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ON THE DIOPHANTINE EQUATION $x^2 + x + 1 = yz$

BY

A. SCHINZEL (Warszawa)

Abstract. All solutions of the equation $x^2 + x + 1 = yz$ in non-negative integers x, y, z are given in terms of an arithmetic continued fraction.

As announced in [2], the following theorem holds.

THEOREM. *All solutions of the equation*

$$x^2 + x + 1 = yz \tag{1}$$

in non-negative integers x, y, z such that $y \le z$ and only those are given by the formulae

$$x = A_{k-1}A_k + B_{k-1}B_k + A_kB_{k-1}, \tag{2}$$

$$y = A_{k-1}^2 + A_{k-1}B_{k-1} + B_{k-1}^2, \tag{3}$$

$$z = A_k^2 + A_kB_k + B_k^2, \tag{4}$$

where A_k, B_k are the numerator and the denominator, respectively, of the continued fraction $[b_0, b_1, \dots, b_k]$, $k \ge 0$ is even, b_0 is an integer, and b_i $(i = 1, \dots, k)$ are positive integers, except if $k = 0$, $b_0 < 0$, when one has to take $x = |b_0| - 1$.

LEMMA 1. *For $k = 0$ the formulae* (2)–(4) *give*

$$x = b_0, \quad y = 1, \quad z = b_0^2 + b_0 + 1.$$

Proof. Clear. ∎

LEMMA 2. *If $k \ge 2$ is even, b_0 is an integer, and b_i $(i = 1, \dots, k)$ are positive integers, then*

$$A_{k-1}A_k + B_{k-1}B_k + A_kB_{k-1} \ge 0.$$

Proof. If $k = 2$, $b_1 = 1$, we have

$$A_1 = b_0 + 1, \quad B_1 = 1, \quad A_2 = b_2(b_0 + 1) + b_0, \quad B_2 = b_2 + 1,$$

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hence by (2),

$$\begin{aligned} x &= (b_2+1)b_0^2 + (3b_2+2)b_0 + 3b_2 + 1 \\ &\ge \frac{4(b_2+1)(3b_2+1) - (3b_2+2)^2}{4} = \frac{3b_2^2+4b_2}{4} > 0. \end{aligned}$$

If $k \ge 2$, $kb_1 > 2$, then $B_{k-1} \ge 2$, $B_k \ge 3$, and taking

$$\xi_0 = b_0 + \frac{1|}{|b_1} + \dots + \frac{1|}{|b_k}, \tag{5}$$

we obtain by [3, Chapter II, Theorem 10],

$$A_{k-1} = B_{k-1}\xi_0 + \rho, \quad |\rho| < \frac{1}{B_{k-1}}, \quad A_k = B_k\xi_0, \tag{6}$$

hence by (2),

$$x = B_{k-1}B_k(\xi_0^2 + \xi_0 + 1) + B_k\rho \ge \frac{3}{2}B_k - \frac{B_k}{2} = B_k \ge 3. \ \blacksquare$$

LEMMA 3. *If $k \ge 1$ and b_0 are integers, and b_i $(i = 1, \dots, k)$ are positive integers, then*

$$A_{k-1}^2 + A_{k-1}B_{k-1} + B_{k-1}^2 \ge A_k^2 + A_kB_k + B_k^2 \tag{7}$$

implies $kb_1 = 1$ and $b_0 < 0$.

Proof. If $kb_1 = 1$, we have

$$\begin{aligned} A_{k-1}^2 + A_{k-1}B_{k-1} + B_{k-1}^2 &= b_0^2 + b_0 + 1, \\ A_k^2 + A_kB_k + B_k^2 &= b_0^2 + 3b_0 + 3, \end{aligned}$$

and thus (7) implies $2b_0 + 2 \le 0$, i.e. $b_0 < 0$.

