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ESTIMATING THE SHAPE PARAMETER OF THE TOPP–LEONE DISTRIBUTION BASED ON TYPE I CENSORED SAMPLES

Abstract. The shape parameter of the Topp–Leone distribution is estimated from classical and Bayesian points of view based on Type I censored samples. The maximum likelihood and the approximate maximum likelihood estimates are derived. The Bayes estimate and the associated credible interval are approximated by using Lindley's approximation and Markov Chain Monte Carlo using the importance sampling technique. Monte Carlo simulations are performed to compare the performances of the proposed methods. Real and simulated data sets have been analyzed for illustrative purposes.

1. Introduction. Topp and Leone [16] introduced a family of distributions with finite support whose cumulative distribution function (cdf) is given by

$$F(x|\theta,\beta)=\begin{cases}0, & x<0,\\ \left(\frac{x}{\beta}\left(2-\frac{x}{\beta}\right)\right)^{\theta}, & 0\le x<\beta, \qquad \theta>0,\\ 1, & x\ge\beta,\end{cases} \tag{1.1}$$

and the probability density function (pdf) is given by

$$f(x\,|\,\theta,\beta)=\frac{2\theta}{\beta}\left(1-\frac{x}{\beta}\right)\left(\frac{x}{\beta}\left(2-\frac{x}{\beta}\right)\right)^{\theta-1}, \quad 0<x\le\beta,\ \theta>0. \tag{1.2}$$

For simplicity, we denote this distribution by $\mathrm{TL}(\theta,\beta)$. The Topp–Leone (T-L) distribution is a continuous unimodal distribution with bounded support; this makes it appropriate for modeling lifetime of distributions with finite support. Topp and Leone [16] did not provide any motivation for this

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family of distributions apart from saying that it could be used to model failure data. Nadarajah and Kotz [12] showed that this distribution exhibits bathtub failure rate functions with widespread applications in reliability. Moreover, Ghitany et al. [8] showed that the T-L distribution possesses some attractive reliability properties such as the bathtub-shape hazard rate, decreasing reversed hazard rate, upside-down mean residual life, and increasing expected inactivity time. Moments for the T-L distribution were derived by Nadarajah and Kotz [12]. Zghoul [19] provided expressions for moments of ordered statistics from the T-L distribution. Bayoud [3] derived admissible minimax estimates for the shape parameter of the T-L distribution under the assumption of non-informative and conjugate priors based on squared and linear-exponential loss functions. Recently, Bayoud [4] studied inferences about the shape parameter of the T-L distribution based on progressive Type II censored samples. A reflected version of the generalized T-L distribution was used by Van Dorp and Kotz [17] to fit the U.S. income data for the year 2001 for Caucasian, Hispanic and Afro-American populations.

Classical and Bayesian inferences about the parameters of the T-L distribution have not yet been studied in the presence of Type I censored samples. In this paper, the shape parameter of the T-L distribution is estimated from classical and Bayesian viewpoints based on the Type I censoring scheme assuming known β.

The Type I censoring scheme can be described as follows. Consider an experiment in which n units are subjected to a life testing experiment. The experiment will terminate at a preselected time T. The Type I censored sample is the set of all times before or equal to this T.

In this paper, we propose the maximum likelihood estimation (MLE), approximate maximum likelihood estimate (AMLE), Bayesian estimates (BE) and empirical Bayes estimates (EBE), and approximate Bayes estimates by using Lindley's approximation and Markov Chain Monte Carlo using the importance sampling technique. Bayes estimates are derived under the assumption of squared error loss function (SELF).

The rest of this paper is organized as follows. Statistical models are presented in Section 2 based on Type I censoring. The MLE and AMLE are derived in Sections 3 and 4, respectively. In Section 5, Bayes inferences including point estimates, approximate Bayes estimates and credible intervals are derived assuming SELF. Simulations and data analysis are presented in Section 6 to investigate the performance of the proposed estimation methods. Finally, Section 7 contains some concluding remarks.

