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ON A PROBLEM OF MAZUR FROM "THE SCOTTISH BOOK" CONCERNING SECOND PARTIAL DERIVATIVES

BY

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Abstract. We comment on a problem of Mazur from "the Scottish Book" concerning second partial derivatives. We prove that if a function $f(x,y)$ of real variables defined on a rectangle has continuous derivative with respect to y and for almost all y the function $F_y(x) := f'_y(x,y)$ has finite variation, then almost everywhere on the rectangle the partial derivative f''_{yx} exists. We construct a separately twice differentiable function whose partial derivative f'_x is discontinuous with respect to the second variable on a set of positive measure. This solves the Mazur problem in the negative.

1. Introduction. By Banach's classical result, for every (Lebesgue) measurable function $f : \mathbb{R} \to \mathbb{R}$ the set D of differentiability points of f is measurable and the derivative f' is measurable on D. Haslam-Jones [4] generalized this result to functions of several variables. More exactly, he established that the set D of differentiability points of a measurable function $f : \mathbb{R}^n \to \mathbb{R}$ is measurable and each of its partial derivatives f'_{x_i} is measurable on D.

Investigation of the existence and measurability of partial derivatives was continued in [13], [6]. In particular, in [13] it was proved that for a measurable function $f(x_1, \ldots, x_n)$, defined on a rectangle P, which is monotone with respect to the ith variable on almost all segments parallel to the ith axis, f'_{x_i} exists almost everywhere. In [6] it was proved that for a measurable function $f(x_1, \ldots, x_n)$ the set of points where f'_{x_i} exists is measurable, under a weaker assumption. Moreover, Serrin [13] has constructed a measurable function f on $[0,1]^2$ which is a.e. differentiable on each horizontal segment as a function of one variable, but for which the set where the partial derivative with respect to the first variable exists is non-measurable.

In the well known "Scottish Book" [8] S. Mazur posed the following question (VII.1935, Problem 66):

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The real function $z = f(x, y)$ of real variables x, y possesses the 1st partial derivatives f'_x, f'_y and the pure 2nd partial derivatives f''_{xx}, f''_{yy}. Do there exist then almost everywhere the mixed 2nd partial derivatives f''_{xy}, f''_{yx}? According to a remark by Mr. Schauder, this theorem is true with the following additional assumptions: The derivatives f'_x, f'_y are absolutely continuous in the sense of Tonelli, and the derivatives f''_{xx}, f''_{yy} are square integrable. An analogous question for n variables.

Mazur's problem has some interest for partial differential equations. If, for example, one considers the equation $f''_{xx} + f''_{yy} = g$, there is a natural question of differentiability properties of its solution (see e.g. [7]). The Mazur problem is a part of a general problem on relations between various partial derivatives in PDE and in the theory of function spaces connected to derivation (see. e.g. [11], [16]). Some results of these theories are formulated for classes of functions in Sobolev spaces, so it is not obvious that they remain valid for individual functions.

From the context of Problem 66 one can suppose that Mazur knew (suspected) that the mixed derivatives need not exist *everywhere*. It may be a surprise, but only in 1958 did Mityagin [9] publish an example which shows that the existence and continuity of the second pure derivatives does not imply the existence of mixed derivatives everywhere. More exactly, he provided a function $f(x, y)$ continuous in a disc with center at zero for which there are continuous derivatives f''_{xx} and f''_{yy}, but $f''_{xy}(0, 0)$ does not exist.

Bugrov [3] improved this result by showing that in a square there exists a harmonic function $f(x, y)$ (i.e. $f''_{xx} = -f''_{yy}$) whose mixed partial derivatives are unbounded. The existence of mixed partial derivatives was investigated by means of Fourier series in Bernstein's fundamental memoir [1]. He obtained the following results.

Theorem 1.1 ([1, Th. 79]). *Let $f(x, y)$ be a function 2π-periodic in both variables, whose partial derivatives $f^{(k)}_{x^k}$ and $f^{(k)}_{y^k}$ can be developed into double trigonometric Fourier series with the sum of the absolute values of the coefficients not greater than c. Then all mixed derivatives of f of order k exist, and are developed into double trigonometric Fourier series with the sums of the absolute values of the coefficients not greater than $2c$.*

Theorem 1.2 ([1, Th. 81]). *Let $f(x, y)$ be a function 2π-periodic in both variables, whose partial derivatives $f^{(k)}_{x^k}$ and $f^{(k)}_{y^k}$ satisfy the Hölder condition with exponent α. Then f has all mixed derivatives of order k, which satisfy the Hölder condition with any exponent $\alpha_1 < \alpha$.*

Theorem 1.3 ([1, Th. 80]). *Let $f(x, y)$ be a function 2π-periodic in both variables which has all second partial derivatives. Moreover suppose that*

$$\int_0^{2\pi}\int_0^{2\pi} (f''_{xx})^2\,dx\,dy \le c \quad \textit{and} \quad \int_0^{2\pi}\int_0^{2\pi} (f''_{yy})^2\,dx\,dy \le c.$$

Then

$$\int_0^{2\pi}\int_0^{2\pi} (f''_{xy})^2\,dx\,dy \le c. \text{ [1]}$$

On the other hand, in the framework of descriptive function theory Tolstov [14] has proved the following statements.

