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ON THE FINITENESS OF THE SEMIGROUP OF CONJUGACY CLASSES OF LEFT IDEALS FOR ALGEBRAS WITH RADICAL SQUARE ZERO

BY

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Abstract. Let A be a finite-dimensional algebra over an algebraically closed field with radical square zero, and such that all simple A-modules have dimension at most two. We give a characterization of those A that have finitely many conjugacy classes of left ideals.

1. Introduction. Let A be a finite-dimensional unital algebra over a field K and let $U(A)$ stand for the group of units of A. Following [18], we denote by $C(A)$ the semigroup of conjugacy classes of left ideals of A, with a binary operation induced by multiplication in A: if $[L]$ is the conjugacy class of a left ideal L of A, then $[L_1][L_2] = [L_1L_2]$. The aim of this paper is to continue the investigation of the following problem, introduced in [14].

PROBLEM 1.1. Let A be a finite-dimensional algebra over an algebraically closed field. Assume that $J(A)^2 = 0$ and the lattice $I(A)$ of two-sided ideals of A is distributive (we then simply say that A is distributive). Find necessary and sufficient conditions on A for $C(A)$ to be finite.

Our study of $C(A)$ is partly motivated by a general program of searching for semigroup invariants of associative algebras [13]. The semigroup $C(A)$ is also related to the subspace semigroup of an associative algebra, studied in [16, 17], which can be seen as an analogue of the semigroup of closed subsets in an algebraic monoid. In the context of ring theory, several related actions of $U(A)$ have been considered on a ring A (see [9], and also [5, 6, 8]), leading to certain finiteness conditions for A.

Several properties of $C(A)$ were obtained in [18], [14] and [13]. These results concern two general problems. First, we try to determine which prop-

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erties of A can be deduced from the structure of $C(A)$. Second, we ask for which algebras A the semigroup $C(A)$ is finite. It is worth mentioning that every algebra of finite representation type has the latter property (see [14, Corollary 1.4]).

The class of algebras A over an algebraically closed field K with the radical $J(A)$ nilpotent of index 2 plays an important role in representation theory of arbitrary algebras (see [19]). On the other hand, within this class the semigroup $C(A)$ determines the algebra A up to isomorphism, provided it is finite (see [13, Theorem 1.2]).

We recall that the lattice $I(A)$ is distributive if and only if $I(A)$ is finite (see [19, §2.2, Exercise 4; and §2.6, Exercise 3]). So, this is a necessary condition for $C(A)$ to be finite. For a basic algebra A (that is, when $A/J(A)$ is a direct product of copies of the field K), Problem 1.1 is solved in [18, Theorem 12].

We note that the finiteness of $C(A)$ is not a Morita invariant. Indeed, the algebra B from [13, Example 4.7] has $C(B)$ finite, but since it is of infinite representation type, the semigroup $C(\mathbb{M}_6(B))$ is infinite. Therefore, the challenge is to find a solution of Problem 1.1 for non-basic algebras with radical square zero. One of the technical tools in our study of such algebras is the separated quiver

$$\Gamma^s(A) = (\Gamma_0^s(A), \Gamma_1^s(A)) \tag{1.2}$$

in the sense of Gabriel [4], defined as follows.

Assume that K is an algebraically closed field and A is a finite-dimensional K-algebra whose Jacobson radical is nonzero and $J(A)^2 = 0$. Fix a maximal subset $\{e_1, \dots, e_n\}$ of a complete set of orthogonal primitive idempotents of A such that $Ae_i \not\simeq Ae_j$ as left A-modules, for any $i \neq j$. The *separated quiver* of A is defined to be the directed graph $\Gamma^s(A) = (\Gamma_0^s(A), \Gamma_1^s(A))$, where:

- the set of vertices of $\Gamma^s(A)$ is
$$\Gamma_0^s(A) = \{1, \dots, n\} \times \{0, 1\},$$
- there exists an arrow $(i, \epsilon') \to (j, \epsilon'')$ in $\Gamma_1^s(A)$ if $\epsilon' = 0$, $\epsilon'' = 1$ and $e_i J(A) e_j \neq 0$. In this case there are precisely d_{ij} arrows $(i, 0) \to (j, 1)$, where $d_{ij} := \dim_{\mathrm{K}} e_i J(A) e_j$.

We recall from [10] and [19, Corollary 2.4c] that if the lattice $I(A)$ is distributive then $\dim_{\mathrm{K}} e_i J(A) e_k \leq 1$ for all $i, j \in \{1, \dots, n\}$. We consider a few examples.

Example 1.3. Assume that $A = \left[\begin{smallmatrix} \mathrm{K} & \mathrm{K}^2 \\ 0 & \mathrm{K} \end{smallmatrix}\right]$ is the Kronecker K-algebra. Then A is a hereditary radical square zero algebra of infinite representation type (see [20, Example 1.5]) and its separated quiver is the four-vertex quiver

$$(2,0),\ (1,0) \rightrightarrows (2,1),\ (1,1).$$

One easily verifies that the semigroup $C(A)$ is infinite. Indeed, consider the family

$$\mathcal{I} = \left\{ I_\alpha = \begin{bmatrix} 0 & \mathrm{K}(1,\alpha) \\ 0 & 0 \end{bmatrix} \;\middle|\; \alpha \in \mathrm{K} \right\}$$

of nilpotent left ideals in A. Since the unit group of A is $\left[\begin{smallmatrix} \mathrm{K}^* & \mathrm{K}^2 \\ 0 & \mathrm{K}^* \end{smallmatrix}\right]$, where $\mathrm{K}^* = \mathrm{K} \setminus \{0\}$, it is easy to see that no two elements of $\mathcal{I}$ belong to the same conjugacy class of A.

Example 1.4. Let $A = \mathbb{M}_2(B)$ be the full two-by-two matrix algebra with coefficients in the radical square zero K-algebra $B = \mathrm{K}[t_1, t_2]/(t_1^2, t_1 t_2, t_2^2)$. Then A is a radical square zero algebra of infinite representation type (by [4]) and its separated quiver is the Kronecker quiver

$$(1,0) \rightrightarrows (1,1).$$

The semigroup $C(A)$ is again infinite. Analogously to the previous example, one may consider the following family of nilpotent left ideals in A:

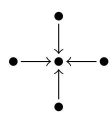
$$\mathcal{I} = \left\{ I_\alpha = \begin{bmatrix} \mathrm{K}(t_1 + \alpha t_2) & 0 \\ \mathrm{K}(t_1 + \alpha t_2) & 0 \end{bmatrix} \;\middle|\; \alpha \in \mathrm{K} \right\}.$$

An easy computation shows that two different members of $\mathcal{I}$ cannot be conjugate in A.

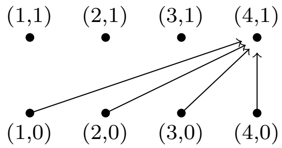
Example 1.5. Let

$$A = \begin{bmatrix} \mathrm{K} & 0 & 0 & 0 & \mathrm{K} \\ 0 & \mathrm{K} & 0 & 0 & \mathrm{K} \\ 0 & 0 & \mathrm{K} & 0 & \mathrm{K} \\ 0 & 0 & 0 & \mathrm{K} & \mathrm{K} \\ 0 & 0 & 0 & 0 & \mathrm{K} \end{bmatrix} \subseteq \mathbb{M}_5(\mathrm{K})$$

be the path algebra of the four-subspace quiver



Then A is a hereditary radical square zero basic algebra of infinite representation type (see [1], [20]) and its separated quiver is



An easy computation shows that the semigroup $C(A)$ is finite. This is also a consequence of Theorem 1.6 below. On the other hand, notice that the dual of Theorem 1.6 implies that the semigroup of conjugacy classes of right ideals of A (defined in the symmetric way) is infinite.

Our main result extends the following result, proved in [18, Theorem 12].

THEOREM 1.6. *Let A be a finite-dimensional basic algebra over an algebraically closed field* K. *Assume that $J(A)^2 = 0$ and the lattice of two-sided ideals of A is distributive. Then $C(A)$ is finite if and only if the separated quiver $\Gamma^s(A)$ has no cycles* (*as an unoriented graph*) *and* $\dim(eJ(A)) \leq 3$ *for every primitive idempotent e of A.*

If the algebra is not basic, not only the structure of $\Gamma^s(A)$ but also the sizes of the simple blocks of $A/J(A)$ will play a key role in the study of the semigroup $C(A)$. In part, this is a consequence of the following result proved in [14, Theorem 1.2].

THEOREM 1.7. *Let A be a finite-dimensional algebra over an algebraically closed field* K. *Assume that $A/J(A) \cong \mathbb{M}_{r_1}(\mathrm{K}) \times \cdots \times \mathbb{M}_{r_k}(\mathrm{K})$ with $r_i \geq 6$ for every i, and $J(A)^2 = 0$. Then the semigroup $C(A)$ is finite if and only if the algebra A is of finite representation type, that is, the set of the isomorphism classes of indecomposable left A-modules is finite.*

By applying the well known theorem of Gabriel [4] and its generalized version proved by Pierce [19, Section 11.8], we derive from Theorem 1.7 the following corollary.

COROLLARY 1.8. *Assume that A is a radical square zero algebra satisfying the conditions of Theorem* 1.7. *The following conditions are equivalent:*

(a) *The semigroup $C(A)$ is finite.*
(b) *The separated quiver $\Gamma^s(A)$, viewed as an unoriented graph, is a disjoint union of simply laced Dynkin diagrams $\mathbb{A}_n$, $n \geq 1$, $\mathbb{D}_n$, $n \geq 4$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$* (*see* [1]).
(c) *The algebra A is of finite representation type.*

Theorems 1.6 and 1.7 deal with two extreme situations: the algebras A such that the simple blocks of $A/J(A)$ are of dimension one, and such that the simple blocks of $A/J(A)$ are of dimension greater than or equal to 36. In contrast to these results, we will show that in general it is no longer possible to express necessary and sufficient conditions for the finiteness of $C(A)$ only in the language of the separated quiver $\Gamma^s(A)$.

In this paper we give a classification of algebras A that satisfy the conditions on the radical and the lattice of two-sided ideals given above and such that the simple blocks of $A/J(A)$ are of dimensions not greater than 4. To state the explicit result we will need some combinatorial structures related to the notions of skeletons and their separated graphs, which were introduced in [14]. The general approach will also be similar. According to [14, Proposition 2.6], it is possible to restate the problem of finiteness of $C(A)$ in the language of certain actions of linear groups on some sets of matrices. This leads to some matrix problems, the treatment of which will require certain new tools.

In Section 2 we briefly recall the notions and results introduced in [14], and we expand them to an appropriate, more general setting. We also formulate our main result, Theorem 2.8, which provides, for an algebra A of the above-mentioned class, conditions equivalent to the finiteness of the semigroup $C(A)$. After stating some preliminary results, we prove that these conditions are indeed necessary. The main difficulties arise in showing that these conditions are also sufficient. The combinatorial methods needed to overcome these obstacles prove to be quite different from the (mostly) geometric arguments used in [14] and in Section 2. Therefore, at the end of Section 2, we introduce certain new notions and results concerning the so-called 0-1 contours. In Section 3 we reduce the proof of the remaining implication of Theorem 2.8 to two matrix problems, formulated in Lemmata 3.9 and 3.10. Since the proofs of these two results require rather technical and computational approach, they are given in separate Sections 4 and 5, respectively. We conclude with some remarks and questions for further study.

2. The main theorem. In this section we recall from [14] the main notions and results needed to reformulate Problem 1.1 in the language of actions of linear groups on certain sets of block matrices. We also introduce notions used in the statement of the main result of the paper, Theorem 2.8.

2.1. Basic notions and preliminary results. In what follows, if not stated otherwise, we assume that A is a finite-dimensional distributive algebra over an algebraically closed field K such that $J(A)^2 = 0$. First, recall basic notation concerning skeletons and contours, coming from [14, Section 2].

By a *skeleton* $\mathcal{S}$ we mean a pair (I, f), where I is a finite set of pairs of positive integers (i, j), called *blocks* of the skeleton, and f is a function $I \to \mathbb{N}^2$, $f(i, j) = (s_i, r_j) \in \mathbb{N}^2$, where s_i is called the *height* of the block (i, j), and r_j is its *width*. The set of all blocks (i, j) with fixed i is referred to as the ith *row* of the skeleton. *Columns* of $\mathcal{S}$ are defined in a similar way.

We assume that all blocks of every row have equal height and all blocks of every column have equal width.

For a skeleton $\mathcal{S} = (I, f)$ and a field K we consider the K-subspace $\mathcal{M}_{\mathcal{S}} \subseteq \mathbb{M}_{(s_1+\cdots+s_t)\times(r_1+\cdots+r_s)}(\mathrm{K})$ consisting of all block matrices (a_{ij}), where a_{ij} is a matrix with s_i rows and r_j columns, and $a_{ij} = 0$ if $(i, j) \notin I$. The elements of $\mathcal{M}_{\mathcal{S}}$ are called *contours of the skeleton* $\mathcal{S}$. One can then speak about blocks of a given contour, and about the height, width and rows and columns of the contour.

It is convenient to present contours of a skeleton $\mathcal{S} = (I, f)$ using only those blocks a_{ij} for which $(i, j) \in I$. A similar convention will be used for presenting skeletons. It will always be clear from the context whether a given diagram represents a contour or a skeleton. For example, if $I = \{(1, 2), (1, 3), (1, 4), (2, 1), (2, 2)\}$ with $f(i, j) = (2, 2)$ for $(i, j) \in I$, then a contour of $\mathcal{S}$ is presented below:

$$\begin{array}{c|c|c|c|}
\cline{2-4}
 & \begin{smallmatrix}1&0\\0&1\end{smallmatrix} & \begin{smallmatrix}1&0\\2&0\end{smallmatrix} & \begin{smallmatrix}0&0\\1&0\end{smallmatrix} \\
\cline{1-4}
\multicolumn{1}{|c|}{\begin{smallmatrix}1&0\\3&0\end{smallmatrix}} & \begin{smallmatrix}1&0\\0&1\end{smallmatrix} & \multicolumn{1}{c}{} & \multicolumn{1}{c}{} \\
\cline{1-2}
\end{array}$$

An important class of contours comes from the family of algebras considered in this paper. Assume that

$$A/J(A) \simeq \mathbb{M}_{r_1}(\mathrm{K}) \times \cdots \times \mathbb{M}_{r_k}(\mathrm{K}) \tag{2.1}$$

for some $k > 0$ and some $r_i > 0$. According to the Wedderburn–Mal'tsev Theorem, we get a linear space decomposition $A = A_1 \oplus \cdots \oplus A_k \oplus J(A)$, where $A_i \simeq \mathbb{M}_{r_i}(\mathrm{K})$. Let e_i be the unit of A_i and let $J_{ij} = e_i J(A) e_j$ for $1 \le i, j \le k$. Define

$$a_i = \sum_{j\colon J_{ij}\neq 0} r_j.$$

Consider the set $I_A = \{(i, j) \mid J_{ij} \neq 0\}$ and the function $f_A : I_A \to \mathbb{N}^2$ such that $f(i, j) := (a_i, r_j)$. Then (I_A, f_A) is called the *skeleton of the algebra* A. The space of contours of this skeleton is denoted by $\mathcal{M}_A$, and its elements are called *contours of the algebra* A. If $\mathcal{S} = \mathcal{S}_A$, then the height of every block of the ith row of $\mathcal{S}$ is equal to the sum of the widths of all blocks of $\mathcal{S}$ lying in the ith row of $\mathcal{S}$. Moreover, it is easy to see that for every skeleton $\mathcal{S}$ satisfying these conditions there exists an algebra A such that $\mathcal{S} = \mathcal{S}_A$.

Recall (from [14, Definition 2.4]) that if $S = (I, f)$ is an arbitrary skeleton and $f(i, j) = (s_i, r_j)$ for $(i, j) \in I$, then one can consider the groups $\mathfrak{H} := \mathrm{Gl}_{s_1}(\mathrm{K}) \times \cdots \times \mathrm{Gl}_{s_t}(\mathrm{K})$ and $\mathfrak{G} := \mathrm{Gl}_{r_1}(\mathrm{K}) \times \cdots \times \mathrm{Gl}_{r_s}(\mathrm{K})$ together with the following action of the group $\mathfrak{H} \times \mathfrak{G}^o$ on $\mathcal{M}_{\mathcal{S}}$ (here $\mathfrak{G}^o$ is the group opposite to $\mathfrak{G}$):

$$\mathfrak{h}\cdot\mathcal{A}\cdot\mathfrak{g} := \begin{bmatrix} h_1a_{11}g_1 & h_1a_{12}g_2 & \dots & h_1a_{1s}g_s \\ h_2a_{21}g_1 & h_2a_{22}g_2 & \dots & h_2a_{2s}g_s \\ \vdots & \vdots & \ddots & \vdots \\ h_ta_{t1}g_1 & h_ta_{t2}g_2 & \dots & h_ta_{ts}g_s \end{bmatrix} \tag{2.2}$$

for $\mathfrak{h} = (h_1, \dots, h_t) \in \mathfrak{H}$ and $\mathfrak{g} = (g_1, \dots, g_s) \in \mathfrak{G}$ (the matrices h_l and g_m will be called the *lth coordinate* of $\mathfrak{h}$ and the *mth coordinate* of $\mathfrak{g}$, respectively, for $1 \le l \le t$ and $1 \le m \le s$), and $\mathcal{A} = (a_{ij}) \in \mathcal{M}_{\mathcal{S}}$. The orbits of the action (2.2) are called the $\mathfrak{H}$-$\mathfrak{G}$-*orbits* on $\mathcal{M}_{\mathcal{S}}$.

If $\mathcal{S} = \mathcal{S}_A$ then the nilpotent elements of $C(A)$ are associated to the $\mathfrak{H}$-$\mathfrak{G}$-orbits on $\mathcal{M}_{\mathcal{A}}$, which leads to the following consequence (see [14, 2.7]):

THEOREM 2.3. *The semigroup $C(A)$ is finite if and only if the number of $\mathfrak{H}$-$\mathfrak{G}$-orbits on the contour space $\mathcal{M}_A$ is finite.*

Let $\mathcal{S} = (I, f)$ be a skeleton. We say that $\Gamma_{\mathcal{S}} = (V_{\mathcal{S}}, E_{\mathcal{S}})$ is the *separated graph* of the skeleton $\mathcal{S}$ if $V_{\mathcal{S}} = (\{1, \dots, t\} \times \{0\}) \cup (\{1, \dots, s\} \times \{1\})$, where t, s denote the number of rows and columns, respectively, of $\mathcal{S}$, while $E_{\mathcal{S}} = \{\{(i,0),(j,1)\} \mid (i,j) \in I\}$. One says that the edge $e \in E_{\mathcal{S}}$ is of *weight* $f(i,j)$ if $e = \{(i,0),(j,1)\}$. The elements $v \in V_{\mathcal{S}}$ that belong to the set $\{1, \dots, t\} \times \{0\}$ are called *row vertices* and the elements of the set $\{1, \dots, s\} \times \{1\}$ are *column vertices* of the graph $\Gamma_{\mathcal{S}}$. If A is an algebra then the separated graph of the skeleton of the algebra A coincides with the (unoriented) separated quiver $\Gamma^s(A) = (\Gamma^s_0(A), \Gamma^s_1(A))$ of A.

One can now introduce a useful relation on the set of all skeletons. Let $\mathcal{S}_1$, $\mathcal{S}_2$ be skeletons. We say that $\mathcal{S}_1$ is *contained* in $\mathcal{S}_2$ if:

(1) $\Gamma_{\mathcal{S}_1} = (V_1, E_1)$ is a subgraph of $\Gamma_{\mathcal{S}_2} = (V_2, E_2)$, which means that there exist embeddings $\phi : V_1 \to V_2$ and $\psi : E_1 \to E_2$ such that $\psi(\{(i,0),(j,1)\}) = \{\phi(i,0), \phi(j,1)\}$,
(2) if for $e_1 \in E_1$, $e_2 \in E_2$ one has $\psi(e_1) = e_2$, and if $(s_1,t_1), (s_2,t_2)$ are the weights of the edges e_1 and e_2, respectively, then $s_1 \le s_2$ and $t_1 \le t_2$.

If for any $e_1 \in E_1$ and $e_2 \in E_2$ the above inequalities are in fact equalities then we say that $\mathcal{S}_1$ is a *subskeleton* of $\mathcal{S}_2$.

Comparing skeletons leads to consequences on the number of orbits of the action (2.2) on the space of contours.

LEMMA 2.4 ([14, Lemma 2.3]). *Assume that a skeleton $\mathcal{S}_1 = (I_1, f_1)$ is contained in a skeleton $\mathcal{S}_2 = (I_2, f_2)$. Consider the sets of contours $\mathcal{M}_{\mathcal{S}_1}$ and $\mathcal{M}_{\mathcal{S}_2}$ and the corresponding groups $\mathfrak{H} \times \mathfrak{G}$ and $\mathfrak{H}' \times \mathfrak{G}'$, acting on these sets as in (2.2). If the number of $\mathfrak{H}'$-$\mathfrak{G}'$-orbits on $\mathcal{M}_{\mathcal{S}_2}$ is finite then the number of $\mathfrak{H}$-$\mathfrak{G}$-orbits on $\mathcal{M}_{\mathcal{S}_1}$ is also finite.*

In order to state the main result of the paper, Theorem 2.8, we also need some further notions.

DEFINITION 2.5. Let $\mathcal{S} = (I, f)$ be a skeleton.

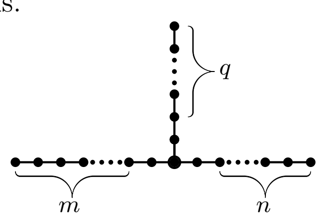
- We say that $\mathcal{S}$ is *connected* if the separated graph $\Gamma_{\mathcal{S}}$ is connected.
- By a *cycle* in $\mathcal{S}$ we mean a sequence $(i_1, j_1), \ldots, (i_{2p}, j_{2p}) \in I$ such that $i_1 = i_2$, $j_2 = j_3$, $i_3 = i_4$, $\ldots$, $i_{2p-1} = i_{2p}$, $i_{2p} = j_1$ and $(i_l, j_l) \neq (i_{l+1}, j_{l+1})$ for all l (with subscripts taken modulo $2p$).
- A block $(i, j) \in I$ is called a *row* [*column*] *terminal block* if it is the only block in its row [column] in the skeleton $\mathcal{S}$. A block is called *terminal* if it is either row or column terminal.
- A block $(i, j) \in I$ is *thick* if it is of size $p \times q$, where $q \geq 2$. A skeleton is *thick* if all of its blocks are thick.
- A block which is not thick is called *flat*. A skeleton is *flat* if all of its blocks are flat.