2. Model assumptions. Assume that n identical units $X_1, \ldots, X_n$ are put to the test and each of them has $\mathrm{TL}(\theta, \beta)$ lifetime distribution with known β. Based on a Type I censoring scheme at time T, we have the data

$D = \{(y_i, \delta_i) : i = 1, \dots, n\}$, where $y_i = \min(x_i, T)$ and

$$\delta_i = \begin{cases} 1, & x_i \le T, \\ 0, & x_i > T. \end{cases} \tag{2.1}$$

The likelihood function of θ and β based on the Type I censored data D is given by

$$L(D \mid \theta, \beta) = \left(\frac{2\theta}{\beta}\right)^k \prod_{i=1}^{n} \left(1 - \frac{y_i}{\beta}\right)^{\delta_i} u_i^{(\theta-1)\delta_i} [1 - u_i^{\theta}]^{1-\delta_i}, \tag{2.2}$$

where $u_i = \frac{y_i}{\beta}(2 - \frac{y_i}{\beta}), 0 < \max y_i \le \beta$ and $\theta > 0$.

Throughout this article it is assumed that n and T are fixed in advance.

3. Maximum likelihood estimate. Based on the Type I censored sample D, the MLE of θ is $\hat{\theta}_{\mathrm{MLE}}$ that satisfies the equation

$$\frac{k}{\hat{\theta}_{\mathrm{MLE}}} + \sum_{i=1}^{n} \delta_i \ln u_i - (1 - \delta_i) \frac{u_i^{\hat{\theta}_{\mathrm{MLE}}}}{1 - u_i^{\hat{\theta}_{\mathrm{MLE}}}} \ln u_i = 0. \tag{3.1}$$

Unfortunately, (3.1) does not have an explicit solution, and so the MLE cannot be obtained in explicit form. Note that (3.1) can be represented as

$$\hat{\theta}_{\mathrm{MLE}} = h(\hat{\theta}_{\mathrm{MLE}}), \tag{3.2}$$

where

$$h(\hat{\theta}_{\mathrm{MLE}}) = \frac{k}{\sum_{i=1}^{n} (\ln u_i) \frac{u_i^{\hat{\theta}_{\mathrm{MLE}}} - \delta_i}{1 - u_i^{\hat{\theta}_{\mathrm{MLE}}}}}.$$

A simple iterative scheme is proposed in this paper as follows: Start with an initial guess of $\hat{\theta}_{\mathrm{MLE}}$, say $\hat{\theta}^{(0)}_{\mathrm{MLE}}$, then obtain $\hat{\theta}^{(1)}_{\mathrm{MLE}} = h(\hat{\theta}^{(0)}_{\mathrm{MLE}})$ and proceed iteratively to obtain the MLE $\hat{\theta}^{(m+1)}_{\mathrm{MLE}} = h(\hat{\theta}^{(m)}_{\mathrm{MLE}})$ such that $|\hat{\theta}^{(m+1)}_{\mathrm{MLE}} - \hat{\theta}^{(m)}_{\mathrm{MLE}}| < \epsilon$, some predetermined tolerance limit.

It should be mentioned here that if β is unknown, then one could easily replace it by its MLE, the maximum ordered observation x_k.

4. Approximate maximum likelihood estimate. The likelihood equation (3.1), as mentioned in the previous section, does not provide an explicit estimator for the shape parameter. Hence, it may be desirable to develop an approximation to the likelihood equation which provides us with an explicit estimator for the unknown parameter. This explicit estimator may also provide us with an excellent starting value for the iterative solution of (3.2). An approximate MLE (AMLE) is derived by expanding the function $g_i(\theta) = \frac{u_i^{\theta}}{1-u_i^{\theta}}$ in (3.1) using first-order Taylor expansion around

$v_i = \frac{\ln p_i}{\ln u_i}$, where $p_i = \frac{i}{n+1}$ for $i = 1, \ldots, n$. A similar procedure was used by Banerjee and Kundu [2] and Balakrishnan and Varadan [1].

