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A QUASISTATIC CONTACT PROBLEM WITH UNILATERAL CONSTRAINT AND SLIP-DEPENDENT FRICTION

Abstract. We consider a mathematical model of a quasistatic contact between an elastic body and an obstacle. The contact is modelled with unilateral constraint and normal compliance, associated to a version of Coulomb's law of dry friction where the coefficient of friction depends on the slip displacement. We present a weak formulation of the problem and establish an existence result. The proofs employ a time-discretization method, compactness and lower semicontinuity arguments.

1. Introduction. Contact problems involving deformable bodies are quite frequent in industry as well as in daily life and play an important role in structural and mechanical systems. Because of the importance of this process a considerable effort has been made in its modelling and numerical simulations. An early attempt to study frictional contact problems within the framework of variational inequalities was made in [7]. The mathematical analysis of unilateral contact problems, including existence and uniqueness results, was widely developed in [8]. The mathematical, mechanical and numerical state of the art can be found in [9, 12].

In this paper we analyze the weak solvability of the quasistatic version of the model of static elastic contact studied recently in [2]. The contact is modelled with unilateral constraint and normal compliance such that the penetration is limited, associated with a slip-dependent version of Coulomb's law of dry friction. The normal compliance condition with unilateral constraint was introduced in [11]; it is a coupling between the Signorini contact condition and the normal compliance, and it models the contact with an elastic-rigid foundation. Examples of normal compliance can be found in [4, 9, 11] for instance.

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We recall that the model of a slip-dependent friction is considered in geophysics and solid mechanics corresponding to a smooth dependence of the friction coefficient on the slip u_τ, i.e. $\mu = \mu(|u_\tau|)$. Several authors were interested to contact problems with slip-dependent friction (see [2, 4, 6, 14] and the references therein). In [2] a static contact problem with unilateral constraint and slip-dependent friction was resolved; numerical results were presented which illustrate both the behavior of the solution and the convergence order of the error estimates. Also in [4] a static contact problem with slip-dependent friction and a prescribed normal stress on the contact surface for elastic materials was studied, while the same model in the quasistatic contact case was studied in [6]. In both references the authors employ the abstract results established in [14].

The contact problem with slip-dependent friction was also studied in dynamic elasticity. By using the Galerkin method and regularization techniques, the authors of [10] have proved the existence of a solution in the two-dimensional case (in-plane and anti-plane problems), hence for the case of the one-dimensional shearing problem, the solution that has been found in two dimensions is unique.

The quasistatic contact problem which uses a normal compliance law has also been studied in [1] by considering incremental problems and in [13] by another method using a time regularization.

Here, as in [15], we continue the study of contact problems with slip-dependent friction. Based on a time discretization method, we prove the existence of a solution for a variational formulation of the quasistatic frictional problem, given in terms of two variational inequalities as in [5]. Thus the method is similar to the one used in [5, 15] in order to study quasistatic contact problems for elastic materials. We construct a sequence of quasi-variational inequalities for which we prove the existence and uniqueness of solution. Then, we interpolate the discrete solution in time and, using compactness and lower semicontinuity, we derive the existence of a solution of the quasistatic contact problem under the smallness assumption on the friction coefficient and the normal compliance.

2. Problem statement and variational formuation. Consider an elastic body represented by a bounded Lipschitzian domain Ω in $\mathbb{R}^d$, $d = 2, 3$. The boundary Γ of Ω is partitioned as $\Gamma = \bar{\Gamma}_1 \cup \bar{\Gamma}_2 \cup \bar{\Gamma}_3$ where Γ_i, $i = 1, 2, 3$, are disjoint and open parts of Γ with $\mathrm{meas}(\Gamma_1) > 0$. The body is acted upon by a volume force of density f_1 on Ω and a surface traction of density f_2 on Γ_2. On Γ_3 the body is in unilateral contact with friction with an obstacle.

Under these conditions the classical formulation of the mechanical problem is the following.

Problem P_1. Find a displacement field $u : \Omega \times [0,T] \to \mathbb{R}^d$ such that

$$\operatorname{div}\sigma(u) = -f_1 \quad \text{in } \Omega \times (0,T), \tag{2.1}$$

$$\sigma(u) = \mathcal{A}\varepsilon(u) \quad \text{in } \Omega \times (0,T), \tag{2.2}$$

$$u = 0 \quad \text{on } \Gamma_1 \times (0,T), \tag{2.3}$$

$$\sigma\nu = f_2 \quad \text{on } \Gamma_2 \times (0,T), \tag{2.4}$$

$$u_\nu \le g, \sigma_\nu + p(u_\nu) \le 0,\ (\sigma_\nu + p(u_\nu))(u_\nu - g) = 0 \text{ on } \Gamma_3 \times (0,T), \tag{2.5}$$

$$\begin{cases} |\sigma_\tau| \le \mu(|u_\tau|)p(u_\nu) \\ |\sigma_\tau| < \mu(|u_\tau|)p(u_\nu) \Rightarrow \dot{u}_\tau = 0 \\ |\sigma_\tau| = \mu(|u_\tau|)p(u_\nu) \Rightarrow \exists\lambda \ge 0 : \sigma_\tau = -\lambda\dot{u}_\tau \end{cases} \quad \text{on } \Gamma_3 \times (0,T), \tag{2.6}$$

$$u(0) = u_0 \quad \text{in } \Omega. \tag{2.7}$$

Here (2.1) represents the equilibrium equation in which $\sigma = \sigma(u)$ denotes the stress tensor, (2.2) is the elastic constitutive law and $\mathcal{A}$ the fourth order tensor of elasticity coefficients, and (2.3) and (2.4) are the displacement-tractions boundary conditions where ν denotes the unit outward normal vector on Γ.