Clearly, if $\mathcal{S}$ is a skeleton then it is acyclic if and only if the separated graph $\Gamma_{\mathcal{S}}$ is acyclic. Moreover, in this case, there is a bijection between the set of connected subgraphs of $\Gamma_{\mathcal{S}}$ and the set of connected subskeletons of $\mathcal{S}$. This will be used later without further comment. Recall also that a *path* in an acyclic graph $\Gamma = (V, E)$ is a set $W = \{e_1, \ldots, e_n\}$ of distinct edges such that e_i is incident to e_j if and only if $|i - j| = 1$.

DEFINITION 2.6. Let $\mathcal{S}$ be a skeleton with the separated graph $\Gamma_{\mathcal{S}} = (V_{\mathcal{S}}, E_{\mathcal{S}})$. Assume that $\Gamma_{\mathcal{S}}$ is connected and acyclic. We say that a vertex $v \in V_{\mathcal{S}}$ is *knotted* if the degree of v (the number of edges incident to v) is at least 3. If the (unique) path between two knotted vertices of $\Gamma_{\mathcal{S}}$ consists only of edges corresponding to thick blocks of $\mathcal{S}$ then the path is called a *thick knotted path*.

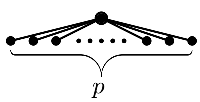
Next, we define two families of graphs: $F_{n,m,q}$ where $n, m, q \geq 0$, and G_p for $p \geq 3$; they are of use when discussing graphs that are not disjoint unions of Dynkin diagrams.

$F_{m,n,q}$:



where $n, m, q \geq 0$,

G_p :



where $p \geq 3$.

As in [14], we use the following notation. Assume that $\Gamma := \Gamma_{\mathcal{S}} = (V_{\mathcal{S}}, E_{\mathcal{S}})$ is the separated graph of a skeleton $\mathcal{S}$, and Γ is $F_{n,m,q}$ or G_k.

In particular, Γ has exactly one knotted vertex. If the knotted vertex of Γ is in the set of row vertices of Γ, then we say that the graph of $\mathcal{S}$ *is of the form* $\overrightarrow{\Gamma}$. If the knotted vertex of Γ is in the set of column vertices of Γ, then we say that the graph of $\mathcal{S}$ *is of the form* $\downarrow\!\Gamma$.

DEFINITION 2.7. Let $\mathcal{S} = (I, f)$ be a skeleton.

(1) We say that $\mathcal{S}$ is a *row n-tuple* if $I = \{(i_1, j), \dots, (i_n, j)\}$. In other words, the separated graph of $\mathcal{S}$ is of the form $\overrightarrow{G_n}$. A *column n-tuple* is defined similarly; its separated graph is of the form $\downarrow\!G_n$.
(2) We say that $\mathcal{S}$ is a *staircase* if $I = \{(i_1, j_1), (i_1, j_2), (i_2, j_2), (i_2, j_3), \dots\}$ or $I = \{(i_1, j_1), (i_2, j_1), (i_2, j_2), (i_3, j_2), \dots\}$, where $i_p \neq i_q$ and $j_p \neq j_q$, for all $p \neq q$.
(3) We say that $\mathcal{S}$ is a *double row staircase* if its separated graph is of the form $\overrightarrow{F_{n,m,0}}$. Similarly, a *double column staircase* is a skeleton whose separated graph is of the form $\downarrow\!F_{n,m,0}$ for some $n, m \geq 1$.
(4) We say that $\mathcal{S}$ is a *triple row staircase* if its separated graph is of the form $\overrightarrow{F_{n,m,q}}$ for some $n, m, q \geq 1$.

See Lemmata 3.9 and 3.10 for a graphical representation of skeletons of types (3) and (4).

2.2. The main result and the proof of necessity. We are now ready to state the main result of this paper.

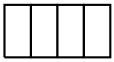
THEOREM 2.8. *Assume that A is a radical square zero* K*-algebra such that each of the matrix block algebras $\mathbb{M}_{r_i}(\mathrm{K})$ in the decomposition* (2.1) *of the residue algebra $A/J(A)$ is of* K*-dimension at most four. Let $\mathcal{S}_A$ be the skeleton of A, and $\mathcal{M}_A$ be the contour space of A. The following three conditions are equivalent:*

(a) *The semigroup $C(A)$ is finite.*
(b) *The number of $\mathfrak{H}$-$\mathfrak{G}$-orbits on the contour space $\mathcal{M}_A$ is finite.*
(c) *The skeleton $\mathcal{S}_A$ is acyclic and:*
   (i) *$\mathcal{S}_A$ does not contain a row* 4*-tuple,*
   (ii) *$\mathcal{S}_A$ does not contain a thick column* 4*-tuple,*
   (iii) *$\mathcal{S}_A$ does not contain a thick skeleton which is a triple row staircase,*
   (iv) *the separated graph of $\mathcal{S}_A$ does not contain a thick knotted path.*

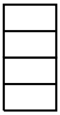
First, we note that (a)$\Leftrightarrow$(b) is a consequence of Theorem 2.3. Our next aim is to prove (b)$\Rightarrow$(c). The proof relies on two propositions coming from [14], asserting that certain specific skeletons cannot be contained in the skeleton of an algebra A with finitely many $\mathfrak{H}$-$\mathfrak{G}$-orbits (according to Lemma 2.4).

Proof of the implication (b)⇒(c) *of Theorem 2.8.* First note that $\mathcal{S}_A$ is acyclic by [14, Lemma 3.4]. To prove the remaining assertions of (c) we show that for a radical square zero algebra A with finitely many $\mathfrak{H}$-$\mathfrak{G}$-orbits on $\mathcal{M}_A$, each of the following conditions (i′)–(iv′) follows from the negation of the respective condition from (i)–(iv):

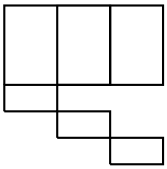
(i′) $\mathcal{S}_A$ contains a skeleton of the form (a) of [14, Proposition 3.6], which consists of four blocks of size 2×1:



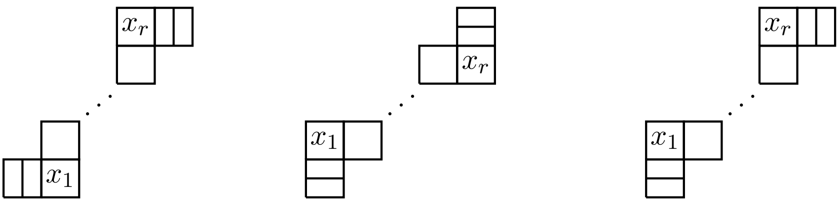
(ii′) $\mathcal{S}_A$ contains a skeleton of the form (a′) of [14, Proposition 3.6], which consists of four blocks of size 1×2:



(iii′) $\mathcal{S}_A$ contains a skeleton of the form (d) of [14, Proposition 3.6], which consists of three blocks of size 3×2 and three blocks of size 1×2:



(iv′) $\mathcal{S}_A$ contains one of the families (i)–(iii) of skeletons of [14, Proposition 3.7], which consist (depending on the choice of the family) of blocks of sizes 1×2, 2×1 and 2×2 (the number of blocks varies with $r \geq 1$):



Then, according to Lemma 2.4, the set of $\mathfrak{H}$-$\mathfrak{G}$-orbits on $\mathcal{M}_A$ is infinite. This will complete the proof.

Assume, to the contrary, that (i) is not satisfied, so that $\mathcal{S}_A$ contains a row 4-tuple in one of its rows, say in the jth row. By the definition of skeleton, this implies that $a_j \geq 4$. Hence, the jth row of $\mathcal{S}_A$ contains a skeleton of the form (a) of [14, Proposition 3.6], so (i′) holds.

Assume that (ii) is not satisfied, so $\mathcal{S}_A$ contains a thick column 4-tuple, say in the lth column. Let x_1, x_2, x_3, x_4 be the corresponding blocks, belonging to rows with indices i_1, i_2, i_3, i_4, respectively. Since the x_j are thick blocks, their sizes are $a_{i_j} \times q$, respectively, where $q \geq 2$. Hence the lth column of $\mathcal{S}_A$ contains a skeleton of the form (a′) of [14, Proposition 3.6]. Therefore (ii′) holds.

Assume that (iii) is not satisfied, so $\mathcal{S}_A$ contains a thick triple row staircase. We may assume that every block of $\mathcal{S}_A$ is thick. The separated graph $\Gamma_{\mathcal{S}_A}$ contains, as a subgraph, a graph of the form $\overrightarrow{F_{n,m,q}}$ for some $m, m, q \geq 1$. In particular, $\Gamma_{\mathcal{S}_A}$ contains $\overrightarrow{F_{1,1,1}}$. Since every block of $\mathcal{S}_A$ is thick, it follows easily that $\mathcal{S}_A$ contains a skeleton of the form (d) of [14, Proposition 3.6]. Hence (iii′) holds.

Finally, assume that (iv) is not satisfied, so the separated graph of $\mathcal{S}_A$ contains a thick knotted path. Then $\mathcal{S}_A$ contains one of the skeletons (i)–(iii) of [14, Proposition 3.7]. Hence (iv′) holds. ■

Our last, and actually the most difficult goal, is to prove that the conditions listed in (c) of Theorem 2.8 imply that assertion (b) of the theorem holds. In the subsequent sections we will show that the action of the group $\mathfrak{H} \times \mathfrak{G}$, as in (2.2), on the contour space $\mathcal{M}_{\mathcal{S}}$ for skeletons $\mathcal{S}$ of certain types has finitely many orbits. This can be accomplished by showing that the $\mathfrak{H}$-$\mathfrak{G}$-orbit of every element of a given contour $\mathcal{A} \in \mathcal{M}_{\mathcal{S}}$ contains a contour $\overline{\mathcal{A}}$, each of whose blocks is a 0-1 *matrix*, that is, every entry is either 0 or 1. Such a contour will be called a 0-1 *contour*. We start with two straightforward observations.

Remark 2.9. Let $\mathcal{S} = (I, f)$ be a skeleton and let $\mathcal{M}^0_{\mathcal{S}}$ be the subset of $\mathcal{M}_{\mathcal{S}}$ consisting of all contours $\mathcal{A} = (a_{ij})$ such that $a_{ij} \neq 0$ for all $(i, j) \in I$. Then the following conditions are equivalent:

- the $\mathfrak{H}$-$\mathfrak{G}$-orbit of every $\mathcal{A} \in \mathcal{M}_{\mathcal{S}}$ contains a 0-1 contour,
- the $\mathfrak{H}$-$\mathfrak{G}$-orbit of every $\mathcal{A} \in \mathcal{M}^0_{\mathcal{S}}$ contains a 0-1 contour.

Remark 2.10. Let $\mathcal{S} = (I, f)$ be a skeleton with subskeletons $\mathcal{S}_i = (I_i, f_i)$, for $1 \leq i \leq q$, such that:

- $I_i \cap I_j = \emptyset$ for every $1 \leq i, j \leq q$,
- $I_1 \cup \cdots \cup I_q = I$,
- if $(a, b) \in I_i$ and $(a', b) \in I_j$ for some a, a', b, then $i = j$,
- if $(c, d) \in I_i$ and $(c, d') \in I_j$ for some c, d, d', then $i = j$.

Assume that $\mathfrak{H} \times \mathfrak{G}^o$ acts on $\mathcal{M}_{\mathcal{S}}$ according to (2.2), and each $\mathfrak{H}_i \times \mathfrak{G}^o_i$ acts on $\mathcal{M}_{\mathcal{S}_i}$. Then the following conditions are equivalent:

- the $\mathfrak{H}$-$\mathfrak{G}$-orbit of every $\mathcal{A} \in \mathcal{M}_{\mathcal{S}}$ contains a 0-1 contour,
- for every $1 \leq i \leq q$, the $\mathfrak{H}_i$-$\mathfrak{G}_i$-orbit of every $\mathcal{A} \in \mathcal{M}_{\mathcal{S}_i}$ contains a 0-1 contour.

The following result follows easily from the proof of Lemma 2.4.

Lemma 2.11. *Assume that a skeleton* $\mathcal{S}_1 = (I_1, f_1)$ *is contained in* $\mathcal{S}_2 = (I_2, f_2)$. *Consider the contour spaces* $\mathcal{M}_{\mathcal{S}_1}$ *and* $\mathcal{M}_{\mathcal{S}_2}$ *and the respective groups* $\mathfrak{H} \times \mathfrak{G}^o$ *and* $\mathfrak{H}' \times (\mathfrak{G}')^o$ *acting on them. If the* $\mathfrak{H}'$-$\mathfrak{G}'$*-orbit of every* $\mathcal{A}' \in \mathcal{M}_{\mathcal{S}_2}$ *contains a* 0-1 *contour, then the* $\mathfrak{H}$-$\mathfrak{G}$*-orbit of every* $\mathcal{A} \in \mathcal{M}_{\mathcal{S}_1}$ *contains a* 0-1 *contour.*

We need another notion essential for the proof of the remaining implication (c)⇒(b) of Theorem 2.8.

Definition 2.12. Let $\mathcal{S} = (I, f)$ be a skeleton such that if $(i, j) \in I$, then the height of the block (i, j) is equal to the sum of the widths of all blocks from the ith row of $\mathcal{S}$. We then say that the skeleton $\mathcal{S}$ has a *reduced form*.

Consider the following subset I' of I:

$$I' := \{(i, j) \in I \mid f(i, j) = (a, b), \text{ where } a \geq 2\}.$$

If $I' = \emptyset$ then we say that the reduced form of $\mathcal{S}$ is *trivial*. If $I' \neq \emptyset$, then consider the skeleton (I', f'), where $f'(i, j) := (a-1, b)$ if $f(i, j) = (a, b)$, for every $(i, j) \in I' \subseteq I$. The skeleton (I', f') will be denoted by $\mathcal{S}^r$ and called the *nontrivial reduced form* of $\mathcal{S}$. If I' is nonempty then the contour space $\mathcal{M}_{\mathcal{S}^r}$ is called the *space of reduced contours* of $\mathcal{S}$.

It is clear that the skeletons of algebras considered in this paper have reduced forms. In some cases these forms can be trivial.

The following remark will often be used in the proofs based on the notion of the reduced form of a skeleton.

Remark 2.13. Let $\mathcal{S}$ be a skeleton with a reduced form. If the skeleton $\mathcal{S}'$ is obtained from $\mathcal{S}$ by deleting certain rows, then $\mathcal{S}'$ also has a reduced form.

The key observation concerning reduced forms is as follows.

Lemma 2.14. *Assume that a skeleton* $\mathcal{S}$ *has a reduced form. If this is a trivial form then the* $\mathfrak{H}$-$\mathfrak{G}$*-orbit of every contour in* $\mathcal{M}_{\mathcal{S}}$ *contains a* 0-1 *contour. If the reduced skeleton* $\mathcal{S}^r$ *of* $\mathcal{S}$ *exists and if the group* $\mathfrak{H}' \times (\mathfrak{G}')^o$ *acts as in* (2.2) *on the space* $\mathcal{M}_{\mathcal{S}^r}$ *of reduced contours, then the following conditions are equivalent:*

(i) *the* $\mathfrak{H}$-$\mathfrak{G}$*-orbit of every* $\mathcal{A} \in \mathcal{M}_{\mathcal{S}}$ *contains a* 0-1 *contour,*
(ii) *the* $\mathfrak{H}'$-$\mathfrak{G}'$*-orbit of every* $\mathcal{A}' \in \mathcal{M}_{\mathcal{S}^r}$ *contains a* 0-1 *contour.*

Proof. (i)⇒(ii). Assume that the reduced form of $\mathcal{S}$ is trivial. Then every row of $\mathcal{S}$ contains exactly one block, which is of size 1×1. It is then clear that the $\mathfrak{H}$-$\mathfrak{G}$-orbit of every contour in $\mathcal{M}_{\mathcal{S}}$ contains a 0-1 contour.

Hence, assume that $\mathcal{S}$ has a nontrivial reduced form $\mathcal{S}^r$. According to Definition 2.12, the skeleton $\mathcal{S}^r$ is contained in $\mathcal{S}$. Hence, (i)⇒(ii) follows from Lemma 2.11.

(ii)⇒(i). We apply induction on the number r of rows of $\mathcal{S}$. Assume first that $r = 1$. Let $\mathcal{A} \in \mathcal{M}_{\mathcal{S}}$. Since $\mathcal{S}$ has a reduced form $\mathcal{S}^r$, the only row of $\mathcal{A}$ can be treated as a square matrix of size $n > 1$. It is enough to show that the $\mathrm{Gl}_n(\mathrm{K})$-$\mathfrak{G}$-orbit of $\mathcal{A}$ contains a 0-1 contour. If the rank of $\mathcal{A}$ is n, then the assertion is clear. If the rank is less than n, then there exists $h' \in \mathrm{Gl}_n(\mathrm{K})$ such that the last row of $h'\mathcal{A}$ is zero. Let $\mathcal{A}'$ be the $(n-1) \times n$-matrix formed by the first $n-1$ rows of $h'\mathcal{A}$. Then $\mathcal{A}' \in \mathcal{M}_{\mathcal{S}^r}$, so that according to (ii) there exist $h'' \in \mathrm{Gl}_{n-1}(\mathrm{K})$ and $\mathfrak{g} \in \mathfrak{G}$ such that $h''\mathcal{A}' \cdot \mathfrak{g}$ is a 0-1 matrix. Hence, $h'''h'\mathcal{A} \cdot \mathfrak{g}$ with $h''' = \left[\begin{smallmatrix} h'' & 0 \\ 0 & 1 \end{smallmatrix}\right] \in \mathrm{Gl}_n(\mathrm{K})$ is a 0-1 matrix, as desired.

Assume now that the assertion holds for every skeleton with less than r rows. Consider a skeleton $\mathcal{S}$ with rows $\mathcal{S}^{(1)}, \ldots, \mathcal{S}^{(r)}$. Let $\mathcal{A} \in \mathcal{M}_{\mathcal{S}}$ with rows $\mathcal{A}^{(1)}, \ldots, \mathcal{A}^{(r)}$, where $\mathcal{A}^{(i)} \in \mathbb{M}_{a_i \times a_i}(\mathrm{K})$ for $1 \le i \le r$, according to the hypothesis that $\mathcal{S}$ has a reduced form. Since this form is nontrivial, we may also assume that $a_i > 1$ for some $1 \le i \le r$. We consider two cases:

- The rank of the matrix $\mathcal{A}^{(i)}$ is less than a_i for every $1 \le i \le r$.

If $a_i = 1$ for some i, then $\mathcal{A}^{(i)} \in \mathbb{M}_{1\times 1}(\mathrm{K})$ is zero. Moreover, the contour $\mathcal{A} \setminus \mathcal{A}^{(i)}$ has skeleton $\mathcal{S} \setminus \mathcal{S}^{(i)}$ with $r - 1$ rows. By Remark 2.13, $\mathcal{S} \setminus \mathcal{S}^{(i)}$ has a reduced form. The assertion then follows by applying the inductive hypothesis to $\mathcal{A} \setminus \mathcal{A}^{(i)}$.

Hence, assume that $a_i > 1$ for every $1 \le i \le r$. Then there exists $\mathfrak{h} = (h_1, \ldots, h_r) \in \mathfrak{H}$ such that the last row of $h_i\mathcal{A}^{(i)}$ is zero. Let $\mathcal{A}'^{(i)}$ in $\mathbb{M}_{(a_i-1)\times a_i}$ be obtained from $h_i\mathcal{A}^{(i)}$ by deleting that row. Then the contour $\mathcal{A}'$ consisting of the rows $\mathcal{A}'^{(i)}$ for $1 \le i \le r$ belongs to the space $\mathcal{M}_{\mathcal{S}^r}$ of reduced contours of $\mathcal{S}$. Hence, by (ii), there exist $\mathfrak{h}' = (h'_1, \ldots, h'_r) \in \mathfrak{H}'$ and $\mathfrak{g}' \in \mathfrak{G}'$ such that $\mathfrak{h}' \cdot \mathcal{A}' \cdot \mathfrak{g}'$ is a 0-1 contour. We set $h''_i = \left[\begin{smallmatrix} h'_i & 0 \\ 0 & 1 \end{smallmatrix}\right] \in \mathrm{Gl}_{a_i}(\mathrm{K})$ for $1 \le i \le r$, and also $\overline{\mathfrak{h}} = (h''_1 h_1, \ldots, h''_r h_r) \in \mathfrak{H}$. It is clear that $\overline{\mathfrak{h}} \cdot \mathcal{A} \cdot \mathfrak{g}$ is a 0-1 contour.

- For some $1 \le i \le r$ the matrix $\mathcal{A}^{(i)}$ is nonsingular.

We may assume that $\mathcal{A}^{(1)} =: g \in \mathrm{Gl}_{a_1}(\mathrm{K})$ is nonsingular. By the inductive hypothesis, we may also assume that the rows $\mathcal{A}^{(i)}$ for $i > 1$ are 0-1 matrices. Indeed, $\mathcal{A} \setminus \mathcal{A}^{(1)}$ has a skeleton with $r-1$ rows, and by Remark 2.13 it has a reduced form. Hence, we set $\mathfrak{h} = (g^{-1}, \mathrm{id}_{a_2}, \ldots, \mathrm{id}_{a_r})$ and we find that $\mathfrak{h} \cdot \mathcal{A}$ is a 0-1 contour. ∎

3. Auxiliary lemmata and the proof of sufficiency. To complete the proof of Theorem 2.8, it remains to prove (c)⇒(b). For this purpose, we define the class of admissible skeletons, which will be the object of our study in this section.

Definition 3.1. Assume that $\mathcal{S}$ is an acyclic skeleton that has a reduced form (see Definition 2.12) and every block of $\mathcal{S}$ is of width not exceeding 2. If conditions (i)–(iv) of Theorem 2.8 are satisfied for $\mathcal{S}$ then we say that $\mathcal{S}$ is *admissible*.

