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INCIDENCE COALGEBRAS OF INTERVAL FINITE POSETS OF TAME COMODULE TYPE

BY

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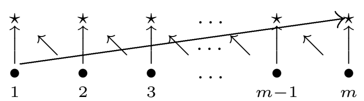
Dedicated to Andrzej Skowroński on the occasion of his 65th birthday

Abstract. The incidence coalgebras $K^{\square}I$ of interval finite posets I and their comodules are studied by means of the reduced Euler integral quadratic form $q^{\bullet} : \mathbb{Z}^{(I)} \to \mathbb{Z}$, where K is an algebraically closed field. It is shown that for any such coalgebra the tameness of the category $K^{\square}I$-comod of finite-dimensional left $K^{\square}I$-modules is equivalent to the tameness of the category $K^{\square}I$-$\mathrm{Comod}_{\mathrm{fc}}$ of finitely copresented left $K^{\square}I$-modules. Hence, the tame-wild dichotomy for the coalgebras $K^{\square}I$ is deduced. Moreover, we prove that for an interval finite $\widetilde{\mathbb{A}}^{*}_{m}$-free poset I the incidence coalgebra $K^{\square}I$ is of tame comodule type if and only if the quadratic form $q^{\bullet}$ is weakly non-negative. Finally, we give a complete list of all infinite connected interval finite $\widetilde{\mathbb{A}}^{*}_{m}$-free posets I such that $K^{\square}I$ is of tame comodule type. In this case we prove that, for any pair of finite-dimensional left $K^{\square}I$-comodules M and N, $\overline{b}_{K^{\square}I}(\mathbf{dim}\, M, \mathbf{dim}\, N) = \sum_{j=0}^{\infty}(-1)^j \dim_K \mathrm{Ext}^j_{K^{\square}I}(M, N)$, where $\overline{b}_{K^{\square}I} : \mathbb{Z}^{(I)} \times \mathbb{Z}^{(I)} \to \mathbb{Z}$ is the Euler $\mathbb{Z}$-bilinear form of I and $\mathbf{dim}\, M$, $\mathbf{dim}\, N$ are the dimension vectors of M and N.

1. Introduction. Throughout this paper, we denote by $\mathbb{Z}$ the ring of integers and by $\mathbb{Z}^{(I)}$ (resp. $\mathbb{Z}^{I}$) the direct sum (resp. direct product) of I copies of $\mathbb{Z}$, where I is any set. We view $\mathbb{Z}^{(I)} \subseteq \mathbb{Z}^{I}$ as abelian groups. Throughout we fix a field K and we assume that $I \equiv (I, \preceq)$ is a poset (not necessarily finite) that is *interval finite* in the sense that the interval

$$[a, b] = \{s \in I;\ a \preceq s \preceq b\}$$

is a finite subposet of I, for all $a \preceq b$ in I (see [46]). A poset I is called $\widetilde{\mathbb{A}}^{*}_{m}$*-free* if it contains no subposet of the form

(1.1) $\widetilde{\mathbb{A}}^{*}_{m}$:  with $m \geq 2$.

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Following [46, 48], we denote by $\widehat{KI}$ the *complete incidence K-algebra* consisting of all square I by I matrices $\lambda = [\lambda_{pq}] \in \mathbb{M}_I(K)$ with $\lambda_{pq} = 0$ if $p \preceq q$, $p, q \in I$, does not hold in I. Since I is interval finite, the product $\lambda' \cdot \lambda'' = [\lambda_{ab}]_{a,b \in I}$ with $\lambda_{ab} = \sum_{j \in I} \lambda'_{aj} \cdot \lambda''_{jb} = \sum_{a \preceq j \preceq b} \lambda'_{aj} \cdot \lambda''_{jb}$ is a well defined matrix lying in $\widehat{KI}$, for any $\lambda' = [\lambda'_{ij}]$ and $\lambda'' = [\lambda''_{ij}]$ in $\widehat{KI}$. Hence, $\widehat{KI}$ is an associative K-algebra and the matrix E, with 1's on the main diagonal and zeros elsewhere, is the identity of $\widehat{KI}$. The *incidence K-algebra* of I is the subalgebra KI of $\widehat{KI}$ consisting of all matrices in $\widehat{KI}$ with at most finitely many non-zero coefficients. It follows that KI is an associative K-subalgebra of the unitary algebra $\widehat{KI}$, and the matrix units e_{pq}, with $p \preceq q$, having the identity in the (p, q) entry and zeros elsewhere, form a K-basis of KI. Given $j \in I$, the matrix unit $e_j = e_{jj} \in KI$ is a primitive idempotent of the K-algebra KI, and $\{e_j\}_{j \in I}$ is a complete set of pairwise orthogonal primitive idempotents of KI. Obviously, KI has an identity element if and only if I is finite.

Following Sweedler [57] (and [38, 46]), given a field K and an interval finite poset I, we define the *incidence K-coalgebra* of I to be the triple

$$K^{\square}I = (KI, \Delta_I, \varepsilon_I), \tag{1.2}$$

where the counit $\varepsilon_I : KI \to K$ and the comultiplication $\Delta_I : KI \to KI \otimes KI$ are defined by

$$\Delta_I(e_{pq}) = \sum_{p \preceq t \preceq q} e_{pt} \otimes e_{tq}, \qquad \varepsilon_I(e_{pq}) = \begin{cases} 0 & \text{for } p \neq q, \\ 1 & \text{for } p = q. \end{cases}$$

Since I is interval finite, Δ_I is well-defined. Obviously, $\dim_K K^{\square}I \le \aleph_0$ if the poset I is of finite width and *connected*, that is, not a disjoint union of two subposets I' and I'' with all pairs $i' \in I'$ and $i'' \in I''$ incomparable in I. If I is connected, the coalgebra $K^{\square}I$ is also *connected*, that is, it is not a direct sum of two non-zero K-coalgebras.

A K-coalgebra C is defined to be *pointed* if every simple subcoalgebra H of C is one-dimensional, or equivalently, if $\dim_K S = \dim_K \operatorname{End}_C S = 1$ for any simple left C-comodule S.

It is shown in [46] that the incidence coalgebra $K^{\square}I$ of an interval finite poset I is pointed (hence basic), $c\ell$-hereditary, and Hom-computable in the sense of [42]. If I is of left locally finite width, then $K^{\square}I$ is left locally artinian (hence left cocoherent).

Here, by the *width* $\mathbf{w}(I)$ of I we mean the maximal number of pairwise incomparable elements of I, if it is finite; otherwise we set $\mathbf{w}(I) = \infty$. We say that I is of *left* (resp. *right*) *locally finite width* if given $b \in I$, the subposet

$$^{\trianglerighteq}b = \{j \in I;\ j \preceq b\} \quad \text{(resp. } b^{\trianglelefteq} = \{j \in I;\ b \preceq j\}),$$

called the *left* (resp. *right*) *cone* at b, is of finite width, that is, has finitely many pairwise incomparable elements. A subposet I' of I is defined to be *convex*, or *interval closed*, if given $a \preceq b$ in I', the interval $[a, b] = \{s \in I;\ a \preceq s \preceq b\} = a^{\trianglelefteq} \cap {}^{\trianglerighteq}b$ is contained in I'.

Throughout this paper we use the coalgebra representation theory notation and terminology introduced in [38–40, 48, 58]. The reader is referred to [4, 25, 48, 57] for the coalgebra and comodule terminology, and to [1, 2, 33, 55, 56] for the representation theory terminology and notation.

Given a K-coalgebra C, we denote by C-Comod and C-comod the categories of left C-comodules and left C-comodules of finite K-dimension, respectively. The corresponding categories of right C-comodules are denoted by Comod-C and comod-C. Further, we denote by C-inj the category of socle finite injective left C-comodules. Given a K-coalgebra C with comultiplication $\Delta : C \to C \otimes C$ and counity $\varepsilon : C \to K$, the coalgebra C^{op} opposite to C is the K-vector space C equipped with the same counity $\varepsilon : C \to K$ and the comultiplication $\Delta^{\mathrm{op}} = \tau \circ \Delta : C \to C \otimes C$, where $\tau : C \otimes C \to C \otimes C$ is the twist map defined by $\tau(x \otimes y) = y \otimes x$ for $x, y \in C$. It is clear that the category Comod-C of right C-comodules is just the category C^{op}-Comod of left C^{op}-comodules.

We recall that a left C-comodule M is *socle-finite* if $\dim_K \operatorname{soc} N$ is finite; M is (socle) *finitely copresented* if there is an exact sequence

$$0 \to M \to E_0 \to E_1 \tag{1.3}$$

in C-Comod with socle-finite injective comodules E_0 and E_1; we call it a *socle-finite injective copresentation* of M. We denote by C-$\mathrm{Comod}_{\mathrm{fc}}$ the full subcategory of C-Comod whose objects are the (socle) finitely copresented $K^{\square}I$-comodules.

We recall from [38, 44, 45] that there are two different notions of tameness of $K^{\square}I$. We define C to be of *tame* (resp. *fc-tame*) *comodule type* if the category C-comod (resp. C-$\mathrm{Comod}_{\mathrm{fc}}$) is of tame representation type; see Section 2 for details.

In this paper we study the tameness and *fc*-tameness of the incidence coalgebras $K^{\square}I$ of interval finite $\widetilde{\mathbb{A}}_m^*$-free posets I, where K is an algebraically closed field (see also [53]). The main results of the paper are presented in Sections 3–4 and are collected in the following two theorems.

THEOREM 1.4. *Assume that K is an algebraically closed field and I is an interval finite $\widetilde{\mathbb{A}}_m^*$-free poset.*

- (a) *The category $K^{\square}I$ is of tame comodule type if and only if it is of fc-tame comodule type.*
- (b) *The tame-wild dichotomy theorem holds for the coalgebras $K^{\square}I$, that is, the category $K^{\square}I$-*comod *is either tame or wild, and these two cases are mutually exclusive.*

THEOREM 1.5. *Assume that I is an interval finite $\widetilde{\mathbb{A}}_m^*$-free poset and K is an algebraically closed field. The following five conditions are equivalent:*

(a) *The coalgebra $K^{\square}I$ is of tame comodule type.*
(a′) *The coalgebra $K^{\square}I$ is of fc-tame comodule type.*
(b) *The reduced Euler quadratic form $q_I^{\bullet} : \mathbb{Z}^{(I)} \to \mathbb{Z}$ (see (3.5)) is weakly non-negative, that is, $q_I^{\bullet}(v) \geq 0$ for all vectors $v \in \mathbb{Z}^{(I)}$ with non-negative coordinates.*
(c) *For any finite subposet J of I, the finite-dimensional incidence K-algebra KJ is of tame representation type.*
(d) *For any finite subposet J of I, the K-algebra KJ is not of wild representation type.*

If, in addition, I is infinite and connected then the coalgebra $K^{\square}I$ is of tame comodule type if and only if I is a subposet of one of the posets listed in Table 4.4.

Some of the results presented in Theorems 1.4 and 1.5 were announced in [48, Section 8.3].

Throughout this paper we use the following definitions and facts (see [4, 25, 38–46, 48]).

A K-coalgebra C is defined to be *basic* if the left C-comodule $\operatorname{soc}_C C$ has a direct sum decomposition

$$\operatorname{soc}_C C = \bigoplus_{j \in I_C} S(j) \tag{1.6}$$

where I_C is a set, $S(j)$ are simple comodules and $S(i) \not\cong S(j)$ for all $i \neq j$. It is shown in [38] that the definition is left-right symmetric. If the field K is algebraically closed, a K-coalgebra C is pointed if and only if it is basic (see [38, 48]).

Following [38, 40–42, 48], we denote by

$$\mathbf{lgth}\, M = (\ell_j(M))_{j \in I_C} \in \mathbb{Z}^{(I_C)} \tag{1.7}$$

the *composition length vector* of a comodule M in C-comod, where $\ell_j(M) \in \mathbb{N}$ is the number of simple composition factors of M isomorphic to the simple comodule $S(j)$. We recall from [38] that the map $M \mapsto \mathbf{lgth}\, M$ extends to a group isomorphism

$$\mathbf{lgth} \colon \mathbf{K}_0(C) \xrightarrow{\simeq} \mathbb{Z}^{(I_C)}, \tag{1.8}$$

where $\mathbf{K}_0(C) = \mathbf{K}_0(C\text{-comod})$ is the Grothendieck group of the category C-comod. If $\dim_K S(j) = 1$, then $\ell_j(M) = \dim_K \operatorname{Hom}_C(M, E(j))$, where $E(j)$ is the injective envelope of $S(j)$ [40, Proposition 2.6]. If C is pointed, we have

$$\mathbf{lgth}\, M = \mathbf{dim}\, M = [\dim_K M e_j]_{j \in I_C}, \tag{1.9}$$

because $\dim_K Me_j = \dim_K \mathrm{Hom}_C(M, Ce_j) = \dim_K \mathrm{Hom}_C(M, E(j)) = \ell_j(M)$ for any $j \in I_C$ (see [5, 14, 40–42, 46, 48]). We call $\mathbf{dim}\, M$ the *dimension vector* of M.

Assume that C is a basic K-coalgebra with a fixed decomposition (1.6). Following [33, 37, 43–45], given a finitely copresented C-comodule M with a minimal injective copresentation (1.3), we define the *coordinate vector* of M to be the bipartite vector

$$(1.10) \quad \mathbf{cdn}(M) = (\mathbf{cdn}_0^M \mid \mathbf{cdn}_1^M) \in \mathbf{K}_0(C) \times \mathbf{K}_0(C) \equiv \mathbb{Z}^{(I_C)} \times \mathbb{Z}^{(I_C)}$$

where $\mathbf{cdn}_0^M = \mathbf{lgth}(\mathrm{soc}\, E_0)$ and $\mathbf{cdn}_1^M = \mathbf{lgth}(\mathrm{soc}\, E_1)$.

The coalgebra C is defined in [42] to be *Hom-computable* (*computable*, for short) if $\dim_K \mathrm{End}_C E$ is finite for any socle-finite direct summand E of ${}_CC$. We call a coalgebra C *left cocoherent* if any finitely cogenerated epimorphic image N of an indecomposable injective C-comodule E is finitely copresented (see [18]). Note that the class of left cocoherent coalgebras contains all right semiperfect coalgebras, all hereditary coalgebras and all left locally artinian coalgebras (i.e., the coalgebras C with all left indecomposable injective comodules artinian) (see [18]).

2. Preliminaries on incidence coalgebras and their comodules. Let $I \equiv (I, \preceq)$ be a poset (finite or infinite). We write $i \prec j$ if $i \preceq j$ and $i \neq j$. We recall that the *Hasse quiver* of I is the quiver Q_I, where the set of points of Q_I is I, and there is a unique arrow $p \to q$ from $p \in I$ to $q \in I$ if and only if $p \prec q$ and there is no $r \in I$ such that $p \prec r \prec q$.

To get a description of $K^\square I$ as the path coalgebra of a bound quiver, we consider the Hasse quiver Q_I and note that the K-algebra homomorphism $KQ_I \to KI$ associating to any arrow $p \to q$ of Q_I the matrix unit $e_{pq} \in KI$ induces a K-algebra isomorphism $KQ_I/\Omega_I \cong KI$, where Ω_I is the two-sided ideal of the path K-algebra KQ_I generated by all commutativity relations, that is, by all differences $w' - w'' \in KQ_I$ of paths w', w'' of length $m \geq 2$ with a common source and a common terminus (see [1, Ch. II] and [33, Ch. 14]). We call (Q_I, Ω_I) the *Hasse bound quiver* of the poset I.

We proved in [46, 48] that there is a coalgebra isomorphism

$$(2.1) \qquad \theta : K^\square I \xrightarrow{\simeq} K^\square(Q_I, \Omega_I),$$

where $K^\square(Q_I, \Omega_I)$ is the path K-coalgebra of the bound quiver (Q_I, Ω_I) defined by the formula

$$(2.2) \qquad K^\square(Q_I, \Omega_I) = \Omega_I^\perp = \{u \in KQ_I;\ \langle u, \Omega_I \rangle = 0\} \subseteq K^\square Q_I$$

(see [19, 38–40, 46, 48] for details).

