A CORSON COMPACT L-SPACE FROM A SUSLIN TREE

BY

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Abstract. The completion of a Suslin tree is shown to be a consistent example of a Corson compact L-space when endowed with the coarse wedge topology. The example has the further properties of being zero-dimensional and monotonically normal.

1. Introduction. In this paper, the coarse wedge topology on trees is used to construct what may be the first consistent example of a Corson compact L-space that is monotonically normal. It is considerably simpler and easier to (roughly!) visualize than the CH example of a Corson compact L-space produced by Kunen [4], or the Corson compact L-space produced by Kunen and van Mill [5] under the hypothesis that $2^\omega_1$ with the product measure is the union of a family of $\aleph_1$ nullsets such that every nullset is contained in some member of the family.

Corson compact L-spaces cannot be constructed in ZFC alone, because MA$_{\omega_1}$ implies there are no compact L-spaces at all. This is one of the earliest applications of MA$_{\omega_1}$ to set-theoretic topology, and one of the few that uses its topological characterization, viz., that a compact ccc space cannot be the union of $\aleph_1$ nowhere dense sets [3], [9, 6.2], [10, p. 16].

Recall that a Corson compact space is a compact Hausdorff space that can be embedded in the $\Sigma$-product of real lines, viz., a subspace of the product space $\mathbb{R}^\Gamma$ (for some set $\Gamma$) consisting of all points which differ from the zero element in only countably many coordinates. Corson compact spaces play a role in functional analysis, especially through their spaces of continuous functions, the Banach space $\langle C(K), \| \cdot \|_\infty \rangle$, and the space $C_p(X)$ of real-valued continuous functions with the relative product topology.

Recall that a topological space is separable if it has a countable dense subset, and Lindelöf if every open cover has a countable subcover. The following terminology is now standard:

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Definition 1.1. An \textit{L-space} is a regular, hereditarily Lindelöf space which has a nonseparable subspace.

For over four decades, one of the best known unsolved problems of set-theoretic topology was whether there is a ZFC example of an L-space. This was solved in an unexpected manner by Justin Tatch Moore, who constructed one with the help of a deep analysis of walks on ordinals \cite{6}. The following problem, motivated by our main example, may still be unsolved:

Problem 1.2. \textit{Is there a ZFC example of an L-space which embeds as a closed subspace in a $\Sigma$-product of real lines?}

Definition 1.3. A space $X$ is \textit{monotonically normal} if there is a function $U(E, F)$ defined on pairs of disjoint closed sets $(E, F)$ such that: (1) $U(E, F)$ is an open set; (2) $E \subset U(E, F)$ and $U(E, F) \cap U(F, E) = \emptyset$; and (3) if $E \subset E'$ and $F \supset F'$, then $U(E, F) \subset U(E', F')$.

A neat feature of our main example is that, being monotonically normal, it is the continuous image of a compact orderable space \cite{11}—and yet every linearly orderable Corson compact space is metrizable \cite{11}. One natural question is whether the main example is actually the continuous image of a compact orderable L-space: such spaces exist iff there is a Suslin tree/line. A much more general pair of contrasting questions is open:

Problem 1.4. \textit{Is the existence of a monotonically normal L-space equivalent to the existence of a Suslin tree?}

Problem 1.5. \textit{Is there a ZFC example of a monotonically normal L-space?}

2. Trees and the coarse wedge topology. The purpose of this section is to make this paper as self-contained as reasonable, and to show that trees with the coarse wedge topology have a property even stronger than being monotonically normal. Readers familiar with the coarse wedge topology might try omitting this section on a first reading. Others with a good understanding of trees might try picking up the reading at Definition 2.6 below.

Definition 2.1. A \textit{tree} is a partially ordered set in which the predecessors of any element are well-ordered. [Given two elements $x < y$ of a poset, we say $x$ is a \textit{predecessor} of $y$ and $y$ is a \textit{successor} of $x$.]

Definition 2.2. If a tree has only one minimal member, it is said to be \textit{rooted} and the minimal member is called the \textit{root} of the tree. A \textit{chain} in a poset is a totally ordered subset. An \textit{antichain} in a tree is a set of pairwise incomparable elements. Maximal members (if any) of a tree are called \textit{leaves}, and maximal chains are called \textit{branches}. 
DEFINITION 2.3. If $T$ is a tree, then $T(0)$ is its set of minimal members. Given an ordinal $\alpha$, if $T(\beta)$ has been defined for all $\beta < \alpha$, then $T|\alpha = \bigcup \{ T(\beta) : \beta < \alpha \}$, while $T(\alpha)$ is the set of minimal members of $T \setminus T|\alpha$. The set $T(\alpha)$ is called the $\alpha$th level of $T$. The height or level of $t \in T$ is the unique $\alpha$ for which $t \in T(\alpha)$, and it is denoted $\ell(t)$. The height of $T$ is the least $\alpha$ such that $T(\alpha) = \emptyset$.

