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A NEW WAY TO ITERATE BRZEZIŃSKI CROSSED PRODUCTS

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Abstract. If $A \otimes_{R,\sigma} V$ and $A \otimes_{P,\nu} W$ are two Brzeziński crossed products and $Q : W \otimes V \to V \otimes W$ is a linear map satisfying certain properties, we construct a Brzeziński crossed product $A \otimes_{S,\theta} (V \otimes W)$. This construction contains as a particular case the iterated twisted tensor product of algebras.

1. Introduction. The *twisted tensor product* of the associative unital algebras A and B is a new associative unital algebra structure built on the linear space $A \otimes B$ with the help of a linear map $R : B \otimes A \to A \otimes B$ called a *twisting map*. This construction, denoted by $A \otimes_R B$, appeared in several contexts and has various applications ([CSV], [VDVK]). Concrete examples come especially from Hopf algebra theory, like for instance the smash product.

It was proved in [JLPV] that twisted tensor products of algebras may be iterated. Namely, if $A \otimes_{R_1} B$, $B \otimes_{R_2} C$ and $A \otimes_{R_3} C$ are twisted tensor products and the twisting maps R_1, R_2, R_3 satisfy the braid relation

$$(\mathrm{id}_A \otimes R_2) \circ (R_3 \otimes \mathrm{id}_B) \circ (\mathrm{id}_C \otimes R_1) = (R_1 \otimes \mathrm{id}_C) \circ (\mathrm{id}_B \otimes R_3) \circ (R_2 \otimes \mathrm{id}_A),$$

then one can define certain twisted tensor products $A \otimes_{T_2} (B \otimes_{R_2} C)$ and $(A \otimes_{R_1} B) \otimes_{T_1} C$ that are equal as algebras (and this algebra is called the iterated twisted tensor product).

The *Brzeziński crossed product*, introduced in [B], is a common generalization of twisted tensor products of algebras and the Hopf crossed product (containing also as a particular case the quasi-Hopf smash product introduced in [BPVO]). If A is an associative unital algebra, V is a linear space endowed with a distinguished element 1_V, and $\sigma : V \otimes V \to A \otimes V$ and $R : V \otimes A \to A \otimes V$ are linear maps satisfying certain conditions, then the Brzeziński crossed product is a certain associative unital algebra structure on $A \otimes V$, denoted by $A \otimes_{R,\sigma} V$.

In [P] it was proved that Brzeziński crossed products may be iterated, in the following sense. One can define first a "mirror version" of the Brzeziński crossed product, denoted by $W \overline{\otimes}_{P,\nu} D$ (where D is an associative unital

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algebra, W is a linear space and P, ν are certain linear maps). Examples are twisted tensor products of algebras and the quasi-Hopf smash product introduced in [BPV]. Then it was proved that, if $W \overline{\otimes}_{P,\nu} D$ and $D \otimes_{R,\sigma} V$ are two Brzeziński crossed products and $Q : V \otimes W \to W \otimes D \otimes V$ is a linear map satisfying some conditions, then one can define certain Brzeziński crossed products $(W \overline{\otimes}_{P,\nu} D) \otimes_{\overline{R},\overline{\sigma}} V$ and $W \overline{\otimes}_{\overline{P},\overline{\nu}} (D \otimes_{R,\sigma} V)$ that are equal as algebras. Iterated twisted tensor products of algebras appear as a particular case of this construction, as also is the so-called quasi-Hopf two-sided smash product $A \# H \# B$ from [BPVO].

The aim of this paper is to show that Brzeziński crossed products may be iterated in a different way, which will also contain as a particular case the iterated twisted tensor product of algebras. Namely, we prove that if $A \otimes_{R,\sigma} V$ and $A \otimes_{P,\nu} W$ are two Brzeziński crossed products and $Q : W \otimes V \to V \otimes W$ is a linear map satisfying certain properties, then we can define two Brzeziński crossed products $A \otimes_{S,\theta} (V \otimes W)$ and $(A \otimes_{R,\sigma} V) \otimes_{T,\eta} W$ that are equal as algebras.