Note that for $i = 1, \ldots, n$,

$$g_i(\theta) \approx \frac{u_i^{v_i}}{1 - u_i^{v_i}} + (\theta - v_i)\frac{u_i^{v_i}}{(1 - u_i^{v_i})^2}\ln u_i. \tag{4.1}$$

Using the approximation (4.1), the likelihood equation (3.1) is approximated by

$$\frac{k}{\theta} + \sum_{i=1}^{n}\delta_i \ln u_i - \sum_{i=1}^{n}(1-\delta_i)\ln u_i\left[\frac{u_i^{v_i}}{1 - u_i^{v_i}}\right] + (\theta - v_i)\frac{u_i^{v_i}}{(1 - u_i^{v_i})^2}\ln u_i = 0. \tag{4.2}$$

From (4.2), the AMLE is the positive solution of the quadratic equation for θ:

$$A\theta^2 + B\theta + k = 0,$$

where

$$A = -\sum_{i=1}^{n}(1 - \delta_i)\frac{u_i^{v_i}}{(1 - u_i^{v_i})^2}[\ln u_i]^2$$

and

$$B = \sum_{i=1}^{n}\delta_i \ln u_i - \sum_{i=1}^{n}(1 - \delta_i)\ln u_i\left[v_i\frac{u_i^{v_i}}{(1 - u_i^{v_i})^2}\ln u_i - \frac{u_i^{v_i}}{1 - u_i^{v_i}}\right].$$

Threfore, the AMLE, say $\hat{\theta}_{\mathrm{AMLE}}$, is obtained as

$$\hat{\theta}_{\mathrm{AMLE}} = \frac{-B - \sqrt{B^2 - 4Ak}}{2A}, \tag{4.3}$$

which is the only positive root.

It is worth mentioning that if $k = n$, then $\delta_i = 1$ for all $i = 1, \ldots, n$, which implies that $A = 0$. In this case $\hat{\theta}_{\mathrm{AMLE}} = -\frac{n}{\sum_{i=1}^{n}\ln u_i}$, which equals the MLE of θ based on the complete sample (see [3]).

5. Bayesian inference. In this section, we discuss the Bayes estimate and the associated credible interval for the shape parameter. The squared error loss function (SELF) is considered, which is defined as

$$L(\hat{\theta}) = (\theta - \hat{\theta})^2,$$

where $\hat{\theta}$ is the estimator of θ.

5.1. Prior and posterior analysis. Suppose that θ has a proper exponential prior with pdf

$$g(\theta) = ae^{-a\theta}, \tag{5.1}$$

where $\theta > 0$ and $a > 0$.

Hence, by using the likelihood function defined in (2.2), the posterior pdf of θ given the Type I censored sample D is given by

$$(5.2)\quad \pi(\theta\,|\,D,\beta)=\frac{L(D\,|\,\theta,\beta)g(\theta)}{\int_0^\infty L(D\,|\,\theta,\beta)g(\theta)\,d\theta}=\frac{\theta^k e^{-a\theta}\prod_{i=1}^n u_i^{\delta_i\theta}[1-u_i^\theta]^{1-\delta_i}}{C},$$

where

$$C=\int_0^\infty \theta^k e^{-a\theta}\prod_{i=1}^n u_i^{\delta_i\theta}[1-u_i^\theta]^{1-\delta_i}\,d\theta,$$

the normalizing constant.

Under the SELF, the Bayes estimate of θ based on Type I censoring scheme, say $\hat{\theta}_B(a)$, is the posterior mean, which is given by

$$(5.3)\qquad \hat{\theta}_B(a)=E_\pi(\theta\,|\,D,\beta)=\frac{1}{C}\int_0^\infty \theta^{k+1}e^{-a\theta}\prod_{i=1}^n u_i^{\delta_i\theta}[1-u_i^\theta]^{1-\delta_i}d\theta.$$

It is worth mentioning that the BE (5.3) is admissible, because it is the posterior mean that arises from a proper prior (see [18]). The empirical Bayes estimate (EBE) is also proposed in this section. The marginal pdf of the variable x is given by

$$(5.4)\qquad k(x)=\int_0^\infty f(x\,|\,\theta)g(\theta)\,d\theta,$$

where $f(x\,|\,\theta)$ is defined in (1.2) and $g(\theta)$ is the prior pdf of θ given in (5.1). Therefore, the marginal pdf in (5.4) is given by

$$(5.5)\qquad k(x)=\frac{2a\left(1-\frac{x}{\beta}\right)}{x\left(2-\frac{x}{\beta}\right)\left[-a+\ln\frac{x}{\beta}\left(2-\frac{x}{\beta}\right)\right]^2},$$

where $0<x<\beta$ and $a>0$.

