COLLOQUIUM MATHEMATICUM

VOL. 141 2015 NO. 2

A CORSON COMPACT L-SPACE FROM A SUSLIN TREE

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Abstract. The completion of a Suslin tree is shown to be a consistent example of a Corson compact L-space when endowed with the coarse wedge topology. The example has the further properties of being zero-dimensional and monotonically normal.

1. Introduction. In this paper, the coarse wedge topology on trees is used to construct what may be the first consistent example of a Corson compact L-space that is monotonically normal. It is considerably simpler and easier to (roughly!) visualize than the CH example of a Corson compact L-space produced by Kunen [4], or the Corson compact L-space produced by Kunen and van Mill [5] under the hypothesis that 2^{ω_1} with the product measure is the union of a family of $\aleph_1$ nullsets such that every nullset is contained in some member of the family.

Corson compact L-spaces cannot be constructed in ZFC alone, because MA_{ω_1} implies there are no compact L-spaces at all. This is one of the earliest applications of MA_{ω_1} to set-theoretic topology, and one of the few that uses its topological characterization, viz., that a compact ccc space cannot be the union of $\aleph_1$ nowhere dense sets [3], [9, 6.2], [10, p. 16].

Recall that a *Corson compact space* is a compact Hausdorff space that can be embedded in the Σ-product of real lines, viz., a subspace of the product space $\mathbb{R}^\Gamma$ (for some set Γ) consisting of all points which differ from the zero element in only countably many coordinates. Corson compact spaces play a role in functional analysis, especially through their spaces of continuous functions, the Banach space $\langle C(K), \|\cdot\|_\infty\rangle$, and the space $C_p(X)$ of real-valued continuous functions with the relative product topology.

Recall that a topological space is *separable* if it has a countable dense subset, and *Lindelöf* if every open cover has a countable subcover. The following terminology is now standard:

2010 *Mathematics Subject Classification*: Primary 46A40, 54A25, 54D35, 54D80; Secondary 54A35, 54B99.

Key words and phrases: Corson compact, Σ-product, L-space, tree, level, chain complete, Dedekind complete, chain completion, coarse wedge topology, Suslin tree, monotonically normal.

DOI: 10.4064/cm141-2-1

Definition 1.1. An *L-space* is a regular, hereditarily Lindelöf space which has a nonseparable subspace.

For over four decades, one of the best known unsolved problems of set-theoretic topology was whether there is a ZFC example of an L-space. This was solved in an unexpected manner by Justin Tatch Moore, who constructed one with the help of a deep analysis of walks on ordinals [6]. The following problem, motivated by our main example, may still be unsolved:

Problem 1.2. *Is there a ZFC example of an L-space which embeds as a closed subspace in a Σ-product of real lines?*

Definition 1.3. A space X is *monotonically normal* if there is a function $U(E,F)$ defined on pairs of disjoint closed sets $\langle E,F\rangle$ such that: (1) $U(E,F)$ is an open set; (2) $E\subset U(E,F)$ and $U(E,F)\cap U(F,E)=\emptyset$; and (3) if $E\subset E'$ and $F\supset F'$, then $U(E,F)\subset U(E',F')$.

A neat feature of our main example is that, being monotonically normal, it is the continuous image of a compact orderable space [11]—and yet every linearly orderable Corson compact space is metrizable [1]. One natural question is whether the main example is actually the continuous image of a compact orderable L-space: such spaces exist iff there is a Suslin tree/line. A much more general pair of contrasting questions is open:

Problem 1.4. *Is the existence of a monotonically normal L-space equivalent to the existence of a Suslin tree?*

Problem 1.5. *Is there a ZFC example of a monotonically normal L-space?*

2. Trees and the coarse wedge topology. The purpose of this section is to make this paper as self-contained as reasonable, and to show that trees with the coarse wedge topology have a property even stronger than being monotonically normal. Readers familiar with the coarse wedge topology might try omitting this section on a first reading. Others with a good understanding of trees might try picking up the reading at Definition 2.6 below.

Definition 2.1. A *tree* is a partially ordered set in which the predecessors of any element are well-ordered. [Given two elements $x<y$ of a poset, we say x is a *predecessor* of y and y is a *successor* of x.]

Definition 2.2. If a tree has only one minimal member, it is said to be *rooted* and the minimal member is called the *root* of the tree. A *chain* in a poset is a totally ordered subset. An *antichain* in a tree is a set of pairwise incomparable elements. Maximal members (if any) of a tree are called *leaves*, and maximal chains are called *branches*.