Our inspiration for looking at this new way of iterating Brzeziński crossed products came from the following result in graded ring theory: If G is a group, R is a G-graded ring, A and B are two finite left G-sets, then there exists a ring isomorphism between the smash products $R \# (A \times B)$ and $(R \# A) \# B$. This result was obtained in [DNN, Corollary 3.2], and it is useful in the study of the von Neumann regularity of rings of the type $R \# A$ (cf. [DNN] again). The smash product $R \# A$ of the G-graded ring R by a (finite) left G-set A was introduced in the paper [NRVO] and it is a particular case of a more general construction. If H is a Hopf algebra, R an H-comodule algebra and C an H-module coalgebra, then we may consider the category ${}_R^C\mathcal{M}(H)$ of Doi–Koppinen Hopf modules (i.e. left R-modules and left C-comodules which satisfy certain compatibility relations). Then the smash product $R \# A$ used in [DNN] is a particular smash product and it is the first example in the category ${}_R^C\mathcal{M}(H)$ (in the case when H is the groupring $k[G]$, R a G-graded ring and C the grouplike coalgebra $k[A]$ on a G-set A).

2. Preliminaries. We work over a commutative field k. All algebras, linear spaces etc. will be over k; unadorned $\otimes$ means $\otimes_k$. By "algebra" we always mean an associative unital algebra. The multiplication of an algebra A is denoted by μ_A or simply μ when there is no danger of confusion, and we usually denote $\mu_A(a \otimes a') = aa'$ for all $a, a' \in A$. The unit of an algebra A is denoted by 1_A or simply 1 when there is no danger of confusion.

We recall from [CSV], [VDVK] that, given two algebras A, B and a k-linear map $R : B \otimes A \to A \otimes B$, with Sweedler-type notation $R(b \otimes a) = a_R \otimes b_R = a_r \otimes b_r$ for $a \in A$, $b \in B$, satisfying the conditions $a_R \otimes 1_R = a \otimes 1$,

$1_R \otimes b_R = 1 \otimes b$, $(aa')_R \otimes b_R = a_R a'_r \otimes (b_R)_r$, $a_R \otimes (bb')_R = (a_R)_r \otimes b_r b'_R$ for all $a, a' \in A$ and $b, b' \in B$, if we define on $A \otimes B$ a new multiplication by $(a \otimes b)(a' \otimes b') = aa'_R \otimes b_R b'$, then this multiplication is associative with unit $1 \otimes 1$. In this case, the map R is called a *twisting map* between A and B, and the new algebra structure on $A \otimes B$ is denoted by $A \otimes_R B$ and called the *twisted tensor product* of A and B afforded by the map R.

We recall from [B] the construction of Brzeziński's crossed product:

Proposition 2.1 ([B]). *Let $(A, \mu, 1_A)$ be an (associative unital) algebra and V a vector space equipped with a distinguished element $1_V \in V$. Then the vector space $A \otimes V$ has the structure of an associative algebra with unit $1_A \otimes 1_V$ and with multiplication such that $(a \otimes 1_V)(b \otimes v) = ab \otimes v$ for all $a, b \in A$ and $v \in V$ if and only if there exist linear maps $\sigma : V \otimes V \to A \otimes V$ and $R : V \otimes A \to A \otimes V$ satisfying the following conditions:*

$$R(1_V \otimes a) = a \otimes 1_V, \quad R(v \otimes 1_A) = 1_A \otimes v, \qquad \forall a \in A,\ v \in V, \tag{2.1}$$

$$\sigma(1_V \otimes v) = \sigma(v \otimes 1_V) = 1_A \otimes v, \qquad \forall v \in V, \tag{2.2}$$