The following example illustrates some fine points of associating ordinals with trees and their elements.

EXAMPLE 2.4. The full $\omega$-ary tree of height $\omega + 1$ is the set $T$ of all sequences of nonnegative integers that are either finite or have domain $\omega$, and in which the order is end extension. Each chain of order type $\omega$ consists of finite sequences whose union is an $\omega$-sequence on level $\omega$. Since this is the last nonempty level of the tree, the tree itself is of height $\omega + 1$. The subtree $T|\omega$ is the full $\omega$-ary tree of height $\omega$.

DEFINITION 2.5. A tree is chain-complete [resp. Dedekind complete] if every chain [resp. chain that is bounded above] has a least upper bound. A tree is complete if it is rooted and chain-complete.

DEFINITION 2.6. For each $t$ in a tree $T$ we let $V_t$ denote the wedge $\{ s \in T : t \leq s \}$. The coarse wedge topology on a tree $T$ is the one whose subbase is the set of all wedges $V_t$ and their complements, where $t$ is either minimal or on a successor level.

Because of the way trees are structured, the nonempty finite intersections of members of the subbase are “notched wedges” of the form

$$W_t^F = V_t \setminus \bigcup \{ V_s : s \in F \} = V_t \setminus V_F$$

where $F$ is a finite set of successors of $t$.

If $t$ is minimal or on a successor level, then a local base at $t$ is formed by the sets $W_t^F$ such that $F$ is a finite set of immediate successors of $t$. If, on the other hand, $t$ is on a limit level, then a local base is formed by the $W_s^F$ such that $s$ is on a successor level below $t$.

A corollary of the following theorem is that every complete tree is compact Hausdorff in the coarse wedge topology.

THEOREM 2.7 ([7, Corollary 3.5]). A tree is compact Hausdorff in the coarse wedge topology iff it is chain-complete and has only finitely many minimal elements.

THEOREM 2.8. A complete tree is Corson compact in the coarse wedge topology iff every chain is countable.

Proof. A necessary and sufficient condition for a compact space being Corson compact is that it have a point-countable $T_0$-separating cover by co-
zero sets—equivalently, open $F_\sigma$-sets \[1\]. If a complete tree has an uncountable chain, then it has a copy of $\omega_1 + 1$ which does not have a point-countable $T_0$-separating open cover of any kind, thanks in part to the Pressing-Down Lemma (Fodor’s Lemma).

Conversely, if every chain is countable, then the clopen sets of the form $V_t$ clearly form a $T_0$-separating, point-countable cover. ■

Hausdorff trees with the coarse wedge topology have a property even stronger than monotone normality; it is the property that results if “clopen” is substituted for “open” in Definition 1.3:

**Definition 2.9.** A space $X$ is *monotonically ultranormal* if there is a function $U(E,F)$ defined on pairs of disjoint closed sets $\langle E,F \rangle$ such that:

1. $U(E,F)$ is a clopen set; 2. $E \subseteq U(E,F)$ and $U(E,F) \cap U(F,E) = \emptyset$; and 3. if $E \subseteq E'$ and $F \supset F'$, then $U(E,F) \subseteq U(E',F')$.

The property in the following theorem is named with the Borges criterion (see below) for monotone normality in mind.

**Theorem 2.10 ([8, Theorem 2.2]).** Every Hausdorff space satisfying the following property is monotonically ultranormal:

Property B+: To each pair $\langle G,x \rangle$ where $G$ is an open set and $x \in G$, it is possible to assign an open set $G_x$ such that $x \in G_x \subset G$ so that $G_x \cap H_y \neq \emptyset$ implies either $x \in H_y$ or $y \in G_x$.

The Borges criterion puts $H$ for $H_y$ and $G$ for $G_x$ in the part of Property B+ after “implies”.

The question of whether every monotonically ultranormal space satisfies Property B+ was posed in [8] and is still open.

**Theorem 2.11.** Every tree with the coarse wedge topology has Property B+.

**Proof.** For each point $t$ and each open neighborhood $G$ of $t$, there exists $s \leq t$ for which there is a basic clopen set $W_{F,s}$ such that $t \subset W_{F,s} \subset G$, and for which $F \subset V_t$. [If $t$ is on a successor level we can let $s = t$, while if $t$ is on a limit level we first find some $s < t$ on a successor level and some finite $F' \subset V_s$ for which $t \subset W_{F',s}$; then let $F = F' \cap V_t$ and choose $s'$ such that $s \leq s' < t$ and all elements of $F' \setminus F$ are incomparable with $s$.]

Now for each $x \in F$ let $x'$ be the immediate successor of $t$ below $x$ and let $F^* = \{ x' : x \in F \}$.