Usually we study the comodule category $K^\square I$-Comod by means of K-linear representations of I (equivalently, representations of (Q_I, Ω_I)), which are the systems $X = (X_p, {}_q\varphi_p)_{p\prec q}$, where X_p is a K-vector space for each $p \in I$, ${}_q\varphi_p : X_p \to X_q$ is a K-linear map for all $p \prec q$, and ${}_s\varphi_q \circ {}_q\varphi_p = {}_s\varphi_p$ for all $p \prec q \prec s$. A morphism $f : X \to X'$ is a system $f = (f_p)_{p\in I}$ of K-linear maps $f_p : X_p \to X'_p$ such that ${}_q\varphi'_p \circ f_p = f_q \circ {}_q\varphi_p$ for $p \prec q$ (see [7, 19, 27, 38, 58]).

We denote by $\mathrm{Rep}_K(I) \cong \mathrm{Rep}_K(Q_I, \Omega_I)$ the Grothendieck K-category of K-linear representations of I, and by $\mathrm{rep}_K(I) \supseteq \mathrm{rep}_K^{\ell f}(I)$ the abelian full subcategories of $\mathrm{Rep}_K(I)$ formed by finitely generated representations and by finitely generated representations of finite length, respectively.

Finally, we denote by $\mathrm{Rep}_K^{\ell f}(I)$ the full Grothendieck subcategory of $\mathrm{Rep}_K(I)$ formed by *locally finite* representations, that is, directed unions of objects from $\mathrm{rep}_K^{\ell f}(I)$; by $\mathrm{nilrep}_K^{\ell f}(I)$ the full subcategory of $\mathrm{rep}_K^{\ell f}(I)$ formed by all nilpotent representations of finite length; and by $\mathrm{Rep}_K^{\ell n\ell f}(I)$ the full subcategory of $\mathrm{Rep}_K^{\ell f}(I)$ formed by all locally nilpotent representations. Since I is a poset, we have $\mathrm{nilrep}_K^{\ell f}(I) = \mathrm{rep}_K^{\ell f}(I)$, and hence $\mathrm{Rep}_K^{\ell n\ell f}(I) = \mathrm{Rep}_K^{\ell f}(I)$ (see [46] for details).

It follows from [46, Proposition 4.3] that there exist category equivalences

$$(2.3)\qquad \begin{array}{ccc} K^{\square}I\text{-Comod} \xrightarrow[F]{\simeq} \mathrm{Rep}_K^{\ell n\ell f}(I) = & \mathrm{Rep}_K^{\ell f}(I) & \cong \mathrm{Rep}_K^{\ell f}(Q_I, \Omega_I) \\ \uparrow & \uparrow & \\ K^{\square}I\text{-comod} \xrightarrow[F]{\simeq} \mathrm{nilrep}_K^{\ell f}(I) = & \mathrm{rep}_K^{\ell f}(I) & \cong \mathrm{rep}_K^{\ell f}(Q_I, \Omega_I) \end{array}$$

We start with the following useful observations.

LEMMA 2.4. *Let K be a field and let $C = K^{\square}I$ be the incidence K-coalgebra of an interval finite poset I. Let $I^{\mathrm{op}} = (I, \preceq^{\mathrm{op}})$ be the poset opposite to $I \equiv (I, \preceq)$, that is, $p \preceq^{\mathrm{op}} q$ if and only if $q \preceq p$.*

(a) *The K-linear map $\widehat{tr} : K^{\square}I \xrightarrow{\simeq} K^{\square}(I^{\mathrm{op}})$ that associates to any matrix λ its transpose matrix $\widehat{tr}(\lambda) = \lambda^{tr}$ defines an isomorphism of the K-coalgebra $K^{\square}(I^{\mathrm{op}})$ with the K-coalgebra C^{op}.*
(b) *The coalgebra isomorphism $(K^{\square}I)^{\mathrm{op}} \cong K^{\square}(I^{\mathrm{op}})$ defined in* (a) *induces category isomorphisms*

$$K^{\square}(I^{\mathrm{op}})\text{-Comod} \cong \text{Comod-}K^{\square}I, \qquad K^{\square}(I^{\mathrm{op}})\text{-comod} \cong \text{comod-}K^{\square}I.$$

(c) *If U is a convex subposet of I then $K^{\square}U$ is a subcoalgebra of $K^{\square}I$ and $K^{\square}U$*-comod *is an extension closed subcategory of $K^{\square}I$*-comod.

Proof. Statements (a) and (b) follow immediately from the definitions of $K^{\square}I$ and $K^{\square}(I^{\mathrm{op}})$. For the proof of (c) we refer to [14, 40, 41, 48]. ■

Now we collect some basic properties of the coalgebra $K^{\square}I$ proved in [38, 42, 46] (see also [5, 7, 14, 26]).

PROPOSITION 2.5. *Let K be a field, let I be an interval finite poset and let $C = K^{\square}I$.*

(a) *The coalgebra $K^{\square}I$ is basic and pointed; it is connected (indecomposable) if and only if the poset I is connected. Moreover,* $\dim_K K^{\square}I \leq \aleph_0$ *if I is of finite width and connected.*

(b) *For each $j \in I_C$, the vector space*

$$S_I(j) = e_j \cdot (KI) \cdot e_j \cong Ke_j$$

is a one-dimensional simple left coideal (and a subcoalgebra) of $K^{\square}I$, the left ideal

$$E_I(j) = KI \cdot e_j$$

of the K-algebra KI is a left coideal of the coalgebra C such that $\operatorname{soc} E_I(j) = S_I(j)$, $\operatorname{End}_C S_I(j) \cong K$, $\operatorname{End}_C E_I(j) \cong K$, *and the vector*

$$\mathbf{lgth}\, E_I(j) = (\ell_{jp})_{p\in I} \in \mathbb{Z}^I \quad \text{is defined by} \quad \ell_{jp} = \begin{cases} 1 & \text{if } p \preceq j, \\ 0 & \text{if } p \not\preceq j. \end{cases}$$

Moreover, there are vector space isomorphisms

$$\operatorname{Hom}_C(E_I(q), E_I(p)) \xrightarrow[\simeq]{\xi_{qp}} \begin{cases} Ke_{pq} & \text{if } p \preceq q, \\ 0 & \text{if } p \not\preceq q. \end{cases} \tag{2.6}$$

(c) *There are left $K^{\square}I$-comodule decompositions*

$$\operatorname{soc} K^{\square}I = \bigoplus_{j\in I} S_I(j) \quad \text{and} \quad K^{\square}I = \bigoplus_{j\in I} E_I(j). \tag{2.7}$$

(d) *The coalgebra C is Hom-computable, its composition length matrix ${}_CF = [\ell_{pq}] \in \mathbb{M}_I^{\preceq}(\mathbb{Z})$ coincides with its Cartan matrix ${}_C\widehat{F} = [\widehat{\ell}_{pq}] \in \mathbb{M}_I^{\preceq}(\mathbb{Z})$ with $\widehat{\ell}_{pq} = \dim_K \operatorname{Hom}_C(E_I(p), E_I(q))$, and ${}_CF^{tr} = \mathbf{C}_I \in \mathbb{M}_I^{\preceq}(\mathbb{Z})$ is the incidence matrix* (3.3) *of the poset I (see Section* 3*).*

(e) *Given $p \in I$, the composition length vector* $\mathbf{lgth}\, E_I(p) = (\ell_{pq})_{q\in I} \in \mathbb{Z}^I$ *is the pth row $(c_{qp})_{q\in I}$ of the incidence matrix $\mathbf{C}_I$.* ■

PROPOSITION 2.8. *Let K be a field and I a connected interval finite poset of left locally finite width.*

(a) *The coalgebra $K^{\square}I$ is Hom-computable and locally left artinian and left cocoherent.*

(b) *The category $K^{\square}I$-*Comod$_{\rm fc}$ *is abelian and coincides with the category of artinian left $K^{\square}I$-comodules. It is closed under taking extensions, contains the categories $K^{\square}I$-*comod *and $K^{\square}I$-*inj, *and every comodule N in $K^{\square}I$-*Comod$_{fc}$ *has an injective resolution lying in $K^{\square}I$-*Comod$_{fc}$.

Proof. Apply [46, Sections 4 and 5]. ■

3. Incidence coalgebras of tame comodule type. Let C be a basic K-coalgebra. We recall from [38] and [39] that C is said to be of *K-wild comodule type* (or *K-wild*, for short) if the category C-comod is of K-wild representation type [33, 38, 56] in the sense that there exists an exact K-linear representation embedding (see [36])

$$T : \operatorname{mod} \varGamma_3(K) \to C\text{-comod},$$

where $\varGamma_3(K) = \begin{bmatrix} K & K^3 \\ 0 & K \end{bmatrix}$. A K-coalgebra C is defined to be of *K-tame comodule type* [38] (or *K-tame*, for short) if C-comod is of K-tame representation type ([33, Section 14.4], [56]), that is, for every vector $v \in \mathbf{K}_0(C) \equiv \mathbb{Z}^{(I_C)}$, there exist C-$K[t]$-bicomodules $L^{(1)}, \ldots, L^{(r_v)}$, which are finitely generated free $K[t]$-modules, such that all but finitely many indecomposable left C-comodules M with $\mathbf{lgth}\, M = v$ are of the form $M \cong L^{(s)} \otimes K^1_\lambda$, where $s \le r_v$ and

$$K^1_\lambda = K[t]/(t - \lambda), \qquad \lambda \in K.$$

Equivalently, there exist a non-zero polynomial $h(t) \in K[t]$ and C-$K[t]_h$-bicomodules $L^{(1)}, \ldots, L^{(r_v)}$, which are finitely generated free $K[t]_h$-modules, such that all but finitely many indecomposable left C-comodules M with $\mathbf{lgth}\, M = v$ are of the form $M \cong L^{(s)} \otimes K^1_\lambda$, where $s \le r_v$ and $K[t]_h = K[t, h(t)^{-1}]$ is a rational K-algebra (see [9] or [33, Section 14.4]). In this case, we say that $L^{(1)}, \ldots, L^{(r_v)}$ form an *almost parametrising family* for the family $\operatorname{ind}_v(C\text{-comod})$ of all indecomposable C-comodules M with $\mathbf{lgth}\, M = v$.

Here, by a C-$K[t]_h$-bicomodule we mean a K-vector space L equipped with a left C-comodule structure and a right $K[t]_h$-module structure satisfying the obvious associativity conditions. In [44, 45], a K-tame-wild dichotomy theorem is proved for left (or right) semiperfect coalgebras and for acyclic hereditary coalgebras over an algebraically closed field K by reducing the problem to the *fc*-tame-wild dichotomy theorem [44, Theorem 2.11], and consequently to the tame-wild dichotomy theorem for bocses and finite-dimensional K-algebras proved in [9].

We recall from [44, 45] that C is of *fc*-tame comodule type if, for every coordinate vector $v = (v' \mid v'') \in \mathbf{K}_0(C) \times \mathbf{K}_0(C)$, the indecomposable finitely copresented C-comodules N such that $\mathbf{cdn}(N) = (v' \mid v'')$ form at most finitely many one-parameter families (see [44] and [48, Section 6] for a precise definition). The reader is referred to [38] and to [44] for the definition of tameness of polynomial growth and *fc*-tameness of polynomial growth, respectively.

The following lemma shows that in the study of the incidence coalgebras $K^\square I$ of tame comodule type, we may assume that $K^\square I$ is left and right locally artinian, hence left and right cocoherent.

LEMMA 3.1. *Assume that K is an algebraically closed field and I is a connected interval finite poset.*

(a) *If J is a connected interval finite subposet of I and the coalgebra $K^{\square}I$ is of tame (resp. fc-tame) comodule type then $K^{\square}J$ is of tame (resp. fc-tame) comodule type.*

(b) *If $K^{\square}I$ is of tame or fc-tame comodule type then:*

  (b1) *I is of left and right locally finite width. More precisely, $\mathbf{w}(b^{\trianglelefteq}) \leq 4$ and $\mathbf{w}({}^{\trianglerighteq}b) \leq 4$, for all $b \in I$.*

  (b2) *$K^{\square}I$ is left and right locally artinian, as well as left and right cocoherent.*

  (b3) *We have $K^{\square}I$-comod $\subseteq K^{\square}I$-Comod$_{\mathrm{fc}}$ and $(K^{\square}I)^{\mathrm{op}}$-comod $\subseteq (K^{\square}I)^{\mathrm{op}}$-Comod$_{\mathrm{fc}}$, and a left (resp. right) $K^{\square}I$-comodule M is finitely copresented if and only if it is artinian.*

Proof. (a) If $K^{\square}I$ is of tame comodule type, we apply the isomorphism (2.1), the equivalences (2.3), and modify the arguments used in the proof of [38, Theorem 6.11(a)]. If $K^{\square}I$ is of *fc*-tame comodule type, we apply the results of [44], in particular [44, Corollary 2.13].

(b) Assume that $K^{\square}I$ is of tame or *fc*-tame comodule type.

To prove (b1), assume to the contrary that $K^{\square}I$ is of tame comodule type and there exists $b \in I$ such that $\mathbf{w}({}^{\trianglerighteq}b) \geq 5$, that is, $\mathbf{w}({}^{\trianglerighteq}b)$ contains five pairwise incomparable elements a_1, a_2, a_3, a_3, a_5. Hence I contains the finite subposet

$$J\colon\quad \begin{array}{ccc} a_2 & & a_1 \\ & \searrow & \downarrow \\ a_3 & \rightarrow & b \\ & \nearrow & \uparrow \\ a_4 & & a_5 \end{array}$$

It follows from [40, 46] that, for the finite-dimensional coalgebra $H := K^{\square}J$, there are representation embeddings

$$H\text{-comod} \to K^{\square}I\text{-comod} \quad \text{and}$$
$$H\text{-comod} = H\text{-Comod}_{\mathrm{fc}} \to K^{\square}I\text{-Comod}_{\mathrm{fc}}$$

preserving the wild representation type. By (2.3), there are equivalences of categories H-comod $\xrightarrow{\sim} \mathrm{rep}_K(J) \cong \mathrm{mod}\, KJ$, preserving wildness. Since the finite-dimensional algebra KJ is wild, the coalgebra H is of wild comodule type and, according to [38, Theorem 6.10] and [48, Theorem 6.7(d)], $K^{\square}I$ is of wild comodule type. This contradicts the weak version of the tame-wild dichotomy for coalgebras proved in [39, Corollary 5.6], [38, Theorem 6.11] and [48, Corollary 6.8], because we have assumed that $K^{\square}I$ is of tame comodule type. Consequently, (b1) follows when $K^{\square}I$ is of tame comodule type, because one proves the second part of (b1) in a similar way. If $K^{\square}I$

is of *fc*-tame comodule type, (b1) can be proved in a similar way by applying [44, Corollary 2.13].

(b2) By (b1), given $b \in I$, the widths of ${}^{\trianglerighteq}b$ and $b^{\trianglelefteq}$ are smaller than 5. Then, by [46, Theorem 5.3], $K^{\square}I$ is left and right locally artinian. Hence, by [18, Proposition 1.3], it is left and right cocoherent.

(b3) Apply (b2), Proposition 2.8, and [46, Theorem 5.7]. ∎

Now we are able to prove that the tame-wild dichotomy [48, (6.10)] holds for the coalgebras $K^{\square}I$ of connected interval finite posets I that are of left locally finite width.

THEOREM 3.2. *Assume that K is an algebraically closed field and I is a connected interval finite poset.*

- (a) *The coalgebra $K^{\square}I$ is of fc-tame comodule type if and only if it is of tame comodule type.*
- (b) *$K^{\square}I$ is fc-tame of polynomial growth* (*resp. of discrete fc-comodule type*) *if and only if it is tame of polynomial growth* (*resp. of discrete comodule type*).
- (c) *$K^{\square}I$ is either of tame or of wild comodule type, and these types are mutually exclusive.*

Proof. (a) By Proposition 2.8, $K^{\square}I$ is computable. If it is of tame or *fc*-tame comodule type then Lemma 3.1 applies. It follows that the category $K^{\square}I\text{-Comod}_{\text{fc}}$ contains $K^{\square}I$-comod, and the *fc*-tameness of $K^{\square}I$ implies its tameness, by [48, Lemma 6.17].

