COLLOQUIUM MATHEMATICUM

VOL. 141 2015 NO. 2

ON THE LINEAR DENJOY PROPERTY OF TWO-VARIABLE CONTINUOUS FUNCTIONS

BY

MIKLÓS LACZKOVICH and ÁKOS K. MATSZANGOSZ (Budapest)

Abstract. The classical Denjoy–Young–Saks theorem gives a relation, here termed the Denjoy property, between the Dini derivatives of an arbitrary one-variable function that holds almost everywhere. Concerning the possible generalizations to higher dimensions, A. S. Besicovitch proved the following: there exists a continuous function of two variables such that at each point of a set of positive measure there exist continuum many directions, in each of which one Dini derivative is infinite and the other three are zero, thus violating the bilateral Denjoy property.

Our aim is to show that for two-variable continuous functions it is possible that on a set of positive measure there exist directions in which even the one-sided Denjoy behaviour is violated. We construct continuous functions of two variables such that (i) both of its one-sided derivatives equal ∞ in continuum many directions on a set of positive measure, and (ii) all four directional Dini derivatives are finite and distinct in continuum many directions on a set of positive measure.

1. Introduction and main results. Let f be a real-valued function defined on $E \subseteq \mathbb{R}$. The function f has the *Denjoy property* at $x \in E$ if one of the following holds:

- The function is differentiable at x.
- $-\infty < D^+f(x) = D_-f(x) < \infty$ and $D^-f(x) = -D_+f(x) = \infty$.
- $-\infty < D^-f(x) = D_+f(x) < \infty$ and $D^+f(x) = -D_-f(x) = \infty$.
- $D^+f(x) = D^-f(x) = -D_-f(x) = -D_+f(x) = \infty$.

The Denjoy–Young–Saks theorem states that any real-valued one-variable function has the Denjoy property at a.e. point.

Let f be a real-valued function defined on $E \subseteq \mathbb{R}^2$. The *directional* (or linear) *Dini derivatives* of f at a point $x \in E$ in a direction $0 \le \eta < 2\pi$ are:

$$\partial^\eta f(x) := \limsup_{E\cap l \ni y \to x} \frac{f(y) - f(x)}{|y-x|},$$

$$\partial_\eta f(x) := \liminf_{E\cap l \ni y \to x} \frac{f(y) - f(x)}{|y-x|},$$

2010 *Mathematics Subject Classification*: Primary 26B05.

Key words and phrases: Denjoy–Young–Saks theorem, two-variable functions.

DOI: 10.4064/cm141-2-2

where l denotes the half-line $l(x,\eta)$ extending from the point x in direction η. If $\partial^\eta f(x) = \partial_\eta f(x)$ then we denote by $f'_\eta(x)$ this common value. We allow $f'_\eta(x) = \pm\infty$.

We denote by $\bar{l}(x,\eta)$ the line $l(x,\eta) \cup l(x,\eta+\pi)$. If the function f restricted to the line $\bar{l}(x,\eta)$ has the Denjoy property at $x \in E$ as a one-variable function, we say that f has the *linear* (or directional) *Denjoy property* at the point x in the direction η.

By a theorem of Ward [W], the linear Denjoy property does hold for two-variable Borel measurable functions at a.e. point in a.e. direction. This is false for Lebesgue measurable functions (see Davies [D2]). On the other hand, Besicovitch [B] showed that the Denjoy property may fail on a set of positive measure, even for continuous functions. In Besicovitch's example the Denjoy property fails by having

$$\partial^{\eta_x} f(x) = \infty, \qquad \partial_{\eta_x} f(x) = 0, \qquad f'_{\eta_x+\pi}(x) = 0$$

on a set of positive measure, where f is a continuous function. (In fact he gives continuum many such directions at each point.)

By the Denjoy–Young–Saks theorem, for one-variable functions, on a set of positive measure it cannot be the case that either any two Dini derivatives are finite and distinct, or a one-sided derivative exists and equals ∞ or $-\infty$. These are the only restrictions holding for one-sided Dini derivatives a.e.; all other cases are possible on a set of positive measure.

Our aim is to show that for two-variable functions, it is possible that on a set of positive measure there exist directions in which this one-sided behaviour is violated. In Besicovitch's example this is not the case. In the following we show that even for continuous functions, it is possible that (i) the one-sided derivative equals ∞ in suitable directions on a set of positive measure, and (ii) all four directional Dini derivatives are finite and distinct in suitable directions on a set of positive measure. Both violate the Denjoy behaviour even in the one-sided sense. More precisely, we prove the following.

Theorem 1.1. *There exists a continuous function $f : [0,1]^2 \to \mathbb{R}$ and a measurable set $M \subset [0,1]^2$ of positive measure such that for each point $x \in M$ there exists a set $F(x) \subset [0,\pi)$ of cardinality continuum such that $f'_\eta(x) = f'_{\eta+\pi}(x) = \infty$ for every $\eta \in F(x)$.*

Theorem 1.2. *There exists a continuous function $f : [0,1]^2 \to \mathbb{R}$ and a measurable set $M \subset [0,1]^2$ of positive measure such that at each point $x \in M$ there exists a set $F(x) \subset [0,\pi)$ of cardinality continuum such that for every $\eta \in F(x)$ the directional Dini derivatives $\partial^\eta f(x)$, $\partial_\eta f(x)$, $\partial^{\eta+\pi} f(x)$, $\partial_{\eta+\pi} f(x)$ are finite and distinct.*

By the theorem of Ward [W] mentioned above, for every Borel measurable function, at a.e. point, the set of non-Denjoy directions is of measure zero. Consequently, at a.e. $x \in M$ the sets $F(x)$ of Theorems 1.1 and 1.2 are of measure zero. It is still reasonable to ask whether they can be of second category. We claim that this is never the case.

For Theorem 1.1, this follows from the theorems on the contingent of subsets of $\mathbb{R}^3$ [S, Theorems 13.7 and 13.11]; we omit the details. For Theorem 1.2, we can argue as follows. If $f\colon G \to \mathbb{R}$ is a continuous function on an open set $G \subseteq \mathbb{R}^2$, then it is easy to check that at any $x \in G$, the set of directions

$$H(x) = \{\eta \in [0, 2\pi) : \partial^\eta f(x) < \infty,\ \partial_\eta f(x) > -\infty\}$$

is an F_σ set. Therefore if $H(x)$ is of second category, it has nonempty interior. Then, by [S, Theorem (14.2)], f is differentiable at a.e. point x for which $H(x)$ is of second category.

It follows from the theorem of Stepanoff that the function of Theorem 1.1 is not locally Lipschitz a.e. on M. Zajíček [Z] has recently constructed a related function, which is Lipschitz on $\mathbb{R}^2$, but at each point of a residual set there exists a direction in which the function is nondifferentiable.

We conclude with the following question. *Does there exist a continuous function $f\colon [0,1]^2 \to \mathbb{R}$ and a measurable set $M \subset [0,1]^2$ of positive measure with the property that for each point $x \in M$ there exists a direction η for which $f'_\eta(x)$ and $f'_{\eta+\pi}(x)$ exist, are finite and distinct?*

2. Proof of Theorem 1.1. By a theorem of R. O. Davies [D1, Theorem 5] there exists a set M of full measure in the plane, each point of which is linearly accessible by $2^{\aleph_0}$ many lines. This means that we can assign to every point $x \in M$ a set $L(x)$ of continuum many lines such that $l \cap M = \{x\}$ for every $l \in L(x)$. (Another construction is given in [F].)

First we show that there exists a compact set $K \subset M$ of positive measure such that the map $x \mapsto L(x)$ $(x \in K)$ can be chosen to be continuous with respect to an appropriate topology.