The EBE of θ is equal to $\hat{\theta}_B(\hat{a}_{\mathrm{MME}})$ where $\hat{a}_{\mathrm{MME}}$ is the method of moment estimate (MME) of a based on the Type I censored sample, which satisfies the following integral equation:

$$E_x(X)=\int_0^\infty \frac{2\hat{a}_{\mathrm{MME}}\left(1-\frac{x}{\beta}\right)}{\left(2-\frac{x}{\beta}\right)\left[-\hat{a}_{\mathrm{MME}}+\ln\frac{x}{\beta}\left(2-\frac{x}{\beta}\right)\right]^2}\,dx=\frac{\sum_i^n y_i\delta_i}{k}.$$

Two approaches are introduced in the next section to approximate the BE in (5.3): Lindley's approximation and Markov Chain Monte Carlo (MCMC) using the importance sampling technique.

5.2. Lindley's approximation. Lindley [11] proposed an approximation procedure to evaluate the ratio of two integrals, such that the Bayes estimate in (5.3) takes a form containing no integrals. This procedure has been used by several authors to obtain the Bayes estimates for various distributions; see, for instance, Press [15]. Based on Lindley's method, the approximate BE of θ under the SELF and based on the prior pdf (5.1) is given by

$$\hat{\theta}_{B,L}(a) \approx \hat{\theta}_{\mathrm{MLE}} + \frac{a}{\hat{l}_{\theta\theta}} + \frac{1}{2}\,\frac{\hat{l}_{\theta\theta\theta}}{\hat{l}^2_{\theta\theta}}, \tag{5.6}$$

where $\hat{\theta}_{\mathrm{MLE}}$ is the MLE of θ, $l = \ln L(D \mid \theta, \beta)$, where $L(D \mid \theta, \beta)$ is the likelihood function defined in (2.2),

$$\hat{l}_{\theta\theta} = \frac{\partial^2 l}{\partial \theta^2}\bigg|_{\theta=\hat{\theta}_{\mathrm{MLE}}} = -\frac{k}{\hat{\theta}^2_{\mathrm{MLE}}} - \sum_{i=1}^{n}(1-\delta_i)[\ln u_i]^2 \frac{u_i^{\hat{\theta}_{\mathrm{MLE}}}}{(1-u_i^{\hat{\theta}_{\mathrm{MLE}}})^2},$$

and

$$\hat{l}_{\theta\theta\theta} = \frac{\partial^3 l}{\partial \theta^3}\bigg|_{\theta=\hat{\theta}_{\mathrm{MLE}}} = \frac{2k}{\hat{\theta}^3_{\mathrm{MLE}}} - \sum_{i=1}^{n}(1-\delta_i)[\ln u_i]^3 \frac{u_i^{\hat{\theta}_{\mathrm{MLE}}}[1+u_i^{\hat{\theta}_{\mathrm{MLE}}}]}{(1-u_i^{\hat{\theta}_{\mathrm{MLE}}})^3}.$$

5.3. MCMC method. It is well known that Lindley's method is not helpful for constructing credible intervals. Therefore, we propose to use the importance sampling technique to generate MCMC samples from the posterior pdf to compute the desired Bayes estimate of θ, and also to construct the associated credible interval. A similar procedure was used, for example, by Chen et al. [5] and Kundu and Pradhan [10], [13], [14]. To implement the importance sampling technique, we rewrite the posterior pdf (5.2) as follows:

$$\pi(\theta \mid D, \beta) \propto f_1(\theta \mid D) f_2(\theta),$$

where

$$f_1(\theta \mid D) = \frac{[a - \sum_{i=1}^{n} \delta_i \ln u_i]^k}{\Gamma(k+1)} \theta^k e^{-\theta[a - \sum_{i=1}^{n} \delta_i \ln u_i]},$$

which is clearly a gamma density function with shape parameter $k+1$ and scale parameter $[a - \sum_{i=1}^{n} \delta_i \ln u_i]^{-1}$, and

$$f_2(\theta) = \prod_{i=1}^{n} [1 - u_i^{\theta}]^{1-\delta_i}.$$

Therefore, (5.3) can be written as

$$\hat{\theta}_B(a) = \frac{\int_0^\infty \theta f_1(\theta \mid D) f_2(\theta)\, d\theta}{\int_0^\infty f_1(\theta \mid D) f_2(\theta)\, d\theta}. \tag{5.7}$$

Assume that n, the censoring time T and the Type I censored sample D are given in advance. The following algorithms are proposed along the lines of Kundu and Pradhan [10] to compute the BE of θ and also to construct the associated credible interval.

