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LOCAL CONVERGENCE FOR A MULTI-POINT FAMILY OF SUPER-HALLEY METHODS IN A BANACH SPACE UNDER WEAK CONDITIONS

Abstract. We present a local multi-point convergence analysis for a family of super-Halley methods of high convergence order in order to approximate a solution of a nonlinear equation in a Banach space. Our sufficient convergence conditions involve only hypotheses on the first and second Fréchet derivative of the operator involved. Earlier studies use hypotheses up to the third Fréchet derivative. Numerical examples are also provided.

1. Introduction. In this study we are concerned with the problem of approximating a solution x^* of the nonlinear equation

$$F(x) = 0, \tag{1.1}$$

where F is a Fréchet-differentiable operator defined on a subset D of a Banach space X with values in a Banach space Y.

Many problems in computational sciences and other disciplines can be brought to the form of equation (1.1) using mathematical modelling [3]. Solutions of (1.1) can rarely be found in closed form. Therefore the solutions are approximated by iterative methods. In particular, the practice of numerical functional analysis for finding such a solution is essentially connected with Newton-like methods [1]–[28]. The study of convergence of iterative procedures is usually of two types: semilocal and local convergence analysis. The semilocal analysis is, based on the information around an initial point, to give conditions ensuring the convergence of the iterative procedure; while the local analysis is, based on the information around a solution, to find estimates of the radii of convergence balls. There exist many studies which deal

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with the local and semilocal convergence analysis of Newton-like methods [1]–[28].

We present the local convergence analysis for the super-Halley method defined for each $n = 0, 1, 2, \dots$ by

$$\begin{aligned} y_n &= x_n - F'(x_n)^{-1}F(x_n), \\ z_n &= y_n + \tfrac{2}{3}F'(x_n)^{-1}F(x_n) = x_n - \tfrac{1}{3}F'(x_n)^{-1}F(x_n), \\ v_n &= y_n - A_n F'(x_n)^{-1}F(x_n), \\ x_{n+1} &= v_n - B_n F'(x_n)^{-1}F(v_n), \end{aligned} \tag{1.2}$$

where x_0 is an initial point, the operator $K : X \to X$ is defined by

$$K(x) = F'(x)^{-1}F''\big(x - \tfrac{1}{3}F'(x)^{-1}F(x)\big)F'(x)^{-1}F(x),$$

and

$$\begin{aligned} &A_n = \tfrac{1}{2}K_n + \tfrac{1}{2}K_n^2 + K_n^\theta P(K_n), \\ &K_n = K(x_n) = F'(x_n)^{-1}F''(z_n)F'(x_n)^{-1}F(x_n), \\ &B_n = I - F'(x_n)^{-1}F''(z_n)(v_n - x_n) + \delta(F'(x_n)^{-1}F''(z_n)(v_n - x_n)^2), \end{aligned}$$

$\theta = 1, 2, \dots,$ δ is a real parameter, and P is a bounded operator on D. The semilocal convergence analysis was presented in [25] under the following conditions $(\mathcal{C})$:

$(\mathcal{C}_1)$ $F'(x_0)^{-1} \in L(Y, X)$ exists and $\|F'(x_0)^{-1}\| \le \beta$;

$(\mathcal{C}_2)$ $\|F'(x_0)^{-1}F(x_0)\| \le \beta_1$;

$(\mathcal{C}_3)$ $\|F'(x_0)^{-1}F''(x)\| \le \beta_2$ for each $x \in D$;

$(\mathcal{C}_4)$ $\|F'(x_0)^{-1}F'''(x)\| \le \beta_3$ for each $x \in D$;

$(\mathcal{C}_5)$ $\|F'(x_0)^{-1}(F'''(x) - F'''(y))\| \le \beta_4\|x - y\|^q$ for each $x, y \in D$ and $q \in [0, 1]$.

The R-order of convergence was shown to be $5 + q$. Notice that in [25], $\theta = 3, 4, \dots,$ $\delta \in [-1, 1]$, whereas in the present paper $\theta = 1, 2, \dots$ and δ is a real parameter. Hence, the applicability of method (1.2) is extended.