We make some comments on the contact conditions (2.5) and (2.6) in which σ_ν denotes the normal stress, p is a prescribed nonnegative function, u_ν is the normal displacement, g is a positive constant which denotes the maximum value of the penetration, σ_τ represents the tangential traction, μ is the coefficient of friction and $\dot{u}_\tau$ represents the tangential velocity.

Indeed, when $u_\nu < 0$, i.e. when there is separation between the body and the obstacle, then condition (2.5) combined with hypothesis (2.13) below shows that the reaction of the obstacle vanishes ($\sigma_\nu = 0$).

When $0 \le u_\nu < g$ then $-\sigma_\nu = p(u_\nu)$, which means that the reaction of the obstacle is uniquely determined by the normal displacement.

When $u_\nu = g$ then $-\sigma_\nu \ge p(g)$ and σ_ν is not uniquely determined.

We note then when $g = 0$ and $p = 0$ then condition (2.5) becomes the classical Signorini contact condition without a gap:

$$u_\nu \le 0, \quad \sigma_\nu \le 0, \quad \sigma_\nu u_\nu = 0,$$

and when $g > 0$ and $p = 0$, condition (2.5) becomes the classical Signorini contact condition with a gap:

$$u_\nu \le g, \quad \sigma_\nu \le 0, \quad \sigma_\nu(u_\nu - g) = 0.$$

The last two conditions are used to model the unilateral conditions with a rigid foundation.

Conditions (2.6) represent a version of Coulomb's law of dry friction in which μ depends on the displacement u_τ. The tangential shear cannot exceed the maximal frictional resistance $\mu(|u_\tau|)p(u_\nu)$. We also point out that when $u_\nu < 0$, conditions (2.5) combined with (2.6) imply that $\sigma_\nu = 0$, $\sigma_\tau = 0$; when $0 \le u_\nu < g$, we obtain a unilateral contact zone with normal compliance associated with the quasistatic version of Coulomb's law of dry friction; also, when the normal displacement u_ν reaches g, i.e. $u_\nu = g$, we obtain a bilateral contact zone described by the quasistatic version of Tresca's friction law. Finally, the function u_0 denotes the initial displacement.

Next, to establish the variational formulation we adopt the following notation. We denote by S_d the space of second order symmetric tensors on $\mathbb{R}^d$ $(d = 2, 3)$. We recall that that the inner products and the corresponding norms are given by

$$u.v = u_i v_i, \qquad |v| = (u.v)^{1/2} \quad \forall u, v \in \mathbb{R}^d,$$
$$\sigma.\tau = \sigma_{ij}\tau_{ij}, \qquad |\tau| = (\tau.\tau)^{1/2} \quad \forall \sigma, \tau \in S_d.$$

The strain tensor is

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \qquad \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \qquad i, j \in \{1, \ldots, d\};$$

$\operatorname{div}\sigma = (\sigma_{ij,j})$ is the divergence of σ where we denote respectively by u and σ the displacement and stress fields in the body.

To proceed with the variational formulation, we consider the following function spaces (the summation convention over repeated indices is used):

$$H = (L^2(\Omega))^d, \qquad H_1 = (H^1(\Omega))^d,$$
$$Q = \{\sigma = (\sigma_{ij}) : \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\},$$
$$Q_1 = \{\sigma \in Q : \operatorname{div}\sigma \in H\}.$$

The spaces H, Q and Q_1 are real Hilbert spaces endowed with the inner products given by

$$\langle u, v\rangle_H = \int_\Omega u_i v_i\,dx, \qquad \langle \sigma, \tau\rangle_Q = \int_\Omega \sigma_{ij}\tau_{ij}\,dx,$$
$$\langle \sigma, \tau\rangle_{Q_1} = \langle \sigma, \tau\rangle_Q + \langle \operatorname{div}\sigma, \operatorname{div}\tau\rangle_H.$$

Keeping in mind the boundary condition (2.3), we introduce the closed subspace of H_1 defined by

$$V = \{v \in H_1 : v = 0 \text{ on } \Gamma_1\}.$$

and let K be the set of admissible displacements given by

$$K = \{v \in V : v_\nu \le g \text{ a.e. on } \Gamma_3\}.$$

Since $\operatorname{meas}(\Gamma_1) > 0$, we have *Korn's inequality* [7]

$$\|\varepsilon(v)\|_Q \ge c_\Omega \|v\|_{H_1} \quad \forall v \in V, \tag{2.8}$$

where the constant c_Ω depends only on Ω and Γ_1. We equip V with the inner product given by

$$(u,v)_V = \langle \varepsilon(u), \varepsilon(v)\rangle_Q$$

and let $\|\cdot\|_V$ be the associated norm. It follows from Korn's inequality (2.8) that the norms $\|\cdot\|_{H_1}$and $\|\cdot\|_V$ are equivalent on V. Therefore $(V,\|\cdot\|_V)$ is a Hilbert space. Moreover by Sobolev's trace theorem, there exists $d_\Omega > 0$ which only depends on the domain Ω, Γ_3 and Γ_1 such that

$$\|v\|_{(L^2(\Gamma_3))^d} \le d_\Omega \|v\|_V \qquad \forall v \in V. \tag{2.9}$$

For every $v \in H_1$, we also write v for the trace of v on Γ, and we denote by v_ν and v_τ the normal and the tangential components of v on Γ given by

$$v_\nu = v.\nu, \qquad v_\tau = v - v_\nu \nu.$$

Similarly, for a function $\sigma \in Q_1$, we denote by σ_ν its normal component or normal stress and σ_τ its tangential component or tangential stress.