Let $K_0 \subset M$ be a compact set of positive measure. It is easy to check that for every $x \in K_0$ the set $H(x) = \{\eta \in [0, 2\pi] : \bar{l}(x, \eta) \cap K_0 = \{x\}\}$ is G_δ. Since it is of cardinality of the continuum by the choice of M, it follows that $H(x)$ contains a nonempty perfect set.

Let $\mathcal{K}$ denote the set of nonempty compact subsets of $[0, 2\pi]$ equipped with the Hausdorff metric. Then $\mathcal{K}$ is a compact metric space. It is easy to check that $\mathcal{P} = \{H \in \mathcal{K} : H \text{ is perfect}\}$ is a G_δ set in $\mathcal{K}$ (see [K, (4.31) Exercise, p. 27]). We claim that the set

$$B = \{(x, H) \in K_0 \times \mathcal{P} : \bar{l}(x, \eta) \cap K_0 = \{x\} \text{ for every } \eta \in H\}$$

is Borel in $K_0 \times \mathcal{K}$. Let $B(x, r)$ denote the open ball of centre $x \in \mathbb{R}^2$ and radius r, and set

$$B_n = \{(x,H) \in K_0 \times \mathcal{K} : (\bar{l}(x,\eta) \setminus B(x,1/n)) \cap K_0 = \emptyset \text{ for every } \eta \in H\}$$

for every n. It is easy to check that B_n is a relatively open subset of $K_0 \times \mathcal{K}$ for every $n = 1,2,\dots$. Since

$$B = (K_0 \times \mathcal{P}) \cap \bigcap_{n=1}^{\infty} B_n,$$

we see that B is a G_δ subset of $K_0 \times \mathcal{K}$.

As seen above, the section $\{H \colon (x,H) \in B\}$ is nonempty for every $x \in K_0$. Therefore, by the uniformization theorem of Jankov and von Neumann [K, (18.1) Theorem, p. 120], there exists a function $F \colon K_0 \to \mathcal{K}$ such that $(x, F(x)) \in B$ for every $x \in K_0$, and F is measurable with respect to the σ-algebra generated by the analytic subsets of K_0. Since all analytic sets are Lebesgue measurable and $\mathcal{K}$ is a separable metric space, it follows that F is Lebesgue measurable. Consequently, there exists a compact set $K \subset K_0$ of positive measure such that F, restricted to K, is continuous. It follows from the choice of F that the set

$$L = \bigcup \{\bar{l}(x,\eta) : x \in K,\ \eta \in F(x)\}$$

is closed.

Lemma 2.1. *There exists a strictly increasing continuous function $h \colon [0,\infty) \to [0,\infty)$ such that*

$$h(t) \le \operatorname{dist}((x_1 + t\cos\eta, x_2 + t\sin\eta), K) \tag{1}$$

for every $x = (x_1,x_2) \in K$, $\eta \in F(x)$ and $t \ge 0$.

Proof. For every $x \in K$, the set

$$L_t(x) = \bigcup \{\bar{l}(x,\eta) : \eta \in F(x)\} \setminus B(x,t)$$

is closed and disjoint from K for every $t > 0$ by the definition of $F(x)$. Therefore, $d_t(x) = \operatorname{dist}(L_t(x), K) > 0$ for every $t > 0$ and every $x \in K$. Since F is continuous, it follows that for every fixed $t > 0$, the map $x \mapsto d_t(x)$ is continuous on K. Then, by compactness, it has a positive minimum on K, which we denote by $m(t)$. Clearly, m is a positive and increasing function on $(0,\infty)$. Let h_n $(n = 1,2,\dots)$ be a strictly decreasing sequence of positive numbers such that $h_n \le m(1/(n+1))$ for every n, and let $h \colon [0,\infty) \to [0,h_1)$ be a strictly increasing continuous function such that $h(1/n) \le h_n$ for every n. Then $h(t) \le m(t)$ for every $t > 0$, and thus (1) holds. ∎

Let $u \colon [0,\infty) \to [0,\infty)$ be defined as follows: u equals the inverse of h on the interval $[0, h(1)]$, and $u(x) = 1$ for every $x \ge h(1)$. Then u is an increasing and continuous function on $[0,\infty)$, and so is $v = \sqrt{u}$.

We define $f(x) = v(\mathrm{dist}(x, K))$ for every $x \in \mathbb{R}^2$. Then f is continuous and $f(x) = 0$ for every $x \in K$.

Let $x = (x_1, x_2) \in K$ and $\eta \in F(x)$ be fixed. We prove that $f'_\eta(x) = f'_{\eta+\pi}(x) = \infty$. If $0 < t < 1$, then by (1) we have

$$f(x_1 + t\cos\eta, x_2 + t\sin\eta) \geq v(h(t)) = \sqrt{u(h(t))} = \sqrt{t}.$$

This implies $f'_\eta(x) = \infty$, and a similar argument proves $f'_{\eta+\pi}(x) = \infty$. ∎

3. Proof of Theorem 1.2. The following recursive construction bears similarities to Davies' construction of an accessible set of positive measure [D1]. See also [Cs] and [T].

We shall use the following notation and terminology.

By the *direction* of a nonhorizontal line ℓ we mean the angle, belonging to $(0, \pi)$, between the halfline $\ell \cap \{(x, y) \in \mathbb{R}^2 : y > 0\}$ and the positive x axis.

All parallelograms considered will have two sides parallel to the x axis; these are referred to as the *bases* of the parallelogram, and the other two sides are its *sides*. The direction determined by the sides of a parallelogram P is called the *direction* of the parallelogram and is denoted by $\theta(P)$. We say that P is *supported by* the parallelogram Q if the bases of P are contained in the respective bases of Q. Therefore, if P is supported by Q then the orthogonal projections of P and Q onto the y axis coincide, and $P \subset Q$.

For every parallelogram P we shall denote by $L(P)$ the union of all lines intersecting both bases of P. Then $L(P)$ is a closed subset of $\mathbb{R}^2$. Note that if $P \subset Q$ are parallelograms, then $L(P) \subset L(Q)$ if and only if every line intersecting both bases of P also intersects both bases of Q. In particular, if P is supported by Q, then $L(P) \subset L(Q)$.

The two-dimensional Lebesgue measure is denoted by m.

The triangle of vertices A, B, C is denoted by $\triangle_{ABC}$. The parallelogram of vertices A, B, C, D is denoted by $\square_{ABCD}$. We always list the vertices A, B, C, D counter-clockwise and in such a way that A is the lower left vertex. Therefore, the bases of the parallelogram are the segments AB and DC.

Let $P = \square_{ABCD}$ be a parallelogram. We divide the sides AD and BC into 10 equal subintervals by points $A = A_0, A_1, \ldots, A_{10} = D$ and $B = B_0, B_1, \ldots, B_{10} = C$. The parallelogram $\square_{A_i B_i B_{i+1} A_{i+1}}$ will be denoted by $P(i)$ $(i = 0, \ldots, 9)$.

We denote by D the set of finite sequences of digits $0, 1, \ldots, 9$. Then $D = \bigcup_{n=0}^{\infty} D_n$, where $D_0 = \{\emptyset\}$ and, for $n = 1, 2, \ldots,$

$$D_n = \{0, \ldots, 9\}^n = \{(a_1, \ldots, a_n) : a_i \in \{0, \ldots, 9\}\ (i = 1, \ldots, n)\}.$$

If $\sigma = (a_1, \ldots, a_n) \in D_n$ $(n \geq 1)$, then we set $d(\sigma) = 0.a_1 \ldots a_n$. We define

$d(\emptyset)=0$. For $\sigma\in D_n$ the rectangle $[0,1]\times[d(\sigma),d(\sigma)+10^{-n}]$ will be denoted by R_σ.