Similar conditions have been used by other authors [1]–[28], on other high convergence order methods. The corresponding conditions for the local convergence analysis are given by simply replacing x_0 by x^* in the $(\mathcal{C})$ conditions. These conditions however are very restrictive. As a motivational example, define a function F on $D = [-1/2, 5/2]$ by

$$F(x) = \begin{cases} x^3 \ln x^2 + x^5 - x^4, & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Then $x^* = 1$,

$$F'(x) = 3x^2 \ln x^2 + 5x^4 - 4x^3 + 2x^2, \qquad F'(1) = 3,$$
$$F''(x) = 6x \ln x^2 + 20x^3 - 12x^2 + 10x,$$
$$F'''(x) = 6 \ln x^2 + 60x^2 - 24x + 22.$$

Obviously, F cannot satisfy condition $(\mathcal{C}_4)$, since F''' is unbounded. In the present paper we only use hypotheses on the first and second order Fréchet derivative (see conditions (2.13)–(2.16)). This way we extend the applicability of method (1.2).

The paper is organized as follows. The local convergence of method (1.2) is analyzed in Section 2, whereas some numerical examples are given in Section 3.

2. Local convergence analysis. Let $U(v, \rho)$, $\overline{U}(v, \rho)$ denote the open and closed balls in X, respectively with center $v \in X$ and radius $\rho > 0$. Let $L_0, L, M, N > 0$, $\alpha \geq 0$, $\delta \in (-\infty, \infty)$ and $\theta = 1, 2, \ldots$ be given parameters. It is convenient to define certain functions on $[0, 1/L_0)$:

$$g_1(r) = \frac{Lr}{2(1 - L_0 r)},$$
$$g_2(r) = \frac{1}{2(1 - L_0 r)}\left(Lr + \frac{4M}{3}\right),$$
$$g_3(r) = \frac{MNr}{(1 - L_0 r)^2},$$
$$g_4(r) = \frac{1}{2}\frac{MNr}{(1 - L_0 r)^2} + \frac{1}{2}\frac{M^2N^2r^2}{(1 - L_0 r)^4} + \frac{(MN)^\theta r^\theta \alpha}{(1 - L_0 r)^{2\theta}},$$
$$g_5(r) = \frac{1}{2(1 - L_0 r)}(Lr + 2M g_4(r)).$$

Suppose that

$$M < 3/2. \tag{2.1}$$

Define

$$r_2 = \frac{2(1 - 2M/3)}{L + 2L_0}. \tag{2.2}$$

Then

$$0 < r_2 < r_R = \frac{2}{3L} \leq r_A = \frac{2}{L + 2L_0}, \tag{2.3}$$

where the last inequality holds for $L_0 \leq L$. Then it follows from the definition of g_2 that

$$0 < g_2(r) < 1 \quad \text{for each } r \in [0, r_2). \tag{2.4}$$

It also follows from the definition of g_1 and (2.5) that

$$0 < g_1(r) < 1 \quad \text{for each } r \in [0, r_2].$$

We need $g_5(r) \in [0,1)$ for each $r \in [0, \bar{r})$ for some $\bar{r}$ to be determined.

Let us define a function p_5 on $[0, 1/L_0]$ by

$$\begin{aligned} p_5(r) = 2\big[&Lr(1-L_0r)^4(1-L_0r)^{2\theta} \\ &+ M^2Nr(1-L_0r)^2(1-L_0r)^{2\theta} + 2M^3N^2r^2(2-L_0r)^{2\theta} \\ &+ 2M^{\theta+1}N^\theta r^\theta \alpha(1-L_0r)^4 - 2(1-L_0r)^5(1-L_0r)^{2\theta}\big]. \end{aligned}$$

Using the definitions of g_4 and g_5, we get

$$\begin{aligned} &g_5(r) - 1 \\ &= \frac{1}{2(1-L_0r)}\left[Lr + 2M\left(\frac{MNr}{2(1-L_0r)^2} + \frac{M^2N^2r^2}{2(1-L_0r)^4} + \frac{(MN)^\theta r^\theta \alpha}{(1-L_0r)^{2\alpha}}\right)\right] - 1 \\ &= \frac{p_5(r)}{4(1-L_0r)^{2\alpha+5}}. \end{aligned}$$