When σ is a regular function, we have $\sigma_\nu = (\sigma\nu).\nu$, $\sigma_\tau = \sigma\nu - \sigma_\nu\nu$, and the following Green's formula holds:

$$\langle \sigma, \varepsilon(v)\rangle_Q + \langle \operatorname{div}\sigma, v\rangle_H = \int_\Gamma \sigma\nu.v\, da \qquad \forall v \in H_1, \tag{2.10}$$

where da represents the surface measure element.

Next, for every real Banach space $(X,\|\cdot\|_X)$ and $T>0$ we write $C([0,T];X)$ for the space of continuous functions from $[0,T]$ to X; it is a real Banach space with the norm

$$\|x\|_{C([0,T];X)} = \max_{t\in[0,T]} \|x(t)\|_X.$$

For $p \in [1,\infty]$, we use the standard notation of $L^p(0,T;V)$ spaces. We also use the Sobolev space $W^{1,\infty}(0,T;V)$ with the norm

$$\|v\|_{W^{1,\infty}(0,T;V)} = \|v\|_{L^\infty(0,T;V)} + \|\dot v\|_{L^\infty(0,T;V)},$$

where a dot now represents the weak derivative with respect to the time variable.

In the study of the contact problem P_1 we assume that the linear elasticity tensor $\mathcal{A} = (a_{ijkh})$ satisfies

$$\begin{cases} \text{(a) } \mathcal{A} : \Omega \times S_d \to S_d; \\ \text{(b) } a_{ijkl} \in L^\infty(\Omega),\ 1 \le i,j,k,l \le d; \\ \text{(c) } \mathcal{A}\sigma.\tau = \sigma.\mathcal{A}\tau \text{ for all } \sigma,\tau \in S_d \text{ and a.e. in } \Omega; \\ \text{(d) there exists } \alpha > 0 \text{ such that } \mathcal{A}\tau.\tau \ge \alpha|\tau|^2 \\ \quad\ \ \text{for all } \tau \in S_d \text{ and a.e. in } \Omega. \end{cases} \tag{2.11}$$

The friction coefficient satisfies

$$(2.12)\qquad \begin{cases} \text{(a)}\ \mu : \varGamma_3 \times \mathbb{R}_+ \to \mathbb{R}_+; \\ \text{(b) there exists } L_\mu > 0 \text{ such that} \\ \qquad |\mu(x,u) - \mu(x,v)| \le L_\mu |u - v| \\ \qquad \text{for all } u, v \in \mathbb{R}_+ \text{ and a.e. } x \in \varGamma_3; \\ \text{(c) there exists } \mu_0 > 0 \text{ such that} \\ \qquad \mu(x,u) \le \mu_0 \text{ for all } u \in \mathbb{R}_+ \text{ and a.e. } x \in \varGamma_3; \\ \text{(d) the function } x \mapsto \mu(x,u) \text{ is Lebesgue measurable on } \varGamma_3 \\ \qquad \text{for all } u \in \mathbb{R}_+. \end{cases}$$

We assume that the normal compliance function satisfies

$$(2.13)\qquad \begin{cases} \text{(a)}\ p : \varGamma_3 \times \mathbb{R} \to \mathbb{R}_+; \\ \text{(b) there exists } L_1 > 0 \text{ such that} \\ \qquad |p(x,r_1) - p(x,r_2)| \le L_1 |r_1 - r_2| \\ \qquad \text{for all } r_1, r_2 \in \mathbb{R} \text{ and a.e. } x \in \varGamma_3; \\ \text{(c)}\ (p(x,r_1) - p(x,r_2))(r_1 - r_2) \ge 0 \\ \qquad \text{for all } r_1, r_2 \in \mathbb{R} \text{ and a.e. } x \in \varGamma_3; \\ \text{(d) there exists } L_2 > 0 \text{ such that} \\ \qquad p(x,g) \le L_2 \text{ for a.e. } x \in \varGamma_3; \\ \text{(e) the function } x \to p(x,r) \text{ is Lebesgue measurable on } \varGamma_3 \\ \qquad \text{for all } r \in \mathbb{R}; \\ \text{(f)}\ p(x,r) = 0 \text{ for all } r \le 0 \text{ and a.e. } x \in \varGamma_3. \end{cases}$$

We assume that the body forces and surface tractions satisfy

(2.14) $f_1 \in W^{1,\infty}(0,T;H)$, $f_2 \in W^{1,\infty}(0,T;(L^2(\varGamma_2))^d)$ and there exists an open subset denoted by S_2 such that $\operatorname{supp}(f_2(t)) \subset S_2 \subset \overline{S}_2 \subset \varGamma_2$ for all $t \in [0,T]$.

Using Riesz's representation theorem we define an element $f(t) \in V$ by

$$(f(t),v)_V = \int_\Omega f_1(t).v\,dx + \int_{\varGamma_2} f_2(t).v\,da \qquad \forall v \in V,\ t \in [0,T].$$

The hypotheses on f_1 and f_2 imply that

$$f \in W^{1,\infty}(0,T;V).$$

Next we define a bilinear symmetric form $a : V \times V \to \mathbb{R}$ by

$$a(u,v) = \langle \mathcal{A}\varepsilon(u), \varepsilon(v)\rangle_Q.$$

By the hypotheses (2.11)(b) & (d) on F, the bilinear form $a(\cdot,\cdot)$ is continuous, that is,

$$(2.15)\qquad \begin{cases} \text{(a) } \exists M>0,\ |a(u,v)|\le M\|u\|_V\|v\|_V \quad \forall u,v\in V,\\ \text{(b) } \exists m>0,\ a(v,v)\ge m\|v\|_V^2 \quad \forall v\in V.\end{cases}$$

Next let us introduce a subset V_0 of H_1 defined by

$$V_0=\{v\in H_1 : \operatorname{div}\sigma(v)\in H\}$$

and let the functionals $j_{\rm c}, j_{\rm fr} : V\times V\to\mathbb{R}$ be given by

$$j_{\rm c}(u,v)=\int_{\Gamma_3} p(u_\nu)v_\nu\,da \qquad \forall (u,v)\in V\times V,$$

$$j_{\rm fr}(u,v)=\int_{\Gamma_3} \mu(|u_\tau|)p(u_\nu)|v_\tau|\,da \qquad \forall (u,v)\in V\times V.$$

We will denote by $\langle\cdot,\cdot\rangle$ the duality pairing on $H^{-1/2}(\Gamma)\times H^{1/2}(\Gamma)$.