An admissible skeleton is clearly a skeleton of an algebra. In order to complete the proof of Theorem 2.8 it is sufficient to show that if $\mathcal{S}$ is an admissible skeleton then the number of $\mathfrak{H}$-$\mathfrak{G}$-orbits on $\mathcal{M}_{\mathcal{S}}$ is finite.

Because of the complex nature of admissible skeletons, our approach will be based on a sequence of reductions of the above problem to skeletons of special simpler types. In particular, the so-called thick components, and their generalization, quasi-thick components, of an admissible skeleton will be introduced. We will first show that the desired assertion can be reduced to the case of connected quasi-thick skeletons, and more precisely to proving Lemma 3.8 below. Next, we will show that the proof of that lemma can be reduced to results on $\mathfrak{H}$-$\mathfrak{G}$-orbits for double row staircase skeletons and triple column staircase skeletons with blocks of width 2, and more precisely to the proofs of Lemmata 3.9 and 3.10. The proofs of these are quite technical and require more tools. They will be given in the subsequent sections.

3.1. Single-valued matrices. For the rest of this section we use the following convention.

Definition 3.2. Let $\mathcal{S}$ be a skeleton and let $\mathcal{A} \in \mathcal{M}_{\mathcal{S}}$. Let $\mathcal{A}'$ be a subcontour of $\mathcal{A}$. Assume that the blocks of $\mathcal{A}'$ are in the rows of $\mathcal{A}$ with indices $i_1, \ldots, i_p$ and in columns of $\mathcal{A}$ with indices $j_1, \ldots, j_q$. Let $\mathfrak{h} = (h_1, \ldots, h_t) \in \mathfrak{H}$ and $\mathfrak{g} = (g_1, \ldots, g_s) \in \mathfrak{G}$. The subcontour of $\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}$ obtained from the blocks of $\mathcal{A}'$ under the action of $\mathfrak{h}, \mathfrak{g}$ will be denoted by $\mathfrak{h} \cdot \mathcal{A}' \cdot \mathfrak{g}$. Moreover, we will say that:

- $\mathfrak{h}$ *acts as identity on* $\mathcal{A}'$ if all h_{i_l} are identity matrices,
- $\mathfrak{g}$ *acts as identity on* $\mathcal{A}'$ if all g_{j_l} are identity matrices,
- $\mathfrak{h}, \mathfrak{g}$ *do not change* $\mathcal{A}'$ *essentially* if $\mathfrak{h}$ acts as identity on $\mathcal{A}'$, and $\mathfrak{g}$ acts as identity on all thick blocks of $\mathcal{A}'$.

In the formulation of our two preliminary results on admissible contours, we need another notion.

Definition 3.3. A matrix $x \in \mathbb{M}_{n\times m}(\mathrm{K})$ is called *single-valued* if $x = sq$, where s is a scalar matrix and q is a 0-1 matrix.

We will see that single-valued matrices sometimes turn out to be more convenient than 0-1 matrices, especially when dealing with skeletons with flat contours.

The following useful remark is straightforward.

REMARK 3.4. Let $\mathcal{A}$ be a contour, with a skeleton $\mathcal{S}$, all of whose thick blocks are 0-1 matrices and all flat blocks are single-valued. If $\mathfrak{h}, \mathfrak{g}$ do not change the contour $\mathcal{A}$ essentially, then all thick blocks of $\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}$ are 0-1 matrices and all flat blocks of $\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}$ are single-valued.

We are now ready to prove two lemmata on admissible contours.

LEMMA 3.5. *Let $\mathcal{S}$ be an admissible skeleton and let $\mathcal{A} \in \mathcal{M}_{\mathcal{S}}$ be a contour such that every thick block of $\mathcal{A}$ is a* 0-1 *matrix and every flat block is a single-valued matrix. Then the $\mathfrak{H}$-$\mathfrak{G}$-orbit of $\mathcal{A}$ contains a* 0-1 *contour.*

LEMMA 3.6. *Let $\mathcal{S}$ be a connected flat skeleton that is admissible and let $\mathcal{A} \in \mathcal{M}_{\mathcal{S}}$. Then the $\mathfrak{H}$-$\mathfrak{G}$-orbit of $\mathcal{A}$ contains a* 0-1 *contour.*

Proof of Lemma 3.5. In view of Remark 2.10 we may assume that $\mathcal{S}$ is connected. We will show that for every $\mathcal{A} \in \mathcal{M}^0_{\mathcal{S}}$ with all blocks nonzero there exist $\mathfrak{h} \in \mathfrak{H}$ and $\mathfrak{g} \in \mathfrak{G}$ such that $\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}$ is a 0-1 contour and the coordinates $g_i \in \mathrm{Gl}_{r_i}(\mathrm{K})$ of $\mathfrak{g}$ are scalar matrices for all i such that the ith column of $\mathcal{A}$ is thick. Then the assertion of the lemma will follow from Remark 2.9.

We proceed by induction on the number t of rows of $\mathcal{S}$. Let $t = 1$. Let $a_1, \dots, a_s$ be the blocks of the unique row of $\mathcal{A}$. If none of these blocks is flat, the assertion is clear (with all coordinates of $\mathfrak{h}, \mathfrak{g}$ equal to the identity matrices of appropriate sizes). Hence, assume that $a_{i_1}, \dots, a_{i_q} \in \mathbb{M}_{s_1 \times 1}(\mathrm{K})$ are the flat blocks of $\mathcal{A}$, for some $i_1, \dots, i_q$. Then $a_{i_1}, \dots, a_{i_q}$ are single-valued matrices, say $a_{i_l} = \alpha_{i_l} \cdot q_{i_l}$, where $\alpha_{i_l} \in \mathrm{K}^*$ and q_{i_l} is a 0-1 matrix. Then we set $\mathfrak{h} = (\mathrm{id}_{s_1})$ and $\mathfrak{g} = (g_1, \dots, g_s)$, where $g_i \in \mathrm{Gl}_{r_i}(\mathrm{K})$ and

$$g_i = \begin{cases} \alpha_{i_l}^{-1} & \text{if } i = i_l \text{ for some } l, \\ \mathrm{id}_{r_i} & \text{otherwise.} \end{cases}$$

Clearly, $\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}$ is the desired 0-1 contour. Moreover, $\mathfrak{g}$ acts as identity on all thick columns of $\mathcal{A}$, so the second part of the inductive assertion is also satisfied.

Thus, assume that the inductive assertion holds for contours with less than t rows. Assume that $\mathcal{S}$ has t rows and consider a contour $\mathcal{A} \in \mathcal{M}^0_{\mathcal{S}}$ such that every thick block of $\mathcal{A}$ is a 0-1 matrix and every flat block is a single-valued matrix. Assume that the blocks in the first row of $\mathcal{A}$ are in columns with indices $m_1, \dots, m_k$. These will be denoted by $a_{m_1}, \dots, a_{m_k}$. Let $\mathcal{S}'$ be the skeleton obtained from $\mathcal{S}$ by deleting the first row. According to Definition 3.1 and Remark 2.13, $\mathcal{S}'$ is admissible. Since $\mathcal{S}$ is connected, every connected component of $\mathcal{S}'$ contains a block lying in one of the columns indexed $m_1, \dots, m_k$. Let $\mathcal{S}_1, \dots, \mathcal{S}_p$ be these components and let $m_{i_1}, \dots, m_{i_p}$ be the indices of the corresponding columns. Let $\mathcal{A}_1, \dots, \mathcal{A}_p$ be the subcontours of $\mathcal{A}$ with skeletons $\mathcal{S}_1, \dots, \mathcal{S}_p$, respectively. It is clear that a connected component of an admissible skeleton is itself admissible. So, $\mathcal{A}_i$ are admissible for $1 \le i \le p$. Each has less than t rows, whence by the inductive hypothesis and by Remark 2.10 there exist $\mathfrak{h}' = (h'_1, \dots, h'_t) \in \mathfrak{H}$ and $\mathfrak{g}' = (g'_1, \dots, g'_s) \in \mathfrak{G}$ such that the subcontours $\mathfrak{h}' \cdot \mathcal{A}_j \cdot \mathfrak{g}'$ of $\mathfrak{h}' \cdot \mathcal{A} \cdot \mathfrak{g}'$ are 0-1 contours for $1 \le j \le p$. Moreover, we may assume that $\mathfrak{h}'$ acts as identity on the first row of $\mathcal{A}$. By the inductive hypothesis, the coordinates g'_l of $\mathfrak{g}'$ are scalar matrices for all l such that column l of $\mathcal{A}$ contains thick blocks of the contour $\mathcal{A}_j$ for some j. We claim that the resulting contour $\mathfrak{h}' \cdot \mathcal{A} \cdot \mathfrak{g}'$ has the following properties:

(i) all blocks of $\mathfrak{h}' \cdot \mathcal{A} \cdot \mathfrak{g}'$ that are not in the first row are 0-1 matrices,
(ii) all blocks of the first row of $\mathfrak{h}' \cdot \mathcal{A} \cdot \mathfrak{g}'$ are single-valued matrices.

The former is clear. So assume, to the contrary, that (ii) does not hold. Then some block $x' = \mathfrak{h}' \cdot x \cdot \mathfrak{g}'$ in the first row and the m_ith column of $\mathfrak{h}' \cdot \mathcal{A} \cdot \mathfrak{g}'$ is not single-valued. We know that $\mathfrak{h}'$ acts as identity on the first row of $\mathcal{A}$, whence $x' = x \cdot \mathfrak{g}' = x g_{m_i}$. Thus, x cannot be a flat block, because otherwise x is a single-valued matrix, so that also $x g_{m_i}$ would be single-valued. Therefore, x is a thick block. If the column of $\mathcal{A}$ containing x also contains a block of one of the contours $\mathcal{A}_j$, then $g_{m_{i_j}}$ is a scalar matrix, by the inductive hypothesis. Since x is a 0-1 matrix by hypothesis, $x' = x g_{m_{i_j}}$ is single-valued, again a contradiction. So, assume that x is the only block of $\mathcal{A}$ in the m_ith column containing x. Then we may assume that $g'_{m_i} = \mathrm{id}_{r_{m_i}}$ and $x' = x$ is a 0-1 matrix. This contradicts our supposition and shows that condition (ii) holds.

So the block $\mathfrak{h}' \cdot x_{m_i} \cdot \mathfrak{g}'$ in the first row and the m_ith column of $\mathfrak{h}' \cdot \mathcal{A} \cdot \mathfrak{g}'$, for $1 \le i \le k$, is of the form $\alpha_i \cdot q_i$, where q_i is a 0-1 matrix and $\alpha_i \in \mathrm{K}^*$. Let $\mathcal{A}'_i$ be the maximal connected subcontour of $\mathfrak{h}' \cdot \mathcal{A} \cdot \mathfrak{g}'$ that contains $\mathfrak{h}' \cdot x_{m_i} \cdot \mathfrak{g}'$ and does not contain blocks $\mathfrak{h}' \cdot x_{m_j} \cdot \mathfrak{g}'$ of the first row of $\mathfrak{h}' \cdot \mathcal{A} \cdot \mathfrak{g}'$ for $j \ne i$. It is clear that every block of $\mathfrak{h}' \cdot \mathcal{A} \cdot \mathfrak{g}'$ belongs to one of the contours $\mathcal{A}'_i$. Moreover, if $\mathcal{A}'_i$ and $\mathcal{A}'_j$ have blocks in the same column then $i = j$. Let J_i be the set of indices of columns containing blocks of $\mathcal{A}'_i$. Then $J_1 \cup \dots \cup J_r = \{1, \dots, s\}$. Let I_i be the set of indices of rows containing blocks of $\mathfrak{h}' \cdot \mathcal{A}_i \cdot \mathfrak{g}'$.

We define $\mathfrak{h}'' = (h''_1, \dots, h''_t) \in \mathfrak{H}$ and $\mathfrak{g}'' = (g''_1, \dots, g''_s) \in \mathfrak{G}$ by setting $h''_1 = \mathrm{id}_{s_1}$, and for $j > 1$,

$$h''_j = \alpha_i \cdot \mathrm{id}_{s_j} \qquad \text{if } j \in I_i,$$
$$g''_j = \alpha_i^{-1} \cdot \mathrm{id}_{r_j} \qquad \text{if } j \in J_i.$$

It is easy to see that $\mathfrak{h}''\mathfrak{h}' \cdot \mathcal{A} \cdot \mathfrak{g}'\mathfrak{g}''$ is a 0-1 matrix. This completes the inductive step and the proof of Lemma 3.5. ∎

Proof of Lemma 3.6. In view of Lemma 3.5, it is enough to show that the $\mathfrak{H}$-$\mathfrak{G}$-orbit of the contour $\mathcal{A} \in \mathcal{M}_{\mathcal{S}}$ contains a contour $\mathcal{A}'$, every block of which is a single-valued matrix. We show that there exist $\mathfrak{h} \in \mathfrak{H}$ and $\mathfrak{g} \in \mathfrak{G}$ such that every coordinate of $\mathfrak{g}$ is an identity matrix (of an appropriate size) and all blocks of $\mathfrak{h}\cdot\mathcal{A}\cdot\mathfrak{g}$ are single-valued. For this, it is enough to assume that $\mathcal{S}$ consists of one row that has, according to condition (i) of Definition 3.1, not more than three blocks.

In view of Lemma 2.14 it is enough to deal with contours in $\mathcal{M}_{\mathcal{S}^r}$. Assume that $\mathcal{S}^r$ has three blocks. Then they are of size 2×1. So we need to show that for any $a, b, c \in \mathbb{M}_{2\times 1}(\mathrm{K})$ there exists $g \in \mathrm{Gl}_2(\mathrm{K})$ such that the matrices ga, gb and gc are single-valued. This is an easy consequence of the elementary row operations on the block matrix $[a\ b\ c]$. If $\mathcal{S}$ has less than three blocks, the assertion is clear. ∎

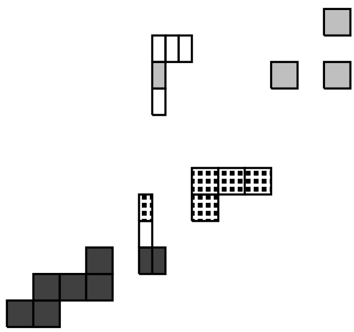
Our next aim is to look deeper into the nature of admissible skeletons in order to see how to reduce our problem to contours considered in Lemmata 3.5 and 3.6.

3.2. Quasi-thick components and the proof of Theorem 2.8. Let $\mathcal{S} = (I, f)$ be a connected and acyclic skeleton. The natural partial order on the set of subsets of I determines a partial order relation on the set of subskeletons of $\mathcal{S}$. We say that a subskeleton $\mathcal{S}'$ is a *thick component* of $\mathcal{S}$ if it is a maximal connected subskeleton of $\mathcal{S}$ consisting of thick blocks.

The main idea is to reduce the proof of the remaining implication of Theorem 2.8 to thick components of admissible skeletons. To do so, it is convenient to introduce another auxiliary notion.

Definition 3.7. Let $\mathcal{S}_1, \dots, \mathcal{S}_m$ be the thick components of an admissible skeleton $\mathcal{S}$, and let $\mathcal{S}'_1, \dots, \mathcal{S}'_m$ be the subskeletons of $\mathcal{S}$ consisting of all rows of $\mathcal{S}$ containing blocks of $\mathcal{S}_1, \dots, \mathcal{S}_m$, respectively. Then the $\mathcal{S}'_i$ are called the *quasi-thick components* of $\mathcal{S}$. Similarly, one defines quasi-thick components of contours in $\mathcal{M}_{\mathcal{S}}$.

It is clear that quasi-thick components of $\mathcal{S}$ are pairwise disjoint. Notice that a quasi-thick component of an admissible skeleton is also admissible. To illustrate this notion, we consider the following example.



The above skeleton has three quasi-thick components. The blocks of each are marked in a different way: grey, dark grey and dotted, respectively.

We will show that the proof of the implication (c)⇒(b) of Theorem 2.8 can be reduced to the following key result.

LEMMA 3.8. *Let $\mathcal{S}$ be a connected admissible skeleton consisting of exactly one quasi-thick component. Let $\mathcal{A} \in \mathcal{M}_{\mathcal{S}}$. Then the $\mathfrak{H}$-$\mathfrak{G}$-orbit of $\mathcal{A}$ contains a* 0-1 *contour.*

The proof of this key result is quite long and technical. Therefore, we first show how the proof of our main result can be derived from Lemma 3.8.

Proof of the implication (c)⇒(b) *of Theorem 2.8.* It is enough to show that if $\mathcal{S}$ is an admissible skeleton then every $\mathfrak{H}$-$\mathfrak{G}$-orbit on $\mathcal{M}_{\mathcal{S}}$ contains a 0-1 contour. We will proceed by induction on the number n of quasi-thick components of $\mathcal{S}$. If $n = 0$, then the assertion follows from Lemma 3.6. So, assume that $n = 1$ and let $\mathcal{A} \in \mathcal{M}_{\mathcal{S}}$. Then $\mathcal{A}$ is a disjoint union $\mathcal{A}' \cup \mathcal{A}''$ of two admissible contours, where $\mathcal{A}'$ is the unique quasi-thick component of $\mathcal{A}$ and $\mathcal{A}''$ is a flat contour (not necessarily connected). In view of Lemma 3.8, we may assume that $\mathcal{A}'$ is a 0-1 contour. By Lemma 3.6 and Remark 2.10 there exist $\mathfrak{h} \in \mathfrak{H}$ and $\mathfrak{g} \in \mathfrak{G}$ such that $\mathfrak{h} \cdot \mathcal{A}'' \cdot \mathfrak{g}$ is a 0-1 contour and $\mathfrak{h}, \mathfrak{g}$ do not change $\mathcal{A}'$ essentially. Hence, according to Remark 3.4, every thick block of $\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}$ is a 0-1 block and every flat block of $\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}$ is single-valued. Thus, Lemma 3.5 implies that the $\mathfrak{H}$-$\mathfrak{G}$-orbit of $\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}$ contains a 0-1 contour. So, the assertion follows in this case.

Next, assume that the assertion holds for admissible skeletons with n quasi-thick components. Let $\mathcal{S}$ be an admissible skeleton with $n+1$ quasi-thick components $\mathcal{S}_1, \ldots, \mathcal{S}_{n+1}$. Let $\Gamma_{\mathcal{S}_t}$, for $1 \le t \le n+1$, be the connected subgraph of the separated graph $\Gamma_{\mathcal{S}}$ corresponding to $\mathcal{S}_t$. We will say that two quasi-thick components $\mathcal{S}_i$, $\mathcal{S}_j$ are *linked in* $\mathcal{S}$ if there exists a path Λ_{ij} in $\Gamma_{\mathcal{S}}$ such that Λ_{ij} contains an edge of $\Gamma_{\mathcal{S}_i}$ and an edge of $\Gamma_{\mathcal{S}_j}$ but it does not contain any edge of $\Gamma_{\mathcal{S}_k}$ for $k \notin \{i, j\}$. By hypothesis, $\mathcal{S}$ does not contain any cycle. Hence, among the $n+1$ quasi-thick components of $\mathcal{S}$ there exists at least one that is linked to at most one (different) quasi-thick component of $\mathcal{S}$. We may assume that $\mathcal{S}_1$ is of this type. Let $\overline{\mathcal{S}}_1$ be the maximal connected subskeleton of $\mathcal{S} \setminus (\mathcal{S}_2 \cup \cdots \cup \mathcal{S}_n)$ containing $\mathcal{S}_1$.

Notice that $\overline{\mathcal{S}}_1$ is an admissible skeleton. Indeed, if a block x belongs to $\overline{\mathcal{S}}_1$, then the entire row of $\mathcal{S}$ containing x is in $\mathcal{S} \setminus (\mathcal{S}_2 \cup \cdots \cup \mathcal{S}_n)$, so in $\overline{\mathcal{S}}_1$. Thus, $\mathcal{S}$ is obtained by deleting some rows from $\mathcal{S}$ and, in view of Definition 3.1 and Remark 2.13, $\overline{\mathcal{S}}_1$ is admissible. Similarly, $\mathcal{S} \setminus \overline{\mathcal{S}}_1$ is admissible.

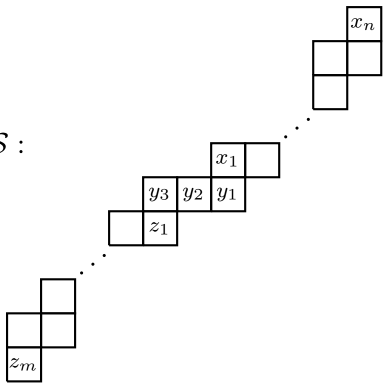
Let $\mathcal{A} \in \mathcal{M}_{\mathcal{S}}$. Then $\mathcal{A}$ is a disjoint union $\mathcal{A}' \cup \mathcal{A}''$ of two admissible skeletons, where $\mathcal{A}' \in \mathcal{M}_{\overline{\mathcal{S}}_1}$, while $\mathcal{A}'' \in \mathcal{M}_{\mathcal{S} \setminus \overline{\mathcal{S}}_1}$ is a contour with exactly n quasi-thick components. By the inductive hypothesis applied to $\mathcal{S} \setminus \overline{\mathcal{S}}_1$, we may assume that $\mathcal{A}''$ is a 0-1 contour. By the first step of the induction, there exist $\mathfrak{h} \in \mathfrak{H}$ and $\mathfrak{g} \in \mathfrak{G}$ such that $\mathfrak{h} \cdot \mathcal{A}' \cdot \mathfrak{g}$ is a 0-1 contour. We claim that $\mathfrak{h}, \mathfrak{g}$ can be chosen so that they do not change $\mathcal{A}''$ essentially. Indeed, by hypothesis $\mathcal{A}' \in \mathcal{M}_{\overline{\mathcal{S}}_1}$, and $\overline{\mathcal{S}}_1$ has exactly one quasi-thick component in $\mathcal{S}$, equal to $\mathcal{S}_1$, that is linked to at most one quasi-thick component of $\mathcal{S}$. It follows that at most one column of $\mathcal{S}$ can contain a block of $\overline{\mathcal{S}}_1$ and a block of $\mathcal{S} \setminus \overline{\mathcal{S}}_1$. If such a column exist, say it is the kth column of $\mathcal{S}$, it must consist of flat blocks. Moreover, the skeletons $\overline{\mathcal{S}}_1$ and $\mathcal{S} \setminus \overline{\mathcal{S}}_1$ have no rows in common. Therefore, one may choose $\mathfrak{h}, \mathfrak{g}$ in such a way that $\mathfrak{h}$ acts as identity on all rows of $\mathcal{A}''$, and $\mathfrak{g}$ acts as identity on all columns of $\mathcal{A}''$ but the kth. Such elements do not change $\mathcal{A}''$ essentially, as claimed.