$$R \circ (\mathrm{id}_V \otimes \mu) = (\mu \otimes \mathrm{id}_V) \circ (\mathrm{id}_A \otimes R) \circ (R \otimes \mathrm{id}_A), \tag{2.3}$$

$$\begin{aligned}(\mu \otimes \mathrm{id}_V) \circ (\mathrm{id}_A \otimes \sigma) \circ (R \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes \sigma) \\ = (\mu \otimes \mathrm{id}_V) \circ (\mathrm{id}_A \otimes \sigma) \circ (\sigma \otimes \mathrm{id}_V),\end{aligned} \tag{2.4}$$

$$\begin{aligned}(\mu \otimes \mathrm{id}_V) \circ (\mathrm{id}_A \otimes \sigma) \circ (R \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes R) \\ = (\mu \otimes \mathrm{id}_V) \circ (\mathrm{id}_A \otimes R) \circ (\sigma \otimes \mathrm{id}_A).\end{aligned} \tag{2.5}$$

If this is the case, the multiplication of $A \otimes V$ is given explicitly by

$$\mu_{A \otimes V} = (\mu_2 \otimes \mathrm{id}_V) \circ (\mathrm{id}_A \otimes \mathrm{id}_A \otimes \sigma) \circ (\mathrm{id}_A \otimes R \otimes \mathrm{id}_V),$$

where $\mu_2 = \mu \circ (\mathrm{id}_A \otimes \mu) = \mu \circ (\mu \otimes \mathrm{id}_A)$. We denote by $A \otimes_{R,\sigma} V$ this algebra structure and call it the crossed product (*or* Brzeziński crossed product) *afforded by the data (A, V, R, σ).*

If $A \otimes_{R,\sigma} V$ is a crossed product, we introduce the following Sweedler-type notation:

$$R : V \otimes A \to A \otimes V, \qquad R(v \otimes a) = a_R \otimes v_R,$$
$$\sigma : V \otimes V \to A \otimes V, \qquad \sigma(v \otimes v') = \sigma_1(v, v') \otimes \sigma_2(v, v'),$$

for all $v, v' \in V$ and $a \in A$. With this notation, the multiplication of $A \otimes_{R,\sigma} V$ reads

$$(a \otimes v)(a' \otimes v') = aa'_R \sigma_1(v_R, v') \otimes \sigma_2(v_R, v'), \qquad \forall a, a' \in A,\ v, v' \in V.$$

A twisted tensor product is a particular case of a crossed product (cf. [DLGG]), namely, if $A \otimes_R B$ is a twisted tensor product of algebras then $A \otimes_R B = A \otimes_{R,\sigma} B$, where $\sigma : B \otimes B \to A \otimes B$ is given by $\sigma(b \otimes b') = 1_A \otimes bb'$ for all $b, b' \in B$.

Remark. The conditions (2.3), (2.4) and (2.5) for R, σ may be written in Sweedler-type notation respectively as

$$(aa')_R \otimes v_R = a_R a'_r \otimes (v_R)_r, \tag{2.6}$$

$$\sigma_1(y,z)_R \sigma_1(x_R, \sigma_2(y,z)) \otimes \sigma_2(x_R, \sigma_2(y,z)) = \sigma_1(x,y)\sigma_1(\sigma_2(x,y),z) \otimes \sigma_2(\sigma_2(x,y),z), \tag{2.7}$$

$$(a_R)_r \sigma_1(v_r, v'_R) \otimes \sigma_2(v_r, v'_R) = \sigma_1(v,v')a_R \otimes \sigma_2(v,v')_R \tag{2.8}$$

for all $a, a' \in A$, $x, y, z, v, v' \in V$, where we also denoted $R(v \otimes a) = a_r \otimes v_r$ for all $a \in A$, $v \in V$.