Conversely, by [44] and [48, Theorem 6.7(e)], the tameness of $K^{\square}I$ implies the tameness of the finite-dimensional coalgebra $K^{\square}U$ for any convex finite subposet U of I, because $K^{\square}U$ is a subcoalgebra of $K^{\square}I$. Hence, the finite-dimensional K-algebra $R_{E_U} = \text{End}_{K^{\square}I} E_U$ is tame for every such U, where $E_U = \bigoplus_{j \in U} E_I(j)$ (see [14, 41] and Proposition 2.5). Then, in view of [48, Corollary 6.28], $K^{\square}I$ is of *fc*-tame comodule type.

(b) If $K^{\square}I$ is of tame or *fc*-tame comodule type then, by Lemma 3.1 and (a), the category $K^{\square}I\text{-Comod}_{\text{fc}}$ contains $K^{\square}I$-comod, and the *fc*-tameness of $K^{\square}I$ is equivalent to its tameness. Then the arguments in the proof of [44, Theorem 3.1] extend almost verbatim to our case and prove (b). The details are left to the reader.

(c) In view of (a), statement (c) is a consequence of the *fc*-tame-wild dichotomy [44, Theorem 2.11] and [48, Theorem 6.26]. ∎

To characterize the coalgebras $K^{\square}I$ for interval finite posets I of tame comodule type, we need some notation introduced in [46]. Assume that I is an interval finite poset (finite or infinite). The set $\mathbb{M}_I(\mathbb{Z})$ of all square $I \times I$

matrices with integer coefficients is viewed as an abelian group with respect to the usual matrix addition. The set

$$\mathbb{M}_I^{\preceq}(\mathbb{Z}) = \{c = [c_{pq}]_{p,q\in I} \in \mathbb{M}_I(\mathbb{Z});\ c_{pq} = 0 \text{ if } p \npreceq q\}$$

is a subgroup of $\mathbb{M}_I(\mathbb{Z})$. Since I is assumed to be interval finite, for any two matrices $c' = [c'_{ij}]$ and $c'' = [c'_{ij}]$ in $\mathbb{M}_I^{\preceq}(\mathbb{Z})$, their product

$$c' \cdot c'' = [c_{ab}]_{a,b\in I}, \quad \text{with} \quad c_{ab} = \sum_{j\in I} c'_{aj} \cdot c''_{jb} = \sum_{a\preceq j\preceq b} c'_{aj} \cdot c''_{jb},$$

is a well defined matrix lying in $\mathbb{M}_I^{\preceq}(\mathbb{Z})$. Hence, $\mathbb{M}_I^{\preceq}(\mathbb{Z})$ is an associative $\mathbb{Z}$-algebra and the matrix E with 1's on the main diagonal and zeros elsewhere is the identity of $\mathbb{M}_I^{\preceq}(\mathbb{Z})$. The relation $\preceq$ is uniquely determined by the *incidence matrix* of I, the integral square $I \times I$ matrix (see [33, 34])

$$(3.3) \qquad \mathbf{C}_I = [c_{ij}]_{i,j\in I} \in \mathbb{M}_I^{\preceq}(\mathbb{Z}) \quad \text{with} \quad c_{ij} = \begin{cases} 1 & \text{for } i \preceq j, \\ 0 & \text{for } i \npreceq j. \end{cases}$$

By [46, Corollary 2.9], $\mathbf{C}_I$ has a unique left and right inverse

$$\overline{\mathbf{C}}_I := \mathbf{C}_I^{-1} = [c_{ij}^{-}]_{i,j\in I} \in \mathbb{M}_I^{\preceq}(\mathbb{Z})$$

defined by [46, (2.11)] and called in [47] the *Euler matrix* of I. Following [47], besides $\overline{\mathbf{C}}_I$, we also associate to I the *reduced Euler matrix*

$$(3.4) \qquad \mathbf{C}_I^{\bullet} = [c_{ij}^{\bullet}] \in \mathbb{M}_I^{\preceq}(\mathbb{Z})$$

with $c_{ij}^{\bullet} = c_{ij}^{-} = 1$ for $i = j$, $c_{ij}^{\bullet} = c_{ij}^{-} = -1$ if $i \to j$, $c_{ij}^{\bullet} = c_{ij}^{-}$ if $i \blacktriangleleft j$, and $c_{ij}^{\bullet} = 0$ in the remaining cases. We recall that we write $i \to j$ if $i \prec j$ and there is no $s \in I$ such that $i \prec s \prec j$. Moreover, we write $a \blacktriangleleft b$ if $a \prec b$ and there are two pairs (a', b'') and (a'', b') of incomparable elements in I such that $a \begin{array}{c} \nearrow\, a' \preceq b' \,\searrow \\ \searrow\, a'' \preceq b'' \,\nearrow \end{array} b$.

In the characterization theorem below we use the *reduced Euler form*

$$(3.5) \qquad q_I^{\bullet}(x) = \sum_{i\in I} x_i^2 - \sum_{i\to j} x_i x_j + \sum_{i\blacktriangleleft j} c_{ij}^{\bullet} x_i x_j = x \cdot \mathbf{C}_I^{\bullet} \cdot x^{tr}, \qquad x \in \mathbb{Z}^{(I)},$$

introduced in [47, (3.4)] (see [5, 26, 53] for some application).

Remark 3.6. We recall from [47] that $q_I^{\bullet}(x)$ coincides with the Tits form in the sense of Bongartz [3] associated with the Hasse bound quiver (Q_I, Ω_I) such that the coalgebra isomorphism (2.1) holds. It follows that, given $a \blacktriangleleft b$ in I, the coefficient $c_{ab}^{\bullet}$ is a positive integer and equals the cardinality of a minimal set generating the ideal in Ω_I generated by all commutative relations starting from a and terminating at b.

Following [47], to any finite poset J we also associate its *reduced Coxeter–Euler matrix*

$$\mathrm{Cox}^{\bullet}_J := -\mathbf{C}^{\bullet}_J \cdot (\mathbf{C}^{\bullet}_J)^{-tr} \in \mathbb{M}_J(\mathbb{Z}) \tag{3.7}$$

and the *reduced Coxeter–Euler polynomial* (cf. [5, 16, 17, 50–53])

$$\mathrm{cox}^{\bullet}_J(t) := \det(t \cdot E - \mathrm{Cox}^{\bullet}_J) \in \mathbb{Z}[t] \tag{3.8}$$

Now we are ready to prove the following useful characterization.

THEOREM 3.9. *Assume that I is a connected $\widetilde{\mathbb{A}}^*_m$-free interval finite poset and K is an algebraically closed field. Then the following conditions are equivalent:*

(a) *The coalgebra $K^{\square}I$ is of tame comodule type.*
(a′) *The coalgebra $K^{\square}I$ of fc-tame comodule type.*
(b) *The reduced Euler form $q^{\bullet}_I : \mathbb{Z}^{(I)} \to \mathbb{Z}$ is weakly non-negative.*
(c) *The coalgebra $K^{\square}U$ is tame for any finite convex subposet U of I.*
(d) *The form $q^{\bullet}_U : \mathbb{Z}^U \to \mathbb{Z}$ is weakly non-negative for any finite convex subposet U of I.*

Proof. The equivalence (a)⇔(a′) is a consequence of Theorem 3.2.

(b)⇔(d). First we observe that, given a connected $\widetilde{\mathbb{A}}^*_m$-free interval finite poset I and a finite convex subposet U of I, we have

$$q^{\bullet}_U(v) = q^{\bullet}_I(\widehat{v}) \tag{3.10}$$

for any vector $v \in \mathbb{Z}^U$, where $\widehat{v} = (\widehat{v}_j)_{j\in I} \in \mathbb{Z}^{(I)}$ is defined by the formula

$$\widehat{v}_j = \begin{cases} v_j & \text{if } j \in U, \\ 0 & \text{if } j \in I \setminus U. \end{cases}$$

Here we apply the fact proved in [46] that the matrix $\mathbf{C}^{\bullet}_U$ is obtained from $\mathbf{C}^{\bullet}_I$ by dropping all rows and columns indexed by $j \in I \setminus U$ (see [46, proof of Corollary 2.9]). Here we use the assumption that U is a convex subposet of I. Hence (b)⇔(d) follows.

(a)⇔(c). By Theorem 3.2, $K^{\square}I$ is of *fc*-tame comodule type if and only if it is of tame comodule type. Then (a)⇔(c) is a consequence of [44, Corollary 2.13].

(c)⇔(d). This follows by applying [47, Theorem 1.5 and Proposition 4.2], proved by using the results of Bongartz [3] and Leszczyński [20, Theorem]. ∎

The following corollary shows that, for $K^{\square}I$ of tame comodule type and of arbitrarily large (finite or infinite) global dimension, the Euler characteristic (see (3.13) below) is a well defined integer and can be computed by using the Euler bilinear form (3.12) of I (see [30, 40, 46, 47]).

Corollary 3.11. *Assume that I is a connected $\widetilde{\mathbb{A}}^*_m$-free interval finite poset, K is an algebraically closed field, and*

$$\overline{b}_I : \mathbb{Z}^{(I)} \times \mathbb{Z}^{(I)} \to \mathbb{Z}$$

is the Euler $\mathbb{Z}$-bilinear form of I defined by

$$\overline{b}_I(u, w) = u \cdot \mathbf{C}^{-1} \cdot w^{tr} \tag{3.12}$$

for all $v, w \in \mathbb{Z}^{(I)}$ (see [46, 47]*). Assume also that $K^\square I$ is of tame comodule type of arbitrarily large (finite or infinite) global dimension* gl.dim $K^\square I$.

(a) *$K^\square I$ is an Euler coalgebra* [42] *and the Euler defect $\partial_{K^\square I} : \mathbb{Z}^{(I)} \times \mathbb{Z}^{(I)} \to \mathbb{Z}$* [42, (4.23)], [49] *of $K^\square I$ is zero.*

(b) *For any M, N in $K^\square I$-*comod*, the Euler characteristic*

$$\chi_{K^\square I}(M, N) = \sum_{j=0}^{\infty} (-1)^j \dim_K \mathrm{Ext}^j_{K^\square I}(M, N) \tag{3.13}$$

is an integer, and

$$\overline{b}_{K^\square I}(\mathbf{lgth}\, M, \mathbf{lgth}\, N) = \overline{b}_{K^\square I}(\mathbf{dim}\, M, \mathbf{dim}\, N) = \chi_{K^\square I}(M, N).$$

(c) *If N is an artinian comodule in $K^\square I$-*Comod *and M is a comodule in $K^\square I$-*comod *then $\mathrm{Ext}^m_{K^\square I}(M, N) = 0$ for $m \gg 0$ sufficiently large, (M, N) is a computable Euler pair in the sense of* [49, Definition 4.1]*, the Euler defect $\widehat{\partial}_{K^\square I}(M, N)$ is zero, and*

$$\widehat{b}_{K^\square I}(\mathbf{lgth}\, M, \mathbf{lgth}\, N) = \sum_{j=0}^{\infty} (-1)^j \dim_K \mathrm{Ext}^j_{K^\square I}(M, N),$$

where $\widehat{b}_{K^\square I} : K_0^+(K^\square I) \times \widehat{K}_0^+(K^\square I) \to \mathbb{Z}$ is the Euler $\mathbb{Z}$-bilinear form defined in [42, (4.11)] *and* [49, (3.5)].

Proof. It follows from Lemma 3.1 that I is of left and right locally finite width, and $K^\square I$ is left and right locally artinian, as well as left and right cocoherent. Hence, by [46, Theorem 5.3(a)], $K^\square I$ is an Euler coalgebra and the Euler defect $\partial_{K^\square I}$ is zero. Moreover, by [46, Corollary 2.9], the incidence matrix $\mathbf{C}_I$ has a unique left and right inverse $\mathbf{C}_I^{-1} \in \mathbb{M}_I^{\preceq}(\mathbb{Z})$ defined by [46, (2.11)]. Hence, the Euler $\mathbb{Z}$-bilinear form (3.12) is well defined. It follows from [40] that, for any finite-dimensional comodules M, N over $K^\square I$,

$$\overline{b}_{K^\square I}(\mathbf{lgth}\, M, \mathbf{lgth}\, N) = \overline{b}_{K^\square I}(\mathbf{dim}\, M, \mathbf{dim}\, N) = \chi_{K^\square I}(M, N) + \partial_{K^\square I}(M, N)$$

(see (1.9)). This finishes the proof of (a) and (b), because $\partial_{K^\square I}(M, N) = 0$ (see also [46, Theorem 5.3(a)]).

(c) Assume that N is an artinian comodule in $K^{\square}I$-Comod and M is a comodule in $K^{\square}I$-comod. Since $K^{\square}I$ is computable by Proposition 2.8, and assumed to be of tame comodule type, Lemma 3.1(b) shows that N and M lie in $K^{\square}I\text{-Comod}_{\rm fc}$ and are computable. Moreover, in view of [46, Corollary 5.6], (M,N) is a computable Euler pair in the sense of [49, Definition 4.1]. In particular, $\mathrm{Ext}^m_{K^{\square}I}(M,N)=0$ for $m \gg 0$ sufficiently large, the Euler defect $\widehat{\partial}_{K^{\square}I}(M,N)$ is well defined, the Euler $\mathbb{Z}$-bilinear form $\widehat{b}_{K^{\square}I}$ is well defined, and [49, Theorem 4.4(b)] yields

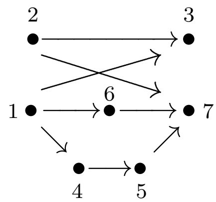
$$\widehat{b}_{K^{\square}I}(\mathbf{lgth}\, M, \mathbf{lgth}\, N) = \chi_{K^{\square}I}(M,N) + \widehat{\partial}_{K^{\square}I}(M,N)$$

(see [49, (4.6)]). Since ${}_{K^{\square}I}F^{tr} = \mathbf{C}_I$ by Proposition 2.5(d) and [46, Theorem 5.3], and $\mathbf{C}_I$ has the two-sided inverse $\mathbf{C}_I^{-1}$, [49, Theorem 4.4(c)] applies. Hence $\widehat{\partial}_{K^{\square}I}(M,N)=0$ and (c) follows (see also [46, Corollary 5.6]). ■

The following example presented in [21, p. 295] shows that Theorem 3.9 does not remain valid for posets I that contain a subposet $\widetilde{\mathbb{A}}^*_m$ with $m \ge 2$.

EXAMPLE 3.14. The reduced Euler form $q^\bullet_J : \mathbb{Z}^7 \to \mathbb{Z}$ of the poset

J :



can be written as follows:

$$\begin{aligned} q^\bullet_J(x) &= x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2+x_7^2-(x_1+x_2)x_3-(x_1+x_5)x_4 \\ &\quad - x_1x_6 + (x_1-x_2-x_5-x_6)x_7 \\ &= \big(x_1 - \tfrac{1}{2}x_3 - \tfrac{1}{2}x_4 - \tfrac{1}{2}x_6 + \tfrac{1}{2}x_7\big)^2 + (x_2 - \tfrac{1}{2}x_3 - \tfrac{1}{2}x_7)^2 \\ &\quad + \tfrac{5}{12}\big(x_3 - \tfrac{2}{5}x_5 - \tfrac{4}{5}x_6 + \tfrac{1}{5}x_7\big)^2 + \tfrac{3}{4}\big(-\tfrac{1}{3}x_3 + x_4 - \tfrac{2}{3}x_5 - \tfrac{1}{3}x_6 + \tfrac{1}{3}x_7\big)^2 \\ &\quad + \tfrac{3}{5}\big(x_5 - \tfrac{1}{2}x_6 - \tfrac{1}{2}x_7\big)^2 + \tfrac{1}{4}(x_6 - x_7)^2. \end{aligned}$$

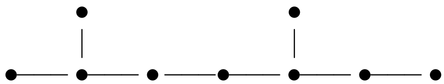
Thus it is non-negative and $\mathrm{Ker}\, q^\bullet_J = \mathbb{Z}\cdot\mathbf{h}$, where $\mathbf{h}=(1,1,1,1,1,1,1)$. This shows that $q^\bullet_J$ is critical in the sense of Ovsienko [29] (see also [24]).