Hence

$$p_5(r) < 0 \;\Rightarrow\; g_5(r) < 1 \quad \text{for each } r \in (0, \bar{r}).$$

The function $p_5(r)$ can also be written as

$$p_5(r) = 2(1-L_0r)^{2\theta}p_6(r) \quad \text{or} \quad p_5(r) = 2(1-L_0r)^4 p_7(r),$$

where

$$\begin{aligned} p_6(r) = {}&Lr(1-L_0r)^4 + M^2Nr(1-L_0r)^2 + 2M^3N^2r^2 \\ &+ 2M^{\theta+1}N^\theta r^\theta \alpha(1-L_0r)^{4-2\theta} - 2(1-L_0r)^5 \end{aligned}$$

and

$$\begin{aligned} p_7(r) = {}&Lr(1-L_0r)^{2\theta} + M^2Nr(1-L_0r)^2(1-L_0r)^{2\theta-4} \\ &+ 2M^3N^2r^2(1-L_0r)^{2\theta-4} \\ &+ 2M^{\theta+1}N^\theta r^\theta \alpha - 2(1-L_0r)^2(1-L_0r)^{2\theta}. \end{aligned}$$

Then

$$\begin{aligned} &p_6(0) = -4 < 0, \quad p_6(1/L_0) = 2M^3N^2(1/L_0)^2 > 0, \qquad \text{for } \theta \le 2, \\ &p_7(0) = -4 < 0, \quad p_7(1/L_0) = 2M^{\theta+1}N^\theta(1/L_0)^2 > 0, \quad \text{for } \theta \ge 2. \end{aligned}$$

It follows that with either choice for θ, the function $g_5(r) - 1$ has zeros in $(0, 1/L_0)$. Denote by r_5 the smallest such zero. Set

$$\overline{r^*} = \min\{r_2, r_5\}. \tag{2.5}$$

Then

$$0 < g_1(r) < 1, \tag{2.6}$$

$$0 < g_2(r) < 1, \tag{2.7}$$

$$0 < g_3(r), \tag{2.8}$$

$$0 < g_4(r), \tag{2.9}$$

$$0 < g_5(r) < 1, \tag{2.10}$$

for each $r \in (0, \overline{r^*})$.

Moreover, define functions g_6 and g_7 on $[0, 1/L_0)$ by

$$g_6(r) = 1 + \frac{MN(1 + g_4(r))r}{(1 - L_0 r)^2} + |\delta| \frac{(MN(1 + g_4(r))r)^2}{(1 - L_0 r)^4},$$

$$g_7(r) = \left(1 + \frac{M g_6(r)}{1 - L_0 r}\right) g_5(r).$$

As in the case of g_5, we show that the function g_7 has zeros in $(0, 1/L_0)$. It follows from the definition of g_5 and g_7 that $r_7 < r_5$ and

$$0 < g_7(r) < 1 \ \Rightarrow \ 0 < g_5(r) < 1 \quad \text{for each } r \in (0, r_7).$$

Set

$$r^* = \min\{r_2, r_7\}. \tag{2.11}$$

Then clearly inequalities (2.6)–(2.10) hold for each $r \in (0, r^*)$. Next, we present the local convergence analysis of method (1.2).

THEOREM 2.1. *Let $F : D \subseteq X \to Y$ be a Fréchet-differentiable operator. Suppose that there exist $x^* \in D$, parameters $L_0, L, N > 0$, $M \in (0, 3/2)$ and $\alpha \geq 0$ such that for all $x \in D$,*

$$F(x^*) = 0, \qquad F'(x^*)^{-1} \in L(Y, X), \tag{2.12}$$

$$\|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq L_0 \|x - x^*\|, \tag{2.13}$$

$$\|F'(x^*)^{-1}[F(x) - F(x^*) - F'(x)(x - x^*)]\| \leq \frac{L}{2} \|x - x^*\|^2, \tag{2.14}$$