The set of finite sequences of 0's and 1's will be denoted by B. Thus $B=\bigcup_{n=0}^{\infty}B_n$, where $B_0=\{\emptyset\}$ and, for $n=1,2,\dots$,

$$B_n=\{0,1\}^n=\{(a_1,\dots,a_n): a_i\in\{0,1\}\ (i=1,\dots,n)\}.$$

The length of a finite sequence σ will be denoted by $|\sigma|$. If σ,τ are finite sequences, then $\sigma<\tau$ will denote that σ is a proper initial segment of τ. The concatenation of the sequence σ and the digit i is denoted by σi.

The proof of Theorem 1.2 consists of two parts. In the first part we construct a set M_0 such that to each point $x\in M_0$ given by coordinates $x=(x_1,x_2)$ we associate continuum many lines $\ell(x,c)$, indexed by infinite 0-1 sequences $c\in 2^\omega$. To each such c there is a sequence of nested parallelograms $P_n=P_{j_n}^{\sigma_n,\varepsilon_n}$ containing x, where $\varepsilon_n\in B_n$ is the nth initial segment of c and $\sigma_n\in D_n$ is the sequence of the first n decimal digits of x_2. The parallelograms are constructed in such a way that each such sequence c defines a different line $\ell(x,c)$ through x intersecting each P_n. Furthermore, we construct sets K^σ $(\sigma\in D)$, with the property that if $\sigma_{n+1}=\sigma_n i$, where $i\in\{0,9\}$, then $\ell(x,c)\cap K^{\sigma_n}\neq\emptyset$, and if $i\in\{1,\dots,8\}$, then $\ell(x,c)\cap K^{\sigma_n}=\emptyset$. The construction is described by Lemmas 3.2 and 3.3.

In the second part of the construction we define the function f using the sets listed above.

We shall construct the parallelograms $P_j^{\sigma,\varepsilon}$ by induction on $|\sigma|=|\varepsilon|$. In order to describe the induction step we shall need a simple lemma on invisible sets. We say that a set $A\subset\mathbb{R}^2$ is *invisible* if its orthogonal projection is of measure zero in almost every direction.

Lemma 3.1. *Let $P=\square_{ABCD}$ be a parallelogram, and let A',B' be inner points of the base AB. Then there exists a closed and invisible set $K(P)\subset\operatorname{int}P(2)$ such that its projection in the direction of P onto the base AB contains the segment $A'B'$.*

Proof. It is well-known that there exists a compact invisible set $K_0\subset[0,1]^2$ such that the orthogonal projection of K_0 onto the x axis equals $[0,1]$ (see [M, 18.12. Lemma, p. 261]). Taking an appropriate similar copy of K_0 we obtain a compact invisible set $K_1\subset\operatorname{int}P(2)$ such that the projection of K_1 in the direction of P onto the base AB is a closed segment. If K_1 is taken small enough, then we can take a finite number of translated copies of K_1 such that their union, K, satisfies the desired condition. ∎

Now we turn to describing one step of induction.

Lemma 3.2. *Let n be a nonnegative integer, and let $\sigma\in D_n$ be fixed. Suppose that $\mathcal{P}$ is a finite set of parallelograms such that every $P\in\mathcal{P}$ is supported by the rectangle R_σ. Then for every $P\in\mathcal{P}$, $i\in\{0,\dots,9\}$ and*

$\nu \in \{0,1\}$ there are finitely many parallelograms $P_j^{i,\nu}$ $(j = 1, \ldots, j(P,i,\nu))$ having the properties listed below, where

$$\Sigma = \{(P,i,\nu,j) : P \in \mathcal{P},\ i \in \{0,\ldots,9\},\ \nu \in \{0,1\},\ 1 \le j \le j(P,i,\nu)\}.$$

(i) *$P_j^{i,\nu}$ is supported by $P(i)$ $((P,i,\nu,j) \in \Sigma)$.*

(ii) *For every $P \in \mathcal{P}$ and $\nu \in \{0,1\}$ the parallelograms $P_j^{i,\nu}$ $(i \in \{0,\ldots,9\},\ j = 1,\ldots,j(P,i,\nu))$ are nonoverlapping.*

(iii) *$L(P_j^{i,\nu}) \subset L(P)$ $((P,i,\nu,j) \in \Sigma)$.*

(iv) *For every $P \in \mathcal{P}$ and $\nu \in \{0,1\}$, the area of the set*

$$P \setminus \bigcup \{P_j^{i,\nu} : i = 0,\ldots,9,\ j = 1,\ldots,j(P,i,\nu)\}$$

is less than $2^{-2n-3}m(P)$.

(v) *The set of directions of the lines contained by any of the sets $L(P_j^{i,0})$ $(P \in \mathcal{P},\ i = 0,\ldots,9,\ j = 1,\ldots,j(P,i,0))$ is disjoint from the set of directions of the lines contained by any of the sets $L(P_j^{i,1})$ $(P \in \mathcal{P},\ i = 0,\ldots,9,\ j = 1,\ldots,j(P,i,1))$.*

Furthermore, there exist open sets U, V such that

(vi) *$U \subset \operatorname{cl} U \subset V \subset (0,1) \times (d(\sigma 2), d(\sigma 3)) = \operatorname{int} R_{\sigma 2}$.*

(vii) *For every $(P,i,\nu,j) \in \Sigma$, if $i \in \{1,\ldots,8\}$, then $L(P_j^{i,\nu}) \cap V = \emptyset$.*

(viii) *For every $(P,i,\nu,j) \in \Sigma$, if $i \in \{0,9\}$ and $\ell \subset L(P_j^{i,\nu})$ is a line, then $\ell \cap U \neq \emptyset$.*

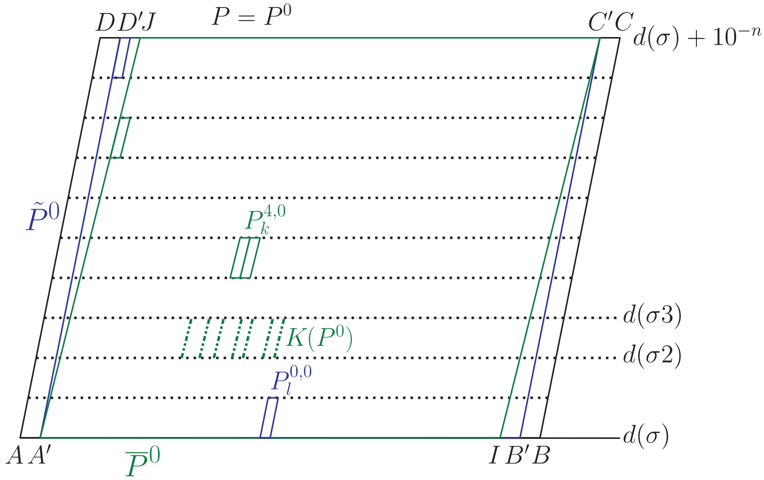

Fig. 1. The parallelograms of Lemma 3.2

Proof. For every $P \in \mathcal{P}$ we choose a parallelogram P^1 such that P^1 is supported by P, the area of $P \setminus P^1$ is less than $2^{-2n-9}m(P)$, and

$$\{\theta(P^1) : P \in \mathcal{P}\} \cap \{\theta(P) : P \in \mathcal{P}\} = \emptyset. \tag{2}$$

We set $P^0 = P$ for every $P \in \mathcal{P}$.