Now we show that the finite-dimensional coalgebra $K^{\square}J$ is of wild comodule type; hence, in view of [48, Corollary 6.8], it is not of tame comodule type if K is algebraically closed. Indeed, let (Q_I, Ω_I) be the Hasse bound quiver of I and let

$$f : (\widetilde{Q}_I, \widetilde{\Omega}_I) \to (Q_I, \Omega_I)$$

be a universal covering of bound quivers. It induces a push-down functor $f_\lambda : K^{\square}(\widetilde{Q}_I, \widetilde{\Omega}_I)\text{-Comod} \to K^{\square}(Q_I,\Omega_I)\text{-Comod}$. One can show that $(\widetilde{Q}_I, \widetilde{\Omega}_I)$ contains a wild subquiver Q of the type

$\widetilde{\widetilde{\mathbb{D}}}_7$:



Hence, the finite-dimensional K-coalgebra $K^{\square}Q$ is a subcoalgebra of $K^{\square}(\widetilde{Q}_I, \widetilde{\Omega}_I)$ and f_λ restricts to a functor

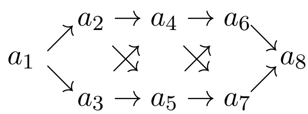
$$f_\lambda^\vee : K^{\square}Q\text{-comod} \to K^{\square}(Q_I, \Omega_I)\text{-comod} \cong K^{\square}I\text{-comod}$$

preserving wildness. We recall from [48, (8.25)] that there exists a coalgebra isomorphism $K^{\square}J \cong K^{\square}(Q_I, \Omega_I)$. It follows that $K^{\square}J$ is of wild comodule type, because $K^{\square}Q$ is, by [48, Theorem 7.21(a) and Corollary 6.8] (see also [33, Chapter 15]). Consequently, $K^{\square}J$ is not of tame comodule type, by [44] and [48, Corollary 6.8].

The following two examples illustrate the difference between the Euler form $q_I(x) = x \cdot \mathbf{C}_I^{-1} \cdot x^{tr}$ of a poset I and its reduced Euler form $q_I^\bullet(x) = x \cdot \mathbf{C}_I^\bullet \cdot x^{tr}$ when $\text{gl.dim}\, KI = \text{gl.dim}\, K^{\square}I > 2$.

EXAMPLE 3.15. Let I be the *completed garland*

I :



We will show that the incidence K-algebra KI and the incidence coalgebra $K^{\square}I$ are tame, by proving that the reduced Euler form $q_I^\bullet : \mathbb{Z}^8 \to \mathbb{Z}$ is weakly non-negative.

First we observe that $\text{gl.dim}\, KI = \text{gl.dim}\, K^{\square}I = 4$, because the simple projective left $K^{\square}I$-comodule $S(a_8)$ has the minimal injective resolution (in the notation of (2.7))

$$0 \to S_I(a_8) \to E_I(a_8) \to E_I(a_7) \oplus E_I(a_6) \to E_I(a_5) \oplus E_I(a_4) \to E_I(a_3) \oplus E_I(a_2) \to E_I(a_1) \to 0,$$

whereas the injective dimension of each of the remaining simple left $K^{\square}I$-comodules $S(a_1), \ldots, S(a_7)$ is less than or equal to three.

The Euler matrix and the reduced Euler matrix of I are

$$\overline{\mathbf{C}}_I = \begin{bmatrix} 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 & -1 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 & -1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{C}_I^\bullet = \begin{bmatrix} 1 & -1 & -1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

A straightforward calculation shows that the Coxeter polynomial and the reduced Coxeter–Euler polynomial of I are (see [13])

$$\begin{aligned} \mathrm{cox}_I(t) &= t^8 + t^7 - 2t^6 - t^5 + 2t^4 - t^3 - 2t^2 + t + 1, \\ \mathrm{cox}_I^\bullet(t) &= t^8 + 4t^7 + 8t^6 + 12t^5 + 14t^4 + 12t^3 + 8t^2 + 4t + 1, \end{aligned}$$

the Coxeter number $\mathbf{c}_I$ equals 6, whereas the reduced Coxeter–Euler number $\mathbf{c}_I^\bullet$ is infinite (see [13, 47, 50–52]).

The Euler form $\overline{q}_I : \mathbb{Z}^8 \to \mathbb{Z}$ is given by

$$\begin{aligned} \overline{q}_I(x) &= x \cdot \mathbf{C}_I^{-1} \cdot x^{tr} \\ &= x_1^2 - (x_1 - x_2)x_2 - (x_1 - x_3)x_3 + (x_1 - x_2 - x_3 + x_4)x_4 \\ &\quad + (x_1 - x_2 - x_3 + x_5)x_5 - (x_1 - x_2 - x_3 + x_4 + x_5 - x_6)x_6 \\ &\quad - (x_1 - x_2 - x_3 + x_4 - x_5 - x_7)x_7 \\ &\quad + (x_1 - x_2 - x_3 + x_4 + x_5 - x_6 - x_7 + x_8)x_8 \\ &= \left(x_1 - \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 + \tfrac{1}{2}x_4 + \tfrac{1}{2}x_5 - \tfrac{1}{2}x_6 - \tfrac{1}{2}x_7 + \tfrac{1}{2}x_8\right)^2 \\ &\quad + \tfrac{3}{4}\left(x_2 - \tfrac{1}{3}x_3 - \tfrac{1}{3}x_4 - \tfrac{1}{3}x_5 + \tfrac{1}{3}x_6 + \tfrac{1}{3}x_7 - \tfrac{1}{3}x_8\right)^2 \\ &\quad + \tfrac{2}{3}\left(x_3 - \tfrac{1}{2}x_4 - \tfrac{1}{2}x_5 + \tfrac{1}{2}x_6 + \tfrac{1}{2}x_7 - \tfrac{1}{2}x_8\right)^2 \\ &\quad + \tfrac{1}{2}(x_4 - x_5)^2 + \tfrac{1}{2}(x_6 - x_7)^2 + \tfrac{1}{2}x_8^2. \end{aligned}$$

It follows that the form $\overline{q}_I$ is non-negative of corank two (see [13]).

The reduced Euler form $q_I^\bullet : \mathbb{Z}^8 \to \mathbb{Z}$ is given by

$$\begin{aligned} q_I^\bullet(x) &= x \cdot \mathbf{C}_I^\bullet \cdot x^{tr} \\ &= x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 + x_8^2 - x_1x_2 - x_1x_3 + x_1x_4 \\ &\quad + x_1x_5 - x_2x_4 - x_2x_5 + x_2x_6 + x_2x_7 - x_3x_4 - x_3x_5 + x_3x_6 + x_3x_7 \\ &\quad - x_4x_6 - x_4x_7 + x_4x_8 - x_5x_6 - x_5x_7 + x_5x_8 - x_6x_8 - x_7x_8 \end{aligned}$$

$$
\begin{aligned}
&= \left(x_1 - \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 + \tfrac{1}{2}x_4 + \tfrac{1}{2}x_5\right)^2 \\
&\quad + \tfrac{3}{4}\left(x_2 - \tfrac{1}{3}x_3 - \tfrac{1}{3}x_4 - \tfrac{1}{3}x_5 + \tfrac{2}{3}x_6 + \tfrac{2}{3}x_7\right)^2 \\
&\quad + \tfrac{2}{3}\left(x_3 - \tfrac{1}{2}x_4 - \tfrac{1}{2}x_5 + x_6 + x_7\right)^2 + \tfrac{1}{2}(x_4 - x_5 + x_8)^2 \\
&\quad - 2\left(x_5 - \tfrac{1}{2}x_6 - \tfrac{1}{2}x_7\right)^2 - (2x_6 + 2x_7)^2 + (2x_6 - 2x_7)^2 \\
&\quad + \tfrac{1}{2}(2x_5 - x_6 - x_7 + x_8)^2.
\end{aligned}
$$

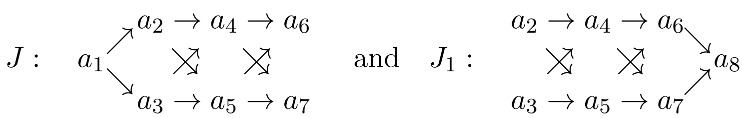
It follows that the form $q_I^\bullet$ is indefinite.

On the other hand, one can check by computer calculation using the algorithm of Dean and de la Peña [8] that $q_I^\bullet$ is weakly non-negative. Alternatively, this follows from the following semi-canonical Lagrange form of $2q_I^\bullet$:

$$
\begin{aligned}
2q_I^\bullet(x) &= (x_1 - x_2 - x_3 + x_4 + x_5 - x_6 - x_7 + x_8)^2 \\
&\quad + (x_2 - x_3)^2 + (x_4 - x_5)^2 + (x_6 - x_7)^2 + (x_1 - x_8)^2 \\
&\quad + 2(x_1x_7 + x_1x_8 + x_2x_8 + x_3x_8).
\end{aligned}
$$

Hence, by [20, 44, 47], and Theorem 3.9, the incidence K-algebra KI and the incidence coalgebra $K^\square I$ are tame.

EXAMPLE 3.16. Let J and J_1 be the following *left completed garland* and the *right completed garland*, respectively:



First we note that $\mathrm{gl.dim}\, KJ = \mathrm{gl.dim}\, K^\square J = 3$ and $\mathrm{gl.dim}\, KJ_1 = \mathrm{gl.dim}\, K^\square J_1 = 3$. We will show that the incidence K-algebras KJ, KJ_1 and the incidence coalgebras $K^\square J$, $K^\square J_1$ are tame, by proving that the reduced Euler forms $q_J^\bullet, q_{J_1}^\bullet : \mathbb{Z}^7 \to \mathbb{Z}$ are weakly non-negative.

For this purpose, we note that the Euler matrix and the reduced Euler matrix of J are

$$
\overline{\mathbf{C}}_J = \begin{bmatrix}
1 & -1 & -1 & 1 & 1 & -1 & -1 \\
0 & 1 & 0 & -1 & -1 & 1 & 1 \\
0 & 0 & 1 & -1 & -1 & 1 & 1 \\
0 & 0 & 0 & 1 & -1 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & -1 & -1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix},
$$

$$\mathbf{C}_J^{\bullet} = \begin{bmatrix} 1 & -1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & -1 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

A straightforward calculation shows that the Coxeter polynomial and the reduced Coxeter–Euler polynomial of J are

$$\operatorname{cox}_J(t) = t^7 + t^6 - t^5 - t^4 - t^3 - t^2 + t + 1,$$
$$\operatorname{cox}_J^{\bullet}(t) = t^7 + 3t^6 + t^5 - 5t^4 - 5t^3 + t^2 + 3t + 1,$$

the Coxeter number $\mathbf{c}_J$ equals 4, whereas the reduced Coxeter–Euler number $\mathbf{c}_J^{\bullet}$ is infinite (see [13, 47, 51, 52]).

The Euler form $\overline{q}_J : \mathbb{Z}^7 \to \mathbb{Z}$ equals

$$\begin{aligned}\overline{q}_J(x) &= x \cdot \mathbf{C}_J^{-1} \cdot x^{tr} \\ &= x_1^2 - (x_1 - x_2)x_2 - (x_1 - x_3)x_3 + (x_1 - x_2 - x_3 + x_4)x_4 \\ &\quad + (x_1 - x_2 - x_3 + x_5)x_5 - (x_1 - x_2 - x_3 + x_4 + x_5 - x_6)x_6 \\ &\quad - (x_1 - x_2 - x_3 + x_4 - x_5 - x_7)x_7 \\ &= \big(x_1 - \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 + \tfrac{1}{2}x_4 + \tfrac{1}{2}x_5 - \tfrac{1}{2}x_6 - \tfrac{1}{2}x_7 + \tfrac{1}{2}x_8\big)^2 \\ &\quad + \tfrac{3}{4}\big(x_2 - \tfrac{1}{3}x_3 - \tfrac{1}{3}x_4 - \tfrac{1}{3}x_5 + \tfrac{1}{3}x_6 + \tfrac{1}{3}x_7\big)^2 \\ &\quad + \tfrac{2}{3}\big(x_3 - \tfrac{1}{2}x_4 - \tfrac{1}{2}x_5 + \tfrac{1}{2}x_6 + \tfrac{1}{2}x_7\big)^2 + \tfrac{1}{2}(x_4 - x_5)^2 + \tfrac{1}{2}(x_6 - x_7)^2.\end{aligned}$$

It follows that the form $\overline{q}_J$ is non-negative of corank two (see [13]).

The reduced Euler form $q_J^{\bullet} : \mathbb{Z}^7 \to \mathbb{Z}$ equals

$$\begin{aligned}q_J^{\bullet}(x) &= x \cdot \mathbf{C}_J^{\bullet} \cdot x^{tr} \\ &= x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 - x_1x_2 - x_1x_3 + x_1x_4 + x_1x_5 \\ &\quad - x_2x_4 - x_2x_5 + x_2x_6 + x_2x_7 - x_3x_4 - x_3x_5 + x_3x_6 + x_3x_7 \\ &\quad - x_4x_6 - x_4x_7 - x_5x_6 - x_5x_7 \\ &= \big(x_1 - \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 + \tfrac{1}{2}x_4 + \tfrac{1}{2}x_5\big)^2 \\ &\quad + \tfrac{3}{4}\big(x_2 - \tfrac{1}{3}x_3 - \tfrac{1}{3}x_4 - \tfrac{1}{3}x_5 + \tfrac{2}{3}x_6 + \tfrac{2}{3}x_7\big)^2 \\ &\quad + \tfrac{2}{3}\big(x_3 - \tfrac{1}{2}x_4 - \tfrac{1}{2}x_5 + x_6 + x_7\big)^2 + \tfrac{1}{2}(x_4 - x_5)^2 \\ &\quad - 2\big(x_5 - \tfrac{1}{2}x_6 - \tfrac{1}{2}x_7\big)^2 - (2x_6 + 2x_7)^2 + (2x_6 - 2x_7)^2 \\ &\quad + \tfrac{1}{2}(2x_5 - x_6 - x_7)^2.\end{aligned}$$

It follows that the form $q_J^{\bullet}$ is indefinite.

On the other hand, $q_J^{\bullet}$ is weakly non-negative, because $2q_J^{\bullet}$ has the following semi-canonical Lagrange form:

$$\begin{aligned}2q_J^{\bullet}(x) = x_1^2 + (x_1 - x_2 - x_3 + x_4 + x_5 - x_6 - x_7)^2 \\ + (x_2 - x_3)^2 + (x_4 - x_5)^2 + (x_6 - x_7)^2 + 2x_1x_7.\end{aligned}$$

It follows that the form $q_J^{\bullet}(x) - \frac{1}{2}x_1^2$ is weakly non-negative and, by [20, 44, 47] and Theorem 3.9, the incidence algebra KJ and the incidence coalgebra $K^{\square}J$ are tame.

Analogously, one shows that the Coxeter polynomial and the reduced Coxeter–Euler polynomial of J_1 are

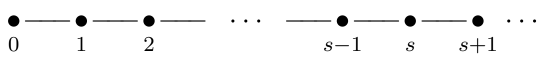
$$\begin{aligned}\operatorname{cox}_{J_1}(t) = \operatorname{cox}_J(t) = t^7 + t^6 - t^5 - t^4 - t^3 - t^2 + t + 1, \\ \operatorname{cox}^{\bullet}_{J_1}(t) = \operatorname{cox}^{\bullet}_J(t) = t^7 + 3t^6 + t^5 - 5t^4 - 5t^3 + t^2 + 3t + 1.\end{aligned}$$

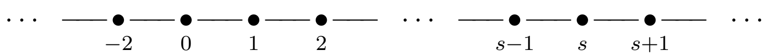
The Coxeter number $\mathbf{c}_{J_1}$ equals 4, whereas the reduced Coxeter–Euler number $\mathbf{c}^{\bullet}_{J_1}$ is infinite. The Euler form $\overline{q}_{J_1} : \mathbb{Z}^7 \to \mathbb{Z}$ is non-negative of corank two, whereas the reduced Euler form $q^{\bullet}_{J_1} : \mathbb{Z}^7 \to \mathbb{Z}$ is indefinite and weakly non-negative. Moreover, one shows that the form $q^{\bullet}_{J_1}(x) - \frac{1}{2}x_8^2$ is weakly non-negative, where $x = (x_2, \ldots, x_8)$. Then, by [20, 44, 47], and Theorem 3.9, the incidence algebra KJ_1 and the incidence coalgebra $K^{\square}J_1$ are tame.