5.3.1. ALGORITHM 1 (BE).

Step 1. Generate a random sample of size M from $f_1(\theta \mid D)$, a gamma density function with shape parameter $k+1$ and scale parameter $[a - \sum_{i=1}^{n} \delta_i \ln u_i]^{-1}$, say $\theta_1, \dots, \theta_M$.
Step 2. Compute $f_2(\theta_j)$, for $j = 1, \dots, M$.
Step 3. Under the assumption of SELF, a simulation consistent estimate of θ can be obtained using the importance sampling technique as

$$\hat{\theta}_{B,IS}(a) = \frac{\sum_{j=1}^{M} \theta_j f_2(\theta_j)}{\sum_{j=1}^{M} f_2(\theta_j)}.$$

Moreover, we can compute a simulation consistent estimate of any function $H(\theta)$ as

$$\hat{H}(\theta) = \frac{\sum_{j=1}^{M} H(\theta_j) f_2(\theta_j)}{\sum_{j=1}^{M} f_2(\theta_j)},$$

provided that $\hat{H}(\theta)$ is defined at all $j = 1, \dots, M$.

Now, to compute the credible interval of θ let, for $0 < p < 1$, θ_p be such that $P(\theta \le \theta_p \mid D, \beta) = \int_0^{\theta_p} \pi(\theta \mid D, \beta)\, d\theta = p$, where $\pi(\theta \mid D, \beta)$ is the posterior pdf defined in (5.2).

5.3.2. ALGORITHM 2 (credible interval). We use the sample $\theta_1, \dots, \theta_M$ obtained from Algorithm 1.

Step 1. Compute $w_j = f_2(\theta_j) / \sum_{j=1}^{M} f_2(\theta_j)$ for $j = 1, \dots, M$.
Step 2. Arrange the set $\{(\theta_1, w_1), (\theta_2, w_2), \dots, (\theta_M, w_M)\}$ as $\{(\theta_{(1)}, w_{[1]}), (\theta_{(2)}, w_{[2]}), \dots, (\theta_{(M)}, w_{[M]})\}$, where $\theta_{(1)} \le \dots \le \theta_{(M)}$.
Step 3. The $100(1-\alpha)\%$ credible interval for θ is given by

$$(\hat{\theta}_{\alpha/2}, \hat{\theta}_{1-\alpha/2}),$$

where $\hat{\theta}_p$ is a simulation consistent Bayes estimate for θ_p, which is given by $\theta_{(M_p)}$ such that M_p is the integer satisfying

$$\sum_{j=1}^{M_p} w_{[j]} \le p < \sum_{j=1}^{M_p+1} w_{[j]}.$$

THEOREM 5.1. *The posterior pdf $\pi(\theta \mid D, \beta)$ in (5.2) is log-concave.*

Proof. Since $u_i = \frac{y_i}{\beta}\left(2 - \frac{y_i}{\beta}\right) > 0$, it is easy to see that

$$\frac{\partial^2 \ln \pi(\theta | D, \beta)}{\partial \theta^2} = -\left[\frac{k}{\theta^2} + \sum_{i=1}^{n} (1 - \delta_i) [\ln u_i]^2 \frac{u_i^{\theta}}{[1 - u_i^{\theta}]^2}\right] < 0$$

for any θ; this proves the result. ∎

Since the posterior distribution (5.2) is log-concave, one can apply Devroye's algorithm [6] to generate a sample from the posterior distribution, say $\theta_1, \ldots, \theta_M$. Based on this sample and under the SELF, the approximate Bayes estimate of θ is given by

$$\hat{\theta}_{\text{MCMC}} = \hat{E}(\theta \mid D) = \frac{1}{M} \sum_{j=1}^{M} \theta_j.$$