It follows that all thick blocks of $\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}$ are 0-1 blocks and all flat blocks of $\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}$ are single-valued. Therefore, the inductive assertion is a consequence of Lemma 3.5. ∎

To complete the proof of Theorem 2.8 it is now enough to prove Lemma 3.8. First, we show that the latter proof can be reduced to the following two results concerning orbits on thick accessible contours of special types.

Lemma 3.9. *Let $\mathcal{S}$ be a double row staircase skeleton that is a disjoint union of a row consisting of blocks y_3, y_2, y_1 of dimensions 5×2 and two staircase skeletons $\mathcal{S}_1$, $\mathcal{S}_2$ with blocks of size 3×2, with ends x_1, x_n and z_1, z_m, respectively, for some n, m. The separated graph of this skeleton is of the form $\overrightarrow{F_{n,m,0}}$. For example, $\mathcal{S}$ can be*

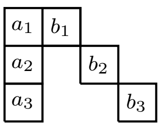
$\mathcal{S}$:



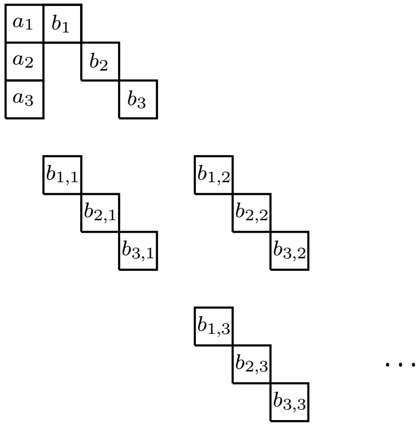
Assume that the group $\mathfrak{H} \times \mathfrak{G}^o$ acts on $\mathcal{M}_{\mathcal{S}}$ as in (2.2)*. Then the $\mathfrak{H}$-$\mathfrak{G}$-orbit of every contour $\mathcal{A} \in \mathcal{M}_{\mathcal{S}}$ contains a 0-1 contour $\overline{\mathcal{A}} \in \mathcal{M}_{\mathcal{S}}$.*

Lemma 3.10. *Consider the following skeleton consisting of six blocks a_i, b_i of size 3×2, for $i = 1, 2, 3$.*

(◇)



Let $\mathcal{S}$ be the triple column staircase skeleton obtained by adjoining to $(\diamond)$ *three staircase skeletons* $\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3$, *where* $\mathcal{S}_i$ *consists of blocks* $b_{i,s}$ *of size* 3×2 *with* $1 \le s \le s_i$ *for some* $s_i > 0$. *The separated graph of* $\mathcal{S}$ *is of the form* $\downarrow F_{s_1,s_2,s_3}$. *For example,* $\mathcal{S}$ *can be*



Assume that the group $\mathfrak{H} \times \mathfrak{G}^o$ *acts on* $\mathcal{M}_{\mathcal{S}}$ *as in* (2.2). *Then the* $\mathfrak{H}$-$\mathfrak{G}$-*orbit of every* $\mathcal{A} \in \mathcal{M}_{\mathcal{S}}$ *contains a* 0-1 *contour* $\overline{\mathcal{A}} \in \mathcal{M}_{\mathcal{S}}$.

Proof of Lemma 3.8. The admissible skeleton $\mathcal{S}$ is obtained by adjoining some flat blocks to the rows of its unique thick component $\mathcal{S}'$. First, we show that $\mathcal{S}'$ is a staircase, a double (row or column) staircase or a triple column staircase.

Indeed, notice that conditions (i), (ii) of Definition 3.1 imply that every row, and every column, of $\mathcal{S}'$ contains at most three blocks. Moreover, by condition (iv) of this definition, the separated graph of $\mathcal{S}$ contains no thick knotted path. Consequently, at most one row or column of the thick connected subskeleton $\mathcal{S}'$ can contain three blocks. Hence, it is easy to see that $\mathcal{S}'$ is a staircase, a double staircase or a triple staircase. Since admissible skeletons cannot contain any thick triple row staircase, by condition (iii) of Definition 3.1, it follows that $\mathcal{S}'$ is of one of the forms stated above.

From the definition of a quasi-thick component and since $\mathcal{S}$ is acyclic, it follows easily that if the blocks in some column of $\mathcal{S}$ are flat then this column contains only one block. As above, condition (iv) of Definition 3.1 implies that at most one row or column of $\mathcal{S}$ can contain three blocks. Since $\mathcal{S}'$ is a staircase, a double staircase or a triple column staircase, all the above restrictions easily imply that $\mathcal{S}$ is also of one of these types.

Next, we derive from $\mathcal{S}$ a new skeleton $\overline{\mathcal{S}}$. If $\mathcal{S} = (I, f)$, where $f(i,j) = (a_i, s_j)$ and a_i is the sum of the widths (equal to s_j for some j) of the blocks in the ith row of $\mathcal{S}$, then the skeleton $\overline{\mathcal{S}}$ is defined as the pair (I, f'), where $f'(i,j) = (a'_i, 2)$ and a'_i is the sum of the widths (equal to 2) of all blocks in the ith row of $\bar{\mathcal{S}}$. It is clear that $\bar{\mathcal{S}}$ contains $\mathcal{S}$. From the definition of $\bar{\mathcal{S}}$ it is also clear that $\bar{\mathcal{S}}$ is admissible because $\mathcal{S}$ was. If $\mathcal{S}$ is a staircase, so is $\bar{\mathcal{S}}$, and similarly in the case where $\mathcal{S}$ is a double or triple staircase.

Notice that it is enough to deal with the case where $\bar{\mathcal{S}}$ is a double row staircase or a triple column staircase. Indeed, every admissible staircase skeleton with blocks of width 2 is contained in an admissible double row staircase with blocks of width 2. Similarly, the case where $\bar{\mathcal{S}}$ is a double column staircase is reduced to the case where $\bar{\mathcal{S}}$ is a triple column staircase.

As in the previous proofs, we may assume that the contours under study have all blocks nonzero. If $\bar{\mathcal{S}}$ is a double column staircase then its blocks are of size:

- 2×2 for rows of $\bar{\mathcal{S}}$ with exactly one block,
- 4×2 for rows of $\bar{\mathcal{S}}$ with exactly two blocks,
- 6×2 for rows of $\bar{\mathcal{S}}$ with exactly three blocks.

By Lemma 2.11 it is then enough to show that the $\overline{\mathfrak{H}}$-$\overline{\mathfrak{G}}$-orbit of every $\mathcal{A} \in \mathcal{M}_{\bar{\mathcal{S}}}$ contains a 0-1 contour, where the appropriate groups $\overline{\mathfrak{H}}, \overline{\mathfrak{G}}$ act on $\mathcal{M}_{\bar{\mathcal{S}}}$ as in (2.2).

Consider the reduced form $\bar{\mathcal{S}}^r$ of $\bar{\mathcal{S}}$ (it is nontrivial because $\bar{\mathcal{S}}$ is a thick contour). In view of Lemma 2.14 it is enough to show that the $\overline{\mathfrak{H}}'$-$\overline{\mathfrak{G}}'$-orbit of every contour (with nonzero blocks) $\mathcal{A}' \in \mathcal{M}_{\overline{\mathcal{S}^r}}$ contains a 0-1 contour, where $\overline{\mathfrak{H}}', \overline{\mathfrak{G}}'$ are the appropriate groups acting on $\mathcal{M}_{\mathcal{S}^r}$. According to Definition 2.12, we see that $\bar{\mathcal{S}}^r$ has blocks of size:

- 1×2 for rows of $\bar{\mathcal{S}}^r$ with exactly one block,
- 3×2 for rows of $\bar{\mathcal{S}}^r$ with exactly two blocks,
- 5×2 for rows of $\bar{\mathcal{S}}^r$ with exactly three blocks.

Hence, $\bar{\mathcal{S}}^r$ is contained in one of the skeletons described in Lemma 3.9. In particular, from Lemmata 3.9 and 2.11 it follows that the $\overline{\mathfrak{H}}'$-$\overline{\mathfrak{G}}'$-orbit of every $\mathcal{A}' \in \mathcal{M}_{\overline{\mathcal{S}^r}}$ contains a 0-1 contour.

Similarly, one deals with the case where $\bar{\mathcal{S}}$ is a triple column staircase. Indeed, in this case $\bar{\mathcal{S}}^r$ is contained in one of the skeletons described in Lemma 3.10. Hence, the assertion follows from Lemmata 3.10 and 2.11. ∎

We have shown that the proof of Lemma 3.8 reduces to Lemmata 3.9 and 3.10. The proofs of the last two results are quite technical and will be given in the two subsequent sections.

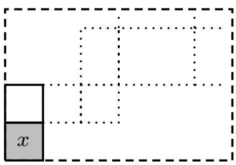
4. Thick double row staircase and the proof of Lemma 3.9. In this section we prove Lemma 3.9. Recall that a matrix x is called a *quasi permutation matrix* if its entries belong to the set $\{0, 1\}$ and if each row and each column of this matrix has at most one nonzero entry. By a *quasi* *permutational contour* we mean a contour with every block being a quasi permutation matrix.

The general idea of the proof is the following. We start by introducing certain auxiliary notions and results concerning orbits of contours with separated graphs isomorphic to Dynkin diagrams of types $\mathbb{A}_n$ and $\mathbb{D}_n$. Then we proceed in three steps. First, we consider a staircase subskeleton $\mathcal{S}_1 \cup \{y_1\}$ of $\mathcal{S}$ and we show that in the orbit of every contour of this subskeleton there exists a quasi permutational contour that admits a certain type of "freeness" in its terminal block—the one that corresponds to the block y_1 of the subskeleton (Corollary 4.3). It allows us to compensate certain actions of subgroups of invertible matrices on this terminal block with the action on the entire contour which, as a result, remains quasi permutational. Next, two blocks y_2 and y_3 are adjoined to $\mathcal{S}_1 \cup \{y_1\}$, and a skeleton with the separated graph of the form $\mathbb{D}_n$ is obtained. In the orbit of every contour of such an extended skeleton we find a 0-1 contour which admits another type of "freeness", this time in the block that corresponds to y_3 (Corollary 4.9). This type of freeness is weaker than in the case of a skeleton contour, as it allows us to compensate actions of a smaller class of subgroups. Until this point, the sizes of the blocks of the skeletons are irrelevant. In the last part we prove that the freeness on the terminal block of the 0-1 contour of the skeleton $\mathcal{S}_1 \cup \{y_1, y_2, y_3\}$ may be preserved even if we successively adjoin pairs z_i, z_{i+1} of blocks of size 3×2 to this skeleton, forming a double row staircase skeleton of arbitrarily many blocks (Lemma 4.12). The orbit of every contour of a skeleton constructed in this manner has a 0-1 contour with freeness in an appropriate terminal block. That allows us to finish the whole proof with an inductive argument, according to Lemma 2.11.

We begin with the definition of two types of freeness mentioned above. The first type will be defined for quasi permutational contours, whereas the second type refers to a larger class of 0-1 contours. By $\mathrm{B}_n(\mathrm{K})$ and $\mathrm{D}_n(\mathrm{K})$ we denote the following subgroups of $\mathrm{Gl}_n(\mathrm{K})$: the group of upper triangular matrices and the group of diagonal matrices, respectively.

DEFINITION 4.1. Let $\mathcal{A}$ be a connected and acyclic quasi permutational contour (respectively, a connected and acyclic 0-1 contour) and let the block $x \in \mathbb{M}_{m\times k}(\mathrm{K})$ of this contour be row terminal (see Definition 2.5). Also, assume that the column of $\mathcal{A}$ that contains x consists of exactly two blocks.

(♠) $\mathcal{A}:$



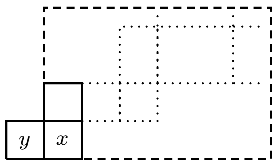
We say that the contour $\mathcal{A}$ *has freeness of type* $\overrightarrow{\mathrm{B}}$ (respectively *of type* $\overrightarrow{\mathrm{D}}$) *in the block* x if there exists a permutation $\sigma \in \Sigma_m$ such that for every $a \in \mathrm{B}_m(\mathrm{K})$ (respectively, for every $a \in \mathrm{D}_m(\mathrm{K})$), there exist $\mathfrak{h} \in \mathfrak{H}$ and $\mathfrak{g} \in \mathfrak{G}$ such that $\mathfrak{h}$ acts as the identity on the row that contains x and if we consider the contour $\mathcal{A}_{\sigma^{-1}a\sigma}$ that is formed from $\mathcal{A}$ by replacing x with $\sigma^{-1}a\sigma x$, then the contour $\mathfrak{h} \cdot \mathcal{A}_{\sigma^{-1}a\sigma} \cdot \mathfrak{g}$ is quasi permutational (respectively, 0-1). In this case, we also say that the left action of the group $\sigma^{-1}\,\mathrm{B}_m(\mathrm{K})\sigma$ (respectively $\sigma^{-1}\,\mathrm{D}_m(\mathrm{K})\,\sigma$) on the terminal block of $\mathcal{A}$ *may be compensated by the action of* $\mathfrak{H}$-$\mathfrak{G}$.

Dually, we define *freeness of type* B$\downarrow$ (respectively, *of type* D$\downarrow$) in the column terminal block x of a connected and acyclic quasi permutational contour (respectively, 0-1 contour) such that the row that contains x consists of exactly two blocks.

The following crucial result explores the connection between freeness of type $\overrightarrow{\mathrm{B}}$ and freeness of type B$\downarrow$ for quasi permutational contours.

Proposition 4.2. *Assume that a connected and acyclic quasi permutational contour* $\mathcal{A}$ *is of the form* (♠) *from Definition* 4.1 *and that* $\mathcal{A}$ *has freeness of type* $\overrightarrow{\mathrm{B}}$ *in the row terminal block* x. *Consider an acyclic contour* $\mathcal{A}' = \mathcal{A} \cup \{y\}$ *which is a disjoint union of* $\mathcal{A}$ *and a single block* $y \in \mathbb{M}_{m\times q}(\mathrm{K})$ *that belongs to the same row as* x.

$\mathcal{A}'$:



Then the block y *is column terminal in* $\mathcal{A}'$ *and the row that contains* y *consists of exactly two blocks. Moreover, if* $\mathfrak{H}, \mathfrak{G}$ *act on* $\mathcal{A}$ *via* (2.2) *then the* $\mathfrak{H}$ - $\mathrm{Gl}_q(\mathrm{K}) \times \mathfrak{G}$*-orbit of* $\mathcal{A}'$ *contains a quasi permutational contour* $\overline{\mathcal{A}}$ *which has freeness of type* B$\downarrow$ *in the block that corresponds to* y.

Before we give a proof, let us note an important application of this result, which simultaneously serves as a good illustration of the notions of freeness and compensation in the case of staircase contours. It is also the first step in the proof of Lemma 3.9.

Corollary 4.3. *If* $\mathcal{A}$ *is a staircase with terminal blocks* x, y, *then the orbit of* $\mathcal{A}$ *contains quasi permutational contours* $\mathcal{A}_1, \mathcal{A}_2$ *such that* $\mathcal{A}_1$ *has freeness of type* $\overrightarrow{\mathrm{B}}$ *or of type* B$\downarrow$ *in the block that corresponds to* x, *and* $\mathcal{A}_2$ *has freeness of type* $\overrightarrow{\mathrm{B}}$ *or* B$\downarrow$ *in the block that corresponds to* y.

Proof. We proceed by induction on the number of blocks in $\mathcal{A}$. First, assume that $\mathcal{A}$ consists of one block $x \in \mathbb{M}_{m\times k}(\mathrm{K})$. Then $\mathfrak{H} = \mathrm{Gl}_m(\mathrm{K})$ and

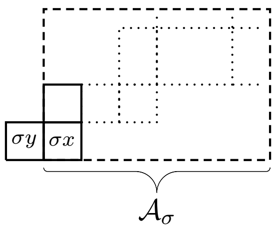
$\mathfrak{G} = \mathrm{Gl}_k(\mathrm{K})$. Thus the orbit of $\mathcal{A}$ contains $\overline{\mathcal{A}} = \{e\}$, where $e = \left[\begin{smallmatrix} \mathrm{id}_s & 0 \\ 0 & 0 \end{smallmatrix}\right]$ for some s. It is easy to see that $\overline{\mathcal{A}}$ has freeness of both types $\overrightarrow{\mathrm{B}}$ and $\mathrm{B}\!\downarrow$. Indeed, for every $b \in \mathrm{B}_m(\mathrm{K})$ we have $beg = e$ for some $g \in \mathrm{Gl}_k(\mathrm{K})$. Thus $\overline{\mathcal{A}}$ has freeness of type $\overrightarrow{\mathrm{B}}$ in e. The argument for freeness of type $\mathrm{B}\!\downarrow$ is similar. Let $j_k \in \Sigma_k$ be the antidiagonal permutation. Then the right action of the group $j_k^{-1}\,\mathrm{B}_k(\mathrm{K}) j_k$ on e can be clearly compensated by the left action of $\mathrm{Gl}_m(\mathrm{K})$. This concludes the base step of induction.

For the inductive step, consider a contour $\mathcal{A}$ that consists of n consecutive blocks $x_1, \dots, x_n$ for $n > 1$. We restrict the argument to the case when x_1 is row terminal and x_n is column terminal. The remaining three cases are proved analogously. Let $\mathcal{A}' = \{x_2, \dots, x_n\}$ and $\mathcal{A}'' = \{x_1, \dots, x_{n-1}\}$ arise from $\mathcal{A}$ by deleting the blocks x_1 and x_n, respectively. According to the inductive hypothesis on the orbits of $\mathcal{A}'$ and $\mathcal{A}''$, there exist quasi permutational contours $\overline{\mathcal{A}'}$ and $\overline{\mathcal{A}''}$ with freeness of type $\mathrm{B}\!\downarrow$ and $\overrightarrow{\mathrm{B}}$ in the blocks x_2 and x_{n-1}, respectively. According to Proposition 4.2 (and its dual version) we know that in the orbits of $\overline{\mathcal{A}'} \cup \{x_1\}$ and $\overline{\mathcal{A}''} \cup \{x_n\}$ there exist quasi permutational contours with freeness of types $\overrightarrow{\mathrm{B}}$ and $\mathrm{B}\!\downarrow$ in the blocks x_1 and x_n, respectively. These are exactly the $\mathcal{A}_1, \mathcal{A}_2$ that we seek. ∎

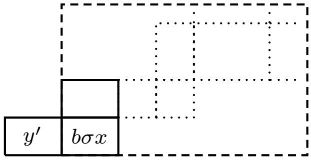
Proof of Proposition 4.2. From the assumption on $\mathcal{A}$ it follows that the row of $\mathcal{A}'$ containing x consists of exactly two blocks: x, y. We also assume that $\mathcal{A}'$ is connected and acyclic. Hence y is a column terminal block.

We prove that the $\mathfrak{H}\text{-}\mathrm{Gl}_q(\mathrm{K}) \times \mathfrak{G}$-orbit of $\mathcal{A}'$ contains a quasi permutational contour. Since we assume that $\mathcal{A}$ has $\overrightarrow{\mathrm{B}}$-freeness in the block x, it follows that there exists a permutation $\sigma \in \Sigma_m$ such that for every $a \in \mathrm{B}_m(\mathrm{K})$ there exist $\mathfrak{h} \in \mathfrak{H}$ and $\mathfrak{g} \in \mathfrak{G}$ such that the contour $\mathfrak{h} \cdot \mathcal{A}_{\sigma^{-1}a\sigma} \cdot \mathfrak{g}$ is quasi permutational, where $\mathcal{A}_{\sigma^{-1}a\sigma}$ is formed by replacing the block x of $\mathcal{A}$ with $\sigma^{-1}a\sigma x$. Consider the contour $\mathcal{A}'' = \mathcal{A}_\sigma \cup \{\sigma y\}$:

$\mathcal{A}''$:

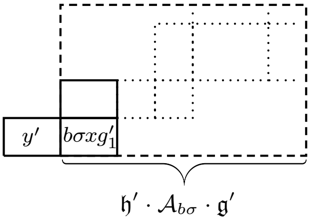


where $\mathcal{A}_\sigma$ is obtained from $\mathcal{A}$ by replacing x with σx, and $y \in \mathbb{M}_{m\times q}(\mathrm{K})$. The contour $\mathcal{A}_\sigma$ is quasi permutational, and from the argument above it follows that the left action of $\mathrm{B}_m(\mathrm{K})$ on σx may be compensated by the action of $\mathfrak{H}$-$\mathfrak{G}$. From [18, Lemma 8] it follows that there exist $b \in \mathrm{B}_m(\mathrm{K})$ and $b' \in \mathrm{B}_q(\mathrm{K})$ such that $b\sigma y b'$ is a quasi permutation matrix. Let i be the rank of this matrix. Then there exists $\sigma' \in \Sigma_q$ such the first i columns of the matrix $y' := b\sigma y b'\sigma'$ are nonzero and the remaining ones (if y' has more than i columns) are zero. Moreover, σ' is such that the nonzero elements of y' (equal to one) are exactly in entries $(1, j_1), \dots, (i, j_i)$, where $j_1 < \cdots < j_i$. Therefore, if we consider the contour



then since the left action of $\mathrm{B}_m(\mathrm{K})$ on the block σx of $\mathcal{A}_\sigma$ can be compensated by the action of $\mathfrak{H}$-$\mathfrak{G}$, there exist $\mathfrak{h}' \in \mathfrak{H}$ and $\mathfrak{g}' \in \mathfrak{G}$ such that the orbit of $\mathcal{A}''$ contains a quasi permutational contour $\mathcal{A}'''$, formed by a disjoint union of $\mathfrak{h}' \cdot \mathcal{A}_{b\sigma} \cdot \mathfrak{g}'$ and the block y', of the following form:

$\mathcal{A}'''$:



where g_1' is the coordinate of $\mathfrak{g}$ that acts on the column of $\mathcal{A}_{b\sigma}$ via (2.2). Thus, setting $\overline{\mathcal{A}} := \mathcal{A}'''$ we obtain the first part of the assertion. Note that since $b \in \mathrm{B}_m(\mathrm{K})$, the action of $\mathrm{B}_m(\mathrm{K})$ on the block $b\sigma x g_1'$ of $\mathfrak{h}' \cdot \mathcal{A}_{b\sigma} \cdot \mathfrak{g}'$ can still be compensated by the action of $\mathfrak{H}$-$\mathfrak{G}$.