If $kb_1 > 1$ and (5) holds, then by [3, Chapter II, Theorem 10] we have (6), hence

$$\begin{aligned} A_{k-1}^2 + A_{k-1}B_{k-1} + B_{k-1}^2 &= B_{k-1}^2(\xi_0^2 + \xi_0 + 1) + B_{k-1}(2\xi_0\rho + \rho) + \rho^2 \\ &< B_{k-1}^2(\xi_0^2 + \xi_0 + 1) + 2\xi_0 + 2, \\ A_k^2 + A_kB_k + B_k^2 &= B_k^2(\xi_0^2 + \xi_0 + 1), \\ A_k^2 + A_kB_k + B_k^2 - A_{k-1}^2 &- A_{k-1}B_{k-1} - B_{k-1}^2 \\ &\ge (B_k^2 - B_{k-1}^2)(\xi_0^2 + \xi_0 + 1) - 2\xi_0 - 2 \\ &\ge 3(\xi_0^2 + \xi_0 + 1) - 2\xi_0 - 2 \ge 3\xi_0^2 + \xi_0 + 1 \ge 11/4. \ \blacksquare \end{aligned}$$

Proof of the Theorem. Necessity. Suppose that (1) holds, where x, y, z are non-negative integers and $y \le z$. Then by [1, Theorem 131] applied with $a = b = c = 1$ we have $G = 1$, $c = f = g = 1$, and there exist integers

u, v, ξ, η such that

$$1 = v\xi - u\eta, \tag{8}$$

$$\text{either } x = \xi u + \eta v + \eta u, \text{ or } x = -\xi u - \eta v - \xi v, \tag{9}$$

$$y = \xi^2 + \xi\eta + \eta^2, \tag{10}$$

$$z = u^2 + uv + v^2. \tag{11}$$

Let $\delta = \operatorname{sgn} v$, $\varepsilon = \operatorname{sgn} \eta$. If $\delta = 0$, from (8) we obtain $u = \pm 1$, $\eta = \pm 1$, ξ arbitrary, thus by (9)–(11),

$$\text{either } x = \pm\xi - 1, \text{ or } x = \mp\xi;$$
$$y = \xi^2 \mp \xi + 1, \qquad z = 1,$$

and since $x \geq 0$, either $x = |\xi| - 1$ ($|\xi| \geq 1$), $y = \xi^2 - |\xi| + 1$, $z = 1$, or $x = |\xi|$, $y = \xi^2 + |\xi| + 1$, $z = 1$. It follows that $y > z$ unless $\xi = 0, \pm 1$, $x = 0$, $y = z = 1$. These values are obtained, by Lemma 1, from formulae (2)–(4) for $k = 0$, $b_0 = 0$.

If $\varepsilon = 0$, we deduce from (8) that $\delta = \pm 1$, $\xi = \pm 1$, u is arbitrary, thus, by (9)–(11),

$$\text{either } x = \pm u, \text{ or } x = \mp u - 1;$$
$$y = 1, \qquad z = u^2 \pm u + 1,$$

and since $x \geq 0$, either $x = |u|$, $y = 1$, $z = u^2 + |u| + 1$, or $x = |u| - 1$ ($|u| \geq 1$), $y = 1$, $z = u^2 - |u| + 1$. These values are obtained, by Lemma 1, from formulae (2)–(4) for $k = 0$, $b_0 = |u|$ or $|u| - 1$, respectively.

If $\varepsilon\delta \neq 0$, from (8) we obtain

$$|v|(\varepsilon\xi) - (\delta u)|\eta| = \varepsilon\delta = \pm 1. \tag{12}$$

If $|v| = |\eta|$, then $|v| = |\eta| = 1$, $v = \delta$, $\eta = \varepsilon$, and since $x \geq 0$ we obtain

$$\text{for } \varepsilon\delta = 1, \quad x = (u+\delta)^2, \quad y = u^2 + 3\delta + 3, \quad z = u^2 + \delta u + 1,$$
$$\text{for } \varepsilon\delta = -1, \quad x = u^2, \quad y = u^2 - \delta u + 1, \quad z = u^2 + \delta u + 1.$$

Since $y \leq z$, for $\varepsilon\delta = 1$ we obtain $\delta u < 0$; for $\varepsilon\delta = -1$, $\delta u \geq 0$. These values of x, y, z are obtained from formulae (2)–(4) for $k = 2$, $b_0 = -1$, $b_1 = |u + \delta|$ or $|u|$, respectively, $b_2 = 1$.