3. The main result and examples

Theorem 3.1. *Let $A \otimes_{R,\sigma} V$ and $A \otimes_{P,\nu} W$ be two crossed products and $Q : W \otimes V \to V \otimes W$ a linear map, written $Q(w \otimes v) = v_Q \otimes w_Q$ for all $v \in V$ and $w \in W$. Assume that the following conditions are satisfied:*

(i) *Q is unital, in the sense that*

$$Q(1_W \otimes v) = v \otimes 1_W, \quad Q(w \otimes 1_V) = 1_V \otimes w, \quad \forall v \in V,\ w \in W. \tag{3.1}$$

(ii) *The braid relation holds for R, P, Q, i.e.*

$$(\mathrm{id}_A \otimes Q) \circ (P \otimes \mathrm{id}_V) \circ (\mathrm{id}_W \otimes R) = (R \otimes \mathrm{id}_W) \circ (\mathrm{id}_V \otimes P) \circ (Q \otimes \mathrm{id}_A), \tag{3.2}$$

or equivalently,

$$(a_R)_P \otimes (v_R)_Q \otimes (w_P)_Q = (a_P)_R \otimes (v_Q)_R \otimes (w_Q)_P \tag{3.3}$$

for all $a \in A, v \in V$, $w \in W$.

(iii) *We have the following hexagonal relation between σ, P, Q:*

$$(\mathrm{id}_A \otimes Q) \circ (P \otimes \mathrm{id}_V) \circ (\mathrm{id}_W \otimes \sigma) = (\sigma \otimes \mathrm{id}_W) \circ (\mathrm{id}_V \otimes Q) \circ (Q \otimes \mathrm{id}_V), \tag{3.4}$$

or equivalently,

$$\sigma_1(v,v')_P \otimes \sigma_2(v,v')_Q \otimes (w_P)_Q = \sigma_1(v_Q, v'_q) \otimes \sigma_2(v_Q, v'_q) \otimes (w_Q)_q \tag{3.5}$$

for all $v, v' \in V$ and $w \in W$, where we also denoted $Q(w \otimes v) = v_q \otimes w_q$ for all $v \in V$, $w \in W$.

(iv) *We have the following hexagonal relation between ν, R, Q:*

$$(R \otimes \mathrm{id}_W) \circ (\mathrm{id}_V \otimes \nu) \circ (Q \otimes \mathrm{id}_W) \circ (\mathrm{id}_W \otimes Q) = (\mathrm{id}_A \otimes Q) \circ (\nu \otimes \mathrm{id}_V), \tag{3.6}$$

or equivalently,

$$\nu_1(w,w') \otimes v_Q \otimes \nu_2(w,w')_Q = \nu_1(w_q, w'_Q)_R \otimes ((v_Q)_q)_R \otimes \nu_2(w_q, w'_Q) \tag{3.7}$$

for all $v \in V$ and $w, w' \in W$, where we also denoted $Q(w \otimes v) = v_q \otimes w_q$ for all $v \in V$, $w \in W$.

Define the linear maps

$$
\begin{aligned}
&S : (V \otimes W) \otimes A \to A \otimes (V \otimes W), \qquad S := (R \otimes \mathrm{id}_W) \circ (\mathrm{id}_V \otimes P),\\
&\theta : (V \otimes W) \otimes (V \otimes W) \to A \otimes (V \otimes W),\\
&\theta := (\mu_A \otimes \mathrm{id}_V \otimes \mathrm{id}_W) \circ (\mathrm{id}_A \otimes R \otimes \mathrm{id}_W) \circ (\sigma \otimes \nu) \circ (\mathrm{id}_V \otimes Q \otimes \mathrm{id}_W),\\
&T : W \otimes (A \otimes V) \to (A \otimes V) \otimes W, \qquad T := (\mathrm{id}_A \otimes Q) \circ (P \otimes \mathrm{id}_V),\\
&\eta : W \otimes W \to (A \otimes V) \otimes W,\\
&\qquad\qquad \eta(w \otimes w') = (\nu_1(w, w') \otimes 1_V) \otimes \nu_2(w, w'), \qquad \forall w, w' \in W.
\end{aligned}
$$