To complete the proof of the proposition it remains to prove that the contour $\overline{\mathcal{A}}$ has freeness of type B$\downarrow$ in the block y'. According to the previous part of the proof the $m \times q$ quasi permutation matrix y' is of the form

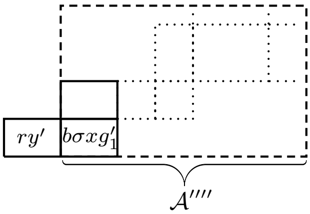
$$(*) \qquad y' = \left[\begin{array}{cccc|c} \vdots & & & & \\ 1 & \leftarrow j_1 & & & \\ \vdots & & & & \\ & 1 & \leftarrow j_2 & & \\ & \vdots & & & 0 \\ & & \ddots & & \\ & 0 & & 1 & \leftarrow j_i \\ & & & \vdots & \end{array}\right].$$

Consider the set $B' \subseteq \mathrm{Gl}_q(\mathrm{K})$ of matrices of the form $\left[\begin{smallmatrix} \mathrm{B}_i(\mathrm{K}) & 0 \\ * & \mathrm{B}_{q-i}(\mathrm{K}) \end{smallmatrix}\right]$. We will show that if $B := \mathrm{B}_m(\mathrm{K})$, then

$$y'B' \subseteq By'. \tag{4.4}$$

Let us first see that (4.4) implies the B$\downarrow$-freeness of $\overline{\mathcal{A}}$ in y'.

Let $\epsilon \in \Sigma_q$ be of the form $\left[\begin{smallmatrix} 0 & \mathrm{id}_i \\ \mathrm{id}_{q-i} & 0 \end{smallmatrix}\right]$. Observe that $B' = \epsilon\, \mathrm{B}_q(\mathrm{K})\epsilon^{-1}$. Let $\mathcal{A}'''' := \mathfrak{h}' \cdot \mathcal{A}_{b\sigma} \cdot \mathfrak{g}'$. We will find a permutation $\tau \in \Sigma_q$ such that for every $a \in \mathrm{B}_q(\mathrm{K})$ the $\mathfrak{H}$ - $\mathrm{Gl}_q(\mathrm{K}) \times \mathfrak{G}$-orbit of the contour of the form $\mathcal{A}'''' \cup \{y'\tau^{-1}a\tau\}$, obtained from $\overline{\mathcal{A}}$ by replacing y' with $y'\tau^{-1}a\tau$, contains a quasi permutational contour. Set $\tau := \epsilon^{-1}$. For each $a \in \mathrm{B}_q(\mathrm{K})$, let $b' \in B'$ be defined by $b' = \epsilon a \epsilon^{-1}$. From (4.4) it follows that $ry' = y'b'$ for some $r \in B$. To establish the B$\downarrow$-freeness of $\overline{\mathcal{A}}$ in y' it is therefore enough to prove that for every $r \in B$ the $\mathfrak{H}$ - $\mathrm{Gl}_q(\mathrm{K}) \times \mathfrak{G}$-orbit of $\mathcal{A}'''' \cup \{ry'\}$ contains a quasi permutational contour.



Let $r^{-1} \in B$ act on the row of $\mathcal{A}'''' \cup \{ry'\}$ containing ry' by left multiplication. Let $\mathcal{A}''''_{r^{-1}}$ be the image of $\mathcal{A}''''$ under this action. The block ry' is mapped to the quasi permutation matrix y'. Also, recall that the action of $\mathrm{B}_m(\mathrm{K})$ on the block $b\sigma x g'_1$ of $\mathcal{A}''''$ may be compensated by the action of $\mathfrak{H}$-$\mathfrak{G}$. Hence there exist $\mathfrak{h}'' \in \mathfrak{H}$ (not changing the row of $\mathcal{A}''''_{r^{-1}}$ containing $r^{-1}b\sigma x g'_1$) and $\mathfrak{g}'' \in \mathfrak{G}$ such that the contour $\mathfrak{h}'' \cdot \mathcal{A}''''_{r^{-1}} \cdot \mathfrak{g}'' \cup \{y'\}$ is quasi permutational. Moreover, this contour belongs to the orbit of $\mathcal{A}'''' \cup \{ry'\}$. We have thus proved that the contour $\overline{\mathcal{A}}$ has B$\downarrow$-freeness in the block y'.

Hence, it remains to prove (4.4). Take $b' \in B'$ of the form $(\beta_{c,d})$, where $1 \le c, d \le q$ and $\beta_{c,d} \in \mathrm{K}$. Then the matrix $y'b'$ is of the form

$$y'b' = \left[\begin{array}{ccccc|c} \vdots & \vdots & \vdots & \ddots & \vdots & \\ \beta_{j_1,1} & \beta_{j_1,2} & * & \cdots & * & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & \beta_{j_2,2} & * & \cdots & * & \mathbf{0} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & \beta_{j_i,i} & \\ \vdots & \vdots & \vdots & \ddots & \vdots & \end{array}\right].$$

For $1 \leq l \leq i$, the nonzero j_lth row of $y'b'$ is exactly the same as the lth row of b'. Moreover, the rows with indices $j_1, \ldots, j_i$ are the only nonzero rows of $y'b'$. Hence, to prove (4.4) it is enough to find $a \in B$ such that $ay' = y'b'$. Let $a = (\alpha_{c,d}) \in \mathbb{M}_{m\times m}(\mathrm{K})$ have the j_lth column equal to the lth column of $y'b'$, for $1 \leq l \leq i$. This matrix is well defined since the j_l are exactly the indices of the rows in $y' \in \mathbb{M}_{m\times q}$, and thus $j_l \leq m$ for all l. The remaining columns of a (with indices $s \neq j_l$) are of the form e_s^T, where e_s is the sth vector of the standard basis of K^m. As a result, a is of the form

$$a = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 & \cdots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\ 0 & \cdots & \beta_{j_1,1} & \cdots & \beta_{j_1,2} & \cdots & * & \cdots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\ 0 & \cdots & 0 & \cdots & \beta_{j_2,2} & \cdots & * & \cdots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & \beta_{j_i,i} & \cdots \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots \end{bmatrix}.$$

Observe that $\alpha_{c,d} = 0$ if $c > d$. Moreover, the diagonal entries of a are either 1 or $\beta_{j_l,l}$, where $1 \leq l \leq i$. The elements $\beta_{j_l,l}$ are therefore on the diagonal of b'. Hence $a \in B$. It is easy to check that $ay' = y'b'$. This concludes the proof of (4.4), and the proposition follows. ∎

We proceed to the second step needed in the proof of Lemma 3.9. We will consider the orbits of contours of skeletons with separated graphs equal to $\overrightarrow{\mathbb{D}_n}$. Our goal is to show that such an orbit contains a 0-1 contour with freeness of type D↓. The following technical result will be useful.

Proposition 4.5. *Let $x \in \mathbb{M}_{m\times k}(\mathrm{K})$ and $y \in \mathbb{M}_{m\times q}(\mathrm{K})$. Then the* $\mathrm{B}_m(\mathrm{K})$ - $\mathrm{Gl}_k(\mathrm{K}) \times \mathrm{Gl}_q(\mathrm{K})$*-orbit of the contour* $\boxed{x \mid y}$ *contains a* 0-1 *contour* $\boxed{\overline{x} \mid \overline{y}}$ *with the following properties:*

(1) *$\overline{x}$ is a quasi permutation matrix and $\overline{y}$ is a* 0-1 *matrix. Every column of $\overline{y}$ contains at most two entries equal to* 1*. Moreover, if a column of $\overline{y}$ contains two ones that belong to rows of indices $i_1 \neq i_2$, then either the i_1th or the i_2th row of $\overline{x}$ is zero.*
(2) *For all $d \in \mathrm{D}_k(\mathrm{K})$ there exist $e \in \mathrm{D}_m(\mathrm{K})$ and $f \in \mathrm{D}_q(\mathrm{K})$ such that $e\overline{x}d = \overline{x}$ and $e\overline{y}f = \overline{y}$.*

Proof. We will prove that the $\mathrm{B}_m(\mathrm{K})$ - $\mathrm{Gl}_k(\mathrm{K}) \times \mathrm{Gl}_q(\mathrm{K})$-orbit of $\boxed{x \mid y}$ contains a 0-1 contour $\boxed{\overline{x} \mid \overline{y}}$ such that condition (1) is satisfied. We proceed by induction on m.

The assertion is clear for $m = 1$. Let $m = 2$. If the ranks $\mathrm{r}(x)$, $\mathrm{r}(y)$ of the matrices x, y are 2, then there is nothing to prove. Assume that $\mathrm{r}(x) = 2$ and $\mathrm{r}(y) \leq 1$. Then the following contour belongs to the $\mathrm{B}_2(\mathrm{K})$ - $\mathrm{Gl}_k(\mathrm{K}) \times \mathrm{Gl}_q(\mathrm{K})$-orbit of $\boxed{x \mid y}$:

$$\begin{array}{|cccc|ccc|} \hline 1 & 0 & \cdots & 0 & \alpha & 0 & \cdots \\ 0 & 1 & \cdots & 0 & \beta & 0 & \cdots \\ \hline \end{array} \tag{4.6}$$

where $\alpha, \beta \in \mathrm{K}$. If either α or β is 0, then the proof is easy. Otherwise, we can act on (4.6) with the following matrices via (2.2):

$$b = \begin{bmatrix} 1 & -\alpha\beta^{-1} \\ 0 & \beta^{-1} \end{bmatrix} \in \mathrm{B}_2(\mathrm{K}), \qquad g = \begin{bmatrix} b^{-1} & 0 \\ 0 & \mathrm{id}_{k-2} \end{bmatrix} \in \mathrm{Gl}_k(\mathrm{K}), \qquad \mathrm{id}_q \in \mathrm{Gl}_q(\mathrm{K}),$$

and obtain $\boxed{\overline{x} \mid \overline{y}}$ that satisfies (1). The argument is similar when $\mathrm{r}(y) = 2$ and $\mathrm{r}(x) \leq 1$. Finally, if both x, y are of rank ≤ 1, then in the orbit of $\boxed{x \mid y}$ one can find a contour

$$\begin{array}{|ccc|ccc|} \hline \alpha_1 & 0 & \cdots & \alpha_2 & 0 & \cdots \\ \beta_1 & 0 & \cdots & \beta_2 & 0 & \cdots \\ \hline \end{array} \tag{4.7}$$

for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathrm{K}$. It is easy to see that the orbit of (4.7) contains a 0-1 contour that satisfies (1). This concludes the first step of induction.

Assume now that $m > 2$. There exist $b \in \mathrm{B}_m(\mathrm{K})$ and $g \in \mathrm{Gl}_k(\mathrm{K})$ such that the matrix $x' := bxg$ is of the form $(*)$ in the proof of Proposition 4.2. Consider $g' \in \mathrm{Gl}_q(\mathrm{K})$ such that $y' = byg'$ is in reduced column echelon form. Let $i, j \leq m$ be the indices of the last nonzero rows in x' and y', respectively. If $i, j < m$, then the assertion can be reduced to one of the previous inductive steps. We can therefore assume that either i or j is equal to m. Consider the following cases.

Case 1: $m = j > i$. After a permutation of columns of x', the contour $\boxed{x' \mid y'}$ takes the form $x' = \left[\begin{smallmatrix} x'' \\ 0 \end{smallmatrix}\right]$, $y' = \left[\begin{smallmatrix} 0 & r \\ y'' & 0 \end{smallmatrix}\right]$, where x'', y'' have i rows, and r is of the form

$$r = \begin{bmatrix} * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 0 & \cdots & 0 \\ & 0 & * & \cdots & * \\ & & & \ddots & \\ & & 0 & & 1 \end{bmatrix}. \tag{$**$}$$

Thus there exists a matrix $b = \left[\begin{smallmatrix} \mathrm{id}_i & 0 \\ 0 & b' \end{smallmatrix}\right] \in \mathrm{B}_m(\mathrm{K})$ such that $b'r$ is a quasi permutation matrix. Moreover, we have $bx' = x'$. Therefore the assertion follows from the inductive hypothesis applied to the contour $\begin{array}{|c|c|}\hline bx'' & by'' \\ \hline\end{array}$.

Case 2: $m = i > j$. Then, after a permutation of columns of x', the contour $\begin{array}{|c|c|}\hline x' & y' \\ \hline\end{array}$ takes the form $x' = \left[\begin{smallmatrix} x'' & 0 \\ 0 & p \end{smallmatrix}\right]$, $y' = \left[\begin{smallmatrix} y'' \\ 0 \end{smallmatrix}\right]$, where x'', y'' are matrices of j rows, and p is a quasi permutation matrix. Therefore, the assertion follows from the inductive hypothesis applied to x'', y''.

Case 3: $i = j = m$. Observe that x' is a quasi permutation matrix and y' is of the form $(**)$. Consider the last column of y'. If it has no nonzero entry off the mth row, then the assertion follows easily from the induction hypothesis, since after a permutation of columns in x', the contour $\begin{array}{|c|c|}\hline x' & y' \\ \hline\end{array}$ takes the form

$$\begin{array}{|ccc|c|ccc|c|}\hline & & & 0 & & & & 0 \\ & x'' & & \vdots & & y'' & & \vdots \\ & & & 0 & & & & 0 \\ \hline 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 \\ \hline\end{array}\ . \tag{4.8}$$

We can therefore assume that there is a nonzero entry in the last column of y' off the mth row. Let $n < m$ be the largest index of a row with such a nonzero entry. Assume that x', y' are chosen such that n is smallest possible, that is, if $\begin{array}{|c|c|}\hline x_1 & y_1 \\ \hline\end{array}$ belongs to the $\mathrm{B}_m(\mathrm{K})$ - $\mathrm{Gl}_k(\mathrm{K}) \times \mathrm{Gl}_q(\mathrm{K})$-orbit of $\begin{array}{|c|c|}\hline x & y \\ \hline\end{array}$, if x_1 is a quasi permutation matrix and if y_1 is of the form $(**)$, then if the last column of y_1 has a nonzero entry in row $n' \neq m$, then $n' \geq n$.

Now we consider two cases. First, assume that the nth and mth rows of x' and y' are of the following form, perhaps after a permutation of columns in x':

$$\begin{array}{|cccc|cccc|}\hline 1 & 0 & 0 & \cdots & 0 & 0 & \cdots & \alpha \\ 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 1 \\ \hline\end{array}$$

for some $\alpha \neq 0$. Then there exist $b \in \mathrm{B}_m(\mathrm{K})$ and $g \in \mathrm{Gl}_k(\mathrm{K})$ such that the nth row of by' is zero, and the remaining rows are the same as the corresponding rows of y'. Moreover, we have $bx'g = x'$. Yet the pair $\begin{array}{|c|c|}\hline x' & by' \\ \hline\end{array}$ belongs to the $\mathrm{B}_m(\mathrm{K})$ - $\mathrm{Gl}_k(\mathrm{K}) \times \mathrm{Gl}_q(\mathrm{K})$-orbit of $\begin{array}{|c|c|}\hline x & y \\ \hline\end{array}$, and by' is of the form $(**)$. Thus, from the choice of n, the first case may be reduced to (4.8), which has already been discussed before.

Now assume that the nth and the mth rows of x', y' are of the following form, perhaps after a permutation of columns in x':

$$\begin{array}{|cccc|cccc|}\hline 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \alpha \\ 0 & 1 & 0 & \cdots & 0 & 0 & \cdots & 1 \\ \hline \end{array}$$

for some $\alpha \neq 0$. Then, using elements of $\mathrm{B}_m(\mathrm{K})$, we can make zero all entries that lie above α in the last column of y'. Thus in the orbit of $\begin{array}{|c|c|}\hline x' & y' \\ \hline\end{array}$ a contour of the following form can be found:

$$\begin{array}{|ccc|c|ccc|c|}\hline & & & 0 & & & & 0 \\ & x'' & & \vdots & & y'' & & \vdots \\ & & & 0 & & & & 0 \\ \hline 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 1 \\ \hline \end{array}\,.$$

The assertion in this case follows easily from the inductive hypothesis. Case 3 and item (1) of the proposition follow.

We will now prove (2). Consider any $d = (\delta_1, \ldots, \delta_k) \in \mathrm{D}_k(\mathrm{K})$. Let $j_1, \ldots, j_t$ be the indices of the nonzero rows in $\overline{x}$. According to (1), every nonzero row of $\overline{x}d$ has exactly one nonzero entry, which is equal to one of δ_i. Let δ'_{j_l} be this nonzero entry in the j_lth row of $\overline{x}d$. Set $e = (\eta_1, \ldots, \eta_m) \in \mathrm{D}_m(\mathrm{K})$, where

$$\eta_s = \begin{cases} (\delta'_s)^{-1} & \text{if } s \in \{j_1, \ldots, j_t\}, \\ 1 & \text{otherwise.} \end{cases}$$

Then $e\overline{x}d = \overline{x}$. Assume now that for some $l \in \{j_1, \ldots, j_t\}$ there exists $l' \neq l$ such that in some column of $\overline{y}$, rows l and l' have entries equal to 1. Then from (1) it follows that the l'th row of $\overline{x}$ is zero.

$$\begin{array}{|ccccc|ccc|l}\hline \vdots & \ddots & \vdots & \ddots & \vdots & & \vdots & & \\ 0 & \cdots & 1 & \cdots & 0 & \cdots & 1 & \cdots & \leftarrow l \\ \vdots & \ddots & \vdots & \ddots & \vdots & & \vdots & & \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 & \cdots & \leftarrow l' \\ \vdots & \ddots & \vdots & \ddots & \vdots & & \vdots & & \\ \cline{1-8} \end{array}$$

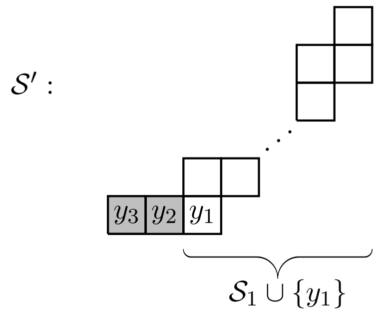
Let $(j_1, j'_1), \ldots, (j_s, j'_s)$ be all pairs of indices with the property of (l, l') as above. Consider the matrix $f = (\gamma_1, \ldots, \gamma_m) \in \mathrm{D}_m(\mathrm{K})$, where

$$\gamma_s = \begin{cases} (\delta'_{j_l})^{-1} & \text{if } s = j'_l \text{ for some } l, \\ 1 & \text{otherwise.} \end{cases}$$

Then $fe\overline{x}d = \overline{x}$. From (1) it follows that every column of $fe\overline{y}$ has at most two nonzero entries. According to the definition of e, f, these nonzero entries must be equal. Hence, one can easily find $d' \in \mathrm{D}_q(\mathrm{K})$ such that $fe\overline{y}d' = \overline{y}$. The assertion follows. ∎

Now we are ready to formulate the result on staircase skeletons with two blocks adjoined, which was mentioned earlier. Here we apply the notation of Lemma 3.9.

Corollary 4.9. *Consider a skeleton $\mathcal{S}' = \mathcal{S}_1 \cup \{y_1, y_2, y_3\}$ which is a disjoint union of a staircase $\mathcal{S}_1 \cup \{y_1\}$ with row terminal block y_1 and two blocks y_3, y_2, of the form*



In the orbit of any $\mathcal{A}' \in \mathcal{M}_{\mathcal{S}'}$ there exists a 0-1 *contour with freeness of type* D↓ *in the block y_3.*

Proof. Assume that $a_2 \in \mathbb{M}_{m\times q}(\mathrm{K})$ and $a_3 \in \mathbb{M}_{m\times k}(\mathrm{K})$ are the blocks of $\mathcal{A}'$ that correspond to the blocks y_2, y_3 of $\mathcal{S}'$. Also, let a_1 be the block that corresponds to y_1. Let $\mathcal{C} \in \mathcal{M}_{\mathcal{S}_1\cup\{y_1\}}$ be the corresponding subcontour of $\mathcal{A}'$. According to Proposition 4.2 we may assume that $\mathcal{C}$ is quasi permutational and has freeness of type $\overrightarrow{\mathrm{B}}$ in a_1. In other words, there exists $\sigma \in \Sigma_m$ such that the left action of $\sigma^{-1}\,\mathrm{B}_m(\mathrm{K})\sigma$ on the block a_1 of $\mathcal{C}$ can be compensated by the action of $\mathfrak{H}$-$\mathfrak{G}$, where $\mathfrak{H}, \mathfrak{G}$ act on the contours of $\mathcal{S}_1 \cup \{y_1\}$ via (2.2). Also, from Proposition 4.5 applied to the contour $\boxed{\sigma a_3}\boxed{\sigma a_2}$ we know that in its $\mathrm{B}_m(\mathrm{K})$ - $\mathrm{Gl}_k(\mathrm{K}) \times \mathrm{Gl}_q(\mathrm{K})$-orbit we can find a 0-1 contour $\boxed{\overline{\sigma a_3}}\boxed{\overline{\sigma a_2}}$ satisfying condition (2) of Proposition 4.5. Let $\overline{\sigma a_3} = b\sigma a_3 g$ and $\overline{\sigma a_2} = b\sigma a_2 h$ for some $b \in \mathrm{B}_m(\mathrm{K})$, $g \in \mathrm{Gl}_k(\mathrm{K})$, $h \in \mathrm{Gl}_q(\mathrm{K})$. If we multiply the row of $\mathcal{A}'$ that contains a_1 by $\sigma^{-1}b\sigma$, then we obtain the contour $\{\sigma^{-1}\overline{\sigma a_3}g^{-1}, \sigma^{-1}\overline{\sigma a_2}h^{-1}\} \cup \mathcal{C}_{\sigma^{-1}b\sigma}$, where $\mathcal{C}_{\sigma^{-1}b\sigma}$ is obtained from $\mathcal{C}$ by replacing a_1 with $\sigma^{-1}b\sigma a_1$. Using the $\overrightarrow{\mathrm{B}}$-freeness of $\mathcal{C}$ we know that the orbit of $\mathcal{A}'$ contains a 0-1 contour $\{\sigma^{-1}\overline{\sigma a_3}, \sigma^{-1}\overline{\sigma a_2}\} \cup \mathfrak{h}\cdot\mathcal{C}_{\sigma^{-1}b\sigma}\cdot\mathfrak{g}$ for some $\mathfrak{h} \in \mathfrak{H}$ and $\mathfrak{g} \in \mathfrak{G}$, where $\mathfrak{h}$ acts as identity on the row that consists of three blocks. All we have to show is that the contour we have obtained has freeness of type D↓ in the block y_3 of $\mathcal{S}'$ (that is, in the block $\sigma^{-1}\overline{\sigma a_3}$).