Let $P^\nu = \square_{ABCD}$, where $P \in \mathcal{P}$ and $\nu \in \{0,1\}$. We choose a parallelogram $\tilde{P^\nu} = \square_{A'B'C'D'}$ supported by P^ν such that $\square_{AA'D'D}$ and $\square_{B'BCC'}$ are nondegenerate parallelograms of area less than $2^{-2n-9}m(P)$. Applying Lemma 3.1 we find an invisible compact set $K(P^\nu) \subset \operatorname{int} P^\nu(2)$ whose projection in the direction of P^ν onto the base AB contains the segment $A'B'$. In this way we have defined the parallelograms $\tilde{P^\nu}$, and the sets $K(P^\nu)$ for every $P \in \mathcal{P}$ and $\nu = 0,1$. Note that since $\square_{AA'D'D}$ and $\square_{B'BCC'}$ are parallelograms, we have $\theta(\tilde{P^\nu}) = \theta(P^\nu)$ for every $P \in \mathcal{P}$ and $\nu \in \{0,1\}$. Therefore, by (2),

$$\{\theta(\tilde{P^0}) : P \in \mathcal{P}\} \cap \{\theta(\tilde{P^1}) : P \in \mathcal{P}\} = \emptyset. \tag{3}$$

We define

$$K = \bigcup \{K(P^0) \cup K(P^1) : P \in \mathcal{P}\}. \tag{4}$$

Then K is an invisible compact set, and

$$K \subset (0,1) \times (d(\sigma 2), d(\sigma 3)) = \operatorname{int} R_{\sigma 2}. \tag{5}$$

Let $P \in \mathcal{P}$ and $\nu \in \{0,1\}$ be fixed, and let $\tilde{P^\nu} = \square_{A'B'C'D'}$. Take a point I in the interior of the segment $A'B'$ close to B', and a point J in the interior of the segment $D'C'$ close to D' such that

- the segments $A'J$ and IC' are parallel to each other,
- the area of $\triangle_{A'JD'}$ and of $\triangle_{IB'C'}$ is less than $2^{-2n-9}m(P)$,
- the projection of K in the direction of $A'J$ and IC' onto the x axis is null.

The existence of I, J follows from the fact that K is invisible, and thus its projection in almost every direction is null. Furthermore, since $\mathcal{P}$ is a finite set and (2) holds, we can also achieve

(ix) the set of directions of the parallelograms $\tilde{P^0}$ and of the corresponding segments $A'J$ (for $P \in \mathcal{P}$) is disjoint from the set of directions of the parallelograms $\tilde{P^1}$ and of the corresponding segments $A'J$ (for $P \in \mathcal{P}$).

Again, let $\tilde{P^\nu} = \square_{A'B'C'D'}$, where $P \in \mathcal{P}$ and $\nu \in \{0,1\}$. Let θ denote the direction of the segments $A'J$ and IC'.

Let $\overline{P^\nu} = \square_{A'IC'J}$. Then $\overline{P^\nu}$ is supported by $\tilde{P^\nu}$, and $\theta(\overline{P^\nu}) = \theta$. Let $\pi(K)$ denote the projection of K in the direction of θ onto the x axis. Then $\pi(K)$ is a compact set of linear measure zero. Thus for every $\varepsilon > 0$ there are open segments $I_1, \dots, I_k \subset \mathbb{R}$ covering $\pi(K)$ of total length $< \varepsilon$. Let S be the union of all lines of direction θ and intersecting $I_1 \cup \dots \cup I_k$. Then S

is open and $K \subset S$. Choosing ε small enough we may assume that the area of $\overline{P^\nu} \cap S$ is less than $2^{-2n-9}m(P)$.

For every $i = 0, \dots, 9$, the components of the set $\overline{P^\nu}(i) \setminus S$ are parallelograms of direction θ such that if a line has direction θ and meets any of these parallelograms, then it is disjoint from K.

We choose a large integer N, and for each $i = 1, \dots, 8$ decompose each of the components of $\overline{P^\nu}(i) \setminus S$ into N congruent nonoverlapping parallelograms supported by $\overline{P^\nu}(i)$. We list the small parallelograms obtained by this construction as $P_j^{i,\nu}$ $(j = 1, \dots, j(P, i, \nu))$.

If N is chosen large enough then $L(P_j^{i,\nu}) \subset L(P)$ for every $P \in \mathcal{P}$, $i = 1, \dots, 8$, $\nu = 0, 1$ and $j = 1, \dots, j(P, i, \nu)$. Indeed, $P_j^{i,\nu}$ is a thin parallelogram of direction θ. Since $P_j^{i,\nu} \subset \overline{P}(i) \subset \tilde{P}(i) \subset \operatorname{int} P$, it is clear that if $P_j^{i,\nu}$ is thin enough and a line ℓ intersects both bases of $P_j^{i,\nu}$, then it intersects both bases of P.

It follows from (ix) that if N is chosen large enough, then the following will be true:

(x) The set of directions of the lines contained in any of the sets $L(P_j^{i,0})$ $(P \in \mathcal{P},\ i = 1, \dots, 8,\ j = 1, \dots, j(P, i, 0))$ and the directions of $\tilde{P^0}$ $(P \in \mathcal{P})$ is disjoint from the set of directions of the lines contained in any $L(P_j^{i,1})$ $(P \in \mathcal{P}, i = 1, \dots, 8,\ j = 1, \dots, j(P, i, 1))$ and the set of directions of $\tilde{P^1}$ $(P \in \mathcal{P})$.

Now we claim that if we choose N large enough, then for every $P \in \mathcal{P}$, $i = 1, \dots, 8$, $\nu = 0, 1$ and $j = 1, \dots, j(P, i, \nu)$ we have

$$L(P_j^{i,\nu}) \cap K = \emptyset. \tag{6}$$

Indeed, the parallelograms $P_j^{i,\nu}$ were obtained by cutting the components of $\overline{P^\nu}(i) \setminus S$ into thin slices of the same direction. From $K \subset S$ it follows that $\overline{P^\nu} \setminus S$ does not intersect K. Since K is compact, its distance from $\overline{P^\nu} \setminus S$ is positive. From this it is clear that if a line ℓ intersects $\overline{P^\nu}(i) \setminus S$ and its direction is close enough to the direction of $\overline{P^\nu}$ and S, then $\ell \cap K = \emptyset$. If the slices are thin enough, that is, if N is large enough, then the lines intersecting the bases of the slices will have directions close to that of $\overline{P^\nu}$, and so (6) will hold. We set

$$\begin{aligned} V = \Big(\bigcup\{\operatorname{int} P(2) : P \in \mathcal{P}\}\Big) \setminus \\ \bigcup\{L(P_j^{i,\nu}) : P \in \mathcal{P},\ i = 1, \dots, 8,\ \nu = 0, 1,\ j = 1, \dots, j(P, i, \nu)\}. \end{aligned} \tag{7}$$

Then V is open, and $K \subset V$ by (4) and (6). It follows from (7) that (vii) is satisfied.

Let U be an open set satisfying $K \subset U \subset \mathrm{cl}\, U \subset V$. Since $P(2) \subset R_{\sigma 2}$ for every $P \in \mathcal{P}$, we have (vi).

Now we define the parallelograms $P_j^{i,\nu}$ with $i \in \{0, 9\}$. Let $P \in \mathcal{P}$ and $\nu \in \{0, 1\}$ be fixed, and let $\tilde{P^\nu} = \square_{A'B'C'D'}$. Let $i \in \{0, 9\}$. We decompose $\tilde{P^\nu}(i)$ into M congruent nonoverlapping parallelograms supported by $P(i)$. We list the small parallelograms obtained by this construction as $P_j^{i,\nu}$ ($j = 1, \dots, j(P, i, \nu)$).