Then we have a crossed product $A \otimes_{S,\theta} (V \otimes W)$ (with respect to $1_{V \otimes W} := 1_V \otimes 1_W$), a crossed product $(A \otimes_{R,\sigma} V) \otimes_{T,\eta} W$ and an algebra isomorphism $A \otimes_{S,\theta} (V \otimes W) \simeq (A \otimes_{R,\sigma} V) \otimes_{T,\eta} W$ given by the trivial identification.

Proof. We first show that $A \otimes_{S,\theta} (V \otimes W)$ is a crossed product, i.e. we prove (2.1)–(2.5) with R replaced by S, σ replaced by θ etc. The relations (2.1) and (2.2) follow immediately by (3.1) and the relations (2.1) and (2.2) for R, σ and P, ν. Note that the maps S and θ are defined explicitly by

$$
\begin{aligned}
S(v \otimes w \otimes a) &= (a_P)_R \otimes v_R \otimes w_P,\\
\theta(v \otimes w \otimes v' \otimes w') &= \sigma_1(v, v'_Q)\nu_1(w_Q, w')_R \otimes \sigma_2(v, v'_Q)_R \otimes \nu_2(w_Q, w')
\end{aligned}
$$

for all $v, v' \in V$, $w, w' \in W$ and $a \in A$. For all $a \in A$, $v \in V$ and $w \in W$, we will denote $R(v \otimes a) = a_R \otimes v_R = a_r \otimes v_r = a_{\mathcal{R}} \otimes v_{\mathcal{R}} = a_{\overline{R}} \otimes v_{\overline{R}}$, $Q(w \otimes v) = v_Q \otimes w_Q = v_q \otimes w_q = v_{\overline{Q}} \otimes w_{\overline{Q}}$ and $P(w \otimes a) = a_P \otimes w_P = a_p \otimes w_p$.

Proof of (2.3).

$$
\begin{aligned}
&S \circ (\mathrm{id}_V \otimes \mathrm{id}_W \otimes \mu_A)(v \otimes w \otimes a \otimes a')\\
&\quad = S(v \otimes w \otimes aa') = ((aa')_P)_R \otimes v_R \otimes w_P \overset{(2.6)}{=} (a_P a'_p)_R \otimes v_R \otimes (w_P)_p\\
&\quad \overset{(2.6)}{=} (a_P)_R (a'_p)_r \otimes (v_R)_r \otimes (w_P)_p\\
&\quad = (\mu_A \otimes \mathrm{id}_V \otimes \mathrm{id}_W)\big((a_P)_R \otimes (a'_p)_r \otimes (v_R)_r \otimes (w_P)_p\big)\\
&\quad = (\mu_A \otimes \mathrm{id}_V \otimes \mathrm{id}_W) \circ (\mathrm{id}_A \otimes S)((a_P)_R \otimes v_R \otimes w_P \otimes a')\\
&\quad = (\mu_A \otimes \mathrm{id}_V \otimes \mathrm{id}_W) \circ (\mathrm{id}_A \otimes S) \circ (S \otimes \mathrm{id}_A)(v \otimes w \otimes a \otimes a'), \quad \text{q.e.d.}
\end{aligned}
$$

Proof of (2.4).