If $|v| \neq |\eta|$, then $|v| < |\eta|$ or $|v| > |\eta|$. In the former case by [3, Chapter II, Theorem 13] there exist integers $k \geq 1$ and b_0 and positive integers b_i ($i = 1, \dots, k$) such that $\delta u = A_{k-1}$, $|v| = B_{k-1}$, $\varepsilon\xi = A_k$, $|\eta| = B_k$, hence $u = \delta A_{k-1}$, $v = \delta B_{k-1}$, $\xi = \varepsilon A_k$, $\eta = \varepsilon B_k$, and by (8),

$$\varepsilon\delta = (-1)^{k-1}. \tag{13}$$

Thus either $x = \varepsilon\delta(A_{k-1}A_k + B_{k-1}B_k + A_k B_{k-1})$, or $x = -\varepsilon\delta(A_{k-1}A_k + B_{k-1}B_k + A_k B_{k-1})$,

$$z = A_{k-1}^2 + A_{k-1}B_{k-1} + B_{k-1}^2, \quad y = A_k^2 + A_k B_k + B_k^2.$$

Since $y \le z$, by Lemma 3 we have $kb_1 = 1$ and $b_0 < 0$, hence by (13), $\varepsilon\delta = 1$. It follows that $x = (b_0+1)^2$, $y = b_0^2+3b_0+3$, $z = b_0^2+b_0+1$. These values of x, y, z are obtained from formulae (2)–(4) for $b'_0 = -1$, $b'_1 = |b_0+1|$, $b'_2 = 1$.

If $|v| > |\eta| > 0$, by [3, Chapter II, Theorem 13] there exist integers $k \ge 1$ and b_0 and positive integers b_i $(i = 1, \ldots, k)$ such that

$$\delta u = A_k, \quad |v| = B_k, \quad \varepsilon\xi = A_{k-1}, \quad |\eta| = B_{k-1},$$

hence

$$u = \delta A_k, \quad v = \delta B_k, \quad \xi = \varepsilon A_{k-1}, \quad \eta = \varepsilon B_{k-1},$$

and by (8),

$$\varepsilon\delta = (-1)^k.$$

If k is even, then

$$\begin{gathered}\text{either} \quad x = \varepsilon\delta(A_{k-1}A_k + B_{k-1}B_k + A_kB_{k-1}) \quad \text{or} \\ x = -\varepsilon\delta(A_{k-1}A_k + B_{k-1}B_k + A_{k-1}B_k), \\ y = A_{k-1}^2 + A_{k-1}B_{k-1} + B_{k-1}^2, \quad z = A_k^2 + A_kB_k + B_k^2,\end{gathered}$$

thus, by Lemma 2, we obtain formulae (2)–(4).

If k is odd, $\varepsilon\delta = -1$, $\xi = 0$, then $k = 1$, $b_0 = 0$; $u = \delta$, $v = \delta b_1$, $\eta = -\varepsilon\delta$; $x = b_1$, $y = 1$, $z = b_1^2 + b_1 + 1$. These values of x, y, z are obtained, by Lemma 1, from formulae (2)–(4) for $k' = 0$, $b'_0 = b_1$.

If k is odd, $\varepsilon\delta = -1$, $u = 0$, then $k = 1$, $b_0 = -1$, $b_1 = 1$; $v = \delta$, $\xi = \delta$, $\eta = -\delta$; $x = 0$, $y = z = 1$. These values of x, y, z are obtained, by Lemma 1, from formulae (2)–(4) for $k' = 0$, $b'_0 = 0$.

If k is odd, $\varepsilon\delta = -1$, $|u| = |\xi|$, then $A_{k-1} = A_k$, $b_0 = -1$, $b_1 = 2$; $u = \xi = \delta$, $v = \eta = -\delta$; $x = 1$, $y = 1$, $z = 3$. These values of x, y, z are obtained, by Lemma 1, from formulae (2)–(4) for $k' = 0$, $b'_0 = 1$.

If k is odd, $\varepsilon\delta = -1$, then $|u| < |\xi|$ is impossible since $|v| > |\eta| > 0$, $v\xi - \eta u = 1$; hence $|u| > |\xi| > 0$, $\operatorname{sgn} A_k = \operatorname{sgn} A_{k-1} = a = \pm 1$, and we apply [3, Chapter II, Theorem 13] to the equation

$$|u|(-\delta a\eta) - |\xi|(-a\varepsilon v) = 1.$$

We infer the existence of integers $l \ge 1$ and b'_0 and positive integers b'_i $(i = 1, \ldots, l)$ such that

$$|u| = B_l, \quad -\varepsilon av = A_l, \quad |\xi| = B_{l-1}, \quad -\delta a\eta = A_{l-1},$$

thus $(-1)^l = 1$, l even.