**Claim.** Letting $G_t = W_{F,s}^{F^*}$ for all $t$, $G$ as above produces an assignment witnessing Property B+.

**Proof.** The notched wedges $W_{F,s}^{F^*}$ clearly have the property that the intersection of any two contains the minimum point of one of them. Let
\( G_x \cap H_y \neq \emptyset \). Assume that the minimum point \( t \) of \( G_x \) is in \( H_y \); in particular, \( t \geq s \). Let \( H_y = W^*_s \).

**Case 1:** \( y < t \). Then \( G_x \subset V_t \subset H_y \), because \( t \) is not in \( V_{z'} \) for any \( z' \in F^* \).

**Case 2:** \( y \) and \( t \) are incomparable. Then \( t > s \), and we again have \( G_x \subset V_t \subset H_y \).

**Case 3:** \( t \leq y \). Then if \( x \) and \( y \) are incomparable, we clearly have \( s < x \in H_y \). This also holds if \( x \leq y \). Finally, if \( x > y \), we must have \( y \in G_x \).

This proves the Claim and hence the theorem. ■

**Corollary 2.12.** Every Hausdorff tree is monotonically normal in the coarse wedge topology.

### 3. The main example

The following construction is utilized in the main example of this paper.

**Example 3.1.** For any tree \( T \), we call a tree a completion of \( T \) if it is formed by adding a supremum to each downwards closed chain that does not already have one. Formally, we define the completion \( \hat{T} \) of \( T \) as follows. If \( T \) is not rooted, we let \( \hat{T} \) be the collection of downwards closed chains (called “paths” by Todorčević), ordered by inclusion. If \( T \) is rooted, we only put the nonempty paths in \( \hat{T} \).

We identify each \( t \in T \) with the path \( P_t = \{ s \in T : s \leq t \} \). Completeness of \( \hat{T} \) follows from rootedness of \( \hat{T} \) and from the easy fact that the supremum of a chain \( C \) of \( \hat{T} \) is the same as the supremum of \( C \cap T \). In particular, if \( C \) is a path in \( \hat{T} \) then \( C \cap T \) is downwards closed in \( T \).

Todorčević called the set of characteristic functions of the paths of \( T \) the path space of \( T \) when it is endowed with the relative topology as a subspace of \( 2^T \) with the product topology. Gruenhage [2] showed that this topology is the coarse wedge topology of \( \hat{T} \).

Recall that a Suslin tree is an uncountable tree in which every chain and antichain is countable. Let us call a tree uniformly \( \omega \)-ary if every nonmaximal point has denumerably many immediate successors. (For instance, the tree in Example 2.4 is uniformly \( \omega \)-ary.)

As is well known, every Suslin tree has a subtree \( T \) in which every point has more than one successor at every level above it. Thus every point of \( T \) has denumerably many successors on the next limit level above it. And so, a uniformly \( \omega \)-ary Suslin tree results when we take the subtree \( S \) of all points on limit levels of \( T \).

**Theorem 3.2.** The completion \( \hat{S} \) of a uniformly \( \omega \)-ary Suslin tree \( S \) is an L-space in the coarse wedge topology.
Proof. Since \( \hat{S} \upharpoonright (\alpha + 1) \) is closed for all \( \alpha < \omega_1 \), we see that \( \hat{S} \) is not separable. In the proof that \( \hat{S} \) is hereditarily Lindelöf, uniform \( \omega \)-arity plays a key role: if the tree were finitary, every point on a successor level would be isolated.

We make use of the elementary fact that a space is hereditarily Lindelöf if (and only if) every open subspace is Lindelöf. Let \( W \) be an open subspace of \( \hat{S} \), and let \( W_0 \) be the set of points \( t \in W \) such that \( V_t \subset W \). If \( t \in W_0 \) is on a limit level, there is also \( s < t \) such that \( V_s \) is clopen and \( s \in W_0 \); see the first paragraph in the proof of Theorem 2.11, and note that here, \( F = \emptyset \). Let \( A = \{ a \in W_0 : a \) is minimal in \( W_0 \} \). Then \( W_0 \) is the disjoint union of the clopen wedges \( V_\alpha (a \in A) \), and \( A \) is countable by the Suslin property.

If \( x \in W \setminus W_0 \), then there is a basic clopen subset of \( W \) of the form \( W_t^F \) where \( F \neq \emptyset \) and \( F \subset V_x \); see the first paragraph in the proof of 2.11 again. There are no more than \( n \) immediate successors of \( x \) below some element of \( F \), and if \( s \) is one of the other immediate successors of \( x \), then \( V_s \subset V_x \setminus V_F \), so \( s \in W_0 \). But then \( s \in A \) also, since any \( V_z \) containing \( V_s \) properly must also contain \( x \), contradicting \( x \in W \setminus W_0 \). So \( W \setminus W_0 \) is countable, and we have countably many basic clopen sets whose union is \( W \). 

The following is now immediate from 2.8, 2.12, and 3.2.

**Corollary 3.3.** If there is a Suslin tree, then there is a Corson compact, monotonically normal L-space.

**REFERENCES**


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