The $100(1-\alpha)\%$ credible interval of θ can be computed by ordering $\theta_1, \ldots, \theta_M$ as $\theta_{(1)} \le \cdots \le \theta_{(M)}$ and taking the interval

$$(\theta_{(M(\alpha/2))}, \theta_{(M(1-\alpha/2))}).$$

6. Simulation study and data analysis

6.1. Simulations. In this section, various simulation studies are presented mainly to observe how the different estimation methods behave for different sample sizes, and for different Type I censoring schemes. The unknown parameter is estimated using the MLE, AMLE, BE and EBE obtained by using Lindley's approximation and by MCMC using the importance sampling technique. Performances of the different estimators are compared with respect to their means and mean squared errors (MSE).

In all cases, the parameter β is assumed without loss of generality to equal 1. Simulations are performed for two values of the shape parameter, namely, $\theta = 0.5$ and $\theta = 1$.

BE is approximated assuming that θ has an exponential prior with a hyper parameter a with pdf given in (5.1). Bayes estimates are computed based on two priors: Prior 0, a non-informative prior with $a \approx 0$, and Prior 1, an informative prior with a predetermined value of a that is assumed based on our information about θ. A good assumption for a is $a \approx 1/\theta$, since $E(\theta) = 1/a$. In the case of $a = 0$ the prior is improper; a small value for the hyper parameter a may be used to make the prior proper. Both values $a = 0$ and $a = 0.0001$ are tried in simulation studies; it was observed that the results are not significantly different based on these values. Accordingly, the results based on $a = 0$ are only reported and the results based on $a = 0.0001$ are omitted.

Different values for the combination (n, T) are considered in order to study their effect on the estimators. The sample sizes $n = 10$, 20 and 30 are considered. Two Type I censored schemes are considered: with T equal to $E(X)$ and $2E(X)$, where $E(X) = \beta(1 - 4^{\theta})\frac{\Gamma^2(1+\theta)}{\Gamma(2+2\theta)}$. The expected values of the corresponding MSE of the proposed estimates are computed over 1000 replications. The results are reported in Tables 1 and 2 assuming the real parameter $\theta = 0.5$ and $\theta = 1$, respectively.

Table 1. Expected value of the proposed estimators and the corresponding MSE when $\theta = 0.5$

				BE assuming $a = 10^{-4}$			BE assuming $a = 2$			Empirical BE		
n	T	$\hat{\theta}_{\text{MLE}}$	$\hat{\theta}_{\text{AMLE}}$	Exact	Lindley's	MCMC	Exact	Lindley's	MCMC	Exact	Lindley's	MCMC
10	0.215	0.591	0.525	0.658	0.627	0.648	0.578	0.544	0.570	0.442	0.312	0.446
		0.065	0.041	0.100	0.077	0.094	0.046	0.029	0.043	0.032	0.285	0.036
	0.430	0.559	0.524	0.620	0.604	0.615	0.552	0.534	0.547	0.517	0.489	0.513
		0.042	0.033	0.066	0.054	0.060	0.032	0.025	0.030	0.035	0.035	0.033
20	0.215	0.553	0.505	0.579	0.570	0.581	0.547	0.536	0.548	0.477	0.446	0.477
		0.027	0.020	0.028	0.030	0.039	0.019	0.020	0.029	0.015	0.017	0.027
	0.430	0.528	0.498	0.554	0.550	0.554	0.525	0.520	0.524	0.508	0.501	0.507
		0.019	0.016	0.020	0.022	0.024	0.014	0.016	0.017	0.014	0.016	0.017
30	0.215	0.543	0.506	0.568	0.554	0.560	0.547	0.533	0.558	0.498	0.475	0.501
		0.016	0.013	0.020	0.017	0.097	0.015	0.013	0.086	0.011	0.011	0.053
	0.430	0.520	0.495	0.544	0.534	0.537	0.525	0.516	0.518	0.514	0.503	0.508
		0.011	0.010	0.014	0.012	0.013	0.010	0.010	0.011	0.011	0.010	0.011