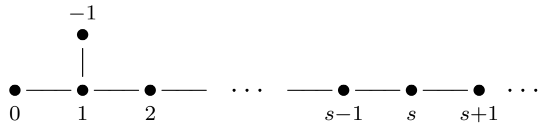
Now we present a description of all infinite connected interval finite posets I such that the coalgebra $K^{\square}I$ is tame of discrete comodule type.

THEOREM 3.17. *Assume that I is an infinite connected interval finite poset and K is an algebraically closed field. Then the following conditions are equivalent:*

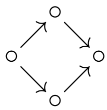
(a) *The coalgebra $K^{\square}I$ is tame of discrete comodule type.*
(a′) *The coalgebra $K^{\square}I$ is fc-tame of discrete comodule type.*
(b) *The Euler form $\overline{q}_I : \mathbb{Z}^{(I)} \to \mathbb{Z}$ is weakly positive.*
(c) *The coalgebra $K^{\square}I$ is left representation-directed* (*see* [43]).
(d) *Given a finite convex subposet U of I, the incidence algebra KU is representation-finite and U is a subposet of one of the representation-finite Loupias–Zavadski–Shkabara posets presented in* [10, 23, 59].
(e) *The poset I has one of the following two properties:*
 (e1) gl.dim $K^{\square}I = 1$ *and I is one of the locally Dynkin posets*

$\mathbb{A}_{\infty}$:


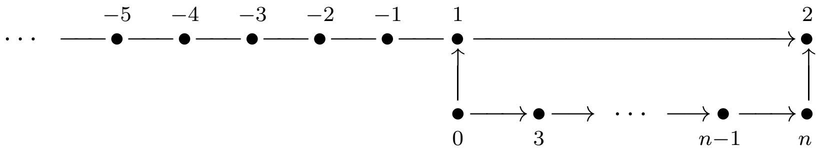
$_{\infty}\mathbb{A}_{\infty}$:


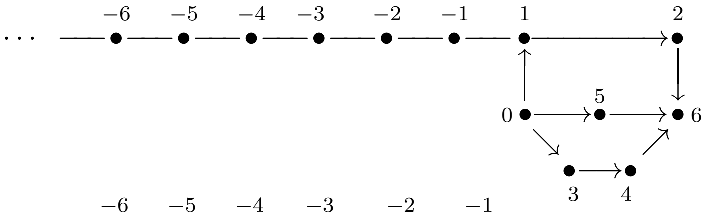
$\mathbb{D}_\infty$: 

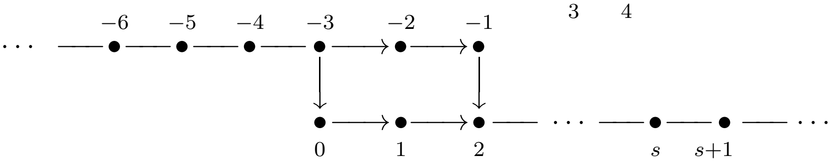
(e2) gl.dim $K^\square I = 2$, *I contains a subposet isomorphic to*



and I or I^{op} is a subposet of any of the following three posets:

${}_\infty\mathbb{D}\mathbb{A}_n$: 

${}_\infty\mathbb{D}\mathbb{D}_5$: 

${}_\infty\mathbb{D}\mathbb{D}_\infty$: 

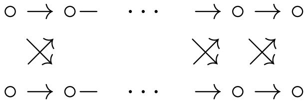
where $n \geq 3$ and $\bullet\!-\!\!-\!\bullet$ *means* $\bullet\!\longrightarrow\!\bullet$ *or* $\bullet\!\longleftarrow\!\bullet$.

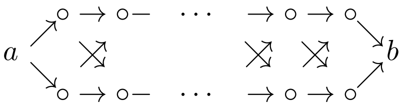
Proof. Apply Theorem 3.2(b) and the results proved in [43]. ■

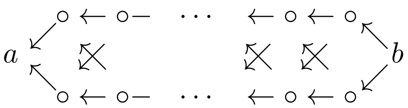
REMARK 3.18. If $K^\square I$ is tame of discrete comodule type as in Theorem 3.16 then gl.dim $K^\square I \leq 2$ and the Euler form $\overline{q}_I$ coincides with the reduced Euler form $q_I^\bullet$.

4. A classification result. In this section we present a complete list of all infinite connected interval finite posets that are $\widetilde{\mathbb{A}}_m^*$-free and have $K^\square I$ of tame comodule type, where K is an algebraically closed field. We need the following notation.

Given $m \geq 1$, by the *non-complete garland* and the *completed garlands* we mean the posets

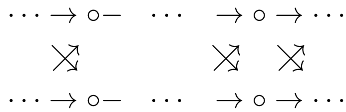
$\mathcal{G}_m$:  ($2m$ points, $m \geq 1$)

$a \to \mathcal{G}_m \to b:$  $(2m+2$ points, $m \geq 1)$

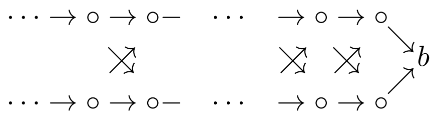
$a \leftarrow \mathcal{G}_m \leftarrow b:$  $(2m+2$ points, $m \geq 1)$

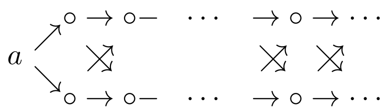
We denote by $a \to \boxed{\mathcal{G}_m} \to b$ (resp. $a \leftarrow \boxed{\mathcal{G}_m} \leftarrow b$) any connected subposet of the completed garland $a \to \mathcal{G}_m \to b$ (resp. $a \leftarrow \mathcal{G}_m \leftarrow b$) that contains its starting point a and the terminal point b. Moreover, by $a \to \boxed{\mathcal{G}_0} \to b$ (resp. $a \leftarrow \boxed{\mathcal{G}_0} \leftarrow b$) we mean the poset $a \to b$ (resp. $a \leftarrow b$).

By the *infinite two-sided unbounded garland* we mean the poset

(4.1) $\mathcal{G}:$ 

By the *infinite left* (resp. *right*) *unbounded garland* we mean the posets

${}_\infty\mathcal{G} \to b:$ 

$a \to \mathcal{G}_\infty:$ 

Now we are able to prove the following classification theorem.

Theorem 4.2. *Assume that I is a connected interval finite poset and K is an algebraically closed field. Moreover, assume that I is infinite and $\widetilde{\mathbb{A}}_m^*$-free.*

(a) *The following conditions are equivalent:*

(a1) *The coalgebra $K^\square I$ is of tame comodule type.*

(a2) *The reduced Euler form $q_I^\bullet : \mathbb{Z}^{(I)} \to \mathbb{Z}$ is weakly non-negative.*

(a3) *The reduced Euler form $q_U^\bullet : \mathbb{Z}^U \to \mathbb{Z}$ is weakly non-negative for any finite convex subposet U of I.*

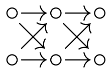
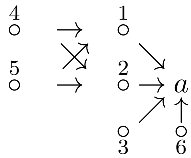
(a4) *I is a connected subposet of one of the posets $\mathcal{G}^{(0)}$, $\mathcal{G}_s^{(1)}$, $\mathcal{G}_{s+1}^{(2)}$, $\mathcal{G}_\infty^{(3)}$, $\mathcal{G}_{s,\infty}^{(4)}$, and $\mathcal{G}_{s+1,\infty}^{(5)}$ in Table 4.4, or of one of their duals ${}^\circ\mathcal{G}_s^{(1)}$, ${}^\circ\mathcal{G}_{s+1}^{(2)}$, ${}^\circ\mathcal{G}_\infty^{(3)}$, ${}^\circ\mathcal{G}_{s,\infty}^{(4)}$, and ${}^\circ\mathcal{G}_{s+1,\infty}^{(5)}$, with some $s \geq 1$.*

(b) *If $K^\square I$ is of tame comodule type then the following three conditions are equivalent:*

(b1) *$K^{\square}I$ is tame of non-polynomial growth.*

(b2) *There exists a finite convex subposet U of I such that the incidence algebra KU is tame of non-polynomial growth.*

(b3) *I contains one of the following two posets:*

(4.3) $\mathcal{G}_3$:  $(\mathcal{NZ})^*$: 

or one of the three pg-critical posets in [20, (2.4a)–(2.4c)] (*see also* [28, 54]), *and I is a connected subposet of one of the posets $\mathcal{G}^{(0)}$, $\mathcal{G}^{(1)}_s$, $\mathcal{G}^{(2)}_{s+1}$, $\mathcal{G}^{(3)}_\infty$, $\mathcal{G}^{(4)}_{s,\infty}$, $\mathcal{G}^{(5)}_{s+1,\infty}$ in Table* 4.4, *or of one of their duals, with some $s \geq 1$.*

Proof. (a) By Theorem 3.9, the coalgebra $K^{\square}I$ is of tame comodule type if and only if, for any finite convex subposet U of I, the coalgebra $K^{\square}U$ is of tame comodule type; equivalently, if and only if the incidence algebra KU is tame (see [38]). Thus the equivalences (a1)⇔(a2)⇔(a3) are immediate consequences of Theorem 3.9.

(a1)⇒(a4). Assume that I is infinite, connected and $\widetilde{\mathbb{A}}^*_m$-free, and $K^{\square}I$ is of tame comodule type. By the observation made earlier, the incidence algebra KU is tame for any finite convex subposet U of I.

Finite connected $\widetilde{\mathbb{A}}^*_m$-free posets with this property are completely described in [20]. It is also proved there that for such a poset U, the incidence algebra KU is tame if and only if the integral Tits quadratic form $q_U : \mathbb{Z}^U \to \mathbb{Z}$ (in the sense of Bongartz [3]) is weakly non-negative. It is shown in [47] that the Tits form q_U coincides with the reduced Euler form $q^{\bullet}_U$. Then, using the description of connected $\widetilde{\mathbb{A}}^*_m$-free posets (given in [20]) with $q^{\bullet}_U$ weakly non-negative, simple combinatorial arguments show that I or I^{op} is a subposet of one of the posets in Table 4.4.

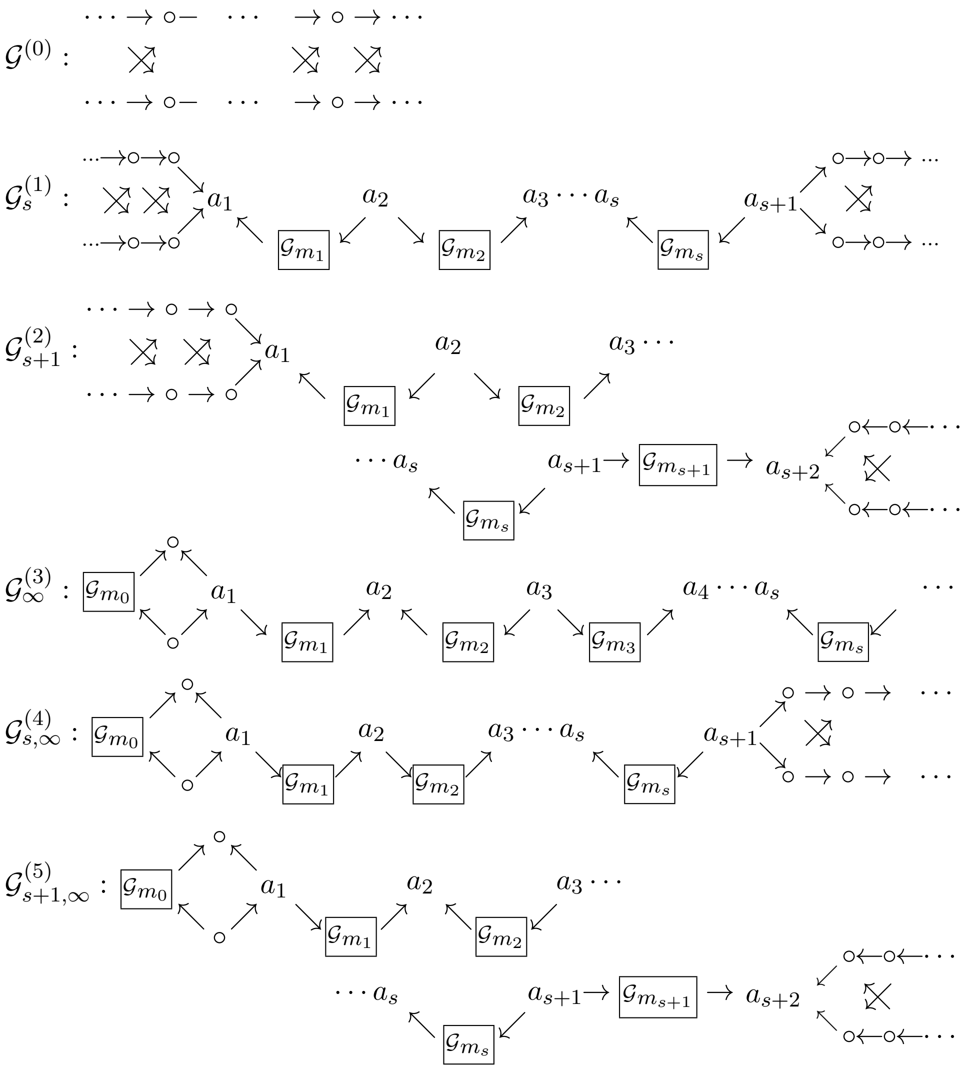
(a4)⇒(a1). Assume that I is a connected $\widetilde{\mathbb{A}}^*_m$-free infinite poset such that I or I^{op} is a subposet of one of the posets in Table 4.4. It follows from the results in [20] that the reduced Euler form $q^{\bullet}_U : \mathbb{Z}^U \to \mathbb{Z}$ is weakly non-negative for any connected convex finite subposet U of I, and so the incidence algebra KU is tame. By [44, Corollary 2.13], this implies that $K^{\square}U$ is of tame comodule type. Hence, by Theorem 3.9, so is $K^{\square}I$. This finishes the proof of (a).

Since the proof essentially depends on the classification given in [20], we give later (in Section 5) an alternative proof of the implication (a4)⇒(a1) by showing that the reduced Euler form is weakly non-negative for any poset of Table 4.4, and for each of its connected subposets.

(b) Assume that J is $\mathcal{G}_3$, $(\mathcal{NZ})^*$, or one of the three *pg*-critical posets in [20, (2.4a)–(2.4c)] (see also [21, 22, 28, 54]). Then $\mathrm{gl.dim}\, KJ = 2$ and the reduced Euler form $q_J^\bullet : \mathbb{Z}^J \to \mathbb{Z}$ coincides with its Euler form $\overline{q}_J$ (see [47]). We recall from [20, Lemma 2.4], [35] and [36, Section 5] that the algebra KJ is tame of non-polynomial growth. Hence, in view of Theorem 3.2, Theorem 3.9, and the results in [44], the implications (b3)⇒(b2)⇔(b1) follow.