$$\|F'(x^*)^{-1} F'(x)\| \leq M, \tag{2.15}$$

$$\|F'(x^*)^{-1} F''(x)\| \leq N, \tag{2.16}$$

$$\|P(K(x))\| \leq \alpha, \tag{2.17}$$

$$\bar{U}(x^*, r^*) \subseteq D, \tag{2.18}$$

where r^* is given by (2.5). Then the sequence $\{x_n\}$ generated by method (1.2) for $x_0 \in U(x^*, r^*)$ is well defined, remains in $U(x^*, r^*)$ for each $n = 0, 1, 2, \ldots$ and converges to x^*. Moreover, the following estimates hold for each $n = 0, 1, 2, \ldots$:

$$\|y_n - x^*\| \leq g_1(\|x_n - x^*\|) \|x_n - x^*\| < \|x_n - x^*\| < r^*, \tag{2.19}$$

$$\|z_n - x^*\| \leq g_2(\|x_n - x^*\|) \|x_n - x^*\| < \|x_n - x^*\|, \tag{2.20}$$

$$\|K_n\| \leq g_3(\|x_n - x^*\|), \tag{2.21}$$

$$\|A_n\| \leq g_4(\|x_n - x^*\|) \tag{2.22}$$

and

$$\|v_n - x^*\| \le g_5(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|, \tag{2.23}$$

$$\|B_n\| \le g_6(\|x_n - x^*\|), \tag{2.24}$$

$$\|x_{n+1} - x^*\| \le g_7(\|x_n - x^*\|)\|x_n - x^*\| < \|x_n - x^*\|. \tag{2.25}$$

Proof. Using (2.13), the definition of r^* and the hypothesis $x_0 \in U(x^*, r^*)$ we get

$$\|F'(x^*)^{-1}(F(x_0) - F(x^*))\| \le L_0\|x_0 - x^*\| < L_0 r^* < 1. \tag{2.26}$$

It follows from (2.26) and the Banach Lemma on invertible operators [3] that $F'(x_0)^{-1} \in L(Y, X)$ and

$$\|F'(x_0)^{-1}F'(x^*)\| \le \frac{1}{1 - L_0\|x_0 - x^*\|} < \frac{1}{1 - L_0 r^*}. \tag{2.27}$$

Hence, y_0 and z_0 are well defined. Using the first substep of method (1.2) for $n = 0$, (2.14) (for $x = x_0$) and the definition of g_1 and g_2, we obtain in turn

$$\begin{aligned} y_0 - x^* &= x_0 - x^* - F'(x_0)^{-1}F(x_0) \\ &= -F'(x_0)^{-1}F'(x^*)F'(x^*)^{-1}[F(x_0) - F(x^*) - F'(x_0)(x_0 - x^*)] \end{aligned}$$

so

$$\begin{aligned} \|y_0 - x^*\| &\le \|F'(x_0)^{-1}F'(x^*)\| \\ &\qquad \times \|F'(x^*)^{-1}[F(x_0) - F(x^*) - F'(x_0)(x_0 - x^*)]\| \\ &\le \frac{L\|x_0 - x^*\|^2}{2(1 - L_0\|x_0 - x^*\|)} \\ &= g_1(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r^*, \end{aligned}$$

which shows (2.19) for $n = 0$. Consequently, from the second substep of method (1.2) for $n = 0$, we have

$$\begin{aligned} \|z_0 - x^*\| &\le \|y_0 - x^*\| + \tfrac{2}{3}\|F'(x_0)^{-1}F'(x^*)\| \\ &\qquad \times \Big\| \int_0^1 F'(x^*)^{-1}F'\big(x^* + t(x_0 - x^*)(x_0 - x^*)\big)\, dt \Big\| \\ &\le \left(\frac{L\|x_0 - x^*\|}{2(1 - L_0\|x_0 - x^*\|)} + \frac{2M}{3(1 - L_0\|x_0 - x^*\|)} \right)\|x_0 - x^*\| \\ &= g_2(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r^*, \end{aligned}$$

which shows (2.20) for $n = 0$ and $z_0 \in U(x^*, r^*)$. Next, using the definition of K_0 and A_0, we obtain in turn

$$\begin{aligned}\|K_0\| &\le \|F'(x_0)^{-1}F'(x^*)\|\,\|F'(x^*)^{-1}F''(z_0)\|\\ &\quad\times \|F'(x_0)^{-1}F'(x^*)\|\left\|\int_0^1 F'(x^*)^{-1}F'(x^*+t(x_0-x^*))\,dt\right\|\|x_0-x^*\|\\ &\le \frac{MN\|x_0-x^*\|}{(1-L_0\|x_0-x^*\|)^2} = g_3(\|x_0-x^*\|),\end{aligned}$$