Table 2. Expected value of the proposed estimators and the corresponding MSE when $\theta = 1.0$

				BE assuming $a = 10^{-4}$			BE assuming $a = 1$			Empirical BE		
n	T	$\hat{\theta}_{\text{MLE}}$	$\hat{\theta}_{\text{AMLE}}$	Exact	Lindley's	MCMC	Exact	Lindley's	MCMC	Exact	Lindley's	MCMC
10	0.333	1.230	1.069	1.296	1.296	1.345	1.058	0.922	1.046	0.994	0.861	1.009
		0.360	0.195	0.414	0.414	0.449	0.119	0.216	0.109	0.145	0.199	0.165
	0.666	1.118	1.064	1.216	1.216	1.229	0.986	0.935	0.988	1.129	1.093	1.108
		0.167	0.134	0.223	0.223	0.236	0.075	0.049	0.070	0.210	0.154	0.164
20	0.333	1.132	1.025	1.197	1.164	1.206	1.065	1.024	1.060	1.041	0.993	1.052
		0.120	0.081	0.150	0.132	0.359	0.067	0.056	0.136	0.076	0.067	0.170
	0.666	1.050	1.009	1.108	1.097	1.103	0.997	0.980	0.992	1.045	1.042	1.048
		0.068	0.058	0.082	0.080	0.083	0.043	0.042	0.046	0.067	0.066	0.068
30	0.333	1.117	1.031	1.151	1.138	1.213	1.076	1.050	1.151	1.049	1.030	1.130
		0.073	0.051	0.086	0.079	1.164	0.055	0.044	1.86	0.050	0.047	0.597
	0.666	1.041	1.003	1.072	1.072	1.076	1.007	0.997	1.003	1.037	1.037	1.039
		0.041	0.035	0.046	0.046	0.048	0.033	0.030	0.030	0.040	0.040	0.041

From Tables 1 and 2 one can see and expect that the MSE decreases, approaching 0, and the average values of the underlying estimates approach the real value as the sample size increases for all estimators under all the censoring schemes. This may prove the consistency of these estimates. For fixed sample size, fixed θ and for any censoring scheme, the informative and empirical Bayes estimates, using Lindley's or (and) MCMC approximations, almost perform, in terms of the MSE, better than the other estimates. It is also clear from these tables that the AMLE performs, in terms of the MSE, better than the MLE. In general, the informative BE, Empirical BE and the AMLEs are quite similar and satisfactory.

6.2. Data analysis. For illustrative purposes, data analysis is presented in this section for real and simulated data sets by using the proposed estimation methods.

6.2.1. *Real data.* In order to discuss the practical applicability of the results obtained in this paper, the following real life data presented by Grubbs

[9] give the failure time (in miles) of eighteen military carriers:

$$162, 200, 271, 302, 393, 508, 539, 629, 706, 777,$$
$$884, 1101, 1182, 1463, 1603, 1984, 2355, 2880.$$

First, it was checked whether the T-L distribution can be used to analyze this data set. The MLE of β is 2880, the maximum order statistic, and the MLE of θ is 1.133 . The Bayes estimate of θ, under the SELF, is 1.125 when $a = 1$ (see [3]). It is obvious that the MLE and the Bayes estimate are almost the same.

The Kolmogorov–Smirnov (KS) distance between the empirical distribution function and the fitted distribution function, using the MLEs, has been used to check the goodness of fit. The KS statistic value is 0.135, and the KS critical value is 0.309 at $n = 18$ and $\alpha = 0.05$. Accordingly, one cannot reject the hypothesis that the data come from the T-L distribution.

Table 3 shows the proposed estimators based on two Type I censoring schemes with $T = 800$ and $T = 1600$. The parameter β is assumed to equal the maximum observed value 2880.