If M is chosen large enough, then (viii) will hold. Indeed, the set $K(P^\nu)$ was constructed in such a way that its projection in the direction of P^ν onto the base AB covers $A'B'$. Therefore, every line which meets $\tilde{P^\nu}$ and has the direction of $\tilde{P^\nu}$ meets K. Since K is compact and $U \supset K$ is open, it follows that U contains a δ-neighbourhood of K for a suitable $\delta > 0$. This implies that if a line ℓ meets $\tilde{P^\nu}$ and if the direction of ℓ is close enough to that of $\tilde{P}$, then ℓ meets U. If M is large enough, then all lines intersecting both bases of the thin slices obtained from $\tilde{P^\nu}(i)$ will have this property, and thus (viii) holds.

It is clear from the construction that (ii) holds.

One can easily check that choosing M large enough we also have (iii). For large enough M we also have (v) by (ix).

We still have to check property (iv). Let $P \in \mathcal{P}$ and $\nu \in \{0, 1\}$ be arbitrary. If $P^\nu = \square_{ABCD}$, then the set in (iv) is covered by $P \setminus P^1$, the parallelograms $\square_{AA'D'D}$, $\square_{B'BCC'}$, the triangles $\triangle_{A'JD'}$, $\triangle_{IB'C'}$, and the set $\overline{P^\nu} \setminus S$. Since each of these sets has area less than $2^{-2n-9} m(P)$, the area of the set in (iv) is less than $2^{-2n-3} m(P)$. This completes the proof of the lemma. ∎

We shall apply the construction of Lemma 3.2 inductively. We set $P_1^{\emptyset,\emptyset} = [0, 1]^2$ and $k(\emptyset, \emptyset) = 1$. Let $n \geq 0$, and suppose that the parallelograms $P_k^{\sigma,\varepsilon}$ ($k = 1, \dots, k(\sigma, \varepsilon)$) have been defined for every $\sigma \in D_n$ and $\varepsilon \in B_n$ such that $P_k^{\sigma,\varepsilon}$ is supported by R_σ for every $\sigma \in D_n$, $\varepsilon \in B_n$ and $k = 1, \dots, k(\sigma, \varepsilon)$. Let

$$\mathcal{P}_\sigma = \{P_k^{\sigma,\varepsilon} : \varepsilon \in B_n,\ k = 1, \dots, k(\sigma, \varepsilon)\}$$

for every $\sigma \in D_n$. Applying Lemma 3.2 with $\mathcal{P} = \mathcal{P}_\sigma$ we obtain the parallelograms

$$(P_k^{\sigma,\varepsilon})_j^{i,\nu} \tag{8}$$

and the open sets U^σ, V^σ. If $\sigma \in D_n$, $i \in \{0, \dots, 9\}$, $\varepsilon \in B_n$, $\nu \in \{0, 1\}$ are fixed, then we arrange the parallelograms in (8), when k and j run through the corresponding finite sets, in a single sequence $P_l^{\sigma i, \varepsilon\nu}$, $l = 1, \dots, k(\sigma i, \varepsilon\nu)$. Then, by Lemma 3.2(i), $P_l^{\sigma i,\varepsilon\nu}$ is supported by $R_{\sigma i}$ for every $\sigma \in D_n$, $i \in \{0, \dots, 9\}$, $\varepsilon \in B_n$, $\nu \in \{0, 1\}$ and $l = 1, \dots, k(\sigma i, \varepsilon\nu)$. By induction, this completes the construction of the open sets U^σ, V^σ and the parallelograms $P_k^{\sigma,\varepsilon}$ $(k = 1, \dots, k(\sigma,\varepsilon))$ for every $\sigma \in D$ and $\varepsilon \in B$ with $|\sigma| = |\varepsilon|$.

We introduce the notation

$$\Pi(\sigma,\varepsilon,k) = \{P_l^{\sigma i,\varepsilon\nu} : i \in \{0,\dots,9\},\ \nu \in \{0,1\},\ l = 1,\dots,k(\sigma i,\varepsilon\nu),\\ P_l^{\sigma i,\varepsilon\nu} \subset P_k^{\sigma,\varepsilon}\}$$

for all $\sigma \in D$, $\varepsilon \in B$ with $|\sigma| = |\varepsilon|$ and $k = 1,\dots,k(\sigma,\varepsilon)$. Set also

$$\Sigma_n = \{(\sigma,\varepsilon,k) : \sigma \in D_n,\ \varepsilon \in B_n,\ k = 1,\dots,k(\sigma,\varepsilon)\}$$

for all n.

LEMMA 3.3. *The parallelograms $P_k^{\sigma,\varepsilon}$ $(k = 1,\dots,k(\sigma,\varepsilon))$ and the open sets U^σ, V^σ constructed above have the following properties:*

(a) *$P_k^{\sigma,\varepsilon}$ is supported by R_σ for all $\sigma \in D$, $\varepsilon \in B$, $|\sigma| = |\varepsilon|$, $k = 1,\dots,k(\sigma,\varepsilon)$.*
(b) *For every $\varepsilon \in B_n$, the parallelograms $P_k^{\sigma,\varepsilon}$ $(\sigma \in D_n,\ k = 1,\dots,k(\sigma,\varepsilon))$ are nonoverlapping.*
(c) *$L(P) \subset L(P_k^{\sigma,\varepsilon})$ for every $P \in \Pi(\sigma,\varepsilon,k)$.*
(d) *The set*

$$M_0 := \bigcap_{n=0}^{\infty} \bigcap_{\varepsilon \in B_n} \bigcup \{\operatorname{int} P_k^{\sigma,\varepsilon} : \sigma \in D_n,\ 1 \le k \le k(\sigma,\varepsilon)\}$$

has positive measure.
(e) *For every $\sigma \in D_{n+1}$, $\varepsilon \in B_n$, $1 \le j \le k(\sigma,\varepsilon 0)$, $1 \le k \le k(\sigma,\varepsilon 1)$, the set of directions of the lines contained in $L(P_j^{\sigma,\varepsilon 0})$ is disjoint from the set of directions of the lines contained in $L(P_k^{\sigma,\varepsilon 1})$.*
(f) *$U^\sigma \subset \operatorname{cl} U^\sigma \subset V^\sigma \subset \operatorname{int} R_{\sigma 2}$ for every $\sigma \in D$.*
(g) *For every $\sigma \in D_n$, $\zeta \in B_{n+1}$ and for every $i \in \{1,\dots,8\}$ and $k = 1,\dots,k(\sigma i,\zeta)$ we have $L(P_k^{\sigma i,\zeta}) \cap V^\sigma = \emptyset$.*
(h) *For every $\sigma \in D_n$, $\zeta \in B_{n+1}$, $i \in \{0,9\}$, $k = 1,\dots,k(\sigma i,\zeta)$ and every line ℓ contained in $L(P_k^{\sigma i,\zeta})$ we have $\ell \cap U^\sigma \neq \emptyset$.*
(j) *The sets V^σ $(\sigma \in D)$ are pairwise disjoint.*
(k) *$M_0 \cap \operatorname{cl} V^\sigma = \emptyset$ for every $\sigma \in D$.*

Proof. Property (a) has already been checked.

Note that if we apply Lemma 3.2 to a set of parallelograms $\mathcal{P}$, and $\mathcal{Q} \subset \mathcal{P}$ is a set of nonoverlapping parallelograms, then for each $\nu \in \{0,1\}$, the parallelograms $P_k^{i,\nu}$ $((P,i,\nu,k) \in \Sigma,\ P \in \mathcal{Q})$ obtained from the lemma are nonoverlapping by property (ii). Now $\mathcal{P}_\emptyset = \{P_1^{\emptyset,\emptyset}\}$ consists of nonoverlapping parallelograms, so for each $\nu \in \{0,1\}$ the resulting $P_j^{i,\nu}$ parallelograms are nonoverlapping, and by induction (b) follows.