THEOREM 1.4. *If a function $f(x,y)$ is separately continuous on a rectangle and has the derivative f''_{xx} everywhere, then f''_{xx} is of the first Baire class.*

THEOREM 1.5. *Let $f(x,y)$ be defined on a rectangle P, and suppose the upper and lower partial derivatives of f'_x, f'_y with respect to each variable are finite on some subset $E \subseteq P$ of positive measure. Then the mixed derivatives f''_{xy} and f''_{yx} exist and are equal a.e. on E.*

Theorem 1.5 implies, in particular, that if $f(x,y)$ has a.e. all second partial derivatives then the mixed derivatives are equal a.e. Moreover, Tolstov [15] has constructed a function of two variables having jointly continuous first partial derivatives and mixed second partial derivatives which are different on a set of positive measure.

In this paper we show that Schauder's remark is valid under significantly weaker additional assumptions. More exactly, we prove that the Mazur problem has a positive answer if f'_x and f'_y have finite variations in the Tonelli sense. As a byproduct, we obtain a new result on measurability of the existence set for a partial derivative (Proposition 2.1). Finally, we solve the Mazur problem in the negative by constructing a separately twice differentiable function f such that f'_x is discontinuous with respect to y at all points of a set of positive measure.

2. Tonelli variation and mixed derivatives. Given a function $f : [a,b]\times[c,d] \to \mathbb{R}$ and $x \in [a,b]$, we denote by $V_1(x)$ the variation of the function $f^x : [c,d] \to \mathbb{R}$, $f^x(y) := f(x,y)$, and given $y \in [c,d]$ we denote by $V_2(y)$ the variation of $f_y : [a,b] \to \mathbb{R}$, $f_y(x) := f(x,y)$ (these variations may be infinite). Note that $V_1(x)$ is lower semicontinuous if f is continuous with respect to x, and similarly for $V_2(y)$. A function f is of *Tonelli bounded variation* [12, p. 169] if $\int_a^b V_1(x)\,dx < \infty$ and $\int_c^d V_2(y)\,dy < \infty$. All integrals we consider are Lebesgue integrals.

PROPOSITION 2.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be continuous with respect to y and let E be the set of all points $(x,y) \in \mathbb{R}^2$ at which f'_x exists. Then E is an $F_{\sigma\delta}$ set.*

[1] The original form of the theorem is different. However, an analysis of Bernstein's proof shows that, in fact, he proved Theorem 1.3.

Proof. Given $m, n \in \mathbb{N}$, denote by $A_{m,n}$ the set of all $(x, y) \in \mathbb{R}^2$ such that for all $u, v \in (x - 1/n,\ x + 1/n)$,

$$|f(u, y) - f(v, y)| \le 1/m,$$

and by $B_{m,n}$ the set of $(x, y) \in \mathbb{R}^2$ such that for all $u, u' \in (x,\ x + 1/n)$ and $v, v' \in (x - 1/n,\ x)$

$$\left| \frac{f(u, y) - f(v, y)}{u - v} - \frac{f(u', y) - f(v', y)}{u' - v'} \right| \le 1/m.$$

All the sets $A_{m,n}$, $B_{m,n}$ are closed, so the sets

$$A = \bigcap_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} A_{m,n} \quad \text{and} \quad B = \bigcap_{m\in\mathbb{N}} \bigcup_{n\in\mathbb{N}} B_{m,n}$$

are $F_{\sigma\delta}$. It remains to note that A is the set of all continuity points of f with respect to x, B is the set of all points (x, y) for which the limit

$$\lim_{(u,v)\to(x+0,x-0)} \frac{f(u, y) - f(v, y)}{u - v}$$

exists and is finite, and $E = A \cap B$. ∎

Note that Proposition 2.1 does not follow from the papers [13], [6] mentioned in the Introduction. Its statement seems to be new, even for functions of one variable (i.e. when f is constant with respect to y). Similar results for continuous functions of one variable can found in [5, p. 309] or [2, p. 228].

PROPOSITION 2.2. *Let $P = [a, b] \times [c, d]$, $f : P \to \mathbb{R}$ be continuous with respect to y and for almost all $y \in [c, d]$ the function $f_y(x) := f(x, y)$ has finite variation. Then f'_x exists a.e. on P.*

Proof. By Proposition 2.1, the set $E = \{(x, y) \in P : f'_x(x, y) \text{ exists}\}$ is measurable. Hence, $F = P \setminus E$ is also measurable.