Table 3. Real life data analysis based on two Type I censoring schemes

			BE assuming $a = 10^{-4}$			BE assuming $a = 1$			Empirical BE		
T	$\hat{\theta}_{\text{MLE}}$	$\hat{\theta}_{\text{AMLE}}$	Exact	Lindley's	MCMC	Exact	Lindley's	MCMC	Exact	Lindley's	MCMC
800	1.225	1.105	1.294	1.264	1.280	1.211	1.180	1.184	1.104	1.052	1.091
1600	1.146	1.043	1.210	1.196	1.207	1.137	1.123	1.136	1.110	1.093	1.108

6.2.2. *Simulated data.* We analyze the following simulated data set presented by Genc [7] assuming $\theta = 0.3$ and $\beta = 1$:

$$0.1425, 0.2707, 0.2783, 0.0718, 0.4537, 0.0615, 0.0047, 0.3454, 0.4428, 0.1909,$$
$$0.1028, 0.0013, 0.0592, 0.5413, 0.2442, 0.0001, 0.0002, 0.0178, 0.0114, 0.5388.$$

Table 4 shows the proposed estimators based on two Type I censoring schemes assuming $T = 0.30$ and $T = 0.50$.

One can see from Table 4 that all estimates are quite similar. The informative and empirical Bayes estimate, and the AMLE, dominate the other when the hyper parameter a is assumed to equal 6.

Table 4. Simulated data analysis based on two Type I censoring schemes

			BE assuming $a = 10^{-4}$			BE assuming $a = 6$			Empirical BE		
T	$\hat{\theta}_{\text{MLE}}$	$\hat{\theta}_{\text{AMLE}}$	Exact	Lindley's	MCMC	Exact	Lindley's	MCMC	Exact	Lindley's	MCMC
0.30	0.441	0.417	0.462	0.457	0.434	0.411	0.402	0.391	0.417	0.409	0.396
0.50	0.415	0.397	0.435	0.433	0.436	0.381	0.382	0.388	0.413	0.410	0.413

The sensitivity of the proposed Bayes estimates of the hyper parameter a is studied for this data and the results are reported in Table 5.

It is clear from Table 5 that the exact Bayes estimate, Lindley's and MCMC approximations have almost the same rate of change with respect

Table 5. Sensitivity of the proposed Bayes estimates of the hyper parameter

	$T = 0.30$				$T = 0.30$			
a	Exact	Lindley's	MCMC	95% CI	Exact	Lindley's	MCMC	95% Cl
0.1	0.461	0.456	0.440	(0.269,0.659)	0.434	0.432	0.432	(0.276,0.629)
1.0	0.453	0.448	0.427	(0.263,0.595)	0.426	0.425	0.427	(0.263,0.616)
1.5	0.448	0.443	0.427	(0.265,0.596)	0.422	0.420	0.421	(0.267,0.621)
2.0	0.443	0.439	0.426	(0.263,0.633)	0.418	0.416	0.419	(0.266,0.628)
3.5	0.431	0.425	0.408	(0.250,0.575)	0.406	0.403	0.405	(0.252,0.561)
4.0	0.427	0.420	0.403	(0.249,0.572)	0.402	0.399	0.401	(0.251,0.590)
6.0	0.411	0.402	0.391	(0.242,0.574)	0.387	0.382	0.388	(0.242,0.561)
10.0	0.382	0.365	0.364	(0.226,0.528)	0.361	0.347	0.361	(0.221,0.532)

to the change in the hyper parameter a, with smaller rate for the estimate obtained by using the MCMC method.

7. Conclusions. In this article, the shape parameter of the T-L distribution was estimated from classical and Bayesian viewpoints based on Type I censored data. It was observed that MLE cannot be derived in explicit form, but it can be obtained numerically. Hence, an approximate MLE was derived in explicit form. The Bayes estimate was considered based on the squared error loss function, and it was observed that it cannot be obtained in explicit form. Lindley's approximation and Markov Chain Monte Carlo using the importance sampling technique were proposed to approximate the Bayes estimate and to construct the associated credible interval. The performance of the proposed estimates was compared by Monte Carlo simulations. It was noticed that the informative and empirical Bayes estimates, using Lindley's or/and MCMC approximations, almost perform, in terms of the MSE, better than the other estimates. It was also observed that the AMLE performs, in terms of the MSE, better than the MLE. Moreover, it was observed from real and simulated data sets analysis that the credible intervals associated with the proposed MCMC Bayes estimates are satisfactory. The sensitivity of the proposed Bayes estimates to the hyper parameter assumption was also studied; it was found that the MCMC method has the smallest rate of change with respect to the hyper parameter change.

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