Let $\overline{\mathcal{C}} = \mathfrak{h} \cdot \mathcal{C}_{\sigma^{-1}b\sigma} \cdot \mathfrak{g}$. According to the definition of $\overrightarrow{\mathrm{B}}$-freeness, this contour is quasi permutational. One can easily see that $\boxed{\sigma^{-1}\overline{\sigma a_3} \mid \sigma^{-1}\overline{\sigma a_2}}$ satisfies the conditions of part (2) of Proposition 4.5. Hence, for any d in $\mathrm{D}_k(\mathrm{K})$ there exist $e \in \mathrm{D}_m(\mathrm{K})$ and $f \in \mathrm{D}_q(\mathrm{K})$ such that $e\sigma^{-1}\overline{\sigma a_3}d = \sigma^{-1}\overline{\sigma a_3}$ and $e\sigma^{-1}\overline{\sigma a_2}f = \sigma^{-1}\overline{\sigma a_2}$. Moreover, $e = \sigma^{-1}e'\sigma$ for some $e' \in \mathrm{D}_m(\mathrm{K})$. Observe, however, that left multiplication of the block of $\overline{\mathcal{C}}$ that is in place of a_1 by any member of $\sigma^{-1}\,\mathrm{B}_m(\mathrm{K})\sigma$ can be compensated by the action of $\mathfrak{H}$-$\mathfrak{G}$. As a result, if $\overline{\mathcal{C}}_e$ is the quasi permutational contour obtained from $\overline{\mathcal{C}}$ by replacing the block x (corresponding to the block y_1 of $\mathcal{S}'$) with ex, then the orbit of the contour $\{e\sigma^{-1}\overline{\sigma a_3}d, e\sigma^{-1}\overline{\sigma a_2}f\} \cup \overline{\mathcal{C}}_e$ still contains a 0-1 contour. Hence, the contour $\{\sigma^{-1}\overline{\sigma a_3}, \sigma^{-1}\overline{\sigma a_2}\} \cup \mathfrak{h} \cdot \mathcal{C}_{\sigma^{-1}b\sigma} \cdot \mathfrak{g}$ has D$\downarrow$-freeness in the block y_3 of $\mathcal{S}'$. The result follows. ∎

The statement of Corollary 4.9 does not rely upon any assumptions on the sizes of blocks of the skeletons considered. To explain this, observe that the separated graph of $\mathcal{S}'$ is the Dynkin diagram $\mathbb{D}_n$. Therefore, according to [18, Theorem 6] and [14, Corollary 2.7], the number of $\mathfrak{H}$-$\mathfrak{G}$-orbits in the contour space of $\mathcal{S}'$ must be finite. As a result, it is not surprising that any of these finitely many orbits contains a 0-1 contour. The key role in our considerations is played by D$\downarrow$-freeness that can be found in the terminal block of certain 0-1 representatives of every orbit of $\mathcal{M}_{\mathcal{S}'}$. This freeness allows us to inductively adjoin consecutive blocks to $\mathcal{S}'$ and obtain two row staircase skeletons that are described in Lemma 3.9 and for which the assertion of the lemma holds. In this part of our proof, the sizes of blocks must also be taken into account.

Definition 4.10. Let $x \in \mathbb{M}_{n\times m}(\mathrm{K})$ be a 0-1 matrix. We will say that x is a *row quasi permutation matrix* if every row of x has at most one nonzero entry, and also if there exists an index l such that every column of x, except the lth, has at most one nonzero entry. Moreover, the lth column of x contains at most two nonzero entries. A column quasi permutation matrix is defined dually.

The following remark is an immediate consequence of Definition 4.10.

Remark 4.11. Assume that $x \in \mathbb{M}_{n\times k}(\mathrm{K})$ is a column quasi permutation (respectively, row quasi permutation) matrix. Then for every $e \in \mathrm{D}_n(\mathrm{K})$ there exists $f \in \mathrm{D}_k(\mathrm{K})$ such that $exf = x$ (respectively, for every $f \in \mathrm{D}_k(\mathrm{K})$ there exists $e \in \mathrm{D}_n(\mathrm{K})$ such that $exf = x$).

Lemma 4.12. *Consider a contour* $\boxed{x \mid y}$, *where* $x, y \in \mathbb{M}_{3\times 2}(\mathrm{K})$. *There exist* $h \in \mathrm{Gl}_3(\mathrm{K})$, $g \in \mathrm{Gl}_2(\mathrm{K})$, $d \in \mathrm{D}_2(\mathrm{K})$ *such that the contour* $\boxed{\overline{x} \mid \overline{y}}$ *is* 0-1, *where* $\overline{x} := hxg$ *and* $\overline{y} := hyd$. *Moreover,* h, g, d *can be chosen in such a way that* $\overline{x}$ *is a row quasi permutation matrix and* $\overline{y}$ *is a column quasi permutation* *matrix. Consequently, for every $e \in \mathrm{D}_2(\mathrm{K})$ there exist $h_e \in \mathrm{D}_3(\mathrm{K})$ and $d_e \in \mathrm{D}_2(\mathrm{K})$ such that the matrices $\overline{x}_d := h_e\overline{x}e$ and $\overline{y}_d := h_e\overline{y}d_e$ are* 0-1.

Proof. By acting on $\boxed{x \mid y}$ on the left with elements of $\mathrm{Gl}_3(\mathrm{K})$, this contour may be reduced to the form $\boxed{x' \mid y'}$, where y' is one of the following matrices:

$$\begin{bmatrix} 1 & \theta \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \tag{4.13}$$

for $\theta \in \mathrm{K}$. Next, one can act on x' with elementary column operations and also with elementary row operations that do not change any of the matrices y' in (4.13). We can therefore assume that x' has one of the following forms:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ \alpha & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ \alpha & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \tag{4.14}$$

for $\alpha \in \mathrm{K}$. In each case, one can easily find a 0-1 contour $\boxed{\overline{x} \mid \overline{y}}$ that belongs to the $\mathrm{Gl}_3(\mathrm{K})$ - $\mathrm{Gl}_2(\mathrm{K}) \times \mathrm{D}_2(\mathrm{K})$-orbit of $\boxed{x' \mid y'}$ and such that $\overline{x}, \overline{y}$ are row and column quasi permutation matrices, respectively. Therefore, the result follows from Remark 4.11. ∎

We have finished the preparations for the proof of Lemma 3.9.

Proof of Lemma 3.9. Let $\mathcal{A} \in \mathcal{M}_{\mathcal{S}}$. Let $\mathcal{A}_1, \mathcal{A}_2$ be subcontours of $\mathcal{A}$ with skeletons $\mathcal{S}_1, \mathcal{S}_2$, respectively, and let a_1, a_2, a_3 be the blocks of $\mathcal{A}$ corresponding to the blocks y_1, y_2, y_3 of $\mathcal{S}$, respectively. According to Corollary 4.9 we can assume that the contour $\mathcal{A}_1 \cup \{a_3, a_2, a_1\}$ is 0-1 and has freeness of type D$\downarrow$ in a_3. Now we proceed by induction on the number of blocks of $\mathcal{A}_2$. We may assume that this number is even. Indeed, if the assertion holds when $\mathcal{A}_2$ has $2n$ blocks, then by Lemma 2.11 it holds for the contour with $2n-1$ blocks for $n > 0$. The induction step now follows directly from Lemma 4.12. ∎

5. Thick triple column staircase and the proof of Lemma 3.10. In this section we prove Lemma 3.10. We begin with some preliminary facts concerning properties of $\mathrm{Gl}_3(\mathrm{K})^3$-$\mathrm{Gl}_2(\mathrm{K})^4$-orbits of contours of the form ($\diamond$). First, we show that these orbits contain 0-1 contours and, as before, we show that the choice of elements of $\mathrm{Gl}_3(\mathrm{K})^3, \mathrm{Gl}_2(\mathrm{K})^4$ to obtain such a form allows some type of "freeness" (Lemma 5.4). Based on this result and some additional technical properties of staircase skeletons (Proposition 5.8), we conclude with the proof of Lemma 3.10.

Remark 5.1. Let $\mathcal{C}$ be a contour of the form ($\diamond$) with nonzero blocks a_i, b_i for $i = 1, 2, 3$. Then in the $\mathrm{Gl}_3(\mathrm{K})^3$-$\mathrm{Gl}_2(\mathrm{K})^4$-orbit of $\mathcal{C}$ there exists a contour $\overline{\mathcal{C}}$ with blocks $\overline{a_i}, \overline{b_i}$ which has the following two properties, up to the order of rows:

(1) the column of blocks $\overline{a_1}, \overline{a_2}, \overline{a_3}$ of $\overline{\mathcal{C}}$ has one of the following forms:

$$\underset{\text{(i)}}{\begin{array}{|cc|}\hline 1&0\\0&0\\0&0\\\hline 1&0\\0&0\\0&0\\\hline 1&0\\0&0\\0&0\\\hline\end{array}}\quad
\underset{\text{(ii)}}{\begin{array}{|cc|}\hline 1&0\\0&0\\0&0\\\hline 1&0\\0&0\\0&0\\\hline 0&1\\0&0\\0&0\\\hline\end{array}}\quad
\underset{\text{(iii)}}{\begin{array}{|cc|}\hline 1&1\\0&0\\0&0\\\hline 1&0\\0&0\\0&0\\\hline 0&1\\0&0\\0&0\\\hline\end{array}}\quad
\underset{\text{(iv)}}{\begin{array}{|cc|}\hline 1&0\\0&1\\0&0\\\hline 1&0\\0&0\\0&0\\\hline 1&0\\0&0\\0&0\\\hline\end{array}}\quad
\underset{\text{(v)}}{\begin{array}{|cc|}\hline 1&0\\0&1\\0&0\\\hline 1&0\\0&0\\0&0\\\hline 0&1\\0&0\\0&0\\\hline\end{array}}\quad
\underset{\text{(vi)}}{\begin{array}{|cc|}\hline 1&0\\0&1\\0&0\\\hline 1&0\\0&1\\0&0\\\hline 1&0\\0&0\\0&0\\\hline\end{array}}\quad
\underset{\text{(vii)}}{\begin{array}{|cc|}\hline 1&0\\0&1\\0&0\\\hline 1&0\\0&1\\0&0\\\hline 1&0\\0&1\\0&0\\\hline\end{array}}$$

(2) every row of $\overline{\mathcal{C}}$ has one of the following forms:

$$\underset{\text{(a)}}{\begin{array}{|cc|cc|}\hline 1&0&1&0\\0&0&0&0\\0&0&0&0\\\hline\end{array}}\quad
\underset{\text{(b)}}{\begin{array}{|cc|cc|}\hline 0&1&1&0\\0&0&0&0\\0&0&0&0\\\hline\end{array}}\quad
\underset{\text{(c)}}{\begin{array}{|cc|cc|}\hline 1&1&1&0\\0&0&0&0\\0&0&0&0\\\hline\end{array}}$$

$$\underset{\text{(d)}}{\begin{array}{|cc|cc|}\hline 1&0&0&0\\0&0&1&0\\0&0&0&0\\\hline\end{array}}\quad
\underset{\text{(e)}}{\begin{array}{|cc|cc|}\hline 0&1&0&0\\0&0&1&0\\0&0&0&0\\\hline\end{array}}\quad
\underset{\text{(f)}}{\begin{array}{|cc|cc|}\hline 1&1&0&0\\0&0&1&0\\0&0&0&0\\\hline\end{array}}$$

$$\underset{\text{(g)}}{\begin{array}{|cc|cc|}\hline 1&0&1&0\\0&0&0&1\\0&0&0&0\\\hline\end{array}}\quad
\underset{\text{(h)}}{\begin{array}{|cc|cc|}\hline 0&1&1&0\\0&0&0&1\\0&0&0&0\\\hline\end{array}}\quad
\underset{\text{(i)}}{\begin{array}{|cc|cc|}\hline 1&1&1&0\\0&0&0&1\\0&0&0&0\\\hline\end{array}}$$

$$\underset{\text{(j)}}{\begin{array}{|cc|cc|}\hline 1&0&0&0\\0&0&1&0\\0&0&0&1\\\hline\end{array}}\quad
\underset{\text{(k)}}{\begin{array}{|cc|cc|}\hline 0&1&0&0\\0&0&1&0\\0&0&0&1\\\hline\end{array}}\quad
\underset{\text{(l)}}{\begin{array}{|cc|cc|}\hline 1&1&0&0\\0&0&1&0\\0&0&0&1\\\hline\end{array}}$$

$$\underset{\text{(m)}}{\begin{array}{|cc|cc|}\hline 1&0&1&0\\0&1&0&0\\0&0&0&0\\\hline\end{array}}\quad
\underset{\text{(n)}}{\begin{array}{|cc|cc|}\hline 1&0&0&0\\0&1&1&0\\0&0&0&0\\\hline\end{array}}\quad
\underset{\text{(o)}}{\begin{array}{|cc|cc|}\hline 1&0&1&0\\0&1&0&1\\0&0&0&0\\\hline\end{array}}\quad
\underset{\text{(p)}}{\begin{array}{|cc|cc|}\hline 1&0&1&0\\0&1&\theta&0\\0&0&0&0\\\hline\end{array}}\quad \text{for } \theta \neq 0$$

$$\underset{\text{(q)}}{\begin{array}{|cc|cc|}\hline 1&0&1&0\\0&1&0&0\\0&0&0&1\\\hline\end{array}}\quad
\underset{\text{(r)}}{\begin{array}{|cc|cc|}\hline 1&0&0&0\\0&1&1&0\\0&0&0&1\\\hline\end{array}}\quad
\underset{\text{(s)}}{\begin{array}{|cc|cc|}\hline 1&0&0&0\\0&1&0&0\\0&0&1&0\\\hline\end{array}}\quad
\underset{\text{(t)}}{\begin{array}{|cc|cc|}\hline 1&0&1&0\\0&1&\theta&0\\0&0&0&1\\\hline\end{array}}\quad \text{for } \theta \neq 0$$

Proof. Consider the block matrix $\mathcal{A}$ consisting of the blocks a_1, a_2, a_3 of the first column of $\mathcal{C}$. It is easy to see that $\mathcal{A}$ can be transformed to one of the forms (i)–(vii) by using only elementary column operations and such elementary row operations that act within the blocks a_i for $i = 1, 2, 3$. This is a standard matrix problem, similar to some well known problems considered in [20, Section 2.1]. As these elementary operations correspond to certain actions of $\mathrm{Gl}_3(\mathrm{K})^3$ and $\mathrm{Gl}_2(\mathrm{K})^4$ on the entire $\mathcal{C}$, it can be assumed that the column $\mathcal{A}$ of $\mathcal{C}$ is already in one of the forms (i)–(vii).

Now, we will show that in the $\mathrm{Gl}_3(\mathrm{K})^3$-$\mathrm{Gl}_2(\mathrm{K})^4$-orbit of $\mathcal{C}$ there exists a contour $\overline{\mathcal{C}}$ such that its first column is equal to $\mathcal{A}$ and each row is of one of the forms (a)–(t). For this, consider two pairs (a, b) and (a, b') of nonzero matrices. We say that (a, b) is *row similar* to (a, b') if the block matrix $[a\ b]$ can be transformed into $[a\ b']$ by elementary row operations that do not change the block a and by elementary operations on the columns that belong to b. It is clear that the assertion will follow if we show that every pair (a, b) that forms a row in $\mathcal{C}$ is row similar to one of the pairs (a)–(t).

First, observe that any pair (a, b) is row similar to (a, b'), where b' is a column echelon form of b. We will consider four cases.

Case 1: $r(a) = r(b') = 1$. Then b' has one of the forms

$$\begin{bmatrix} 1 & 0 \\ \alpha & 0 \\ \beta & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ \gamma & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \tag{5.2}$$

where $\alpha, \beta, \gamma \in \mathrm{K}$. Since the second row and the third row of a are zero, because the column $\mathcal{A}$ of $\mathcal{C}$ is in one of the forms (i)–(vii), the following elementary operations do not change a: interchanging the second and third rows and subtracting either the second or the third row from any other row. Thus if b' is of the first form in (5.2) then (a, b') is row similar either to one of the pairs (a)–(c) if $\alpha = \beta = 0$, or to one of the pairs (d)–(f) if $\alpha \neq 0$ or $\beta \neq 0$. In the other cases of (5.2) the pair (a, b') is row similar to (d), (e) or (f).

Case 2: $r(a) = 2$, $r(b') = 1$. Then b' is again of one of the forms in (5.2). In this case the block a does not change under the elementary operations of subtraction of the third row from either the first or the second row. Thus if either β or γ is nonzero, then (a, b') is row similar to (s). Now, let $\beta = \gamma = 0$. If b' is of the first form in (5.2), then (a, b') is row similar either to (m), if $\alpha = 0$, or to one of the pairs (p), if $\alpha \neq 0$. In the other cases of (5.2), (a, b') is row similar to either (n) or (s).

Case 3: $r(a) = 1$, $r(b') = 2$. Now, b' has one of the forms

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ \alpha & \beta \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ \gamma & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \tag{5.3}$$

where $\alpha, \beta, \gamma \in \mathrm{K}$. Hence, as in Case 1, if b' is of the first form in (5.3) then (a, b') is row similar either to one of the pairs (g)–(i), for $\alpha = 0$, or to one of (j)–(l), for $\alpha \neq 0$. If b' is of the second form in (5.3), then (a, b') is row similar either to one of the pairs (g)–(i), for $\gamma = 0$, or to one of (j)–(l), for $\gamma \neq 0$. In the last case of (5.3), (a, b') is row similar to one of the pairs (j)–(l).

CASE 4: $r(a) = r(b') = 2$. Then b' is again in one of the forms listed in (5.3). As in Case 2, the matrix a does not change under the elementary operations of subtraction of the third row from either the first or the second row. Thus if b' is of the first form in (5.3), then (a, b') is row similar either to one of the pairs (t), for α, β nonzero, or to one of the pairs (r), (q), (o), if either α or β is 0. In the other cases of (5.3), (a, b') is row similar to (q), (r) or (t). ∎

Next, we present the following key result, to which the proof of Lemma 3.10 will be later reduced.

LEMMA 5.4. *Let $\mathcal{K}$ be a contour of the form* ($\diamond$) *with rows of blocks a_i, b_i, and let $(\alpha_i, \beta_i) \neq (0,0)$ be pairs of elements of the field* K *for $i = 1, 2, 3$. If $\mathfrak{H} = \mathrm{Gl}_3(\mathrm{K})^3$ and $\mathfrak{G} = \mathrm{Gl}_2(\mathrm{K})^4$ act on $\mathcal{K}$ via* (2.2), *then there exist $\mathfrak{h} = (h_1, h_2, h_3) \in \mathfrak{H}$, $\mathfrak{g} = (t, g_1, g_2, g_3) \in \mathfrak{G}$ and $\gamma_1, \gamma_2, \gamma_3 \in \mathrm{K}^*$ such that:*

- $\mathfrak{h} \cdot \mathcal{K} \cdot \mathfrak{g}$ *is a* 0-1 *contour,*
- *the matrices $\gamma_i \cdot [\alpha_i\ \beta_i] \cdot g_i$ are* 0-1 *for $i = 1, 2, 3$.*

The proof will be preceded with the following observation.

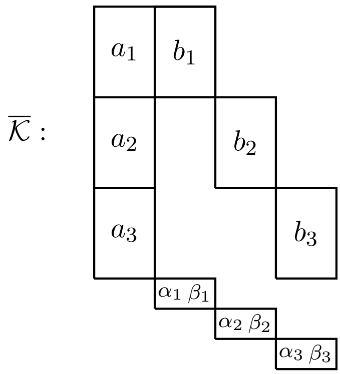
REMARK 5.5. Let (a, b) be one of the pairs of matrices of size 3×2 listed in (a)–(t) of Remark 5.1 and let x be a nonzero matrix of size 1×2. Then as long as (a, b) is not of the form (o), there exist $f \in \mathrm{Gl}_3(\mathrm{K})$, $g \in \mathrm{Gl}_2(\mathrm{K})$ and $\gamma \in \mathrm{K}^*$ such that

$$(5.6) \qquad fa = a, \quad fbg = b, \quad \gamma x g = x',$$

where x' is a 0-1 matrix.

Proof. Let $x = [\alpha_1\ \alpha_2]$. If either α_1 or α_2 is zero, then the assertion is clear. Assume that $\alpha_1, \alpha_2 \neq 0$. First, consider the case where the second column of b is zero. Then, to get (5.6) it is enough to set $f = id_3$, $g = \left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha_1\alpha_2^{-1} \end{smallmatrix}\right]$ and $\gamma = \alpha_1^{-1}$. Therefore, the assertion follows when (a, b) is one of the pairs (a)–(f), (m)–(p) or (s). On the other hand, in each of the cases (g)–(l) and (q), (r), (t), we can find a nonzero row of b, say the lth, such that the lth row of a is zero. Moreover, the lth row of b is $[0\ 1]$ in all these cases. As a result, it is enough to define $f \in \mathrm{Gl}_3(\mathrm{K})$ as the matrix of the elementary operation of multiplication of the lth row by $\alpha_1^{-1}\alpha_2$, to define $g \in \mathrm{Gl}_2(\mathrm{K})$ as the matrix $\left[\begin{smallmatrix} 1 & 0 \\ 0 & \alpha_1\alpha_2^{-1} \end{smallmatrix}\right]$, and $\gamma = \alpha_1^{-1}$. The result follows. ∎

Proof of Lemma 5.4. The assertion is clearly equivalent to the following statement. *Assume that three blocks of size* 1×2 *are adjoined to the contour* $\mathcal{K}$ *in order to form a new contour* $\overline{\mathcal{K}}$:



Let $\overline{\mathfrak{H}} = \mathrm{Gl}_3(\mathrm{K})^3 \times (\mathrm{K}^*)^3$ *and* $\overline{\mathfrak{G}} = \mathrm{Gl}_2(\mathrm{K})^4$. *Then in the* $\overline{\mathfrak{H}}$-$\overline{\mathfrak{G}}$*-orbit of* $\overline{\mathcal{K}}$ *there exists a* 0-1 *contour.*

We proceed to the proof. We will denote the coordinates of the elements $\overline{\mathfrak{h}} \in \overline{\mathfrak{H}}$ and $\overline{\mathfrak{g}} \in \overline{\mathfrak{G}}$ that act on $\overline{\mathcal{K}}$ according to (2.2) by

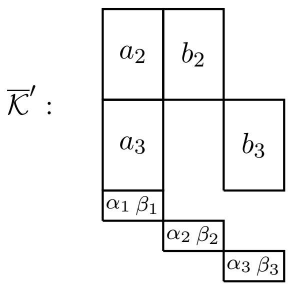
$$\overline{\mathfrak{h}} = (f_1, f_2, f_3, \gamma_1, \gamma_2, \gamma_3), \qquad \overline{\mathfrak{g}} = (t, g_1, g_2, g_3), \tag{5.7}$$

where $f_i \in \mathrm{Gl}_3(\mathrm{K})$, $t, g_i \in \mathrm{Gl}_2(\mathrm{K})$ and $\gamma_i \in \mathrm{K}^*$, for $i = 1, 2, 3$.