Let $t\in[0,T]$. For a function $v\in H_1$ such that $\operatorname{div}\sigma(v)=-f_1(t)$, we define the normal stress $\sigma_\nu(v)\in H^{-1/2}(\Gamma)$ by

$$(2.16)\quad \langle\sigma_\nu(v),w_\nu\rangle=a(v,w)-\langle f_1(t),w\rangle_H \qquad \forall w\in H_1 \text{ with } w_\tau=0 \text{ on } \Gamma.$$

We shall use the notation

$$\langle\rho\sigma_\nu(v),w_\nu\rangle=\langle\sigma_\nu(v),\rho w_\nu\rangle \qquad \forall \rho\in C_0^1(\mathbb{R}^d).$$

We also assume that the initial data u_0 satisfies

$$(2.17)\qquad \begin{cases} u_0\in K,\\ a(u_0,v-u_0)+j_{\rm c}(u_0,v-u_0)+j_{\rm fr}(u_0,v-u_0)\\ \qquad \ge (f(0),v-u_0)_V \quad \forall v\in K.\end{cases}$$

Now, using Green's formula (2.10), it is straighforward to see that if u is a sufficiently regular function which satisfies (2.1)–(2.6), then for almost all $t\in(0,T)$, we have

$$\begin{aligned} &a(u(t),v-\dot u(t))+j_{\rm fr}(u(t),v)-j_{\rm fr}(u(t),\dot u(t))\\ &\qquad\qquad \ge (f(t),v-\dot u(t))_V+(\sigma_\nu(u(t)),v_\nu-\dot u_\nu(t))_{L^2(\Gamma_3)} \qquad \forall v\in V,\\ &\big(\sigma_\nu(u(t))+p(u_\nu(t)),z_\nu-u_\nu(t)\big)_{L^2(\Gamma_3)}\ge 0 \qquad \forall z\in K.\end{aligned}$$

Finally we define the cut-off function $\theta\in C_0^\infty(\mathbb{R}^d)$, $0\le\theta\le 1$, such that $\theta=1$ in a neighbourhood of $\overline{\Gamma}_3$ and 0 in a neighbourhood of $\overline{S}_2$. Therefore, using (2.7) and the inequalities above leads to the following variational formulation of Problem P_1.

Problem P_2. Find a displacement field $u\in W^{1,\infty}(0,T;V)$ such that $u(0)=u_0$ in Ω, $u(t)\in K\cap V_0$ for all $t\in[0,T]$, and for almost all $t\in(0,T)$,

$$\text{(2.18)}\qquad a(u(t), v-\dot u(t)) + j_{\rm fr}(u(t), v) - j_{\rm fr}(u(t), \dot u(t)) \ge (f(t), v-\dot u(t))_V + \langle \sigma_\nu(u(t)), \theta(v_\nu - \dot u_\nu(t))\rangle \quad \forall v\in V,$$

$$\text{(2.19)}\qquad \langle \sigma_\nu(u(t)) + p(u_\nu(t)), \theta(z_\nu - u_\nu(t))\rangle \ge 0 \quad \forall z\in K.$$

The main result of this paper, to be proved in the next section, is the following.

THEOREM 2.1. *Let* (2.11)–(2.14) *and* (2.17) *hold. If*

$$(L_2L_\mu + \mu_0 L_1)d_\Omega^2 < m,$$

then Problem P_2 has at least one solution u.

3. The time-discretized formulation. In order to solve Problem P_2, we adopt the following time discretization. For all $n\in\mathbb{N}^*$, we set $\Delta t = T/n$ and $t_i = i\Delta t$, $0\le i\le n$. We denote by u^i the approximation of u at time t_i and set $\Delta u^i = u^{i+1} - u^i$. For $w\in C([0,T];X)$ where X is a Banach space, we use the notation $w^i = w(t_i)$. Then we obtain a sequence of incremental time-discretized problems P_n^i defined for $u^0 = u_0$ by

PROBLEM P_n^i. Find $u^{i+1}\in K\cap V_0$ such that

$$\begin{cases} a(u^{i+1}, w-u^{i+1}) + j_{\rm fr}(u^{i+1}, w-u^i) - j_{\rm fr}(u^{i+1}, \Delta u^i) \\ \qquad \ge (f^{i+1}, w-u^{i+1})_V + \langle \sigma_\nu(u^{i+1}), \theta(w_\nu - u_\nu^{i+1})\rangle \quad \forall w\in V, \\ \langle \sigma_\nu(u^{i+1}) + p(u_\nu^{i+1}), \theta(z_\nu - u_\nu^{i+1})\rangle \ge 0 \quad \forall z\in K. \end{cases}$$

LEMMA 3.1. *Problem P_n^i is equivalent to the following:*

PROBLEM Q_n^i. Find $u^{i+1}\in K\cap V_0$ such that

$$\text{(3.1)}\qquad \begin{cases} (Au^{i+1}, w-u^{i+1})_V + j_{\rm fr}(u^{i+1}, w-u^i) - j_{\rm fr}(u^{i+1}, \Delta u^i) \\ \qquad \ge (f^{i+1}, w-u^{i+1})_V \quad \forall w\in K, \end{cases}$$

where the operator $A: V\to V$ is defined as

$$(Au, v)_V = a(u,v) + j_{\rm c}(u,v) \quad \forall u, v\in V.$$

Proof. We refer the reader to [5]. ■

Now we can prove the following result.