On the other hand,

$$u = \delta aB_l, \quad v = -\varepsilon avA_l, \quad \xi = \varepsilon aB_{l-1}, \quad \eta = -\delta aA_{l-1},$$

hence using $\varepsilon\delta = -1$, we get

$$x = -B_{l-1}B_l - A_{l-1}A_l - A_{l-1}B_l \quad \text{or} \quad x = B_{l-1}B_l + A_{l-1}A_l + B_{l-1}A_l,$$
$$y = A_{l-1}^2 + A_{l-1}B_{l-1} + B_{l-1}^2, \quad z = A_l^2 + A_lB_l + B_l^2.$$

Since $x \geq 0$ we have

$$x = A_{l-1}A_l + B_{l-1}B_l + B_{l-1}A_l.$$

Indeed, by Lemma 2,

$$B_{l-1}B_l + A_{l-1}A_l + A_{l-1}B_l = A_{l-1}A_l + B_{l-1}B_l + A_lB_{l-1} + 1 > 0.$$

The proof is complete.

Sufficiency. It has been proved in [2] that formulae (2)–(4) give for every even k and b_i ($i = 0, \dots, k$) solutions of equation (1). The same follows from the following identity, given by the referee:

$$(a^2+ab+b^2)(A^2+AB+B^2) - \big((aA+bB+bA)^2 + (aA+bB+bA)+1\big) = (aB-Ab-1)(aA+aB+bB+1)$$

and from the well known formula $A_{k-1}B_k - A_kB_{k-1} = (-1)^k$. The inequalities $x \geq 0$ and $0 \leq y \leq z$ follow from Lemmas 2 and 3. ∎

It is not enough to assume, as I originally conjectured, that $b_i \geq 0$ ($i = 0, \dots, k$). Take the example $x = 67$, $y = 49$, $z = 93$. If all $b_i \geq 0$, then all $A_i \geq 0$ and all $B_i \geq 0$, hence if (2)–(4) hold, then $A_{k-1} = 3$, $B_{k-1} = 5$, $A_k = 4$, $B_k = 7$. If all $b_i = 0$, then

$$A_i = \frac{1-(-1)^i}{2}, \qquad B_i = \frac{1+(-1)^i}{2},$$

which is impossible. Let j be the greatest $i \leq k$ such that $b_i > 0$. Then for all $i \geq j-1$,

$$A_i = \begin{cases} 3 & \text{for } i \text{ odd}, \\ 4 & \text{for } i \text{ even}, \end{cases} \qquad B_i = \begin{cases} 5 & \text{for } i \text{ odd}, \\ 7 & \text{for } i \text{ even}. \end{cases}$$

If j is odd, then $A_j < A_{j-1}$, which contradicts $A_j = b_jA_{j-1} + A_{j-2}$. If j is even, then $b_j = 1$, $A_{j-2} = 1$, $B_{j-2} = 3$, $j \geq 4$, and since $A_{-1} = 1$ we have

$$\begin{aligned} \text{either} \quad & b_0 = 1, \quad b_i = 0 \quad (0 < i \leq j-2), \\ \text{or} \quad & b_i = 0 \quad (0 \leq i < j-2), \quad b_{j-2} = 1. \end{aligned}$$

These cases give $B_i = (1+(-1)^i)/2$ $(0 \leq i \leq j-2)$ or $B_i = 0$ $(0 \leq i \leq j-2)$, respectively, which contradicts $B_{j-2} = 3$.

In a similar way one may find all integral solutions of the equation $ax^2 + bx + c = yz$, where a, b, c are given integers such that $b^2 - 4ac$ is not a perfect square. The resulting formulae for x, y, z will be in general more complicated than those given in the Theorem.

REFERENCES

[1] L. E. Dickson, *Modern Elementary Theory of Numbers*, Univ. of Chicago Press, Chicago, 1939.

[2] J. Florek, *Equations relating factors in decomposition into factors of some family of plane triangulations, and applications* (with an appendix by A. Schinzel), Colloq. Math. 138 (2015), 23–42.

[3] O. Perron, *Die Lehre von den Kettenbrüchen*, 2nd ed., Teubner, 1927; reprint Chelsea, New York, 1950.

A. Schinzel
Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-656 Warszawa, Poland
E-mail: schinzel@impan.pl

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