According to Remark 5.1, we may assume that every row (a_i, b_i) of $\mathcal{K}$ is in one of the forms (a)–(t). First, consider the case when no pair (a_i, b_i) is of the form (o) for $i = 1, 2, 3$. Then to each triple $a_i, b_i, [\alpha_i\ \beta_i]$ we can apply Remark 5.5 to obtain matrices $f_i \in \mathrm{Gl}_3(K)$, $g_i \in \mathrm{Gl}_2(\mathrm{K})$ and elements $\gamma_i \in \mathrm{K}^*$ such that (5.6) is satisfied. Hence, if we set $t = \mathrm{id}_2$ and then act with elements $\overline{\mathfrak{h}}, \overline{\mathfrak{g}}$ of the form (5.7) on $\overline{\mathcal{K}}$ via (2.2), we will obtain a 0-1 contour that belongs to the $\overline{\mathfrak{H}}$-$\overline{\mathfrak{G}}$-orbit of $\overline{\mathcal{K}}$.

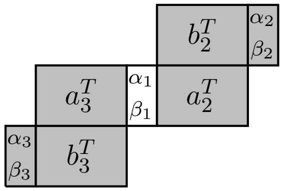
Next, we consider the following cases:

Case 1: *Exactly one of the rows* (a_i, b_i) *of* $\overline{\mathcal{K}}$ *is of the form* (o). We may assume that it is (a_1, b_1). Consider the following contour $\overline{\mathcal{K}}'$ with blocks a_2, b_2, a_3, b_3 of size 3×2 and also with blocks $[\alpha_i\ \beta_i]$ of size 1×2 for $i = 1, 2, 3$:



Now we show that if $\mathfrak{H}' = \mathrm{Gl}_3(\mathrm{K})^2 \times (\mathrm{K}^*)^3$ and $\mathfrak{G}' = \mathrm{Gl}_2(\mathrm{K})^3$, then in the $\mathfrak{H}'$-$\mathfrak{G}'$-orbit of $\overline{\mathcal{K}}'$ there exists a 0-1 contour. Observe that by changing the order of rows and columns of $\overline{\mathcal{K}}'$, and after a transposition, we obtain a contour $\overline{\mathcal{K}}''$ with blocks of sizes 2×3 and 2×1 of the form

$\overline{\mathcal{K}}''$:



Set $\mathfrak{H}'' = \mathrm{Gl}_2(\mathrm{K})^3$ and $\mathfrak{G}'' = \mathrm{K}^* \times \mathrm{Gl}_3(\mathrm{K}) \times \mathrm{K}^* \times \mathrm{Gl}_3(\mathrm{K}) \times \mathrm{K}^*$. It is clear that the existence of a 0-1 contour in the $\mathfrak{H}'$-$\mathfrak{G}'$-orbit of $\overline{\mathcal{K}}'$ is equivalent to the existence of a 0-1 contour in the $\mathfrak{H}''$-$\mathfrak{G}''$-orbit of $\overline{\mathcal{K}}''$. The latter is, however, an easy consequence of Corollary 4.3. Indeed, using this corollary we can assume that all blocks of $\overline{\mathcal{K}}''$, except $[\alpha_1\ \beta_1]^T$, are quasi permutational matrices. Then it is clear that the contour of this form has a 0-1 contour in its orbit. It is obtained by the action of certain elements of $\mathfrak{H}''$, $\mathfrak{G}''$ via (2.2) with all coordinates diagonal. Thus, in the $\mathfrak{H}''$-$\mathfrak{G}''$-orbit of $\overline{\mathcal{K}}''$ we can find a 0-1 contour, and as argued above, it follows that we can also find a 0-1 contour in the $\mathfrak{H}'$-$\mathfrak{G}'$-orbit of $\overline{\mathcal{K}}'$.

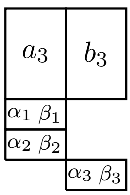
Now, let $f_2, f_3 \in \mathrm{Gl}_3(\mathrm{K})$, $\gamma_1, \gamma_2, \gamma_3 \in \mathrm{K}^*$, $t, g_2, g_3 \in \mathrm{Gl}_2(\mathrm{K})$ be such that $\mathfrak{h}' \cdot \overline{\mathcal{K}}' \cdot \mathfrak{g}'$ is a 0-1 contour for $\mathfrak{h}' = (f_2, f_3, \gamma_1, \gamma_2, \gamma_3) \in \mathfrak{H}'$ and $\mathfrak{g}' = (t, g_2, g_3) \in \mathfrak{G}'$. Also, consider

$$g_1 = t \in \mathrm{Gl}_2(\mathrm{K}), \quad f_1 = \begin{bmatrix} g_1^{-1} & 0 \\ 0 & 1 \end{bmatrix} \in \mathrm{Gl}_3(\mathrm{K}).$$

Then, if we define $\overline{\mathfrak{h}} = (f_1, f_2, f_3, \gamma_1, \gamma_2, \gamma_3) \in \overline{\mathfrak{H}}$ and $\overline{\mathfrak{g}} = (t, g_1, g_2, g_3) \in \overline{\mathfrak{G}}$, we can easily see that $\overline{\mathfrak{h}} \cdot \overline{\mathcal{K}} \cdot \overline{\mathfrak{g}}$ is a 0-1 contour, since its first row is of the form (o). The result follows in this case.

Case 2. *Exactly two rows (a_i, b_i) of $\overline{\mathcal{K}}$ are in the form* (o). We may assume that these are $(a_1, b_1), (a_2, b_2)$. Consider the contour

$\overline{\mathcal{K}}'''$:



It is contained in the contour $\overline{\mathcal{K}}'$, thus from Lemma 2.11 and Case 1 it follows that if $\mathfrak{H}''' = \mathrm{Gl}_3(\mathrm{K}) \times (\mathrm{K}^*)^3$ and $\mathfrak{G}''' = \mathrm{Gl}_2(\mathrm{K})^2$, then there exist

$\mathfrak{h}''' = (f_3, \gamma_1, \gamma_2, \gamma_3) \in \mathfrak{H}'''$ and $\mathfrak{g}''' = (t, g_3) \in \mathfrak{G}'''$ such that $\mathfrak{h}''' \cdot \overline{\mathcal{K}}''' \cdot \mathfrak{g}'''$ is a 0-1 contour.

Therefore, it is enough to set $g_1 = g_2 = t$ and $f_1 = f_2 = \left[\begin{smallmatrix} t^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right]$. Then, if $\overline{\mathfrak{h}} = (f_1, f_2, f_3, \gamma_1, \gamma_2, \gamma_3)$ and $\overline{\mathfrak{g}} = (t, g_1, g_2, g_3)$, we can easily see that the contour $\overline{\mathfrak{h}} \cdot \overline{\mathcal{K}} \cdot \overline{\mathfrak{g}}$ is 0-1.

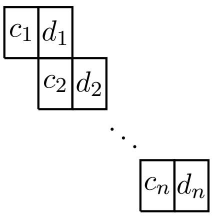
Case 3: *All three rows* (a_i, b_i) *of* $\overline{\mathcal{K}}$ *are of the form* (o). This case is treated similarly to the previous cases, by noticing that the assertion can be reduced to the problem of finding a 0-1 contour in the $(\mathrm{K}^*)^3$-$\mathrm{Gl}_2(\mathrm{K})$ orbit of the contour

α_1	β_1
α_2	β_2
α_3	β_3

This concludes the proof. ∎

The final step needed in completing the proof of Lemma 3.10 is to show that it can be reduced to the assertion of Lemma 5.4. This requires proving a certain technical result on staircase contours.

Proposition 5.8. *Consider the following staircase contour* $\mathcal{A}$*, formed from the consecutive blocks* $c_1, d_1, \ldots, c_n, d_n$ *of size* 3×2*:*



where c_1, d_n *are column terminal blocks. There exist* $\mathfrak{h} = (h_1, \ldots, h_n) \in \mathfrak{H} = \mathrm{Gl}_3(\mathrm{K})^n$ *and* $\mathfrak{g} = (\mathrm{id}_2, g_1, \ldots, g_n) \in \mathfrak{G} = \mathrm{Gl}_2(\mathrm{K})^{n+1}$ *such that:*

(i) *the matrix* h_1c_1 *is equal to the product* qe *of a column quasi permutation matrix* q *and* $e \in \mathrm{D}_2(\mathrm{K})$,
(ii) $h_id_ig_i$ *are row quasi permutation matrices for* $1 \le i \le n$,
(iii) $h_ic_ig_{i-1}$ *are column quasi permutation matrices for* $1 < i \le n$.

Moreover:

(a) *if* $e \in \mathrm{D}_2(\mathrm{K})$ *is a scalar matrix, then there exist* $\mathfrak{h}' \in \mathfrak{H}$ *and* $\mathfrak{g}' \in \mathfrak{G}$ *such that* $\mathfrak{h}'\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}\mathfrak{g}'$ *is a* 0-1 *contour and* $\mathfrak{g}'$ *acts as identity on* h_1c_1,
(b) *there exists a pair* $(\alpha', \beta') \neq (0, 0)$ *of elements of* K *such that for every* $\gamma \in \mathrm{K}^*$ *and* $g \in \mathrm{Gl}_2(\mathrm{K})$ *for which* $\gamma[\alpha'\ \beta']g$ *is a* 0-1 *matrix, there exist* $\mathfrak{h}''(\gamma, g) \in \mathfrak{H}$ *and* $\mathfrak{g}''(\gamma, g) \in \mathfrak{G}$ *(that depend on* γ, g*) and such that* $\mathfrak{h}''(\gamma, g)\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}\mathfrak{g}''(\gamma, g)$ *is a* 0-1 *contour and the first coordinate of* $\mathfrak{g}\mathfrak{g}''(\gamma, g)$ *that acts on* h_1c_1 *is equal to* g.

Proof. We prove assertions (i)–(iii) by induction on n.

For $n = 1$, the statement follows from Lemma 4.12.

Assume now that for some $n \geq 2$ the assertions hold for all $m < n$ and that the contour $\mathcal{A}$ has $2n$ blocks. From the inductive hypothesis it follows that there exist $h'_1, \dots, h'_{n-1} \in \mathrm{Gl}_3(\mathrm{K})$ and $g'_1, \dots, g'_{n-1} \in \mathrm{Gl}_2(\mathrm{K})$ such that (i)–(iii) hold for the blocks $h'_1 c_1, h'_1 d_1 g'_1, \dots, h'_{n-1} c_{n-1} g'_{n-2}, h'_{n-1} d_{n-1} g'_{n-1}$. Consider the matrices $c_n g'_{n-1}$, d_n. Again, Lemma 4.12 allows us to find $h \in \mathrm{Gl}_3(\mathrm{K})$, $g \in \mathrm{Gl}_2(\mathrm{K})$, $e_n \in \mathrm{D}_2(\mathrm{K})$ such that $q_n := h c_n g'_{n-1} e_n^{-1}$ is a column quasi permutation matrix and $h d_n g$ is a row quasi permutation matrix. Therefore, for $\mathfrak{h}_n = (h'_1, \dots, h'_{n-1}, h) \in \mathfrak{H}$ and $\mathfrak{g}_n = (\mathrm{id}, g'_1, \dots, g'_{n-1}, g) \in \mathfrak{G}$, we have:

- (i′) in place of the blocks c_1, c_n of $\mathcal{A}$, the contour $\mathfrak{h}_n \cdot \mathcal{A} \cdot \mathfrak{g}_n$ has blocks that are products of a column quasi permutation matrix and a matrix that belongs to $\mathrm{D}_2(\mathrm{K})$,
- (ii′) in place of the blocks d_i, the contour $\mathfrak{h}_n \cdot \mathcal{A} \cdot \mathfrak{g}_n$ has row quasi permutation matrices, for $1 \leq i \leq n$,
- (iii′) in place of the blocks c_i, the contour $\mathfrak{h}_n \cdot \mathcal{A} \cdot \mathfrak{g}_n$ has column quasi permutation matrices, for $1 < i < n$.

By using Remark 4.11, we can find $e_{n-1} \in \mathrm{D}_3(\mathrm{K})$ and $f_n \in \mathrm{D}_2(\mathrm{K})$ that satisfy the following conditions:

$$f_n = e_n^{-1}, \quad (h c_n g'_{n-1}) f_n = q_n, \quad e_{n-1}(h'_{n-1} d_{n-1} g'_{n-1}) f_n = h'_{n-1} d_{n-1} g'_{n-1},$$

and such that $e_{n-1}(h'_{n-1} c_{n-1} g'_{n-2})$ is a product of $e_{n-1} \in \mathrm{D}_3(\mathrm{K})$ and a column quasi permutation matrix $h'_{n-1} c_{n-1} g'_{n-2}$. We proceed in this manner, applying Remark 4.11 $n-1$ times; in the ith step bringing the set of the last i columns (for $i = 2, \dots, n$) to 0-1 forms (by acting with the invertible diagonal matrices e_{n+1-i}, f_{n+2-i}). Then we obtain a contour such that in place of c_1 we have $e_1 h'_1 c_1$, and the remaining blocks are either row or column quasi permutation matrices, precisely as required in (ii) and (iii). Finally, due to Remark 4.11, $e_1 h'_1 c = h'_1 c e'$ for some $e' \in \mathrm{D}_2(\mathrm{K})$. Thus (i) is satisfied and the inductive step follows. This concludes the proof of the first part of the proposition.

Next, we prove (a). Consider a contour $\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}$ that satisfies (i)–(iii) and assume that $e \in \mathrm{D}_2(\mathrm{K})$ is a scalar matrix. Consider scalar matrices $a_1, \dots, a_n \in \mathrm{D}_3(\mathrm{K})$ and $b_1, \dots, b_n \in \mathrm{D}_2(\mathrm{K})$ such that:

- $a_1(h_1 c_1) = a_1 q e = q$,
- $a_i(h_i c_i g_{i-1}) b_{i-1} = h_i c_i g_{i-1}$ for $i = 2, \dots, n$,
- $a_i(h_i d_i g_i) b_i = h_i d_i g_i$ for $i = 1, \dots, n$.

If we set $\mathfrak{h}' = (a_1, \dots, a_n)$ and $\mathfrak{g}' = (\mathrm{id}_2, b_1, \dots, b_n)$, then it is clear that the contour $\mathfrak{h}'\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}\mathfrak{g}'$ is 0-1. Thus (a) holds.

Finally, we prove (b). Once again, consider a contour $\mathfrak{h}\cdot\mathcal{A}\cdot\mathfrak{g}$ that satisfies (i)–(iii). According to (i), the block that belongs to the first row and the first column of this contour is of the form qe, where q is a column quasi permutation matrix of size 3×2 and $e\in \mathrm{D}_2(\mathrm{K})$ is $\left[\begin{smallmatrix}\alpha & 0\\ 0 & \beta\end{smallmatrix}\right]$ for some $\alpha,\beta\in \mathrm{K}^*$.

We will consider some special cases, corresponding to the possible forms of q. To reduce the number of cases we will begin by showing that if we can prove (b) when the first block of $\mathfrak{h}\cdot\mathcal{A}\cdot\mathfrak{g}$ is of the form qe, then we can also prove (b) when the first block is of the form $(\sigma q\epsilon)e$ for any permutation matrices $\sigma\in\Sigma_3$ and $\epsilon\in\Sigma_2$.

So assume that we have a contour $\mathfrak{h}\cdot\mathcal{A}\cdot\mathfrak{g}$ that satisfies (i)–(iii) with the first block being a quasi permutation matrix $(\sigma q\epsilon)e$. Consider a new contour $\mathcal{K}=\mathfrak{h}_\sigma\mathfrak{h}\cdot\mathcal{A}\cdot\mathfrak{g}\mathfrak{g}_\epsilon$, where

$$\mathfrak{h}_\sigma=(\sigma^{-1},\mathrm{id}_3,\ldots,\mathrm{id}_3)\quad\text{and}\quad \mathfrak{g}_\epsilon=(e^{-1}\epsilon^{-1}e,\mathrm{id}_2,\ldots,\mathrm{id}_2).$$

Clearly, it also satisfies (i)–(iii). Moreover, its first block is of the form qe. Thus, by our assumption there exists $(\alpha',\beta')\neq(0,0)$ such that for any $\gamma\in\mathrm{K}^*$ and $g\in\mathrm{Gl}_2(\mathrm{K})$ such that $\gamma[\alpha'\ \beta']g$ is a 0-1 matrix, there exist $\mathfrak{h}''(\gamma,g)\in\mathfrak{H}$ and $\mathfrak{g}''(\gamma,g)\in\mathfrak{G}$ such that the first coordinate of $\mathfrak{g}''(\gamma,g)$ is g and the contour $\mathfrak{h}''(\gamma,g)\cdot\mathcal{K}\cdot\mathfrak{g}''(\gamma,g)$ is 0-1. Consider another pair $\alpha'',\beta''\in\mathrm{K}$ such that $[\alpha''\ \beta'']=[\alpha'\ \beta']\cdot e^{-1}\epsilon e$. Then for any $\gamma\in\mathrm{K}^*$ and $g\in\mathrm{Gl}_2(\mathrm{K})$ such that $\gamma[\alpha''\ \beta'']g$ is a 0-1 matrix, the matrix $\gamma[\alpha'\ \beta'](e^{-1}\epsilon eg)$ is 0-1 as well. Again, by our assumption there exist $\mathfrak{h}''(\gamma,e^{-1}\epsilon eg)\in\mathfrak{H}$ and $\mathfrak{g}''(\gamma,e^{-1}\epsilon eg)\in\mathfrak{G}$ such that $\mathfrak{h}''(\gamma,e^{-1}\epsilon eg)\cdot\mathcal{K}\cdot\mathfrak{g}''(\gamma,e^{-1}\epsilon eg)$ is a 0-1 contour. The first coordinate of $\mathfrak{g}''(\gamma,e^{-1}\epsilon eg)$ equals $(e^{-1}\epsilon e)g$. From the definition of $\mathcal{K}$ it follows that the contour

$$\begin{aligned}&\mathfrak{h}''(\gamma,e^{-1}\epsilon eg)(\mathfrak{h}_\sigma\mathfrak{h}\cdot\mathcal{A}\cdot\mathfrak{g}\mathfrak{g}_\epsilon)\mathfrak{g}''(\gamma,e^{-1}\epsilon eg)\\ &\qquad\qquad\qquad=\mathfrak{h}''(\gamma,e^{-1}\epsilon eg)\mathfrak{h}_\sigma(\mathfrak{h}\cdot\mathcal{A}\cdot\mathfrak{g})\mathfrak{g}_\epsilon\mathfrak{g}''(\gamma,e^{-1}\epsilon eg)\end{aligned}$$

is 0-1 and the first coordinate of $\mathfrak{g}_\epsilon\mathfrak{g}''(\gamma,e^{-1}\epsilon eg)$ is $(e^{-1}\epsilon^{-1}e)(e^{-1}\epsilon eg)=g$. We have thus proved (b) for the contour $\mathfrak{h}\cdot\mathcal{A}\cdot\mathfrak{g}$ with the first block equal to $(\sigma q\epsilon)e$. It follows that when proving (b) we may, without loss of generality, ignore the order of rows and columns of the matrix q.

First we consider the case when q is of rank 1. As argued above, it is enough to assume that qe has one of the forms

$$\begin{bmatrix}\alpha & \beta\\ 0 & 0\\ 0 & 0\end{bmatrix},\qquad \begin{bmatrix}\alpha & 0\\ 0 & 0\\ 0 & 0\end{bmatrix}. \tag{5.9}$$

Let $g\in\mathrm{Gl}_2(\mathrm{K})$ and $\gamma\in\mathrm{K}^*$ be such that the matrix $\gamma[\alpha\ \beta]g$ is 0-1. We consider two cases listed in (5.9).

In the first case, set $\alpha' := \alpha$ and $\beta' := \beta$. Also, let $(g, \mathrm{id}_2, \ldots, \mathrm{id}_2) \in \mathfrak{G}$. Consider the contour $\mathfrak{h} \cdot \mathcal{A} \cdot \overline{\mathfrak{g}}$, where $\overline{\mathfrak{g}} = \mathfrak{g}(g, \mathrm{id}_2, \ldots, \mathrm{id}_2)$. It clearly satisfies (i)–(iii). Moreover, the block qeg of this contour is of the form $q'e'$, where q' is a column quasi permutation matrix and $e' \in \mathrm{D}_2(\mathrm{K})$. It follows from (a) that there exist $\mathfrak{h}' \in \mathfrak{H}'$ and $\mathfrak{g}' \in \mathfrak{G}'$ such that $\mathfrak{h}'\mathfrak{h} \cdot \mathcal{A} \cdot \overline{\mathfrak{g}}\mathfrak{g}'$ is a 0-1 contour. Thus, the assertion follows for $\mathfrak{h}''(\gamma, g) := \mathfrak{h}'$ and $\mathfrak{g}''(\gamma, g) := (g, \mathrm{id}_2, \ldots, \mathrm{id}_2)\mathfrak{g}'$. In the second case of (5.9), the argument is almost the same. Now we set $\alpha' := \alpha$ and $\beta' := 0$. Thus, (b) follows when q is of rank 1.