PROPOSITION 3.2. *If*

$$(L_2L_\mu + \mu_0 L_1)d_\Omega^2 < m,$$

then problem Q_n^i has a unique solution.

To show Proposition 3.2 we introduce an auxiliary problem. We consider the nonempty closed subset $L^2(\Gamma_3)_+$ defined as

$$L^2(\Gamma_3)_+ = \{s\in L^2(\Gamma_3) : s\ge 0 \text{ a.e. on } \Gamma_3\}.$$

For $\eta \in L^2(\Gamma_3)_+$ define a mapping $\varphi : K \to \mathbb{R}$ by

$$\varphi(w) = \int\limits_{\Gamma_3} \eta |w_\tau - u^i_\tau| \, da \qquad \forall w \in K.$$

Then we define the following contact problem with given friction bound.

PROBLEM $Q^i_{n\eta}$. Find $u_\eta \in K$ such that

$$(3.2) \qquad (Au_\eta, w - u_\eta)_V + \varphi(w) - \varphi(u_\eta) \geq (f^{i+1}, w - u_\eta)_V \qquad \forall w \in K.$$

We have the lemma below.

LEMMA 3.3. *Problem $Q^i_{n\eta}$ has a unique solution.*

Proof. We use (2.13)(a) & (b) and (2.15)(a) & (b) to see that the operator A is Lipschitz continuous and strongly monotone, K is a nonempty closed convex of V, and φ is convex and lower semicontinuous. Then it follows (see [3]) that for every $\eta \in L^2(\Gamma_3)_+$, Problem $Q^i_{n\eta}$ has a unique solution u_η. ■

Next, we prove the following lemma.

LEMMA 3.4. *Let $\Phi : L^2(\Gamma_3)_+ \to L^2(\Gamma_3)_+$ be defined by*

$$\Phi(\eta) = \mu(|u_{\eta\tau}|)p(u_{\eta\nu}).$$

If

$$(L_2 L_\mu + \mu_0 L_1) d^2_\Omega < m,$$

then Φ has a unique fixed point η^*, and u_{η^*} is a unique solution of Problem Q^i_n.

Proof. It suffices to show that Φ is a contraction. For simplicity we write $u_{\eta i} = u_i$, $i = 1, 2$. Then

$$\|\Phi(\eta_1) - \Phi(\eta_2)\|_{L^2(\Gamma_3)} = \|\mu(|u_{1\tau}|)p(u_{1\nu}) - \mu(|u_{2\tau}|)p(u_{2\nu})\|_{L^2(\Gamma_3)}$$

Using (2.9), (2.12)(b) & (c) and (2.13)(b) & (d) we obtain

$$\begin{aligned} &\|\mu(|u_{1\tau}|)p(u_{1\nu}) - \mu(|u_{2\tau}|)p(u_{2\nu})\|_{L^2(\Gamma_3)} \\ &\quad = \|(\mu(|u_{1\tau}|) - \mu(|u_{2\tau}|))p(u_{1\nu}) + \mu(|u_{2\tau}|)(p(u_{1\nu}) - p(u_{2\nu}))\|_{L^2(\Gamma_3)} \\ &\quad \leq (L_2 L_\mu + \mu_0 L_1) d_\Omega \|u_1 - u_2\|_V. \end{aligned}$$

On the other hand, setting $v = u_1$ in $Q^i_{n\eta_1}$ and $v = u_2$ in $Q^i_{n\eta_2}$ and adding the relevant inequalities, by using (2.9) and (2.15)(b), we get

$$\|u_1 - u_2\|_V \leq \frac{d_\Omega}{m} \|\eta_1 - \eta_2\|_{L^2(\Gamma_3)}.$$

Hence

$$\|\Phi(\eta_1) - \Phi(\eta_2)\|_{L^2(\Gamma_3)} \leq \frac{(L_2 L_\mu d_\Omega + \mu_0 L_1) d^2_\Omega}{m} \|\eta_1 - \eta_2\|_{L^2(\Gamma_3)}.$$

Thus if $(L_2 L_\mu d_\Omega + \mu_0 L_1) d^2_\Omega < m$, we deduce that Φ is a contraction, so it has a unique fixed point η^* and $u_{\eta *}$ is a unique solution of Problem Q^i_n. ■

Now, in order to prove the existence of a solution, we first need to establish the following estimates.

LEMMA 3.5. *We have*

$$\|u^{i+1}\|_V \le \big(\mu_0 L_2 d_\Omega \sqrt{\operatorname{meas}(\Gamma_3)} + \|f^{i+1}\|_V\big)/m, \tag{3.3}$$

and if

$$(L_2 L_\mu + \mu_0 L_1) d_\Omega^2 < m,$$

then there exists a constant $c > 0$ such that

$$\|\Delta u^i\|_V \le c\|\Delta f^i\|_V. \tag{3.4}$$

Proof. We take $w = 0$ in inequality (3.1) to deduce

$$(Au^{i+1}, u^{i+1})_V \le j_{\mathrm{fr}}(u^{i+1}, u^{i+1}) + (f^{i+1}, u^{i+1})_V.$$

Using (2.12)(c) & (b) and (2.13)(e) we have

$$j_{\mathrm{fr}}(u^{i+1}, u^{i+1}) \le d_\Omega \mu_0 L_2 \sqrt{\operatorname{meas}(\Gamma_3)}\|u^{i+1}\|_V.$$

Using (2.15)(b) and (2.13)(c), we deduce

$$m\|u^{i+1}\|_V^2 \le d_\Omega \mu_0 L_2 \sqrt{\operatorname{meas}(\Gamma_3)}\|u^{i+1}\|_V + \|f^{i+1}\|_V \|u^{i+1}\|_V,$$

from which we conclude that (3.3) holds.