which shows (2.21) for $n=0$. Moreover, we have

$$\begin{aligned}\|A_0\| &\le \tfrac12\|K_0\| + \tfrac12\|K_0\|^2 + \|K_0\|^\theta\|P(K(x_0))\|\\ &\le \frac{MN\|x_0-x^*\|}{2(1-L_0\|x_0-x^*\|)^2} + \frac{(MN\|x_0-x^*\|)^2}{2(1-L_0\|x_0-x^*\|)^4}\\ &\quad + \frac{(MN)^\theta\|x_0-x^*\|\alpha}{(1-L_0\|x_0-x^*\|)^{2\theta}} = g_4(\|x_0-x^*\|),\end{aligned}$$

which shows (2.22) for $n=0$. Then, using the third substep in method (1.2) for $n=0$, we get

$$\begin{aligned}\|v_0-x^*\| &\le \|y_0-x^*\| + \|A_0\|\,\|F'(x_0)^{-1}F'(x^*)\|\\ &\quad\times\left\|\int_0^1 F'(x^*)^{-1}F'(x^*+t(x_0-x^*))(x_0-x^*)\,dt\right\|\\ &\le \frac{1}{2(1-L_0\|x_0-x^*\|)}[L\|x_0-x^*\| + 2Mg_4(\|x_0-x^*\|)]\|x_0-x^*\|\\ &= g_5(\|x_0-x^*\|)\|x_0-x^*\| < \|x_0-x^*\| < r^*,\end{aligned}$$

which shows (2.23) for $n=0$. Using the definition of B_0 and g_6, (2.15), (2.16), (2.21), (2.22), (2.27), and the estimate

$$\begin{aligned}\|F'(x^*)^{-1}F(x_0)\| &= \left\|\int_0^1 F'(x^*)^{-1}F'(x^*+t(x_0-x^*))(x_0-x^*)\,dt\right\|\\ &\le M\|x_0-x^*\|,\end{aligned}$$

we get

$$\begin{aligned}\|B_0\| &\le \|I\| + \|F'(x_0)^{-1}F'(x^*)\|\,\|F'(x^*)^{-1}F''(z_0)\|\\ &\quad\times(1+\|A_0\|)\|F'(x_0)^{-1}F'(x^*)\|\,\|F'(x^*)^{-1}F(x_0)\|\\ &\quad+|\delta|\big(\|F'(x_0)^{-1}F'(x^*)\|\,\|F'(x^*)^{-1}F''(z_0)\|\\ &\quad\times(1+\|A_0\|)\|F'(x_0)^{-1}F'(x^*)\|\,\|F'(x^*)^{-1}F(x_0)\|\big)^2\\ &\le 1 + \frac{MN(1+g_4(\|x_0-x^*\|)\|x_0-x^*\|}{(1-L_0\|x_0-x^*\|)^2}\\ &\quad+|\delta|\frac{(MN(1+g_4(\|x_0-x^*\|)\|x_0-x^*\|)^2}{(1-L_0\|x_0-x^*\|)^4}\\ &= g_6(\|x_0-x^*\|),\end{aligned}$$

which shows (2.24) for $n = 0$. Hence, using the definition of g_7, (2.15), (2.23) and the preceding estimate we get

$$\begin{aligned}\|x_1 - x^*\| &\le \|v_0 - x^*\| + \|B_0\| \, \|F'(x_0)^{-1}F'(x^*)\| \\ &\qquad \times \Big\| \int_0^1 F'(x^*)^{-1}F'(x^* + t(x_0 - x^*))(v_0 - x^*)\, dt \Big\| \\ &\le \bigg[1 + \frac{M g_6(\|x_0 - x^*\|)}{1 - L_0\|x_0 - x^*\|}\bigg] \|v_0 - x^*\| \\ &\le \bigg[1 + \frac{M g_6(\|x_0 - x^*\|)}{1 - L_0\|x_0 - x^*\|}\bigg] g_5(\|x_0 - x^*\|)\|x_0 - x^*\| \\ &= g_7(\|x_0 - x^*\|)\|x_0 - x^*\| < \|x_0 - x^*\| < r^*,\end{aligned}$$

which shows (2.25) for $n = 0$.