The case when q is of rank 2 is more technical. Again, if we ignore the order of rows and columns, it suffices to consider the case when qe is of the form

$$\begin{bmatrix} \alpha & 0 \\ 0 & \beta \\ 0 & 0 \end{bmatrix}.$$

First, observe that if $\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}$ consists of exacly one block qe then the assertion is clear. Indeed, let $\alpha' = \beta' = 1$. Then for any $g \in \mathrm{Gl}_2(\mathrm{K})$ and $\gamma \in \mathrm{K}^*$ such that $\gamma[1\ 1]g$ is a 0-1 matrix, we set $\mathfrak{h}''(\gamma, g) := (h)$, where $h = \left[\begin{smallmatrix} g^{-1}e^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right] \in \mathrm{Gl}_3(\mathrm{K})$ and $\mathfrak{g}''(\gamma, g) := (g)$. As a result, $\mathfrak{h}''(\gamma, g)\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}\mathfrak{g}''(\gamma, g) = q$, which is 0-1, and the result follows. Therefore, we now assume that $\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}$ consists of at least two blocks.

Consider the row quasi permutation matrix $h_1 d_1 g_1$ that belongs to the same row of $\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}$ as the block qe. We can assume (perhaps after a permutation of the columns in 0-1 matrices that belong to the second column of $\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}$; see the previous argument on permuting rows and columns of qe) that this matrix is one of

$$\text{(5.10)} \qquad \begin{bmatrix} * & 0 \\ * & 0 \\ * & * \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

It is clear that the elementary operation of subtraction of the third row from either the second or the first row, performed at the same time on both qe and $h_1 d_1 g_1$ (when $h_1 d_1 g_1$ has one of the last two forms in (5.10)), reduces $h_1 d_1 g_1$, up to the order of columns, to one of the first two matrices in (5.10). Therefore, it is enough to prove (b) for these.

First, assume that $h_1 d_1 g_1$ is the first type in (5.10). Let $\theta_1, \theta_2 \in \mathrm{K}$ be such that

$$\begin{bmatrix} \alpha^{-1} & 0 & 0 \\ 0 & \beta^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot h_1 d_1 g_1 = \begin{bmatrix} \theta_1 & 0 \\ \theta_2 & 0 \\ * & * \end{bmatrix}.$$

If $(\theta_1, \theta_2) = (0, 0)$, then the result follows easily: namely, we set $\alpha' = \beta' = 1$

and consider any $\gamma \in \mathrm{K}^*$ and $g \in \mathrm{Gl}_2(\mathrm{K})$ such that $\gamma[1\ 1]g$ is a 0-1 matrix. Then the conditions of (b) are satisfied by $\mathfrak{h}''(\gamma, g) := (h, \mathrm{id}_3, \dots, \mathrm{id}_3) \in \mathfrak{H}$ and $\mathfrak{g}''(\gamma, g) := (g, \mathrm{id}_2, \dots, \mathrm{id}_2) \in \mathfrak{G}$ where

$$h = \begin{bmatrix} g^{-1} & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \alpha^{-1} & 0 & 0 \\ 0 & \beta^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The contour $\mathfrak{h}''(\gamma, g)\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}\mathfrak{g}''(\gamma, g)$ is clearly 0-1, and (b) follows.

Now assume that $(\theta_1, \theta_2) \neq (0, 0)$ and let $\alpha' = -\theta_2$ and $\beta' = \theta_1$. Let $g = \left[\begin{smallmatrix} \tau_1 & \tau_2 \\ \tau_3 & \tau_4 \end{smallmatrix}\right] \in \mathrm{Gl}_2(\mathrm{K})$ and $\gamma \in \mathrm{K}^*$ be such that $\gamma[-\theta_2\ \theta_1]g$ is a 0-1 matrix of the form $[\rho_1\ \rho_2]$. Then

$$\begin{cases} \rho_1 = \gamma(-\tau_1\theta_2 + \tau_3\theta_1), \\ \rho_2 = \gamma(-\tau_2\theta_2 + \tau_4\theta_1). \end{cases}$$

Also, set $g' = \det(g) \cdot g^{-1} = \begin{bmatrix} \tau_4 & -\tau_2 \\ -\tau_3 & \tau_1 \end{bmatrix}$. Then

$$\gamma g' \cdot \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} \rho_2 \\ -\rho_1 \end{bmatrix}. \tag{5.11}$$

Consider the elements $\overline{\mathfrak{h}} \in \mathfrak{H}$ and $\overline{\mathfrak{g}} \in \mathfrak{G}$ of the form $\mathfrak{h} = (h, \mathrm{id}_3, \dots, \mathrm{id}_3)$ and $\mathfrak{g} = (g, \mathrm{id}_2, \dots, \mathrm{id}_2)$ such that

$$h = \begin{bmatrix} \gamma g' & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \alpha^{-1} & 0 & 0 \\ 0 & \beta^{-1} & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then, if we act with $\overline{\mathfrak{h}}, \overline{\mathfrak{g}}$ on $\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}$ via (2.2), only the first row of $\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}$ is modified: namely, from (5.11) it follows that in place of blocks qe and $h_1 d_1 g_1$ there are

$$\begin{bmatrix} \gamma \cdot \det(g) & 0 \\ 0 & \gamma \cdot \det(g) \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} \rho_2 & 0 \\ -\rho_1 & 0 \\ \rho_3 & \rho_4 \end{bmatrix},$$

respectively, where $\rho_1, \rho_2, \rho_3, \rho_4 \in \{0, 1\}$. If $\rho_1 = 0$, then the result follows. Therefore, we will consider two cases when $\rho_1 = 1$.

Let $\rho_1 = 1$ and $\rho_2 = 0$. Then the contour $\overline{\mathfrak{h}}\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}\overline{\mathfrak{g}}$ satisfies conditions (i)–(iii), and in place of the block qe of $\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}$ there is a matrix qe', where q is a column quasi permutation matrix and $e' \in \mathrm{D}_2(\mathrm{K})$. Thus, according to (a), there exist $\mathfrak{h}' \in \mathfrak{H}'$ and $\mathfrak{g}' \in \mathfrak{G}$ such that $\mathfrak{h}'\overline{\mathfrak{h}}\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}\overline{\mathfrak{g}}\mathfrak{g}'$ is a 0-1 contour. Notice that the first coordinate of $\overline{\mathfrak{g}}\mathfrak{g}'$ is g. Thus, if we set $\mathfrak{h}''(\gamma, g) := \mathfrak{h}'\overline{\mathfrak{h}}$ and $\mathfrak{g}'(\gamma, g) := \overline{\mathfrak{g}}\mathfrak{g}'$, the statement of (b) follows.

If $\rho_1 = \rho_2 = 1$, then we multiply the matrices in the first row of $\overline{\mathfrak{h}}\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}\overline{\mathfrak{g}}$ on the left by

$$h' = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

As a result, we obtain matrices of the form

$$\begin{bmatrix} \gamma \cdot \det(g) & 0 \\ \gamma \cdot \det(g) & \gamma \cdot \det(g) \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ \rho_3 & \rho_4 \end{bmatrix},$$

where $\rho_3, \rho_4 \in \{0, 1\}$. The first is a product of a 0-1 matrix and a scalar matrix that belongs to $\mathrm{D}_2(\mathrm{K})$. Thus, consider the contour $\overline{\mathfrak{h}}'\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}\overline{\mathfrak{g}}$, where $\overline{\mathfrak{h}}' = (h', \mathrm{id}_3, \dots, \mathrm{id}_3)\overline{\mathfrak{h}}$. The first block of this contour is again a product of a 0-1 matrix and a scalar (invertible) matrix. The remaining blocks are 0-1. We proceed as in the proof of (a) (where it is enough to assume that q is a 0-1 matrix, but not necessarily a quasi permutation matrix). We prove that there exist $\mathfrak{h}' \in \mathfrak{H}$ and $\mathfrak{g}' \in \mathfrak{G}$ such that the first coordinate of $\mathfrak{g}'$ is an identity matrix, and the contour $\mathfrak{h}'\overline{\mathfrak{h}}'\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}\overline{\mathfrak{g}}\mathfrak{g}'$ is 0-1. Therefore, we define $\mathfrak{h}''(\gamma, g) := \mathfrak{h}'\overline{\mathfrak{h}}'$ and $\mathfrak{g}''(\gamma, g) := \overline{\mathfrak{g}}\mathfrak{g}'$, and (b) follows.

We have finished the proof for $(\theta_1, \theta_2) \neq (0, 0)$. Therefore, (b) follows when $h_1 d_1 g_1$ is in the first form of (5.10).

It remains to prove (b) when $h_1 d_1 g_1$ is in the second form of (5.10). First, this is easy if $\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}$ consists of exactly two blocks: we then set $\alpha' = \beta' = 1$, and for any $g \in \mathrm{Gl}_2(\mathrm{K})$ and $\gamma \in \mathrm{K}^*$ such that $\gamma[1\ 1]g$ is a 0-1 matrix, we define $\mathfrak{h}''(\gamma, g) := (h) \in \mathfrak{H}$, $\mathfrak{g}''(\gamma, g) := (g, eg) \in \mathfrak{G}$, where $h = \left[\begin{smallmatrix} g^{-1}e^{-1} & 0 \\ 0 & 1 \end{smallmatrix}\right] \in \mathrm{Gl}_3(\mathrm{K})$. As a result, $\mathfrak{h}''(\gamma, g)\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}\mathfrak{g}''(\gamma, g)$ consists of two 0-1 blocks of the form q and $h_1 d_1 g_1$, and the assertion follows.

We can therefore assume that $\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}$ has more than two blocks. In this case it is easy to see that the statement of (b) is equivalent for two contours: $\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}$ and the staircase contour that consists of the blocks $(h_2 c_2 g_1)e, h_2 d_2 g_2, h_3 c_3 g_2, h_3 d_3 g_3, \dots$ and is obtained from $\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}$ by deleting the first row of matrices, qe and $h_1 d_1 g_1$, and by multiplying the column that contains the quasi permutation block $h_2 c_2 g_1$ by $e \in \mathrm{D}_2(\mathrm{K})$. Thus, if we prove (b) by induction on the number of blocks in $\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}$, we can see that the case under consideration can be reduced to one of the following cases considered before: where $\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}$ consists of one or two blocks, or where the first row of the contour is of the form considered before.

Consequently, (b) follows also in the case where $h_1 d_1 g_1$ is of the second form in (5.10). We have thus proved (b) for q of rank 2. This concludes the entire proof of (b), and the result follows. ∎

Proof of Lemma 3.10. According to Lemma 2.11, we may assume that s_i is an even integer $2t_i$ for $i = 1, 2, 3$. Let $\mathcal{A} \in \mathcal{M}_{\mathcal{S}}$. We may also assume (due to Remark 2.9) that all blocks of $\mathcal{A}$ are nonzero. Let $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3$ stand for the subcontours of $\mathcal{A}$ that consist of the blocks $b_{i,j}$ for $i = 1, 2, 3$ respectively and for $j \le s_i$. Observe that if blocks x, y of $\mathcal{A}$ belong to the same row or column and if $x \in \mathcal{A}_i$ and $y \in \mathcal{A}_j$, for some $i, j \in \{1, 2, 3\}$, then $i = j$. Therefore, if the group acting on $\mathcal{A}_i$ via (2.2) is denoted by $\mathfrak{H}_i \times \mathfrak{G}_i \simeq \mathrm{Gl}_3(\mathrm{K})^{t_i} \times \mathrm{Gl}_2(\mathrm{K})^{t_i+1}$, for $i = 1, 2, 3$, then

$$(5.12) \qquad \mathfrak{H} \simeq \mathrm{Gl}_3(\mathrm{K})^3 \times \mathfrak{H}_1 \times \mathfrak{H}_2 \times \mathfrak{H}_3, \quad \mathfrak{G} \simeq \mathrm{Gl}_2(\mathrm{K}) \times \mathfrak{G}_1 \times \mathfrak{G}_2 \times \mathfrak{G}_3$$

(we may assume that the first coordinates of elements of $\mathfrak{G}_i$ act on the column that contains the block b_i of $\mathcal{A}$, for $i = 1, 2, 3$).

By Proposition 5.8, applied separately to each of the contours $\mathcal{A}_i$, we may assume that:

- $b_{i,1}$ is a product of a column quasi permutation matrix and a matrix that belongs to $\mathrm{D}_2(\mathrm{K})$, for $i = 1, 2, 3$,
- $b_{i,2_j}$ are row quasi permutation matrices for $1 \le j \le t_i$ and $i = 1, 2, 3$,
- $b_{i,2j-1}$ are column quasi permutation matrices for $1 \le j \le t_i$ and $i = 1, 2, 3$.

Thus, as in the proof of (b) in Proposition 5.8, to each contour $\mathcal{A}_i$ we may assign a pair of scalars $(\alpha_i, \beta_i) \neq (0, 0)$ such that for any $\gamma_i' \in \mathrm{K}^*$ and $g_i' \in \mathrm{Gl}_2(\mathrm{K})$ for which $\gamma_i'[\alpha_i\ \beta_i]g_i'$ is a 0-1 matrix, there exist $\mathfrak{h}(\gamma_i', g_i') \in \mathfrak{H}_i$ and $\mathfrak{g}(\gamma_i', g_i') \in \mathfrak{G}_i$ such that $\mathfrak{h}(\gamma_i', g_i') \cdot \mathcal{A}_i \cdot \mathfrak{g}(\gamma_i', g_i')$ is a 0-1 contour and the first coordinate of $\mathfrak{g}(\gamma_i', g_i')$ is g_i'. Now we apply Lemma 5.4 to find $f_1, f_2, f_3 \in \mathrm{Gl}_3(\mathrm{K})$, $\gamma_1, \gamma_2, \gamma_3 \in \mathrm{K}^*$, and $t, w_1, w_2, w_3 \in \mathrm{Gl}_2(\mathrm{K})$ such that the matrices $f_i a_i t, f_i b_i w_i$ and $\gamma_i[\alpha_i\ \beta_i]w_i$ are all 0-1 for $i = 1, 2, 3$. As said before, there exist $\mathfrak{h}(\gamma_i, w_i) \in \mathfrak{H}_i$ and $\mathfrak{g}(\gamma_i, w_i) \in \mathfrak{G}_i$ such that $\mathfrak{h}(\gamma_i, w_i) \cdot \mathcal{A}_i \cdot \mathfrak{g}(\gamma_i, w_i)$ is a 0-1 contour and the first coordinate of $\mathfrak{g}(\gamma_i, w_i)$ is w_i. As a result, we can define $\mathfrak{h} \in \mathfrak{H}$ and $\mathfrak{g} \in \mathfrak{G}$ such that $\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}$ is a 0-1 contour. These elements are of the following form:

- $g_1 := t$ and $h_i := f_i$ for $i = 1, 2, 3$,
- the coordinates $\mathfrak{g}$ that act on the columns of $\mathcal{A}$ containing b_1, b_2, b_3 are w_1, w_2, w_3, respectively,
- if $\phi_i : \mathfrak{H} \to \mathfrak{H}_i$ and $\psi_i : \mathfrak{G} \to \mathfrak{G}_i$ are the natural projections, in accordance with (5.12), then $\phi_i(\mathfrak{h}) = \mathfrak{h}(\gamma_i, w_i)$ and $\psi_i(\mathfrak{g}) = \mathfrak{g}(\gamma_i, w_i)$ for $i = 1, 2, 3$.

With this definition, in the contour $\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}$, 0-1 matrices appear in place of blocks a_i, b_i, according to Lemma 5.4. Moreover, from the same lemma it follows that the coordinates w_i of $\mathfrak{g}$ that act on $b_{i,1}$ are such that the matrices $\gamma_i[\alpha_i\ \beta_i]w_i$ are 0-1. From (b) in Proposition 5.8 it follows that $\mathfrak{h}(\gamma_i, w_i) \cdot \mathcal{A}_i \cdot$

$\mathfrak{g}(\gamma_i, w_i)$ is a 0-1 contour for $i = 1, 2, 3$. Therefore, all blocks of $\mathfrak{h} \cdot \mathcal{A} \cdot \mathfrak{g}$ are 0-1, as claimed. ∎

6. Concluding remarks and open problems. We conclude with a few remarks and questions. First of all, it should be noted that although Problem 1.1 is not fully solved in this paper, Theorem 2.8 along with [14, Theorem 1.2] provide potential methods and intuition for tackling the difficulties of the general case. It follows from Lemma 2.11 that these two results impose necessary conditions on skeletons of arbitrary algebras A with $C(A)$ finite. One can hope that, by further computational effort, Problem 1.1 can be solved in general.

A possible next step might be to try to understand the finiteness problem for the semigroup $C(A)$ for algebras A with the Jacobson radical of nilpotency index greater than 2. Clearly, the results obtained so far in the context of Problem 1.1 provide some necessary conditions for the finiteness of $C(A/J(A)^2)$. For certain classes of algebras, partial results can be obtained: see Corollary 6.3 below for Frobenius algebras with $J(A)^3 = 0$. It also seems of interest to take up the finiteness problem in the class of all basic algebras.

One of the referees of my Ph.D. thesis [12] has pointed out that, in the representation theory of algebras, the study of modules over any radical square zero algebra A reduces to the study of the radical square zero upper block triangular algebra

$$(6.1) \qquad \Lambda_A := \begin{bmatrix} \overline{A} & J \\ 0 & \overline{A} \end{bmatrix}, \quad \text{where } \overline{A} = A/J(A) \text{ and } J = J(A).$$

The algebra Λ_A is hereditary, i.e. all left ideals of Λ_A are projective. We recall from [4] that there exists a K-linear functor $\mathbb{F}$: A-mod $\to \Lambda_{\mathrm{A}}$-mod defining a bijection between the isoclasses of indecomposable left A-modules and the isoclasses of indecomposable left Λ_A-modules that are not simple projective (see also [21] for more details). In particular, Λ_A is of finite representation type if and only if A is of finite representation type. It is clear that the Jacobson radical of Λ_A is equal to $\left[\begin{smallmatrix} 0 & J \\ 0 & 0 \end{smallmatrix}\right]$, and therefore Theorem 1.1 in [13] easily implies that the finiteness of $C(\Lambda_A)$ is equivalent to the finiteness of $C(A)$. We have the following useful consequence of Corollary 1.8.

COROLLARY 6.2. *Assume that A is a radical square zero* K*-algebra satisfying the conditions of Theorem* 1.7. *The following conditions are equivalent:*

(a) *The semigroup $C(A)$ is finite.*
(b) *The semigroup $C(\Lambda_A)$ is finite.*
(c) *The separated quiver $\Gamma^s(A)$ of A, viewed as an unoriented graph, is a disjoint union of simply laced Dynkin diagrams $\mathbb{A}_n$, $n \geq 1$, $\mathbb{D}_n$, $n \geq 4$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$.*

Now we are able to extend Corollary 6.2 from radical square zero K-algebras to Frobenius (or self-injective) radical cube zero ones, as follows.

Corollary 6.3. *Assume that A is a Frobenius* K*-algebra such that $J(A)^3 = 0$ and the quotient algebra $A/J(A)$ satisfies the conditions of Theorem* 1.7. *The socle* $\mathrm{soc}_A A$ *of A is a two-sided ideal, the quotient* K*-algebra $B := A/\mathrm{soc}_A A$ is a radical square zero algebra, and the following conditions are equivalent:*

(a) *The semigroup $C(A)$ is finite.*
(b) *The algebra A is of finite representation type.*
(c) *The semigroup $C(B)$ is finite.*
(d) *The algebra B is of finite representation type.*
(e) *The separated quiver $\Gamma^s(B)$ of B, viewed as an unoriented graph, is a disjoint union of simply laced Dynkin diagrams $\mathbb{A}_n$, $n \geq 1$, $\mathbb{D}_n$, $n \geq 4$, $\mathbb{E}_6$, $\mathbb{E}_7$, $\mathbb{E}_8$.*

We only sketch the proof. Since A is Frobenius, it is self-injective and every injective left A-module is projective. The left ideal $\mathfrak{U} := \mathrm{soc}_A A$ is a two-sided ideal, because $\mathrm{soc}_A A = \mathrm{soc}\, A_A$ (see [22, Theorem 6.13]). Since $J(A)^3 = 0$, we have $J(A)^2 \subseteq \mathfrak{U}$, and therefore $J(B)^2 = 0$.

First we prove the equivalence (b)$\Leftrightarrow$(d), by applying the argument used in [3, pp. 788–789]. Since (b)$\Rightarrow$(d) is obvious, we need to prove (b)$\Leftarrow$(d). We do this by showing that every indecomposable left A-module M is projective (then a direct summand Ae of ${}_AA$m where e is a primitive idempotent) or is annihilated by $\mathfrak{U}$, i.e. M is a module over the quotient algebra $B := A/\mathfrak{U}$. Assume that M is an indecomposable left A-module such that $\mathfrak{U}M \neq 0$, and let S be a simple submodule of $\mathfrak{U}M$. Then the injective envelope $P := E(S)$ is indecomposable projective. By the injectivity of P, there is $f \in \mathrm{Hom}_A(M, P)$ such that the restriction of f to S is the embedding $S \hookrightarrow P$. We recall from [1, Section I.1.5] that P has a unique maximal submodule $J(A)P$. Note that $\mathrm{Im}\, f$ is not contained in $J(A)P$, because the inclusions $S \subseteq P$, $S \subseteq \mathfrak{U}M$ and $\mathrm{Im}\, f \subseteq J(A)M$ imply $0 \neq f(S) \subseteq f(\mathfrak{U}M) \subseteq \mathfrak{U}J(A)P = 0$, a contradiction. It follows that $\mathrm{Im}\, f + J(A)P = P$, and the Nakayama lemma yields $\mathrm{Im}\, f = P$. By the projectivity of P, the homomorphism f is bijective, because M is indecomposable. Consequently, M is projective. This finishes the proof of (b)$\Leftrightarrow$(d).

The implication (a)$\Rightarrow$(c) follows from the fact that the canonical algebra surjection $A \to B$ induces a semigroup surjection $C(A) \to C(B)$. Since (b)$\Rightarrow$(a) is well known (see [18, Corollary 4]) and the equivalences (c)$\Leftrightarrow$(d)$\Leftrightarrow$(e) are a consequence of Corollary 6.2 with A and B interchanged, the proof is complete. ∎

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