To show the estimate (3.4) we consider the translated inequality of (3.1) at time t_i, that is,

$$(Au^i, w - u^i)_V + j_{\mathrm{fr}}(u^i, w - u^{i-1}) - j_{\mathrm{fr}}(u^i, u^i - u^{i-1}) \ge (f^i, w - u^i)_V \quad \forall w \in V. \tag{3.5}$$

Taking $w = u^i$ in (3.1) and $w = u^{i+1}$ in (3.5) and adding up the results, one obtains

$$-(Au^{i+1} - Au^i, \Delta u^i)_V - j_{\mathrm{fr}}(u^{i+1}, \Delta u^i) + j_{\mathrm{fr}}(u^i, u^{i+1} - u^{i-1}) - j_{\mathrm{fr}}(u^i, u^i - u^{i-1}) \ge (-\Delta f^i, \Delta u^i)_V.$$

Then using the inequality

$$\big||u_\tau^{i+1} - u_\tau^{i-1}| - |u_\tau^i - u_\tau^{i-1}|\big| \le |u_\tau^{i+1} - u_\tau^i|,$$

we have

$$j_{\mathrm{fr}}(u^i, u^{i+1} - u^{i-1}) - j_{\mathrm{fr}}(u^i, u^i - u^{i-1}) \le j_{\mathrm{fr}}(u^i, \Delta u^i).$$

Therefore

$$(Au^{i+1} - Au^i, \Delta u^i)_V + j_{\mathrm{fr}}(u^{i+1}, \Delta u^i) - j_{\mathrm{fr}}(u^i, \Delta u^i) \le (\Delta f^i, \Delta u^i)_V. \tag{3.6}$$

Using (2.9), (2.12)(b) & (c) and (2.13)(b) & (d) we obtain

$$|j_{\mathrm{fr}}(u^{i+1}, \Delta u^i) - j_{\mathrm{fr}}(u^i, \Delta u^i)| \le (L_2 L_\mu + \mu_0 L_1) d_\Omega^2 \|\Delta u^i\|_V^2.$$

Applying (2.15)(b), (2.13)(c) and the estimate above we deduce from (3.6) that

$$m\|\Delta u^i\|_V^2 \le (L_2L_\mu + \mu_0 L_1)d_\Omega^2\|\Delta u^i\|_V^2 + \|\Delta f^i\|_V\|\Delta u^i\|_V.$$

Then we deduce that if $(L_2L_\mu+\mu_0L_1)d_\Omega^2 < m$, there exists a constant $c>0$ such that

$$\|\Delta u^i\|_V \le c\|\Delta f^i\|_V. \ \blacksquare$$

4. Existence of a solution for Problem P_2. In this section we shall prove Theorem 2.1. The weak solution for Problem P_2 is obtained as a limit of the interpolate function in time of the discrete solution. For this, we shall define a sequence of functions $u^n : [0,T] \to V$ by

$$u^n(t) = u^i + \frac{t-t_i}{\Delta t}\Delta u^i \quad \text{ on } [t_i,t_{i+1}], i=0,\ldots,n-1.$$

As in [15] we have the following lemma.

LEMMA 4.1. *There exists $u \in W^{1,\infty}(0,T;V)$ and a subsequence of (u^n), still denoted (u^n), such that*

$$u^n \to u \quad \textit{weak}^* \textit{ in } W^{1,\infty}(0,T;V).$$

Proof. From (3.3), we deduce that the sequence (u^n) is bounded in $C([0,T];V)$ and there exist positive constants c_1 and c_2 such that

$$\max_{0\le t\le T}\|u^n(t)\|_V \le c_1\|f\|_{C([0,T];V)} + c_2.$$

From (3.4), the sequence $(\dot{u}^n)$ is bounded in $L^\infty(0,T;V)$ and there exists $c_3>0$ such that

$$\|\dot{u}^n\|_{L^\infty(0,T;V)} = \max_{0\le i\le n-1}\left\|\frac{\Delta u^i}{\Delta t}\right\|_V \le c_3\|\dot{f}\|_{L^\infty(0,T;V)}.$$

Consequently, (u^n) is bounded in $W^{1,\infty}(0,T;V)$. Therefore, there exists a function $u \in W^{1,\infty}(0,T;V)$ and a subsequence, still denoted by (u^n), such that

$$u^n \to u \quad \text{weak}^* \text{ in } W^{1,\infty}(0,T;V) \text{ as } n\to\infty. \ \blacksquare$$

REMARK 4.2. As $W^{1,\infty}(0,T;V) \hookrightarrow C([0,T];V)$ we have $u^n(t) \to u(t)$ weakly in V for all $t \in [0;T]$.

Now let us introduce piecewise constant functions $\widetilde{u}^n$, $\widetilde{f}^n : [0,T] \to V$, defined as follows:

$$\widetilde{u}^n(t) = u^{i+1}, \qquad \widetilde{f}^n(t) = f(t_{i+1}), \qquad \forall t \in (t_i,t_{i+1}],\ i=0,\ldots,n-1.$$

As in [15] we have the following result.

