)^{1/2}(n-i-2)\cdots(n-k+2)(n-k+3)^{i-1}\big]$$
$$\leq \frac{M}{3(k-1)}\,\frac{1}{(n+3)(n+2)^{1/2}} \qquad (n > N_0 > k)$$

For $i = k-2$, we have

$$I_1 = \frac{|(n-k+2)b_{n-k+2}a_{1,0} + \cdots + b_1 a_{1,n-k+1}|}{(n+1)n\cdots(n+2-k)} \tag{8.8}$$
$$\leq \frac{M}{(n+1)n\cdots(n+2-k)}\left[\frac{n-k+2}{(n-k+4)(n-k+3)^{1/2}} \times 2^{k-3}\right.$$
$$\left. + \cdots + \frac{1}{3\times 2^{1/2}} \times (n-k+3)^{k-3}\right]$$
$$\leq \frac{M(n-k+3)^{k-3}(n-k+2)}{(n+1)n\cdots(n+2-k)}\left[\frac{1}{(n-k+3)^{3/2}} + \cdots + \frac{1}{2^{3/2}}\right]$$
$$\leq \frac{M}{(n+3)(n+2)^{1/2}}\,\frac{(n+3)(n+2)^{1/2}(n-k+3)^{k-3}(n-k+2)}{(n+1)n\cdots(n+2-k)}\sum_{i=2}^{\infty}\frac{1}{i^{3/2}}$$
$$\leq \frac{M}{3(k-1)}\,\frac{1}{(n+3)(n+2)^{1/2}} \qquad (n > N_0 > k).$$

For $i = k-1$, we have

$$I_0 = \frac{|b_{n-k+1}a_{0,0} + \cdots + b_0 a_{0,n-k+1}|}{(n+1)n\cdots(n+2-k)} \tag{8.9}$$
$$\leq \frac{M}{(n+1)n\cdots(n+2-k)}\left[\frac{1}{(n-k+3)(n-k+2)^{1/2}} \times 2^{k-2}\right.$$
$$\left. + \cdots + \frac{1}{2\times 1^{1/2}} \times (n-k+3)^{k-2}\right]$$
$$\leq \frac{M}{(n+3)(n+2)^{1/2}}\,\frac{(n+3)(n+2)^{1/2}(n-k+3)^{k-2}}{(n+1)n\cdots(n+2-k)}\sum_{i=1}^{\infty}\frac{1}{i^{3/2}}$$
$$\leq \frac{M}{3(k-1)}\,\frac{1}{(n+3)(n+2)^{1/2}} \qquad (n > N_0 > k).$$

Inequalities (8.4)–(8.9) show that

$$|b_{n+1}| \leq \frac{M}{(n+3)(n+2)^{1/2}}.$$

Therefore, $|b_n| \leq \frac{M}{(n+2)(n+1)^{1/2}}$ for all $n \geq 0$, and so $\sum_{n=0}^{\infty} b_n z^n$ is absolutely convergent on $\mathbb{D}$. Hence f is analytic in $\mathbb{D}$ and $f \in \mathcal{D}$ since $\sum_{n=1}^{\infty} n|b_n|^2 < \infty$. This completes the proof of Theorem 1.12.

Acknowledgments. This work is supported by the National Natural Science Foundation of China (Nos. 11301232, 11171119) and the Natural Science Foundation of Jiangxi Province (No. 20132BAB211009). We are grateful to the referee for suggestions which improved the presentation of the paper.

References

[1] R. Aulaskari, J. Xiao and R. Zhao, *On subspaces and subsets of BMOA and UBC*, Analysis 15 (1995), 101–121.

[2] S. Bank, *A general theorem concering the growth of solutions of first-order algebraic differential equations*, Compos. Math. 25 (1972), 61–70.

[3] S. Bank and I. Laine, *On the oscillation theory of* $f'' + Af = 0$, *where* A *is entire*, Trans. Amer. Math. Soc. 273 (1982), 351–363.

[4] I. Chyzhykov, G. Gundersen and J. Heittokangas, *Linear differential equations and logarithmic derivative estimates*, Proc. London Math. Soc. 86 (2003), 735–754.

[5] P. Duren, *Theory of* H^p *Spaces*, Academic Press, New York, 1970.

[6] P. Duren and A. Schuster, *Bergman Spaces*, Amer. Math. Soc., Providence, RI, 2004.

[7] J. Gröhn and J. Heittokangas, *New findings on the Bank–Sauer approach in oscillation theory*, Constr. Approx. 35 (2012), 345–361.

[8] J. Heittokangas, *Solutions of* $f'' + A(z)f = 0$ *in the unit disc having Blaschke sequences as the zeros*, Comput. Methods Funct. Theory 5 (2005), 49–63.

[9] J. Heittokangas, *Blaschke-oscillatory equations of the form* $f'' + A(z)f = 0$, J. Math. Anal. Appl. 318 (2006), 120–133.

[10] J. Heittokangas, *On complex differential equations in the unit disc*, Ann. Acad. Sci. Fenn. Math. Diss. 122 (2000), 1–54.

[11] J. Heittokangas, *Growth estimates for logarithmic derivatives of Blaschke products and of functions in the Nevanlinna class*, Kodai Math J. 30 (2007), 263–279.

[12] J. Heittokangas, *A survey on Blaschke-oscillatory differential equations, with updates*, in: J. Mashreghi and E. Fricain (eds.), Blaschke Products and Their Applications, Fields Inst. Comm. 65, 2012, 43–98.

[13] J. Heittokangas, *On interpolating Blaschke products and Blaschke-oscillatory equations*, Constr. Approx. 34 (2011), 1–21.

[14] J. Heittokangas, R. Korhonen and J. Rättyä, *Linear differential equations with coefficients in weighted Bergman and Hardy spaces*, Trans. Amer. Math. Soc. 360 (2008), 1035–1055.

[15] J. Heittokangas and J. Rättyä, *Zero distribution of solutions of complex linear differential equations determines growth of coefficients*, Math. Nachr. 284 (2011), 412–420.

[16] W. J. Kim, *The Schwarzian derivative and multivalence*, Pacific J. Math. 31 (1969), 717–724.

[17] B. Korenblum, *An extension of the Nevanlinna theory*, Acta Math. 135 (1975), 187–219.

[18] I. Laine, *Nevanlinna Theory and Complex Differential Equations*, de Gruyter, Berlin, 1993.
[19] H. Li and H. Wulan, *Linear differential equations with solutions in the Q_k spaces*, J. Math. Anal. Appl. 375 (2011), 478–489.
[20] M. Li and L. Xiao, *Solutions of linear differential equations in the unit disc*, J. Math. Res. Appl. 34 (2014), 729–735.
[21] D. London, *On the zeros of solutions of $\omega''(z) + p(z)\omega(z) = 0$*, Pacific J. Math. 12 (1962), 979–991.
[22] Z. Nehari, *The Schwarzian derivative and schlicht functions*, Bull. Amer. Math. Soc. 55 (1949), 545–551.
[23] B. Schwarz, *Complex nonoscillation theorems and criteria of univalence*, Trans. Amer. Math. Soc. 80 (1955), 159–186.

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Received 25.2.2015
and in final form 22.6.2015 (3641)