We claim that if $(\sigma,\varepsilon,k)\in\Sigma_n$ is given and $P\in\Pi(\sigma,\varepsilon,k)$, then P is obtained from $P_k^{\sigma,\varepsilon}$ by applying Lemma 3.2. Indeed, if P is obtained from another parallelogram $P_l^{\sigma,\varepsilon}$, then, by property (b), $\operatorname{int} P_k^{\sigma,\varepsilon}\cap\operatorname{int} P_l^{\sigma,\varepsilon}=\emptyset$. Hence, from $P\subset P_l^{\sigma,\varepsilon}$ we have $P\cap\operatorname{int} P_k^{\sigma,\varepsilon}=\emptyset$ and $P\notin\Pi(\sigma,\varepsilon,k)$, which is a contradiction. Thus property (iii) translates as (c).

By (iv), for every $(\sigma,\varepsilon,j)\in\Sigma_n$ and $\nu\in\{0,1\}$, the area of the set

$$P_j^{\sigma,\varepsilon}\setminus\bigcup\{P_l^{\sigma i,\varepsilon\nu}\in\Pi(\sigma,\varepsilon,j)\} \tag{9}$$

is less than $2^{-2n-3}m(P_j^{\sigma,\varepsilon})$. Let

$$A_\varepsilon=\bigcup_{\sigma\in D_n}\bigcup_{j=1}^{k(\sigma,\varepsilon)}P_j^{\sigma,\varepsilon}$$

for every $\varepsilon\in B_n$. Since M_0 and $\bigcap_{n=0}^\infty\bigcap_{\varepsilon\in B_n}A_\varepsilon$ only differ in a countable union of line segments (the boundaries of the parallelograms $P_j^{\sigma,\varepsilon}$), it is enough to show that $\bigcap_{n=0}^\infty\bigcap_{\varepsilon\in B_n}A_\varepsilon$ is of positive measure. We prove that

$$\bigcap_{\varepsilon\in B_n}A_\varepsilon\setminus\bigcap_{\xi\in B_{n+1}}A_\xi\subset\bigcup_{\varepsilon\in B_n}\bigcup_{\nu=0}^{1}(A_\varepsilon\setminus A_{\varepsilon\nu}) \tag{10}$$

for every n. Indeed, if x is in the left hand side of (10), then $x\notin\bigcap_{\xi\in B_{n+1}}A_\xi$, and thus there is a $\xi\in B_{n+1}$ such that $x\notin A_\xi$. Let $\xi=\varepsilon\nu$, where $\varepsilon\in B_n$ and $\nu\in\{0,1\}$. Then $x\in A_\varepsilon$, hence $x\in A_\varepsilon\setminus A_{\varepsilon\nu}$, which proves (10).

Let $\varepsilon\in B_n$ and $\nu\in\{0,1\}$ be fixed. It is clear from the definition of A_ε that $A_\varepsilon\setminus A_{\varepsilon\nu}$ is covered by the sets (9), where σ runs through D_n and $1\le j\le k(\sigma,\varepsilon)$. We have just seen that the area of the set in (9) is less than $2^{-2n-3}m(P_j^{\sigma,\varepsilon})$, and the total area of the parallelograms $P_j^{\sigma,\varepsilon}$ is at most 1 by (b), so the area of $A_\varepsilon\setminus A_{\varepsilon\nu}$ is at most 2^{-2n-3}.

Therefore, by (10), the area of the left hand side of (10) is at most $2^{n+1}\cdot 2^{-2n-3}=2^{-n-2}$. Since $A_\emptyset=P_0^{\emptyset,\emptyset}=[0,1]^2$, the measure of $\bigcap_{n=0}^\infty\bigcap_{\varepsilon\in B_n}A_\varepsilon$ is at least $1-\sum_{n=0}^\infty 2^{-n-2}=1/2$. This proves $m(M_0)\ge 1/2$.

Properties (v)–(viii) translate directly into (e)–(h).

We prove (j). Let $\sigma,\tau\in D$ be different sequences. If σ and τ are incompatible, that is, neither $\sigma<\tau$ nor $\tau<\sigma$, then the intervals $(d(\sigma),d(\sigma)+10^{-|\sigma|})$ and $(d(\tau),d(\tau)+10^{-|\tau|})$ are disjoint, and thus (f) yields

$$V^\sigma\cap V^\tau=\emptyset. \tag{11}$$

Suppose that $\sigma<\tau$. Then there is an $i\in\{0,\dots,9\}$ such that $\sigma i=\tau$ or $\sigma i<\tau$. Suppose $V^\sigma\cap V^\tau\ne\emptyset$, and let $x\in V^\sigma\cap V^\tau$. By the definition (7) of V the relation $x\in V^\tau$ implies that there are $\zeta\in B$, $|\zeta|=|\tau|$ and $1\le k\le k(\tau,\zeta)$ such that

$$x\in P_j^{\tau,\zeta}(2)\subset P_j^{\tau,\zeta}\subset P_k^{\sigma i,\varepsilon\nu}$$

with suitable $\varepsilon \in B$, $|\varepsilon| = |\sigma|$, $\nu \in \{0,1\}$ and $1 \le k \le k(\sigma i, \varepsilon\nu)$. If $i \ne 2$ then (f) yields (11), which is impossible. If $i = 2$ then, by the construction of V^σ (see (7)), we have $V^\sigma \cap L(P_k^{\sigma i,\varepsilon\nu}) = \emptyset$. Since $P_k^{\sigma i,\varepsilon\nu} \subset L(P_k^{\sigma i,\varepsilon\nu})$, we obtain $V^\sigma \cap P_k^{\sigma i,\varepsilon\nu} = \emptyset$, which is impossible. This proves (j).

Let $\sigma \in D_n$ and $x \in M_0 \cap \operatorname{cl} V^\sigma$. Fix an $\varepsilon \in B_n$. Then there are $\tau \in D_n$ and a $k = 1, \ldots, k(\tau, \varepsilon)$ such that $x \in \operatorname{int} P_k^{\tau,\varepsilon}$. Hence we must have $\tau = \sigma$ by (f). Also, $x \in \operatorname{int} P_k^{\xi,\varepsilon 0}$ with a suitable $\xi \in D_{n+1}$ and $1 \le k \le k(\xi, \varepsilon 0)$. Then $\xi = \sigma 2$ by (f). However, (7) implies $\operatorname{cl} V^\sigma \cap \operatorname{int} P_k^{\sigma 2,\varepsilon 0} = \emptyset$, which is a contradiction. This proves (k). ∎

Finally, we prove Theorem 1.1 using Lemma 3.3. We construct a continuous function $f : [0,1]^2 \to \mathbb{R}$ with $\operatorname{supp} f \subset \bigcup_{\sigma \in D} V^\sigma$. Using the parallelograms $P_k^{\sigma,\varepsilon}$ we shall construct the lines $\ell(x,c)$ for a.e. $x \in M_0$ and for every 0-1 sequence c. Then we shall use properties (g) and (h) to obtain bounds on the differential quotients of f along $\ell(x,c)$.

By Tietze's theorem, for every $\sigma \in D_n$ there exists a continuous function $f^\sigma : \mathbb{R}^2 \to [0, 10^{-n}]$ such that $f^\sigma(x) = 10^{-n}$ if $x \in \operatorname{cl} U^\sigma$, and $f^\sigma(x) = 0$ if $x \notin V^\sigma$. We put

$$f = \sum_{n=0}^{\infty} (-1)^n \sum_{\sigma \in D_n} f^\sigma.$$

Since f^σ is zero outside V^σ and the open sets V^σ ($\sigma \in D_n$) are pairwise disjoint, it follows that $f_n = \sum_{\sigma \in D_n} f^\sigma$ is continuous, and its range equals $[0, 10^{-n}]$. Therefore, the series $\sum_{n=0}^\infty (-1)^n f_n$ is uniformly convergent, and thus $f = \sum_{n=0}^\infty (-1)^n f_n$ is continuous on $\mathbb{R}^2$.