$$
\begin{aligned}
&(\mu_A \otimes \mathrm{id}_V \otimes \mathrm{id}_W) \circ (\mathrm{id}_A \otimes \theta) \circ (S \otimes \mathrm{id}_V \otimes \mathrm{id}_W)\\
&\qquad\qquad \circ (\mathrm{id}_V \otimes \mathrm{id}_W \otimes \theta)(v \otimes w \otimes v' \otimes w' \otimes v'' \otimes w'')
\end{aligned}
$$

$$
\begin{aligned}
&= (\mu_A \otimes \mathrm{id}_V \otimes \mathrm{id}_W) \circ (\mathrm{id}_A \otimes \theta) \circ (S \otimes \mathrm{id}_V \otimes \mathrm{id}_W) \\
&\quad \big(v \otimes w \otimes \sigma_1(v', v''_Q)\nu_1(w'_Q, w'')_R \otimes \sigma_2(v', v''_Q)_R \otimes \nu_2(w'_Q, w'')\big) \\
&= (\mu_A \otimes \mathrm{id}_V \otimes \mathrm{id}_W) \circ (\mathrm{id}_A \otimes \theta) \\
&\quad \big(([\sigma_1(v', v''_Q)\nu_1(w'_Q, w'')_R]_P)_r \otimes v_r \otimes w_P \otimes \sigma_2(v', v''_Q)_R \otimes \nu_2(w'_Q, w'')\big) \\
&= ([\sigma_1(v', v''_Q)\nu_1(w'_Q, w'')_R]_P)_r \sigma_1(v_r, (\sigma_2(v', v''_Q)_R)_q) \\
&\quad \nu_1((w_P)_q, \nu_2(w'_Q, w''))_{\mathcal{R}} \otimes \sigma_2(v_r, (\sigma_2(v', v''_Q)_R)_q)_{\mathcal{R}} \\
&\quad \otimes \nu_2((w_P)_q, \nu_2(w'_Q, w'')) \\
&\overset{(2.6)}{=} (\sigma_1(v', v''_Q)_P(\nu_1(w'_Q, w'')_R)_p)_r \sigma_1(v_r, (\sigma_2(v', v''_Q)_R)_q) \\
&\quad \nu_1(((w_P)_p)_q, \nu_2(w'_Q, w''))_{\mathcal{R}} \otimes \sigma_2(v_r, (\sigma_2(v', v''_Q)_R)_q)_{\mathcal{R}} \\
&\quad \otimes \nu_2(((w_P)_p)_q, \nu_2(w'_Q, w'')) \\
&\overset{(2.6)}{=} (\sigma_1(v', v''_Q)_P)_{\overline{R}}((\nu_1(w'_Q, w'')_R)_p)_r \sigma_1((v_{\overline{R}})_r, (\sigma_2(v', v''_Q)_R)_q) \\
&\quad \nu_1(((w_P)_p)_q, \nu_2(w'_Q, w''))_{\mathcal{R}} \otimes \sigma_2((v_{\overline{R}})_r, (\sigma_2(v', v''_Q)_R)_q)_{\mathcal{R}} \\
&\quad \otimes \nu_2(((w_P)_p)_q, \nu_2(w'_Q, w'')) \\
&\overset{(3.3)}{=} (\sigma_1(v', v''_Q)_P)_{\overline{R}}((\nu_1(w'_Q, w'')_p)_R)_r \sigma_1((v_{\overline{R}})_r, (\sigma_2(v', v''_Q)_q)_R) \\