In view of the equivalence (a1)⇔(a4), the implication (b1)⇒(b3) is a consequence of [35] and the description of finite $\widetilde{\mathbb{A}}_m^*$-free connected representation-tame posets in [20–22] (see also [53]). ∎

Table 4.4. Infinite connected $\widetilde{\mathbb{A}}_m^*$-free posets I with weakly non-negative reduced Euler form $q_I^\bullet : \mathbb{Z}^{(I)} \to \mathbb{Z}$



We also add to Table 4.4 the posets ${}^\circ\mathcal{G}_s^{(1)}$, ${}^\circ\mathcal{G}_{s+1}^{(2)}$, ${}^\circ\mathcal{G}_\infty^{(3)}$, ${}^\circ\mathcal{G}_{s,\infty}^{(4)}$, and ${}^\circ\mathcal{G}_{s+1,\infty}^{(5)}$ dual to $\mathcal{G}_s^{(1)}$, $\mathcal{G}_{s+1}^{(2)}$, $\mathcal{G}_\infty^{(3)}$, $\mathcal{G}_{s,\infty}^{(4)}$, and $\mathcal{G}_{s+1,\infty}^{(5)}$, respectively.

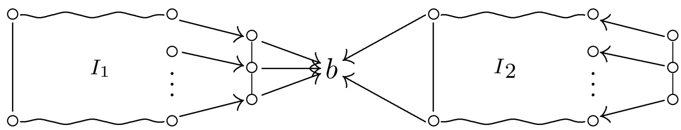
REMARK 4.5. (a) If I is one of the five posets $\mathcal{G}^{(0)}$, $\mathcal{G}_s^{(1)}$, $\mathcal{G}_{s+1}^{(2)}$, $\mathcal{G}_{s,\infty}^{(4)}$, and $\mathcal{G}_{s+1,\infty}^{(5)}$, then gl.dim $K^\square I$ is infinite, whereas gl.dim $K^\square \mathcal{G}_\infty^{(3)}$ is finite.

(b) If I is the infinite garland $\mathcal{G}^{(0)}$, then the category $K^\square I$-Comod has no non-zero projective objects and no non-zero flat objects (see [6, 11, 27, 40].

Now we show that the reduced Euler form is weakly non-negative for any poset of Table 4.4, and for each of its connected subposets. We start with the following reduction lemma that is analogous to the peak reflection result proved in [31] and applied in [12].

LEMMA 4.6. *Assume that I is a poset and $b \in I$ is a point such that the left cone ${}^{\trianglerighteq}b$ is of the form*

${}^{\trianglerighteq}b$:

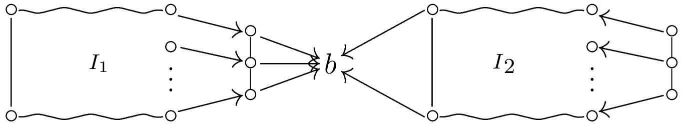


and all points $j_1 \in I_1 \setminus \{b\}$ are incomparable with all points $j_2 \in I_2 \setminus \{b\}$. Moreover, assume that $I = I_1 \cup J$ and J is a disjoint union $I_2 \cup T$ with T such that $I_1 \cap T$ is empty. Denote by $I'_b = I_1 \cup J^{\mathrm{op}}$ the poset obtained from I by replacing the subposet J with its opposite J^{op}.

(a) *The reduced Euler form $q_I^\bullet : \mathbb{Z}^{(I)} \to \mathbb{Z}$ coincides with the form $q_{I'_b}^\bullet$: $\mathbb{Z}^{(I'_b)} \to \mathbb{Z}$ under the obvious identification $\mathbb{Z}^{(I)} \equiv \mathbb{Z}^{(I'_b)}$. In particular, $q_{I'_b}^\bullet$ is weakly non-negative if and only if $q_I^\bullet$ is.*

(b) *If I is finite, the reduced Coxeter–Euler polynomials $\mathrm{cox}_I^\bullet(t)$ and $\mathrm{cox}_{I'_b}^\bullet(t)$ coincide.*

Proof. By our assumption, I is of the shape ${}^{\trianglerighteq}b \cup T$, that is,



$\cup\, T$

Without loss of generality we may assume that I is finite, the points of I_1 are numbered by $1, \ldots, m$, $b = m$, the points of I_2 are numbered by $m = b$, $m+1, \ldots, m+s$, and the points of T are numbered by $m+s+1, m+s+2, \ldots, r$. Moreover, we assume that $j_1 \prec j'_1$ implies $j_1 <_{\mathbb{N}} j'_1$ in the natural order for

$j_1, j_1' \in I_1$, and $j_2 \prec j_2'$ implies $j_2 >_{\mathbb{N}} j_2'$ for $j_2, j_2' \in J := I_2 \cup T$. It follows that the reduced Euler matrix $\mathbf{C}_I^\bullet \in \mathbb{M}_{m+s+r}(\mathbb{Z})$ is of the form

$$(4.7)\qquad \mathbf{C}_I^\bullet = \left[\begin{array}{ccc|cccccc} \bar{c}_{1\,1} & \cdots & \bar{c}_{1\,b-1} & \bar{c}_{1\,b} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \bar{c}_{b-1\,1} & \cdots & \bar{c}_{b-1\,b-1} & \bar{c}_{b-1\,b} & 0 & 0 & \cdots & 0 & 0 \\ \hline \bar{c}_{b\,1} & \cdots & \bar{c}_{b\,b-1} & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \cdots & 0 & & 1 & & & & \\ 0 & \cdots & 0 & & & 1 & & & \mathbf{O} \\ \vdots & \ddots & \vdots & & & & \ddots & & \\ 0 & \cdots & 0 & & & & & 1 & \\ 0 & \cdots & 0 & \mathbf{C}_J^\bullet & & & & & 1 \end{array}\right].$$

By applying the definition of the reduced Euler matrix we can show that $\mathbf{C}_{I_b'}^\bullet$ is obtained from $\mathbf{C}_I^\bullet$ by replacing its lower right corner

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ & 1 & & & & \\ & & 1 & & & \mathbf{O} \\ & & & \ddots & & \\ & & & & 1 & \\ \mathbf{C}_J^\bullet & & & & & 1 \end{bmatrix}$$

with $(\mathbf{C}_J^\bullet)^{tr} = \mathbf{C}_{J^{\mathrm{op}}}^\bullet$. When J is the chain

$$b \leftarrow m+1 \leftarrow m+2 \leftarrow \cdots \leftarrow m+s,$$

a detailed proof is given in [12, pp. 88–89].

It follows that $\mathbf{C}_{I_b'}^\bullet + (\mathbf{C}_{I_b'}^\bullet)^{tr} = \mathbf{C}_I^\bullet + (\mathbf{C}_I^\bullet)^{tr}$ and $2q_{I_b'}^\bullet(x) = x \cdot \mathbf{C}_{I_b'}^\bullet \cdot x^{tr} = x \cdot (\mathbf{C}_{I_b'}^\bullet + (\mathbf{C}_{I_b'}^\bullet)^{tr}) \cdot x^{tr} = x \cdot (\mathbf{C}_I^\bullet + (\mathbf{C}_I^\bullet)^{tr}) \cdot x^{tr} = x \cdot \mathbf{C}_I^\bullet \cdot x^{tr} = 2q_I^\bullet(x)$. Hence (a) follows.

To prove (b), assume that I is finite and note that $\mathbf{C}_{I_b'}^\bullet = T \cdot \mathbf{C}_I^\bullet \cdot T$, where

$$T = \left[\begin{array}{ccc|cccc} 1 & & \mathbf{O} & 0 & 0 & \cdots & 0 \\ & \ddots & & \vdots & \vdots & \ddots & \vdots \\ \mathbf{O} & & 1 & 0 & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & \mathbf{O} & & & 1 \\ 0 & \cdots & 0 & & & 1 & \\ \vdots & \ddots & \vdots & & \iddots & & \\ 0 & \cdots & 0 & 1 & & & \mathbf{O} \end{array}\right].$$

Since $T^{tr} = T^{-1} = T$, we have $\mathbf{C}_{I_b'}^\bullet = T^{tr} \cdot \mathbf{C}_I^\bullet \cdot T$ and a simple calculation shows that $\mathrm{Cox}_{I_b'}^\bullet = T \cdot \mathrm{Cox}_I^\bullet \cdot T$. Hence $\mathrm{cox}_{I_b'}^\bullet(t) = \mathrm{cox}_I^\bullet(t)$. ■

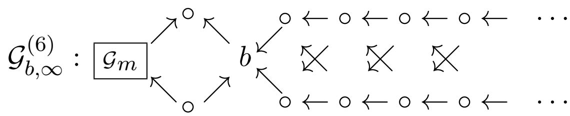
Corollary 4.8. *Assume that I is a finite connected subposet of one of the posets $\mathcal{G}^{(0)}$, $\mathcal{G}_s^{(1)}$, $\mathcal{G}_{s+1}^{(2)}$, $\mathcal{G}_\infty^{(3)}$, $\mathcal{G}_{s,\infty}^{(4)}$, $\mathcal{G}_{s+1,\infty}^{(5)}$ in Table 4.4, or of one of their duals, with some $s \geq 1$.*

(a) *The reduced Euler form $q_I^\bullet : \mathbb{Z}^I \to \mathbb{Z}$ is weakly non-negative.*

(b) *There exists a finite connected subposet I' of $\mathcal{G}^{(0)}$ or of $\mathcal{G}_{b,\infty}^{(6)}$ given below such that $\mathrm{cox}_I^\bullet(t) = \mathrm{cox}_{I'}^\bullet(t)$.*

Proof. First we assume that I is a finite connected subposet of $\mathcal{G}_s^{(1)}$ or $\mathcal{G}_{s+1}^{(2)}$, or their dual posets. By Lemma 4.6, the weak non-negativity of $q_I^\bullet$ reduces to the weak non-negativity of $q_{I'}^\bullet$ for a finite connected subposet I' of $\mathcal{G}^{(0)}$ with $\mathrm{cox}_I^\bullet(t) = \mathrm{cox}_{I'}^\bullet(t)$.

Next we assume that I is a finite connected subposet of $\mathcal{G}_\infty^{(3)}$, $\mathcal{G}_{s,\infty}^{(4)}$, or $\mathcal{G}_{s+1,\infty}^{(5)}$, or of one of their duals. By applying an obvious extension of Lemma 4.6, one reduces the weak non-negativity for I to the weak non-negativity for a finite connected subposet I' of

(4.9) 

with $\mathrm{cox}_I^\bullet(t) = \mathrm{cox}_{I'}^\bullet(t)$. Consequently, it remains to prove that if I is a finite connected subposet of $\mathcal{G}^{(0)}$ or of $\mathcal{G}_{b,\infty}^{(6)}$ then $q_I^\bullet$ is weakly non-negative. This is implicitly proved in [20]. On the other hand, one can prove it by applying the Dean–de la Peña algorithm [8], or directly by induction on the number of points in I. For the convenience of the reader we present a proof in Corollary 5.3 below. ∎

5. Weak non-negativity of the reduced Euler form of garlands. The aim of this section is to prove that the reduced Euler form $q_I^\bullet : \mathbb{Z}^{(I)} \to \mathbb{Z}$ of any connected subposet I of any of the infinite posets of Table 4.4 is weakly non-negative. The problem obviously reduces to the case when I is finite. Moreover, it was shown in the proof of Corollary 4.8 that the problem reduces to the case when I is a subposet of the two-sided infinite garland $\mathcal{G}$ or of $\mathcal{G}_{b,\infty}^{(6)}$. The proof uses the following two technical (but useful) propositions.

Proposition 5.1. *Assume that I is a connected subposet of the garland $\mathcal{G}_m$.*

(a) *For any $m \geq 1$, the form $q_I^\bullet : \mathbb{Z}^I \to \mathbb{Z}$ is weakly non-negative.*

(b) *If J is a connected subposet of the completed garland $\widehat{\mathcal{G}}_m : a \to \boxed{\mathcal{G}_m} \to b$ with $m \geq 1$, and J contains the point a (resp. b), then the* *forms* $q_J^\bullet : \mathbb{Z}^I \to \mathbb{Z}$ *and* $q_J^\bullet(x) - \frac{1}{2}x_a^2$ *(resp.* $q_J^\bullet(x) - \frac{1}{2}x_b^2$*) are weakly non-negative.*

(c) *If* I *is a connected subposet of* $\widehat{\mathcal{G}}_m : a \to \boxed{\mathcal{G}_m} \to b$ *with* $m \geq 1$*, and* I *contains the points* a *and* b*, then the forms* $q_I^\bullet : \mathbb{Z}^I \to \mathbb{Z}$ *and* $q_I^\bullet(x) - \frac{1}{2}x_a^2 - \frac{1}{2}x_b^2$ *are weakly non-negative.*

Proof. (a1) First we prove (a) for $I = \mathcal{G}_m$. Assume that $m \geq 1$ and the points of the garland $\mathcal{G}_m$ are labelled as follows:

$$\mathcal{G}_m : \quad \begin{array}{ccccccccc} \overset{1}{\circ} & \to & \overset{3}{\circ} & - & \cdots & \to & \overset{2m-3}{\circ} & \to & \overset{2m-1}{\circ} \\ & \boxtimes & & & & \boxtimes & & \boxtimes & \\ \underset{2}{\circ} & \to & \underset{4}{\circ} & - & \cdots & \to & \underset{2m-2}{\circ} & \to & \underset{2m}{\circ} \end{array} \qquad (2m \text{ points}, m \geq 1)$$

We proceed by induction on $m \geq 1$. For $m = 1$, the form $q_{\mathcal{G}_1}^\bullet : \mathbb{Z}^2 \to \mathbb{Z}$ is positive definite, because $q_{\mathcal{G}_1}^\bullet(x_1, x_2) = x_1^2 + x_2^2$. For $m = 2$, the form $q_{\mathcal{G}_2}^\bullet : \mathbb{Z}^4 \to \mathbb{Z}$ is positive semi-definite, because $2q_{\mathcal{G}_2}^\bullet(x_1, x_2, x_3, x_4) = (x_1 + x_2 - x_3 - x_4)^2 + (x_1 - x_2)^2 + (x_3 - x_4)^2$.

For $m = 3$, $I = \mathcal{G}_3$ is the interval closed subposet of the garland $a_1 \to \boxed{\mathcal{G}_3} \to a_8$ of Example 3.15. Since the reduced Euler form of $\mathcal{G}_3$ is obtained from that of $a_1 \to \boxed{\mathcal{G}_3} \to a_8$ by the substitutions $x_1 = 0$ and $x_8 = 0$, the form $q_{\mathcal{G}_3}^\bullet : \mathbb{Z}^6 \to \mathbb{Z}$ of $\mathcal{G}_3$ is weakly non-negative, because it is shown in Example 3.15 that the reduced Euler form of $a_1 \to \boxed{\mathcal{G}_3} \to a_8$ is weakly non-negative. Hence the form $q_I^\bullet : \mathbb{Z}^I \to \mathbb{Z}$ is also weakly non-negative.

Assume that $m \geq 4$ and the claim is proved for all garlands $\mathcal{G}_s$ such that $s \leq m - 1$. Assume, to the contrary, that $q_{\mathcal{G}_m}^\bullet : \mathbb{Z}^{2m} \to \mathbb{Z}$ is not weakly non-negative, with $m \geq 4$ minimal possible. Let $v = (v_1, \ldots, v_{2m}) \in \mathbb{N}^{2m}$ be a non-zero vector such that $q_{\mathcal{G}_m}^\bullet(v) < 0$.