LEMMA 4.3. *After passing to a subsequence, again denoted $(\widetilde{u}^n)$, we have*

(i) $\widetilde{u}^n \to u$ *weak* in* $L^\infty(0,T;V)$,
(ii) $\widetilde{u}^n(t) \to u(t)$ *weakly in* V *for a.e.* $t\in[0,T]$,
(iii) $u(t) \in K\cap V_0$ *for all* $t\in[0,T]$.

Now we have all the ingredients to prove the following proposition.

PROPOSITION 4.4. *The function u is a solution to Problem P_2.*

Proof. In the first inequality of Problem P_n^i, for $v \in V$ set $w = u^i + v\Delta t$ and divide by Δt to obtain

$$a\Big(u^{i+1}, v - \frac{\Delta u^i}{\Delta t}\Big) + j_{\mathrm{fr}}(u^{i+1}, v) - j_{\mathrm{fr}}\Big(u^{i+1}, \frac{\Delta u^i}{\Delta t}\Big) \geq \Big(f(t_{i+1}), v - \frac{\Delta u^i}{\Delta t}\Big)_V + \Big\langle \sigma_\nu(u^{i+1}), \theta\Big(v_\nu - \frac{\Delta u_\nu^i}{\Delta t}\Big)\Big\rangle.$$

Hence for any $v \in L^2(0,T;V)$, we have

$$a(\widetilde{u}^n(t), v(t) - \dot{u}^n(t)) + j_{\mathrm{fr}}(\widetilde{u}^n(t), v(t)) - j_{\mathrm{fr}}(\widetilde{u}^n(t), \dot{u}^n(t)) \geq (\widetilde{f}^n(t), v(t) - \dot{u}^n(t))_V + \langle \theta\sigma_\nu(\widetilde{u}^n(t)), v_\nu(t) - \dot{u}_\nu^n(t))\rangle, \quad \text{a.e. } t \in [0,T].$$

Integrating both sides on $(0,T)$ gives

$$\int_0^T a(\widetilde{u}^n(t), v(t) - \dot{u}^n(t))\,dt + \int_0^T j_{\mathrm{fr}}(\widetilde{u}^n(t), v(t))\,dt - \int_0^T j_{\mathrm{fr}}(\widetilde{u}^n(t), \dot{u}^n(t))\,dt \geq \int_0^T (\widetilde{f}^n(t), v(t) - \dot{u}^n(t))_V\,dt + \int_0^T \langle \theta\sigma_\nu(\tilde{u}^n(t)), v_\nu(t) - \dot{u}_\nu^n(t))\rangle\,dt. \tag{4.1}$$

Before passing to the limit as $n \to \infty$ in (4.1), we need to prove the lemma below.

LEMMA 4.5. *We have the following properties:*

$$\limsup_{n\to\infty} \int_0^T a(\widetilde{u}^n(t), v(t) - \dot{u}^n(t))\,dt \leq \int_0^T a(u(t), v(t) - \dot{u}(t))\,dt \quad \forall v \in L^2(0,T;V), \tag{4.2}$$

$$\liminf_{n\to\infty} \int_0^T j_{\mathrm{fr}}(\widetilde{u}^n(t), \dot{u}^n(t))\,dt \geq \int_0^T j_{\mathrm{fr}}(u(t), \dot{u}(t))\,dt, \tag{4.3}$$

$$\lim_{n\to\infty} \int_0^T j_{\mathrm{fr}}(\widetilde{u}^n(t), v(t))\,dt = \int_0^T j_{\mathrm{fr}}(u(t), v(t))\,dt \quad \forall v \in L^2(0,T;V), \tag{4.4}$$

$$\lim_{n\to\infty} \int_0^T (\widetilde{f}^n(t), v(t) - \dot{u}^n(t))_V\,dt = \int_0^T (f(t), v(t) - \dot{u}(t))_V\,dt \quad \forall v \in L^2(0,T;V), \tag{4.5}$$

(4.6)
$$\lim_{n\to\infty}\int_0^T\langle\theta\sigma_\nu(\tilde{u}^n(t)),v_\nu(t)-\dot{u}^n_\nu(t))\rangle\,dt$$
$$=\int_0^T\langle\theta\sigma_\nu(u(t)),v_\nu(t)-\dot{u}_\nu(t))\rangle\,dt \qquad \forall v\in L^2(0,T;V).$$

Proof. For (4.2) we refer the reader to [5]. For the proof of (4.3) we have

$$j_{\rm fr}(\widetilde{u}^n(t),\dot{u}^n(t)) = \int_{\Gamma_3}\big(\mu(|\tilde{u}^n_\tau(t)|)-\mu(|u_\tau(t)|)\big)p(\tilde{u}^n_\nu(t))|\dot{u}^n_\tau(t)|\,da + \int_{\Gamma_3}\mu(|u_\tau(t)|)p(\tilde{u}^n_\nu(t))|\dot{u}^n_\tau(t)|\,da.$$

Using the hypotheses (2.12)(b) and (2.13)(c), we get

$$\Big|\int_{\Gamma_3}(\mu(|\tilde{u}^n_\tau(t)|)-\mu(|u_\tau(t)|))p(\tilde{u}^n_\nu(t))|\dot{u}^n_\tau(t)|\,da\Big| \le L_2L_\mu\|\tilde{u}^n_\tau(t)-u_\tau(t)\|_{(L^2(\Gamma_3))^d}\|\dot{u}^n_\tau(t)\|_{(L^2(\Gamma_3))^d}.$$