By simply replacing y_0, z_0, v_0, x_1 by y_k, z_k, v_k, x_{k+1} in the preceding estimates we arrive at (2.19)–(2.25). Finally, from the estimate $\|x_{k+1} - x^*\| < \|x_k - x^*\|$ we deduce that $\lim_{k\to\infty} x_k = x^*$. ∎

Remark 2.2. 1. In view of (2.13) and the estimate

$$\begin{aligned}\|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\| \\ &\le 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \le 1 + L_0\|x - x^*\|\end{aligned}$$

condition (2.15) can be dropped and M can be replaced by

$$M(r) = 1 + L_0 r.$$

Moreover, condition (2.14) can be replaced by the popular but stronger conditions

$$\|F'(x^*)^{-1}(F'(x) - F'(y))\| \le L\|x - y\| \quad \text{for each } x, y \in D \tag{2.28}$$

or

$$\|F'(x^*)^{-1}(F'(x^* + t(x - x^*)) - F'(x))\| \le L(1 - t)\|x - x^*\| \quad \text{for each } x, y \in D \text{ and } t \in [0, 1].$$

2. The results obtained here can be used for operators F satisfying autonomous differential equations [3] of the form

$$F'(x) = T(F(x))$$

where T is a continuous operator. Then since $F'(x^*) = T(F(x^*)) = T(0)$, we can apply the results without actually knowing x^*. For example, let $F(x) = e^x - 1$. Then we can choose $T(x) = x + 1$.

3. The local results obtained here can be used for projection methods such as Arnoldi's method, the generalized minimum residual method (GMRES), the generalized conjugate method (GCR) for combined Newton/finite projection methods, and in connection with the mesh independence principle

can be used to develop the cheapest and most efficient mesh refinement strategies [3, 4].

The parameter r_A given in (2.3) was shown by us to be the convergence radius of Newton's method [3, 4]

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \quad \text{for each } n = 0, 1, 2, \dots \tag{2.29}$$

under conditions (2.14) and (2.28). It follows from (2.6) and (2.12) that the convergence radius r^* of method (1.2) cannot be larger than the convergence radius r_A of the second order Newton's method (2.29). As already noted in [3, 4], r_A is at least as large as the convergence radius given by Rheinboldt [3, 4]

$$r_R = \frac{2}{3L}.$$

In particular, for $L_0 < L$ we have

$$r_R < r_A \quad \text{and} \quad r_R/r_A \to 1/3 \quad \text{as } L_0/L \to 0.$$

That is, our convergence radius r_A is at most three times larger than Rheinboldt's. The same value for r_R was given by Traub [3, 4].

4. It is worth noticing that method (1.2) does not chang when we use the conditions of Theorem 2.1 instead of the stronger $(\mathcal{C})$ conditions used in [25]. Moreover, we can compute the computational order of convergence (COC) defined by

$$\xi = \ln\left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|}\right) \Big/ \ln\left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|}\right),$$

or the approximate computational order of convergence

$$\xi_1 = \ln\left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|}\right) \Big/ \ln\left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|}\right).$$

Thus we obtain in practice the order of convergence in a way that avoids the bounds given in [25] involving estimates up to the third Fréchet derivative of the operator F.

3. A numerical example. We present a numerical example in this section.

Example 3.1. Let $X = Y = \mathbb{R}^2$, $D = \bar{U}(0, 1)$, $x^* = 0$, and define a function F on D by

$$F(x) = \left(\sin x, \tfrac{1}{4}(e^x + 3x - 1)\right). \tag{3.1}$$

Let $P = 0$. Then we can choose $\alpha = 0$. Using (2.12)–(2.17), we get $L_0 = L = 1$, $M = \frac{1}{4}(e + 3)$, $N = e/4$, $\theta = 3$, $\delta = 0.1$. Then $r_2 = 0.0313$ and $r_7 = 0.0720$, so by (2.11) we obtain

$$r^* = 0.0313 < r_R = r_A = 0.6667.$$

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