Definition 2.3. If T is a tree, then $T(0)$ is its set of minimal members. Given an ordinal α, if $T(\beta)$ has been defined for all $\beta < \alpha$, then $T\restriction\alpha = \bigcup\{T(\beta) : \beta < \alpha\}$, while $T(\alpha)$ is the set of minimal members of $T \setminus T\restriction\alpha$. The set $T(\alpha)$ is called *the αth level of* T. The *height* or *level* of $t \in T$ is the unique α for which $t \in T(\alpha)$, and it is denoted $\ell(t)$. The *height of* T is the least α such that $T(\alpha) = \emptyset$.

The following example illustrates some fine points of associating ordinals with trees and their elements.

Example 2.4. The *full ω-ary tree of height* $\omega + 1$ is the set T of all sequences of nonnegative integers that are either finite or have domain ω, and in which the order is end extension. Each chain of order type ω consists of finite sequences whose union is an ω-sequence on level ω. Since this is the last nonempty level of the tree, the tree itself is of height $\omega+1$. The subtree $T\restriction\omega$ is the *full ω-ary tree of height ω*.

Definition 2.5. A tree is *chain-complete* [resp. *Dedekind complete*] if every chain [*resp.* chain that is bounded above] has a least upper bound. A tree is *complete* if it is rooted and chain-complete.

Definition 2.6. For each t in a tree T we let V_t denote the wedge $\{s \in T : t \leq s\}$. The *coarse wedge topology* on a tree T is the one whose subbase is the set of all wedges V_t and their complements, where t is either miminal or on a successor level.

Because of the way trees are structured, the nonempty finite intersections of members of the subbase are "notched wedges" of the form

$$W_t^F = V_t \setminus \bigcup\{V_s : s \in F\} = V_t \setminus V_F$$

where F is a finite set of successors of t.

If t is minimal or on a successor level, then a local base at t is formed by the sets W_t^F such that F is a finite set of immediate successors of t. If, on the other hand, t is on a limit level, then a local base is formed by the W_s^F such that s is on a successor level below t.

A corollary of the following theorem is that every complete tree is compact Hausdorff in the coarse wedge topology.

Theorem 2.7 ([7, Corollary 3.5]). *A tree is compact Hausdorff in the coarse wedge topology iff it is chain-complete and has only finitely many minimal elements.*

Theorem 2.8. *A complete tree is Corson compact in the coarse wedge topology iff every chain is countable.*

Proof. A necessary and sufficient condition for a compact space being Corson compact is that it have a point-countable T_0-separating cover by cozero sets—equivalently, open F_σ-sets [1]. If a complete tree has an uncountable chain, then it has a copy of ω_1+1 which does not have a point-countable T_0-separating open cover of any kind, thanks in part to the Pressing-Down Lemma (Fodor's Lemma).

Conversely, if every chain is countable, then the clopen sets of the form V_t clearly form a T_0-separating, point-countable cover. ∎

Hausdorff trees with the coarse wedge topology have a property even stronger than monotone normality; it is the property that results if "clopen" is substituted for "open" in Definition 1.3:

Definition 2.9. A space X is *monotonically ultranormal* if there is a function $U(E,F)$ defined on pairs of disjoint closed sets $\langle E,F\rangle$ such that: (1) $U(E,F)$ is a clopen set; (2) $E\subset U(E,F)$ and $U(E,F)\cap U(F,E)=\emptyset$; and (3) if $E\subset E'$ and $F\supset F'$, then $U(E,F)\subset U(E',F')$.

The property in the following theorem is named with the Borges criterion (see below) for monotone normality in mind.

Theorem 2.10 ([8, Theorem 2.2]). *Every Hausdorff space satisfying the following property is monotonically ultranormal:*

Property B+: *To each pair $\langle G,x\rangle$ where G is an open set and $x\in G$, it is possible to assign an open set G_x such that $x\in G_x\subset G$ so that $G_x\cap H_y\neq\emptyset$ implies either $x\in H_y$ or $y\in G_x$.*

The Borges criterion puts H for H_y and G for G_x in the part of Property B+ after "implies".

The question of whether every monotonically ultranormal space satisfies Property B+ was posed in [8] and is still open.

Theorem 2.11. *Every tree with the coarse wedge topology has Property B+.*

Proof. For each point t and each open neighborhood G of t, there exists $s\le t$ for which there is a basic clopen set W^F_s such that $t\subset W^F_s\subset G$, and for which $F\subset V_t$. [If t is on a successor level we can let $s=t$, while if t is on a limit level we first find some $s<t$ on a successor level and some finite $F'\subset V_s$ for which $t\subset W^{F'}_s$; then let $F=F'\cap V_t$ and choose s' such that $s\le s'<t$ and all elements of $F'\setminus F$ are incomparable with s'.]