For simplicity of the presentation, we set $\widehat{v}_1 := v_1 + v_2$, $\widehat{v}_3 := v_3 + v_4$, $\widehat{v}_5 := v_5 + x_6, \ldots$. View $I = \mathcal{G}_m$ as the usual extension of the disjoint union

$$\mathcal{G}_1 \cup J : \quad \begin{array}{ccccccccccc} \overset{1}{\circ} & & \overset{3}{\circ} & \to & \circ & \to & \cdots & \to & \overset{2m-3}{\circ} & \to & \overset{2m-1}{\circ} \\ & & & \boxtimes & & \boxtimes & & & \boxtimes & \boxtimes & \\ \underset{2}{\circ} & & \underset{4}{\circ} & \to & \circ & \to & \cdots & \to & \underset{2m-2}{\circ} & \to & \underset{2m}{\circ} \end{array}$$

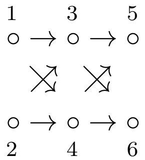
of two subposets, where $J = \mathcal{G}_{m-1}$ is obtained from $\mathcal{G}_m$ by removing $\circ_1$ and $\circ_2$. It follows from our assumption and the definition of $q_{\mathcal{G}_m}^\bullet(x)$ that

$$\begin{aligned} 0 > q_{\mathcal{G}_m}^\bullet(v) &= v_1^2 + v_2^2 + q_J^\bullet(v^-) + \widehat{v}_1 \cdot \widehat{v}_5 - \widehat{v}_1 \cdot \widehat{v}_3 \\ &= v_1^2 + v_2^2 + q_J^\bullet(v^-) + \widehat{v}_1 \cdot (\widehat{v}_5 - \widehat{v}_3), \end{aligned}$$

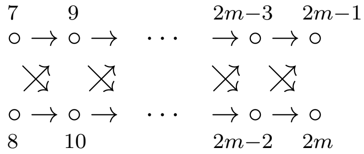
where $v^- = (v_3, \ldots, v_{2m}) \in \mathbb{N}^{2m-2}$. Hence, $\widehat{v}_1 \cdot (\widehat{v}_5 - \widehat{v}_3) < 0$ and $\widehat{v}_5 < \widehat{v}_3$, because $v = (v_1, \ldots, v_{2m}) \in \mathbb{N}^{2m}$ is non-zero and $v_1^2 + v_2^2 + q_J^\bullet(v^-) \geq 0$, by the inductive assumption.

On the other hand, we can view $I = \mathcal{G}_m$ as the usual extension of the disjoint union $J_1 \cup J_2$ of its two subposets

J_1 :



and J_2 :



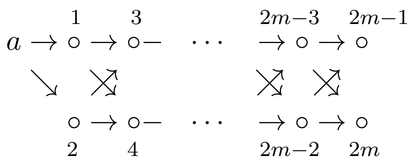
It follows from our assumption and the definition of $q_{\mathcal{G}_m}^\bullet(x)$ that

$$\begin{aligned} 0 > q_{\mathcal{G}_m}^\bullet(v) &= q_{J_1}^\bullet(v') + q_{J_2}^\bullet(v'') + \widehat{v}_3 \cdot \widehat{v}_7 + \widehat{v}_5 \cdot \widehat{v}_8 - \widehat{v}_5 \cdot \widehat{v}_7 \\ &= q_{J_1}^\bullet(v') + q_{J_2}^\bullet(v'') + (\overline{v}_3 - \overline{v}_5) \cdot \widehat{v}_7 + \widehat{v}_5 \cdot \widehat{v}_9 \\ &\geq q_{J_1}^\bullet(v') + q_{J_2}^\bullet(v'') + (\overline{v}_3 - \overline{v}_5) \cdot \widehat{v}_7, \end{aligned}$$

where $v' = (v_1, v_2, v_3, v_4) \in \mathbb{N}^4$ and $v'' = (v_5, \ldots, v_{2m}) \in \mathbb{N}^{2m-4}$. Hence $q_{J_1}^\bullet(v') + q_{J_2}^\bullet(v'') + (\overline{v}_3 - \overline{v}_5) \cdot \widehat{v}_7 < 0$. Since the inductive hypothesis yields $q_{J_1}^\bullet(v') + q_{J_2}^\bullet(v'') \geq 0$, we have $(\overline{v}_3 - \overline{v}_5) \cdot \widehat{v}_7 < 0$ and consequently $\overline{v}_3 < \overline{v}_5$, contrary to $\widehat{v}_5 < \widehat{v}_3$ obtained earlier. Here we note that $\widehat{v}_7 > 0$, because otherwise $v_7 = 0$, $v_8 = 0$, and we can replace the garland J_2 by a smaller one, and the inductive assumption applies. This finishes the inductive step, and therefore (a) follows for $I = \mathcal{G}_m$.

(b1) Next we prove (b) for the left completed garland $J = (a \to \mathcal{G}_m)$ and for the right completed garland $J = (\mathcal{G}_m \to b)$. Since the second case is dual to the first, we prove (b1) for $J = (a \to \mathcal{G}_m)$. Assume that $m \geq 1$ and the points of $J = (a \to \mathcal{G}_m)$ are labelled as follows:

$a \to \mathcal{G}_m$:



($2m + 1$ points, $m \geq 1$)

Now we prove by induction on $m \geq 1$ that the form $q_J^\bullet(x) - \frac{1}{2}x_a^2$ is weakly non-negative. For $m = 2$, we have

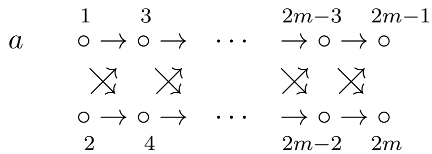
$$\begin{aligned} q_J^\bullet(x_a, x_1, x_2, x_3, x_4) &= \overline{q}_J(x_a, x_1, x_2, x_3, x_4) \\ &= x_a^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_a x_1 - x_a x_2 \\ &\quad - (x_1 + x_2)(x_3 + x_4) + x_a x_3 + x_a x_4 \\ &= \tfrac{1}{2}(x_a - x_1 - x_2 + x_3 + x_4)^2 \\ &\quad + \tfrac{1}{2}x_a^2 + \tfrac{1}{2}(x_1 - x_2)^2 + \tfrac{1}{2}(x_3 - x_4)^2. \end{aligned}$$

It follows that the form $q^\bullet_J(x) - \frac{1}{2}x_a^2$ is positive semi-definite, hence weakly non-negative. Thus (b1) follows for $m = 2$ and also for $m = 1$.

Assume that $m \geq 3$, J is $a \to \mathcal{G}_m$ and the claim is proved for all $a \to \mathcal{G}_s$ such that $s \leq m-1$. Assume, to the contrary, that $q^\bullet_J : \mathbb{Z}^{2m+1} \to \mathbb{Z}$ is not weakly non-negative, with $m \geq 3$ minimal possible. Let $v = (v_a, v_1, \dots, v_{2m}) \in \mathbb{N}^{2m+1}$ be a non-zero vector such that $q^\bullet_J(v) < 0$. Now we follow the proof of (a1) keeping the notation introduced there.

View $J = (a \to \mathcal{G}_m)$ as the usual extension of the disjoint union

$\{a\} \cup \mathcal{G}_m$:



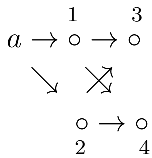
of two subposets, where $\mathcal{G}_m$ is obtained from J by removing a. It follows from (a1) and the definition of $q^\bullet_J(x)$ that

$$0 > q^\bullet_J(v) - \tfrac{1}{2}v_a^2 = \tfrac{1}{2}v_a^2 + q^\bullet_{\mathcal{G}_m}(v^-) + v_2 \cdot (\widehat{v}_2 - \widehat{v}_1),$$

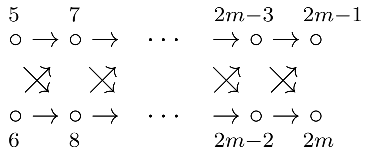
where $v^- = (v_1, \dots, v_{2m}) \in \mathbb{N}^{2m}$. Hence, $v_2 \cdot (\widehat{v}_2 - \widehat{v}_1)$ and $\widehat{v}_2 < \widehat{v}_1$, because $v = (v_1, \dots, v_{2m}) \in \mathbb{N}^{2m}$ is non-zero and $\frac{1}{2}v_a^2 + q^\bullet_{\mathcal{G}_m}(v^-) \geq 0$, by (a1).

On the other hand, we can view $J = (a \to \mathcal{G}_m)$ as the usual extension of the disjoint union $J_1 \cup J_2$ of

J_1 :



and J_2 :



It follows from (a1) and the definitions of $q^\bullet_J(x)$ that

$$\begin{aligned} 0 > q^\bullet_J(v) &= q^\bullet_{J_1}(v') - \tfrac{1}{2}v_a^2 + q^\bullet_{J_2}(v'') + \widehat{v}_1 \cdot \widehat{v}_3 - \widehat{v}_2 \cdot \widehat{v}_3 + \widehat{v}_2 \cdot \widehat{v}_5 \\ &= q^\bullet_{J_1}(v') - \tfrac{1}{2}v_a^2 + q^\bullet_{J_2}(v'') + (\widehat{v}_1 - \widehat{v}_2) \cdot \widehat{v}_3 + \widehat{v}_2 \cdot \widehat{v}_5 \\ &\geq q^\bullet_{J_1}(v') - \tfrac{1}{2}v_a^2 + q^\bullet_{J_2}(v'') + (\widehat{v}_1 - \widehat{v}_2) \cdot \widehat{v}_3, \end{aligned}$$

where $v' = (v_a, v_1, v_2, v_3, v_4) \in \mathbb{N}^5$ and $v'' = (v_5, \dots, v_{2m}) \in \mathbb{N}^{2m-2}$. Hence $q^\bullet_{J_1}(v') - \frac{1}{2}v_a^2 + q^\bullet_{J_2}(v'') + (\overline{v}_1 - \overline{v}_2) \cdot \widehat{v}_3 < 0$. Since (a) and the inductive hypothesis yield $q^\bullet_{J_1}(v') - \frac{1}{2}v_a^2 + q^\bullet_{J_2}(v'') \geq 0$, we have $(\widehat{v}_1 - \widehat{v}_2) \cdot \widehat{v}_3 < 0$ and so $\widehat{v}_1 < \widehat{v}_2$, contrary to $\widehat{v}_2 < \widehat{v}_1$ obtained earlier. This finishes the inductive step, and thus (a2) follows for $J = (a \to \mathcal{G}_m)$ with $m \geq 1$.

(c3) Now we prove (c) when I is the completed garland $\widehat{\mathcal{G}}_m : a \to \boxed{\mathcal{G}_m} \to b$ with $m \geq 1$. More precisely, we prove by induction on $m \geq 1$ that the form $q^\bullet_I(x) - \frac{1}{2}x_a^2 - \frac{1}{2}x_b^2$ is weakly non-negative.

For $m = 2$, $I = \widehat{\mathcal{G}}_2$ can be viewed as the garland

$$I: \quad \begin{array}{ccccccc} & & & a_2 & \to & a_4 & \to & a_6 \\ a_1 & \nearrow & & & \times & & \nearrow & \\ & \searrow & & a_3 & \to & a_5 & & \end{array}$$

obtained from the garland J of Example 3.16 by removing a_7. Hence, $q_I^\bullet(x)$ is obtained from $q_J^\bullet(x)$ by the substitution $x_7 = 0$. Then, in view of Example 3.16, the form $q_I^\bullet : \mathbb{Z}^6 \to \mathbb{Z}$ is weakly non-negative, because $2q_I^\bullet$ has the canonical Lagrange form

$$\begin{aligned} 2q_I^\bullet(x) = x_1^2 &+ (x_1 - x_2 - x_3 + x_4 + x_5 - x_6)^2 \\ &+ (x_2 - x_3)^2 + (x_4 - x_5)^2 + x_6^2. \end{aligned}$$

Moreover, $q_I^\bullet(x) - \frac{1}{2}x_1^2 - \frac{1}{2}x_6^2$ is weakly non-negative. Hence (c3) follows for $m = 1$.

Assume that $m \geq 3$, I is the completed garland $\widehat{\mathcal{G}}_m : a \to \boxed{\mathcal{G}_m} \to b$ and (c3) is proved for all $\widehat{\mathcal{G}}_s$ with $s \leq m - 1$. Assume, to the contrary, that $q_I^\bullet : \mathbb{Z}^{2m+2} \to \mathbb{Z}$ is not weakly non-negative, with $m \geq 3$ minimal possible.

By a simple modification of the arguments used in the proof of (b1), with $q_J^\bullet(v) - \frac{1}{2}v_a^2$ and $q_{\widehat{\mathcal{G}}_m}^\bullet(v) - \frac{1}{2}v_a^2 - \frac{1}{2}v_b^2$ interchanged, we will get a contradiction. We modify the proof of (b1) by replacing the first disjoint union $\{a\} \cup \mathcal{G}_m$ with $\{a\} \cup (\mathcal{G}_m \to b)$, and the second disjoint union $J_1 \cup J_2 = J_1 \cup \mathcal{G}_m$ with $J_1 \cup (\mathcal{G}_m \to b)$. This proves the inductive step and completes the proof of (c3).

Now we will show that, for each connected subposet I of a garland $\mathcal{G}_m$, statements (a) and (b) are consequences of (c). For this purpose, denote by $\widehat{I} := (a_0 \to I \to b_0)$ the poset obtained from I by adding a unique minimal point a_0 and a unique maximal point b_0. Then the form $q_{\widehat{I}}^\bullet : \mathbb{Z}^{\widehat{I}} \to \mathbb{Z}$ is weakly non-negative, by (c). Hence, $q_I^\bullet : \mathbb{Z}^I \to \mathbb{Z}$ is weakly non-negative, because it is the restriction of $q_{\widehat{I}}^\bullet$ to the interval closed subposet I. In case I is as in (b), the proof is analogous. Consequently, to finish the proof, it remains to prove (c).

(c) Assume that I is a connected subposet of the completed garland $\widehat{\mathcal{G}}_m : a \to \boxed{\mathcal{G}_m} \to b$ containing a and b.

CASE 1°. Assume that I is the chain $a \to b_1 \to \cdots \to b_n \to b$. Then $q_I^\bullet : \mathbb{Z}^{n+2} \to \mathbb{Z}$ is positive definite since

$$q_I^\bullet(x) = \frac{1}{2}[x_a^2 + (x_a - x_{b_1})^2 + (x_{b_1} - x_{b_2})^2 + \cdots + (x_{b_n} - x_b)^2 + x_a^2].$$

Hence $q_I^\bullet(x) - \frac{1}{2}x_a^2 - \frac{1}{2}x_b^2$ is weakly non-negative and (c) follows.

Case 2°. Assume that I is the completed garland $\widehat{\mathcal{G}}_m : a \to \boxed{\mathcal{G}_m} \to b$ with $m \geq 1$. Then (c) follows from (c1).

Case 3°. Assume that I is not a chain and is a connected subposet of $\widehat{\mathcal{G}}_m$ containing a and b. We will prove that $q_I^\bullet(x) - \frac{1}{2}x_a^2 - \frac{1}{2}x_b^2$ is weakly non-negative by induction on the cardinality $|I|$ of I. Note that $|I| \geq 4$, because I is not a chain.

If $|I| = 4$ then I is the completed garland $\widehat{\mathcal{G}}_1$, and (c) follows by Case 2°. Assume that $|I| \geq 5$ and (c) has been proved for all connected subposets J of $\widehat{\mathcal{G}}_m : a \to \boxed{\mathcal{G}_m} \to b$ such that J contains a and b, and $|J| < |I|$. We may assume that I is not a completed garland, because otherwise $q_I^\bullet(x) - \frac{1}{2}x_a^2 - \frac{1}{2}x_b^2$ is weakly non-negative, by Case 2°. It follows that I has a waist point $c \notin \{a, b\}$ in the sense of [31], that is, $a \prec c \prec b$ and $I = {}^{\trianglerighteq}c \cup c^{\triangleleft}$. Consequently, I has the waist splitting form (see [32])

$$I = (a \to I_1 \to c \to c_1 \to \cdots \to c_s \to I_2 \to b),$$

where $c = c_0 \to c_1 \to \cdots \to c_s$ is a chain with $s \geq 0$, whereas $\widehat{I}_1 = (a \to I_1 \to c)$ and $\widehat{I}_2 = (c \to c_1 \to \cdots \to c_s \to I_2 \to b)$ are connected subposets of the completed garlands $a \to \boxed{\mathcal{G}_{s_1}} \to c$ and $c \to \boxed{\mathcal{G}_{s_2}} \to b$, respectively. Obviously, $|\widehat{I}_1| < |I|$ and $|\widehat{I}_2| < |I|$.

Since I is not a chain, it contains a pair of incomparable elements; we may assume that they lie in the subposet I_2 and the inductive hypothesis applies to $\widehat{I}_2$.