&\quad \nu_1(((w_P)_q)_p, \nu_2(w'_Q, w''))_{\mathcal{R}} \otimes \sigma_2((v_{\overline{R}})_r, (\sigma_2(v', v''_Q)_q)_R)_{\mathcal{R}} \\
&\quad \otimes \nu_2(((w_P)_q)_p, \nu_2(w'_Q, w'')) \\
&\overset{(2.8)}{=} (\sigma_1(v', v''_Q)_P)_{\overline{R}} \sigma_1(v_{\overline{R}}, \sigma_2(v', v''_Q)_q)(\nu_1(w'_Q, w'')_p)_R \\
&\quad \nu_1(((w_P)_q)_p, \nu_2(w'_Q, w''))_{\mathcal{R}} \otimes (\sigma_2(v_{\overline{R}}, \sigma_2(v', v''_Q)_q)_R)_{\mathcal{R}} \\
&\quad \otimes \nu_2(((w_P)_q)_p, \nu_2(w'_Q, w'')) \\
&\overset{(3.5)}{=} \sigma_1(v'_{\overline{Q}}, (v''_Q)_q)_{\overline{R}} \sigma_1(v_{\overline{R}}, \sigma_2(v'_{\overline{Q}}, (v''_Q)_q))(\nu_1(w'_Q, w'')_p)_R \\
&\quad \nu_1(((w_{\overline{Q}})_q)_p, \nu_2(w'_Q, w''))_{\mathcal{R}} \otimes (\sigma_2(v_{\overline{R}}, \sigma_2(v'_{\overline{Q}}, (v''_Q)_q))_R)_{\mathcal{R}} \\
&\quad \otimes \nu_2(((w_{\overline{Q}})_q)_p, \nu_2(w'_Q, w'')) \\
&\overset{(2.7)}{=} \sigma_1(v, v'_{\overline{Q}}) \sigma_1(\sigma_2(v, v'_{\overline{Q}}), (v''_Q)_q)(\nu_1(w'_Q, w'')_p)_R \nu_1(((w_{\overline{Q}})_q)_p, \nu_2(w'_Q, w''))_{\mathcal{R}} \\
&\quad \otimes (\sigma_2(\sigma_2(v, v'_{\overline{Q}}), (v''_Q)_q)_R)_{\mathcal{R}} \otimes \nu_2(((w_{\overline{Q}})_q)_p, \nu_2(w'_Q, w'')) \\
&\overset{(2.6)}{=} \sigma_1(v, v'_{\overline{Q}}) \sigma_1(\sigma_2(v, v'_{\overline{Q}}), (v''_Q)_q)[\nu_1(w'_Q, w'')_p \nu_1(((w_{\overline{Q}})_q)_p, \nu_2(w'_Q, w''))]_R \\
&\quad \otimes \sigma_2(\sigma_2(v, v'_{\overline{Q}}), (v''_Q)_q)_R \otimes \nu_2(((w_{\overline{Q}})_q)_p, \nu_2(w'_Q, w'')) \\
&\overset{(2.7)}{=} \sigma_1(v, v'_{\overline{Q}}) \sigma_1(\sigma_2(v, v'_{\overline{Q}}), (v''_Q)_q)[\nu_1((w_{\overline{Q}})_q, w'_Q)\nu_1(\nu_2((w_{\overline{Q}})_q, w'_Q), w'')]_R \\
&\quad \otimes \sigma_2(\sigma_2(v, v'_{\overline{Q}}), (v''_Q)_q)_R \otimes \nu_2(\nu_2((w_{\overline{Q}})_q, w'_Q), w'')
\end{aligned}
$$