Now for each $x\in F$ let x' be the immediate successor of t below x and let $F^*=\{x':x\in F\}$.

Claim. *Letting $G_t=W^{F^*}_s$ for all t,G as above produces an assignment witnessing Property* B+.

Proof. The notched wedges W^F_t clearly have the property that the intersection of any two contains the minimum point of one of them. Let

$G_x \cap H_y \neq \emptyset$. Assume that the minimum point t of G_x is in H_y; in particular, $t \geq s$. Let $H_y = W_s^{F^*}$.

CASE 1: $y < t$. Then $G_x \subset V_t \subset H_y$, because t is not in $V_{z'}$ for any $z' \in F^*$.

CASE 2: y and t are incomparable. Then $t > s$, and we again have $G_x \subset V_t \subset H_y$.

CASE 3: $t \leq y$. Then if x and y are incomparable, we clearly have $s < x \in H_y$. This also holds if $x \leq y$. Finally, if $x > y$, we must have $y \in G_x$.

This proves the Claim and hence the theorem. ■

COROLLARY 2.12. *Every Hausdorff tree is monotonically normal in the coarse wedge topology.*

3. The main example. The following construction is utilized in the main example of this paper.

EXAMPLE 3.1. For any tree T, we call a tree *a completion* of T if it is formed by adding a supremum to each downwards closed chain that does not already have one. Formally, we define *the completion* $\hat{T}$ *of* T as follows. If T is not rooted, we let $\hat{T}$ be the collection of downwards closed chains (called "paths" by Todorčević), ordered by inclusion. If T is rooted, we only put the nonempty paths in $\hat{T}$.

We identify each $t \in T$ with the path $P_t = \{s \in T : s \leq t\}$. Completeness of $\hat{T}$ follows from rootedness of $\hat{T}$ and from the easy fact that the supremum of a chain C of $\hat{T}$ is the same as the supremum of $C \cap T$. In particular, if C is a path in $\hat{T}$ then $C \cap T$ is downwards closed in T.

Todorčević called the set of characteristic functions of the paths of T the *path space* of T when it is endowed with the relative topology as a subspace of 2^T with the product topology. Gruenhage [2] showed that this topology is the coarse wedge topology of $\hat{T}$.

Recall that a *Suslin tree* is an uncountable tree in which every chain and antichain is countable. Let us call a tree *uniformly* ω*-ary* if every nonmaximal point has denumerably many immediate successors. (For instance, the tree in Example 2.4 is uniformly ω-ary.)

As is well known, every Suslin tree has a subtree T in which every point has more than one successor at every level above it. Thus every point of T has denumerably many successors on the next limit level above it. And so, a uniformly ω-ary Suslin tree results when we take the subtree S of all points on limit levels of T.

THEOREM 3.2. *The completion* $\hat{S}$ *of a uniformly* ω*-ary Suslin tree* S *is an L-space in the coarse wedge topology.*

Proof. Since $\hat{S}\restriction(\alpha+1)$ is closed for all $\alpha < \omega_1$, we see that $\hat{S}$ is not separable. In the proof that $\hat{S}$ is hereditarily Lindelöf, uniform ω-arity plays a key role: if the tree were finitary, every point on a successor level would be isolated.

We make use of the elementary fact that a space is hereditarily Lindelöf if (and only if) every open subspace is Lindelöf. Let W be an open subspace of $\hat{S}$, and let W_0 be the set of points $t \in W$ such that $V_t \subset W$. If $t \in W_0$ is on a limit level, there is also $s < t$ such that V_s is clopen and $s \in W_0$: see the first paragraph in the proof of Theorem 2.11, and note that here, $F = \emptyset$. Let $A = \{a \in W_0 : a \text{ is minimal in } W_0\}$. Then W_0 is the disjoint union of the clopen wedges V_α $(a \in A)$, and A is countable by the Suslin property.

If $x \in W \setminus W_0$, then there is a basic clopen subset of W of the form W_t^F where $F \neq \emptyset$ and $F \subset V_x$: see the first paragraph in the proof of 2.11 again. There are no more than n immediate successors of x below some element of F, and if s is one of the other immediate successors of x, then $V_s \subset V_x \setminus V_F$, so $s \in W_0$. But then $s \in A$ also, since any V_z containing V_s properly must also contain x, contradicting $x \in W \setminus W_0$. So $W \setminus W_0$ is countable, and we have countably many basic clopen sets whose union is W. ■

The following is now immediate from 2.8, 2.12, and 3.2.

COROLLARY 3.3. *If there is a Suslin tree, then there is a Corson compact, monotonically normal L-space.*

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Received 2 January 2014 (6114)