In the remaining part of the proof of Theorem 1.1 we show that at almost every point $x \in M_0$ there exist continuum many directions η such that the directional Dini derivatives $\partial^\eta f(x)$, $\partial_\eta f(x)$, $\partial^{\eta+\pi} f(x)$, $\partial_{\eta+\pi} f(x)$ are finite and distinct.

By property (k) ($M_0 \cap \overline{V^\sigma} = \emptyset$), we have $f(x) = 0$ for every $x \in M_0$.

Let 2^ω denote the set of infinite 0-1 sequences. We shall construct, for every $x \in M_0$ and $c \in 2^\omega$, a line $\ell(x,c)$ going through x. Let ε_n denote the nth initial segment of c. Then for every n there are a sequence $\sigma_n \in D_n$ and an index $1 \le j_n \le k(\sigma_n, \varepsilon_n)$ such that $x \in \operatorname{int} P_{j_n}^{\sigma_n,\varepsilon_n}$. This implies that

$$\operatorname{int} P_{j_n}^{\sigma_n,\varepsilon_n} \cap \operatorname{int} P_{j_{n+1}}^{\sigma_{n+1},\varepsilon_{n+1}} \ne \emptyset, \tag{12}$$

and thus by (a), σ_{n+1} is a continuation of σ_n for every n. Therefore, there is an infinite sequence $(a_1, a_2, \ldots)$ such that $\sigma_n = (a_1, \ldots, a_n)$ for every n. Let $x = (x_1, x_2)$. It is clear from (a) that $d(\sigma_n) < x_2 < d(\sigma_n) + 10^{-n}$ for every n. Therefore, $0.a_1 a_2 \ldots$ is the decimal expansion of x_2.

For brevity, we shall write P_n for $P_{j_n}^{\sigma_n,\varepsilon_n}$. Then, by (b) applied with $\varepsilon = \varepsilon_n$ and by (12), we have $P_{n+1} \in \Pi(\sigma_n, \varepsilon_n, j_n)$ for every n. Thus $L(P_{n+1}) \subset L(P_n)$ follows from (c) for every n.

We denote by $\ell_n(x,c)$ the line containing x and parallel to the sides of P_n. Then $\ell_n(x,c)$ intersects both bases of P_n, and so $\ell_n(x,c)\subset L(P_n)$. We select a line $\ell(x,c)$ so that it contains x and its slope is a point of accumulation of the sequence of the slopes of P_n $(n=0,1,\ldots)$. Then $\ell(x,c)\subset L(P_n)$ for every $n=0,1,\ldots$. In particular, $\ell(x,c)$ intersects both bases of $P_0=[0,1]^2$.

Let $c,d\in 2^\omega$ be distinct, and let ε_n and ζ_n denote their respective nth initial segments. Since $c\neq d$, there is an n and an $\varepsilon\in B_n$ such that $\varepsilon_{n+1}=\varepsilon 0$ and $\zeta_{n+1}=\varepsilon 1$ or the other way around. This implies by (e) that the directions of the lines $\ell(x,c)$ and $\ell(x,d)$ are different. This proves that for every $x\in M_0$, the lines $\ell(x,c)$ $(c\in 2^\omega)$ are distinct.

Fix $x\in M_0$ and $c\in 2^\omega$, and let θ denote the direction of the line $\ell(x,c)$. Our next aim is to prove that

$$-10\le \partial_\theta f(x)\le 0\le \partial^\theta f(x)\le 10, \tag{13}$$

$$-5/3\le \partial_{\theta+\pi} f(x)\le 0\le \partial^{\theta+\pi} f(x)\le 5/3. \tag{14}$$

Let $\ell^+(x,c)\subset\ell(x,c)$ and $\ell^-(x,c)\subset\ell(x,c)$ denote the halflines having x as endpoint, and intersecting the upper and lower base of $[0,1]^2$ (that is, $[0,1]\times\{1\}$ and $[0,1]\times\{0\}$), respectively. We have proved above that $x\notin \operatorname{cl} V^\sigma$ for every $\sigma\in D$. This implies that $\ell^+(x,c)\setminus\bigcup_{\sigma\in D}V^\sigma$ is nonempty. Moreover, it contains a sequence converging to x. Indeed, otherwise an initial open segment of $\ell^+(x,c)$ would be covered by the pairwise disjoint open sets V^σ. Since the segment is connected, it would be covered by one of the sets V^σ, implying $x\in\operatorname{cl} V^\sigma$, which is impossible.

Therefore, $f(x_i)=0$ for a suitable sequence of points $x_i\in\ell^+(x,c)$ converging to x, which implies $\partial_\theta f(x)\le 0\le\partial^\theta f(x)$. A similar argument gives $\partial_{\theta+\pi} f(x)\le 0\le \partial^{\theta+\pi} f(x)$.

Suppose that $f(y)\neq 0$ for some $y=(y_1,y_2)\in\ell^+(x,c)$. Then $y\in V^\tau$ for some $\tau\in D_n$, so by (f), we have

$$d(\tau 2)<y_2<d(\tau 3).$$

Since $y\in\ell^+(x,c)$, we have

$$d(\sigma_n)<x_2<y_2<d(\tau 3)<d(\tau)+10^{-n},$$

hence $d(\sigma_n)\le d(\tau)$.

Suppose $d(\sigma_n)=d(\tau)$; then $\tau=\sigma_n$. Since $\ell(x,c)\subset L(P_{n+1})$ and $y\in\ell(x,c)\cap V^\tau$, by (g) we have $a_{n+1}\in\{0,9\}$. However, since $x_2<y_2<d(\sigma_n 3)$, we must have $a_{n+1}=0$. Thus $x_2<d(\tau 1)<d(\tau 2)<y_2$ and $y_2-x_2>10^{-n-1}$.

If $d(\tau)>d(\sigma_n)$, then

$$x_2<d(\sigma_n)+10^{-n}\le d(\tau)<d(\tau 2)<y_2.$$

In both cases, $y_2-x_2>10^{-n-1}$. Since $y\in V^\tau$, we have $|f(y)|\le 10^{-n}$.

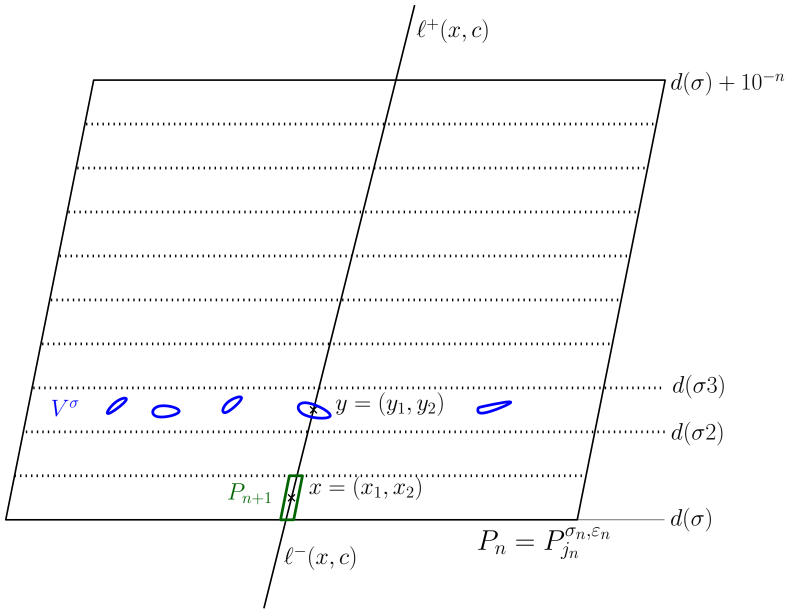

Fig. 2. Estimates on $|y-x|$

Therefore
$$\frac{|f(y)-f(x)|}{|y-x|} \le \frac{10^{-n}-0}{y_2-x_2} \le \frac{10^{-n}}{10^{-n-1}} = 10.$$
It follows that $\partial^\theta f(x) \le 10$ and $\partial_\theta f(x) \ge -10$.