From (2.9) and $\|\dot{u}^n\|_{L^\infty(0,T;V)}\le c\|f\|_{W^{1,\infty}(0,T.V)}+c'$, where $c'>0$, we deduce that

$$\Big|\int_0^T\int_{\Gamma_3}\big(\mu(|\tilde{u}^n_\tau(t)|)-\mu(|u_\tau(t)|)\big)p(\tilde{u}^n_\nu(t))|\dot{u}^n_\tau(t)|\,da\,dt\Big| \le c_4\|\tilde{u}^n_\tau-u_\tau\|_{L^2(0,T;(L^2(\Gamma_3))^d)}$$

for some constant $c_4>0$. Now, for $t\in(0,T)$ we write

$$\|\tilde{u}^n_\tau(t)-u_\tau(t)\|_{L^2(\Gamma_3)^d}\le\|\tilde{u}^n_\tau(t)-u^n_\tau(t)\|_{(L^2(\Gamma_3))^d}+\|u^n_\tau(t)-u_\tau(t)\|_{(L^2(\Gamma_3))^d},$$

and using (2.9) we obtain, for every $t\in(0,T)$,

$$\|\tilde{u}^n_\tau(t)-u^n_\tau(t)\|_{L^2(\Gamma_3)^d}\le d_\Omega\frac{T}{n}\|\dot{u}^n(t)\|_V\le c_3d_\Omega\frac{T}{n}\|\dot{f}\|_{L^\infty(0,T;V)}.$$

On the other hand,

$$u^n_\tau(t)\to u_\tau(t)\quad \text{strongly in } (L^2(\Gamma_3))^d \text{ for all } t\in[0,T],$$

from which we deduce that

$$\tilde{u}^n_\tau\to u_\tau\quad\text{strongly in } L^2(0,T;(L^2(\Gamma_3))^d),$$

and we conclude that

$$\lim_{n\to\infty}\int_0^T\int_{\Gamma_3}\big(\mu(|\tilde{u}^n_\tau(t)|)-\mu(|u_\tau(t)|)\big)p(\tilde{u}^n_\nu(t))|\dot{u}^n_\tau(t)|\,da\,dt=0.$$

Finally, as by Mazur's lemma we have

$$\liminf_{n\to\infty}\int_0^T j_{\mathrm{fr}}(u(t),\dot u^n(t))\,dt \ge \int_0^T j_{\mathrm{fr}}(u(t),\dot u(t))\,dt,$$

we obtain

$$\liminf_{n\to\infty}\int_0^T j_{\mathrm{fr}}(\tilde u^n(t),\dot u^n(t))\,dt \ge \int_0^T j_{\mathrm{fr}}(u(t),\dot u(t))\,dt.$$

The same proof of (4.3) is used to prove (4.4). To prove (4.5), it suffices to use the fact that $\tilde f^n \to f$ strongly in $L^2(0,T;V)$ (see [15]). Finally, for the proof of (4.6) use (2.16) and see also [5]. ∎

Now using (2.16) and passing to the limit in (4.1), we get

$$\int_0^T \big(a(u(t),v(t)-\dot u(t)) + j_{\mathrm{fr}}(u(t),v(t)) - j_{\mathrm{fr}}(u(t),\dot u(t))\big)\,dt \\ \ge \int_0^T (f(t),v(t)-\dot u(t))_V\,dt + \int_0^T \langle\theta\sigma_\nu(u(t)), v_\nu(t)-\dot u_\nu(t)\rangle\,dt.$$

In this inequality we take $v\in L^2(0,T;V)$ defined by

$$v(s)=\begin{cases} z & \text{for } s\in[t,t+\lambda],\\ \dot u(s) & \text{elsewhere,}\end{cases}$$

and dividing by λ, we obtain

$$\frac{1}{\lambda}\int_t^{t+\lambda} \big(a(u(s),z-\dot u(s)) + j_{\mathrm{fr}}(u(s),z) - j_{\mathrm{fr}}(u(s),\dot u(s))\big)\,ds \\ \ge \frac{1}{\lambda}\int_t^{t+\lambda} (f(s),z-\dot u(s))_V ds + \frac{1}{\lambda}\int_t^{t+\lambda} \langle\theta\sigma_\nu(u(s)), z_\nu-\dot u_\nu(s)\rangle\,ds.$$

Letting $\lambda\to 0_+$, by Lebesgue's theorem we infer that u satisfies (2.18).

To complete the proof, we deduce from (3.1) that

$$(A\widetilde u^n(t), v-\widetilde u^n(t))_V + j_{\mathrm{fr}}(\widetilde u^n(t), v-\widetilde u^n(t)) \\ \ge (\widetilde f^n(t), v-\widetilde u^n(t))_V \quad \forall v\in K,\ \text{a.e. } t\in[0,T].$$

Integrating both sides gives

$$\int_0^T (A\widetilde u^n(t), v(t)-\widetilde u^n(t))_V\,dt + \int_0^T j_{\mathrm{fr}}(\widetilde u^n(t), v(t)-\widetilde u^n(t))\,dt \\ \ge \int_0^T (\widetilde f^n(t), v(t)-\widetilde u^n(t))_V\,dt \quad \forall v\in L^2(0,T;V);\, v(t)\in K,\ \text{a.e. } t\in[0,T].$$

Letting $n \to \infty$, we get

$$\int_0^T (Au(t), v(t) - u(t))_V \, dt + \int_0^T j_{\mathrm{fr}}(u(t), v(t) - u(t)) \, dt$$
$$\geq \int_0^T (f(t), v(t) - u(t))_V \, dt \quad \forall v \in L^2(0,T;V); v(t) \in K, \text{ a.e. } t \in [0,T].$$

Proceeding in a similar way, we deduce that

$$(Au(t), v - u(t))_V + j_{\mathrm{fr}}(u(t), v - u(t))$$
$$\geq (f(t), v - u(t))_V \quad \forall v \in K, \ \text{a.e. } t \in [0,T].$$

Using Green's formula, as in [5], we conclude that u satisfies the inequality (2.19) and consequently u is a solution of Problem P_2.

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