Since c is a waist point, we have $q_I^\bullet(v) = q_{\widehat{I}_1}^\bullet(v|_{\widehat{I}_1}) + q_{\widehat{I}_2}^\bullet(v|_{\widehat{I}_2}) - v_c^2$ for any $v \in \mathbb{Z}^I$, where $v|_{\widehat{I}_1} \in \mathbb{Z}^{\widehat{I}_1}$ and $v|_{\widehat{I}_2} \in \mathbb{Z}^{\widehat{I}_2}$ are the corresponding restrictions.

When $\widehat{I}_1$ is a chain, the form $q_{\widehat{I}_1}^\bullet(x|_{\widehat{I}_1}) - \frac{1}{2}x_a^2 - \frac{1}{2}x_c^2$ is weakly non-negative by Case 1°. If $\widehat{I}_1$ is not a chain, this form is weakly non-negative by Case 2° and the inductive assumption.

In both cases, we have

$$\begin{aligned} q_I^\bullet(v) - \tfrac{1}{2}v_a^2 - \tfrac{1}{2}v_b^2 &= q_{\widehat{I}_1}^\bullet(v|_{\widehat{I}_1}) + q_{\widehat{I}_2}^\bullet(v|_{\widehat{I}_2}) - v_c^2 - \tfrac{1}{2}v_a^2 - \tfrac{1}{2}v_b^2 \\ &= \big(q_{\widehat{I}_1}^\bullet(v|_{\widehat{I}_1}) - \tfrac{1}{2}v_a^2 - \tfrac{1}{2}v_c^2\big) + \big(q_{\widehat{I}_2}^\bullet(v|_{\widehat{I}_2}) - \tfrac{1}{2}v_c^2 - \tfrac{1}{2}v_b^2\big) \geq 0 \end{aligned}$$

for every $v \in \mathbb{Z}^I$ with non-negative coefficients. This finishes the inductive step and completes the proof of (c). ■

Proposition 5.2. *If I is the poset*

$$\begin{array}{ccccccc} & \nearrow & a_2 & \nwarrow & & & \\ \boxed{\mathcal{G}_m} & & & & b & \leftarrow & \boxed{\mathcal{G}_s} \\ & \nwarrow & a_1 & \nearrow & & & \end{array} \qquad \textit{with } m, s \geq 1,$$

then the reduced Euler form $q_I^\bullet : \mathbb{Z}^I \to \mathbb{Z}$ is weakly non-negative.

Proof. Denote by J the subposet obtained from I by removing $\boxed{\mathcal{G}_s}$, and let T be the subposet $b \leftarrow \boxed{\mathcal{G}_s}$ of I. By our assumption on I, there is a minimal commutativity relation between a_1 and a_2. Then the (a_1, a_2) entry $c^\bullet_{a_1a_2}$ of the reduced Euler matrix $\mathbf{C}^\bullet_I$ equals 1 if $m \neq 1$, and 2 if $m = 1$. Let $J_1 = J \setminus \{b\}$ be the subposet $a_1 \to \boxed{\mathcal{G}_m} \to a_2$ of J. Then, by applying the definition of $q^\bullet_J$, we get

$$\begin{aligned} q^\bullet_J(x) &= q^\bullet_{J_1}(x) - x_{a_1}x_b - x_{a_2}x_b + c^\bullet_{a_1a_2}x_{a_1}x_{a_2} + x_b^2 \\ &\geq \big(q^\bullet_{J_1}(x) - \tfrac{1}{2}x_{a_1}^2 - \tfrac{1}{2}x_{a_2}^2\big) \\ &\quad + \big(\tfrac{1}{2}x_{a_1}^2 + \tfrac{1}{2}x_{a_2}^2 - x_{a_1}x_b - x_{a_2}x_b + x_{a_1}x_{a_2} + \tfrac{1}{2}x_b^2\big) + \tfrac{1}{2}x_b^2 \\ &= \tfrac{1}{2}x_b^2 + \big(q^\bullet_{J_1}(x) - \tfrac{1}{2}x_{a_1}^2 - \tfrac{1}{2}x_{a_2}^2\big) + \tfrac{1}{2}(x_{a_1} + x_{a_2} - x_b)^2. \end{aligned}$$

It follows that $q^\bullet_J(x) - \frac{1}{2}x_b^2 \geq (q^\bullet_{J_1}(x) - \frac{1}{2}x_{a_1}^2 - \frac{1}{2}x_{a_2}^2) + \frac{1}{2}(x_{a_1} + x_{a_2} - x_b)^2$. Thus the form $q^\bullet_J(x) - \frac{1}{2}x_b^2$ is weakly non-negative, because $q^\bullet_{J_1}(x) - \frac{1}{2}x_{a_1}^2 - \frac{1}{2}x_{a_2}^2$ is, by Proposition 5.1(c).

Given $v \in \mathbb{Z}^I$, by the definition of $q^\bullet_I$, we get

$$\begin{aligned} q^\bullet_I(v) &= q^\bullet_J(v|_J) + q^\bullet_T(v|_T) - v_b^2 \\ &= q^\bullet_J(v|_J) - \tfrac{1}{2}v_b^2 + q^\bullet_T(v|_T) - \tfrac{1}{2}v_b^2. \end{aligned}$$

It follows that the form $q^\bullet_I(x)$ is weakly non-negative, because $q^\bullet_J(x) - \frac{1}{2}x_b^2$ and $q^\bullet_T(x|_T) - \frac{1}{2}x_b^2$ are (the latter by Proposition 5.1(b)). ∎

Corollary 5.3. *If I or I^{op} is any of the infinite posets of Table* 4.4 *then the reduced Euler form of I is weakly non-negative.*

Proof. It was shown in the proof of Corollary 4.8 that the problem reduces to the case when I is a connected finite subposet of the two-sided infinite garland $\mathcal{G}$ or of the poset $\mathcal{G}^{(6)}_{b,\infty}$ of (4.9). Then the corollary is a consequence of Propositions 5.1 and 5.2. ∎

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REFERENCES

[1] I. Assem, D. Simson and A. Skowroński, *Elements of the Representation Theory of Associative Algebras, Volume 1. Techniques of Representation Theory*, London Math. Soc. Student Texts 65, Cambridge Univ. Press, Cambridge, 2006.

[2] M. Auslander, I. Reiten and S. Smalø, *Representation Theory of Artin Algebras*, Cambridge Stud. Adv. Math. 36, Cambridge Univ. Press, 1995.

[3] K. Bongartz, *Algebras and quadratic forms*, J. London Math. Soc. 28 (1983), 461–469.

[4] W. Chin and S. Montgomery, *Basic coalgebras*, AMS/IP Studies in Adv. Math. 4 (1997), 41–47.
[5] W. Chin and D. Simson, *Coxeter transformation and inverses of Cartan matrices for coalgebras*, J. Algebra 324 (2010), 2223–2248.
[6] J. Cuadra and D. Simson, *Flat comodules and perfect coalgebras*, Comm. Algebra 35 (2007), 3164–3194.
[7] S. Dascalescu, M. C. Iovanov and C. Nastasescu, *Quiver algebras, path coalgebras and coreflexivity*, Pacific J. Math. 262 (2013), 49–80.
[8] A. Dean and J. A. de la Peña, *Algorithms for weakly nonnegative quadratic forms*, Linear Algebra Appl. 235 (1996), 35-47.
[9] Yu. A. Drozd, *Tame and wild matrix problems*, in: Representations and Quadratic Forms, Akad. Nauk USSR, Inst. Mat., Kiev, 1979, 39–74 (in Russian).
[10] G. Drozdowski and D. Simson, *Remarks on posets of finite representation type*, preprint, Toruń, 1978, 23 pp.; http://www.users.mat.umk.pl/simson/DrozdowskiSimson1978.pdf.
[11] S. Estrada and M. Saorin, *Locally finitely presented categories with no flat objects*, Forum Math. 27 (2015), 269–301.
[12] M. Gąsiorek and D. Simson, *A computation of positive one-peak posets that are Tits-sincere*, Colloq. Math. 127 (2012), 83–103.
[13] M. Gąsiorek, D. Simson and K. Zając, *On Coxeter type study of non-negative posets using matrix morsifications and isotropy groups of Dynkin and Euclidean diagrams*, Eur. J. Combin. 48 (2015), 127–142.
[14] P. Jara, L. Merino and G. Navarro, *Localization and representation theory of coalgebras*, J. Pure Appl. Algebra 211 (2007), 342–359.
[15] S. A. Joni and G. C. Rota, *Coalgebras and bialgebras in combinatorics*, in: Contemp. Math. 6, Amer. Math. Soc., 1982, 1–47.
[16] S. Kasjan and D. Simson, *Mesh algorithms for Coxeter spectral classification of Cox-regular edge-bipartite graphs with loops, I. Mesh root systems*, Fund. Inform. 139 (2015), 153–184.
[17] S. Kasjan and D. Simson, *Mesh algorithms for Coxeter spectral classification of Cox-regular edge-bipartite graphs with loops, II. Application to Coxeter spectral analysis*, Fund. Inform. 139 (2015), 185–209.
[18] M. Kleiner and I. Reiten, *Abelian categories, almost split sequences and comodules*, Trans. Amer. Math. Soc. 357 (2005), 3201–3214.
[19] J. Kosakowska and D. Simson, *Hereditary coalgebras and representations of species*, J. Algebra 293 (2005), 457–505.
[20] Z. Leszczyński, *The completely separating incidence algebras of tame representation type*, Colloq. Math. 94 (2002), 243–262.
[21] Z. Leszczyński, *Representation-tame incidence algebras of finite posets*, Colloq. Math. 96 (2003), 293–305.
[22] Z. Leszczyński and A. Skowroński, *Tame triangular matrix algebras*, Colloq. Math. 86 (2000), 259–303.
[23] M. Loupias, *Indecomposable representations of finite ordered sets*, in: Representations of Algebras, Lecture Notes in Math. 488, Springer, Berlin, 1975, 201–209.
[24] G. Marczak, A. Polak and D. Simson, *P-critical integral quadratic forms and positive forms. An algorithmic approach*, Linear Algebra Appl. 433 (2010), 1873–1888.
[25] S. Montgomery, *Indecomposable coalgebras, simple comodules and pointed Hopf algebras*, Proc. Amer. Math. Soc. 123 (1995), 2343–2351.
[26] G. Navarro, *Valued Gabriel quiver of a wedge product and semiprime coalgebras*, Front. Math. China 8 (2013), 1157–1183.

[27] S. Nowak and D. Simson, *Locally Dynkin quivers and hereditary coalgebras whose left comodules are direct sums of finite dimensional comodules*, Comm. Algebra 30 (2002), 405–476.

[28] R. Nörenberg and A. Skowroński, *Tame minimal non-polynomial growth simply connected algebras*, Colloq. Math. 73 (1997), 301–330.

[29] S. A. Ovsienko, *Integral weakly positive forms*, in: Schur Matrix Problems and Quadratic Forms, Inst. Mat. Akad. Nauk USSR, preprint 78.25 (1978), 3–17 (in Russian).

[30] C. M. Ringel, *Tame Algebras and Integral Quadratic Forms*, Lecture Notes in Math. 1099, Springer, Berlin, 1984.

[31] D. Simson, *Peak reductions and waist reflection functors*, Fund. Math. 137 (1991), 115–144.

[32] D. Simson, *A splitting theorem for multipeak path algebras*, Fund. Math. 138 (1991), 113–137.

[33] D. Simson, *Linear Representations of Partially Ordered Sets and Vector Space Categories*, Algebra Logic Appl. 4, Gordon & Breach, 1992.

[34] D. Simson, *Posets of finite prinjective type and a class of orders*, J. Pure Appl. Algebra 90 (1993), 77–103.

[35] D. Simson, *Triangles of modules and non-polynomial growth*, C. R. Acad. Sci. Paris Sér. I 321 (1995), 33–38.

[36] D. Simson, *Representation embedding problems, categories of extensions and prinjective modules*, in: Representation Theory of Algebras, CMS Conf. Proc. 18, Amer. Math. Soc., 1996, 601–639.

[37] D. Simson, *Prinjective modules, propartite modules, representations of bocses and lattices over orders*, J. Math. Soc. Japan 49 (1997), 31–68.

[38] D. Simson, *Coalgebras, comodules, pseudocompact algebras and tame comodule type*, Colloq. Math. 90 (2001), 101–150.

[39] D. Simson, *Path coalgebras of quivers with relations and a tame-wild dichotomy problem for coalgebras*, in: Lecture Notes in Pure Appl. Math. 236, Dekker, 2005, 465–492.

[40] D. Simson, *Path coalgebras of profinite bound quivers, cotensor coalgebras of bound species and locally nilpotent representations*, Colloq. Math. 109 (2007), 307–343.

[41] D. Simson, *Localising embeddings of comodule categories with applications to tame and Euler coalgebras*, J. Algebra 312 (2007), 455–494.

[42] D. Simson, *Hom-computable coalgebras, a composition factors matrix and the Euler bilinear form of an Euler coalgebra*, J. Algebra 315 (2007), 42–75.

[43] D. Simson, *Representation-directed incidence coalgebras of interval finite posets*, Comm. Algebra 36 (2008), 2764–2784.

[44] D. Simson, *Tame-wild dichotomy for coalgebras*, J. London Math. Soc. 78 (2008), 783–797.

[45] D. Simson, *Tame comodule type, Roiter bocses, and a geometry context for coalgebras*, Ukrain. Math. J. 61 (2009), 810–833.

[46] D. Simson, *Incidence coalgebras of intervally finite posets, their integral quadratic forms and comodule categories*, Colloq. Math. 115 (2009), 259–295.

[47] D. Simson, *Integral bilinear forms, Coxeter transformations and Coxeter polynomials of finite posets*, Linear Algebra Appl. 433 (2010), 699–717.

[48] D. Simson, *Coalgebras of tame comodule type, comodule categories and tame-wild dichotomy problem*, in: Trends in Representation Theory of Algebras and Related Topics (Tokyo, 2010), A. Skowroński and K. Yamagata (eds.), Ser. Congress Reports, Eur. Math. Soc., Zürich, 2011/12, 561–660.

[49] D. Simson, *The Euler characteristic and Euler defect for comodules over Euler coalgebras*, J. K-Theory 7 (2011), 91–113.
[50] D. Simson, *Algorithms determining matrix morsifications, Coxeter polynomials and mesh geometries of roots for Dynkin diagrams*, Fund. Inform. 123 (2013), 447–490.
[51] D. Simson, *A framework for Coxeter spectral analysis of edge-bipartite graphs, their rational morsifications and mesh geometries of root orbits*, Fund. Inform. 124 (2013), 309–338.
[52] D. Simson, *A Coxeter–Gram classification of simply laced edge-bipartite graphs*, SIAM J. Discrete Math. 27 (2013), 827–854.
[53] D. Simson, *Tame-wild dichotomy of Birkhoff type problems for nilpotent linear operators*, J. Algebra 424 (2015), 254–293.
[54] D. Simson and A. Skowroński, *The Jacobson radical power series of module categories and the representation type*, Bol. Soc. Mat. Mexicana 5 (1999), 223–236.
[55] D. Simson and A. Skowroński, *Elements of the Representation Theory of Associative Algebras, Volume 2. Tubes and Concealed Algebras of Euclidean Type*, London Math. Soc. Student Texts 71, Cambridge Univ. Press, Cambridge, 2007.
[56] D. Simson and A. Skowroński, *Elements of the Representation Theory of Associative Algebras, Volume 3. Representation-Infinite Tilted Algebras*, London Math. Soc. Student Texts 72, Cambridge Univ. Press, Cambridge, 2007.
[57] M. E. Sweedler, *Hopf Algebras*, Benjamin, New York, 1969.
[58] D. Woodcock, *Some categorical remarks on the representation theory of coalgebras*, Comm. Algebra 25 (1997), 2775–2794.
[59] A. G. Zavadski and A. S. Shkabara, *Commutative quivers and matrix algebras of finite type*, preprint IM-76-3, Akad. Nauk USSR, Inst. Mat., Kiev, 1976, 52 pp. (in Russian).

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