Now suppose that $f(y) \ne 0$ for some $y=(y_1,y_2) \in \ell^-(x,c)$. Then $y \in V^\tau$ for some $\tau \in D_n$, so by (f),
$$d(\tau 2) < y_2 < d(\tau 3).$$
Since $y \in \ell^-(x,c)$, we have
$$d(\tau) < d(\tau 2) < y_2 < x_2 < d(\sigma_n) + 10^{-n},$$
so there are two cases: either $d(\tau) = d(\sigma_n)$, or $d(\tau) < d(\sigma_n)$.

Suppose $\tau = \sigma_n$. Since $\ell(x,c) \subset L(P_{n+1})$ and $y \in \ell(x,c) \cap V^\tau$, by (g) we have $a_{n+1} \in \{0,9\}$. However, as $d(\tau 2) < y_2 < x_2$, we must have $a_{n+1} = 9$ and
$$y_2 < d(\tau 3) < d(\tau 9) < x_2.$$
If $d(\tau) < d(\sigma_n)$, then
$$y_2 < d(\tau 3) < d(\tau) + 10^{-n} \le d(\sigma_n) < x_2.$$
In both cases, $x_2 - y_2 > 6 \cdot 10^{-n-1}$. Since $y \in V^\tau$, we have $|f(y)| \le 10^{-n}$. Therefore
$$\frac{|f(y)-f(x)|}{|y-x|} \le \frac{10^{-n}-0}{x_2-y_2} \le \frac{10^{-n}}{6\cdot 10^{-n-1}} = \frac{5}{3}.$$
It follows that $\partial^{\theta+\pi} f(x) \le 5/3$ and $\partial_{\theta+\pi} f(x) \ge -5/3$.

Let T denote the set of numbers $t \in [0,1]$ such that in the decimal expansion $0.a_1a_2\dots$ of t there are infinitely many odd as well infinitely many even indices n with $a_n = 0$, and there are infinitely many odd as well infinitely many even indices n with $a_n = 9$. Since a.e. number is normal to base 100, it follows that T is of full measure in $[0,1]$. We set $M = M_0 \cap ([0,1] \times T)$. Then M is a measurable set of positive measure. We prove that

$$-10 \le \partial_\theta f(x) \le -2, \qquad 2 \le \partial^\theta f(x) \le 10, \tag{15}$$

$$-5/3 \le \partial_{\theta+\pi} f(x) \le -5/6, \quad 5/6 \le \partial^{\theta+\pi} f(x) \le 5/3, \tag{16}$$

for every $x \in M$.

Let $x = (x_1, x_2) \in M$, and let the decimal expansion of x_2 be $0.a_1a_2\dots$. Then $x \in \operatorname{int} P_n = \operatorname{int} P_{j_n}^{\sigma_n,\varepsilon_n}$ for every n, where $\sigma_n = (a_1, \dots, a_n)$.

Suppose that $a_{n+1} = 0$ and n is even. Since $\sigma_{n+1} = \sigma_n 0$, it follows from (h) that if a line intersects both bases of P_{n+1}, then it intersects U^{σ_n}. Now $\ell(x,c)$ is such a line, and thus we can select $y \in \ell^+(x,c) \cap U^{\sigma_n}$. Then $y_2 - x_2 \le 3 \cdot 10^{-n-1}$, and thus

$$\frac{f(y)-f(x)}{|y-x|} = \frac{10^{-n}-0}{|y-x|} \ge \frac{10^{-n}}{\sqrt{2}\,(y_2-x_2)} \ge \frac{10^{-n}}{5 \cdot 10^{-n-1}} = 2.$$

Here we have used $|y-x| \le \sqrt{2}\,(y_2-x_2)$, which follows from the fact that $\ell(x,c)$ intersects both bases of the unit square $[0,1]^2$.

Since there are infinitely many even n with $a_{n+1} = 0$, this implies $\partial^\theta f(x) \ge 2$. The same argument shows $\partial_\theta f(x) \le -2$, which proves (15).

The proof of (16) is similar. Suppose that $a_{n+1} = 9$ and n is even. Since $\sigma_{n+1} = \sigma_n 9$, it follows from (h) that if a line intersects both bases of P_{n+1}, then it intersects U^{σ_n}. Now $\ell(x,c)$ is such a line, and thus we can select $y \in \ell^-(x,c) \cap U^{\sigma_n}$. Then $x_2 - y_2 \le 8 \cdot 10^{-n-1}$, and thus

$$\frac{f(y)-f(x)}{|y-x|} = \frac{10^{-n}-0}{|y-x|} \ge \frac{10^{-n}}{\sqrt{2}\,(x_2-y_2)} \ge \frac{10^{-n}}{12 \cdot 10^{-n-1}} = \frac{10}{12}.$$

Since there are infinitely many even n with $c_{n+1} = 9$, this implies $\partial^{\theta+\pi} f(x) \ge 5/6$. The same argument shows $\partial_{\theta+\pi} f(x) \le -5/6$. This proves (16). Now (15) and (16) imply that the four Dini derivatives $\partial^\theta f(x)$, $\partial_\theta f(x)$, $\partial^{\theta+\pi} f(x)$, $\partial_{\theta+\pi} f(x)$ are finite and distinct. This completes the proof of Theorem 1.2. ∎

Acknowledgements. This research was partly supported by the Hungarian National Foundation for Scientific Research, Grant No. K104178.

REFERENCES

[B] A. S. Besicovitch, *On differentiation of functions of two variables*, Math. Z. 41 (1936), 402–404.

[Cs] M. Csörnyei, *On the visibility of invisible sets*, Ann. Acad. Sci. Fenn. Math. 25 (2000), 417–421.

[D1] R. O. Davies, *On accessibility of plane sets and differentiation of functions of two real variables*, Math. Proc. Cambridge Philos. Soc. 48 (1952), 215–232.

[D2] R. O. Davies, *A note on linear derivates of measurable functions*, Math. Proc. Cambridge Philos. Soc. 52 (1956), 153–155.

[F] K. J. Falconer, *Sets with prescribed projections and Nikodym sets*, Proc. London Math. Soc. 53 (1986), 48–64.

[K] A. S. Kechris, *Classical Descriptive Set Theory*, Grad. Text in Math. 156, Springer, 1995.

[M] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces*, Cambridge Univ. Press, 1995.

[S] S. Saks, *Theory of the Integral*, Dover, 1964.

[T] M. Talagrand, *Sur la mesure de la projection d'un compact et certaines familles de cercles*, Bull. Sci. Math. 104 (1980), 225–231.

[W] A. J. Ward, *The linear derivates and approximate linear derivates of a function of two variables*, Proc. London Math. Soc. (2) 42 (1937), 266–273.

[Z] L. Zajíček, *A Lipschitz function which is* C^∞ *on a.e. line need not be generically differentiable*, Colloq. Math. 131 (2013), 29–39.

Miklós Laczkovich
Department of Analysis
Eötvös Loránd University
Budapest, Pázmány Péter sétány 1/C
1117 Hungary
E-mail: laczk@cs.elte.hu

Ákos K. Matszangosz
Department of Mathematics and its Applications
Central European University
Budapest, Nádor utca 9
1051 Hungary
E-mail: matszangosz.akos@gmail.com

Received 7 July 2014 (6312)