By the assumptions of Proposition 2.2, we can choose a subset $A \subseteq [c, d]$ with Lebesgue measure $\mu(A) = d - c$ such that each f_y, $y \in A$, has finite variation on $[a, b]$. It is well known (see e.g. [10, Ch. VIII, §2, Th. 4]) that monotone functions (hence functions of finite variation) have derivative a.e. So, $\mu(F \cap ([a, b] \times \{y\})) = 0$ for each $y \in A$. Now, by the Fubini theorem (or by [10, Ch. XI, §5, Th. 1]), $\mu(F) = 0$. ∎

The next corollary shows that in Schauder's remark the assumption of square integrability of f''_{xx} and f''_{yy} is superfluous.

COROLLARY 2.3. *Let $P = [a, b] \times [c, d]$, let $f : P \to \mathbb{R}$ be continuously differentiable with respect to y, and suppose that for almost all $y \in [c, d]$ the function $F_y(x) := f'_y(x, y)$ has finite variation (e.g. let f'_y have finite Tonelli variation). Then f''_{yx} exists a.e. on P.*

Proof. Use Proposition 2.2 with f'_y in place of f. ∎

3. Example. For a real valued function f, denote $\operatorname{supp} f = \{x \in \mathbb{R} : f(x) \neq 0\}$.

LEMMA 3.1. *Let $I_n = (a_n, b_n)$ be pairwise disjoint intervals and let $\psi_n : \mathbb{R} \to \mathbb{R}$ be differentiable functions such that $\operatorname{supp} \psi_n \subset I_n$, $n = 1, 2, \ldots$, and $\sup_{\mathbb{R}} \psi'_n(x) \to 0$ as $n \to \infty$. Then the function $g(x) = \sum_{n=1}^{\infty} \psi_n(x)$ is differentiable, and*

$$g'(x) = \sum_{n=1}^{\infty} \psi'_n(x).$$

The lemma follows easily from the theorem on series differentiability.

The next theorem gives a negative answer to Mazur's problem.

THEOREM 3.2. *There exists a twice separately differentiable function $f : [0,1]^2 \to \mathbb{R}$ and a measurable subset $E \subset [0,1]^2$ with $\mu(E) > 0$ such that f'_x is discontinuous with respect to y at all points of E, in particular f''_{xy} does not exist on E.*

Proof. Let $B \subset [0,1]$ be a closed set of positive measure without isolated points whose complement $[0,1] \setminus B$ is dense in $[0,1]$. Take intervals $I_n = (a_n, b_n)$ such that $[0,1] \setminus B = \bigsqcup_{n=1}^{\infty} I_n$. Let $\psi : \mathbb{R} \to \mathbb{R}^+$ be an arbitrary twice continuously differentiable function with $\operatorname{supp} \psi = (0,1)$ and

$$\psi_n(y) := \psi\left(\frac{y - a_n}{b_n - a_n}\right), \qquad n = 1, 2, \ldots$$

Take $\varepsilon_n, \delta_n > 0$ so that

$$\lim_{n\to\infty} \frac{\varepsilon_n}{(b_n - a_n)^2} = 0, \tag{3.1}$$

$$\sum_{n=1}^{\infty} \delta_n < \infty. \tag{3.2}$$

Choose twice differentiable functions $\varphi_n : [0,1] \to [0, \varepsilon_n]$ such that

$$\mu(A_n) > 1 - \delta_n, \qquad n = 1, 2, \ldots, \tag{3.3}$$

where $A_n = \{x \in [0,1] : |\varphi'_n(x)| \geq 1\}$.

Let $f : [0,1]^2 \to \mathbb{R}$ be given by

$$f(x,y) = \sum\nolimits_{n=1}^{\infty} \varphi_n(x)\psi_n(y).$$

It is easy to see that f''_{xx} exists. Moreover, by (3.1) and Lemma 3.1, f''_{yy} exists. Set

$$A = \bigcup_{m=1}^{\infty} \bigcap_{n \geq m} A_n.$$

Then, by (3.3) and (3.2), $\mu(A) = 1$. We will show that f'_x is discontinuous with respect to y at every point of $E = A \times B$. Fix $(x_0, y_0) \in E$ and $\delta > 0$. Choose m so that $x_0 \in \bigcap_{n\geq m} A_n$. Since B has no isolated points, there exists $k > m$ such that $|y - y_0| < \delta$ for all $y \in I_k$. Take $y_k \in I_k$ so that $\psi_k(y_k) = \max_{\mathbb{R}} \psi(y)$. Now we have $|y_k - y_0| < \delta$, $x_0 \in A_k$ and

$$|f'_x(x_0, y_k) - f'_x(x_0, y_0)| = |\varphi'_k(x_0)|\psi_k(y_k) \geq \max_{\mathbb{R}} \psi(y). \blacksquare$$

Note that in the above example the partial derivative f''_{yy} is bounded and f''_{xx} is not absolutely integrable.

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