$$\begin{aligned}
&\overset{(2.8)}{=} \sigma_1(v, v'_{\overline{Q}})\{[\nu_1((w_{\overline{Q}})_q, w'_Q)\nu_1(\nu_2((w_{\overline{Q}})_q, w'_Q), w'')]_R\}_r \\
&\quad \sigma_1(\sigma_2(v, v'_{\overline{Q}})_r, ((v''_Q)_q)_R) \otimes \sigma_2(\sigma_2(v, v'_{\overline{Q}})_r, ((v''_Q)_q)_R) \\
&\quad \otimes \nu_2(\nu_2((w_{\overline{Q}})_q, w'_Q), w'') \\
&\overset{(2.6)}{=} \sigma_1(v, v'_{\overline{Q}})[\nu_1((w_{\overline{Q}})_q, w'_Q)_R \nu_1(\nu_2((w_{\overline{Q}})_q, w'_Q), w'')_{\mathcal{R}}]_r \\
&\quad \sigma_1\big(\sigma_2(v, v'_{\overline{Q}})_r, (((v''_Q)_q)_R)_{\mathcal{R}}\big) \otimes \sigma_2\big(\sigma_2(v, v'_{\overline{Q}})_r, (((v''_Q)_q)_R)_{\mathcal{R}}\big) \\
&\quad \otimes \nu_2(\nu_2((w_{\overline{Q}})_q, w'_Q), w'') \\
&\overset{(3.7)}{=} \sigma_1(v, v'_{\overline{Q}})[\nu_1(w_{\overline{Q}}, w')\nu_1(\nu_2(w_{\overline{Q}}, w')_Q, w'')_{\mathcal{R}}]_r \sigma_1(\sigma_2(v, v'_{\overline{Q}})_r, (v''_Q)_{\mathcal{R}}) \\
&\quad \otimes \sigma_2(\sigma_2(v, v'_{\overline{Q}})_r, (v''_Q)_{\mathcal{R}}) \otimes \nu_2(\nu_2(w_{\overline{Q}}, w')_Q, w'') \\
&\overset{(2.6)}{=} \sigma_1(v, v'_{\overline{Q}})\nu_1(w_{\overline{Q}}, w')_R(\nu_1(\nu_2(w_{\overline{Q}}, w')_Q, w'')_{\mathcal{R}})_r \sigma_1((\sigma_2(v, v'_{\overline{Q}})_R)_r, (v''_Q)_{\mathcal{R}}) \\
&\quad \otimes \sigma_2((\sigma_2(v, v'_{\overline{Q}})_R)_r, (v''_Q)_{\mathcal{R}}) \otimes \nu_2(\nu_2(w_{\overline{Q}}, w')_Q, w'') \\
&\overset{(2.8)}{=} \sigma_1(v, v'_{\overline{Q}})\nu_1(w_{\overline{Q}}, w')_R \sigma_1(\sigma_2(v, v'_{\overline{Q}})_R, v''_Q)\nu_1(\nu_2(w_{\overline{Q}}, w')_Q, w'')_r \\
&\quad \otimes \sigma_2(\sigma_2(v, v'_{\overline{Q}})_R, v''_Q)_r \otimes \nu_2(\nu_2(w_{\overline{Q}}, w')_Q, w'') \\
&= (\mu_A \otimes \mathrm{id}_V \otimes \mathrm{id}_W)\big(\sigma_1(v, v'_{\overline{Q}})\nu_1(w_{\overline{Q}}, w')_R \otimes \sigma_1(\sigma_2(v, v'_{\overline{Q}})_R, v''_Q) \\
&\quad \nu_1(\nu_2(w_{\overline{Q}}, w')_Q, w'')_r \otimes \sigma_2(\sigma_2(v, v'_{\overline{Q}})_R, v''_Q)_r \\
&\quad \otimes \nu_2(\nu_2(w_{\overline{Q}}, w')_Q, w'')\big) \\
&= (\mu_A \otimes \mathrm{id}_V \otimes \mathrm{id}_W) \circ (\mathrm{id}_A \otimes \theta)(\sigma_1(v, v'_{\overline{Q}})\nu_1(w_{\overline{Q}}, w')_R \\
&\quad \otimes \sigma_2(v, v'_{\overline{Q}})_R \otimes \nu_2(w_{\overline{Q}}, w') \otimes v'' \otimes w'') \\
&= (\mu_A \otimes \mathrm{id}_V \otimes \mathrm{id}_W) \circ (\mathrm{id}_A \otimes \theta) \circ (\theta \otimes \mathrm{id}_V \otimes \mathrm{id}_W)(v \otimes w \\
&\quad \otimes v' \otimes w' \otimes v'' \otimes w''), \quad \text{q.e.d.}
\end{aligned}$$

Proof of (2.5).

$$\begin{aligned}
&(\mu_A \otimes \mathrm{id}_V \otimes \mathrm{id}_W) \circ (\mathrm{id}_A \otimes \theta) \circ (S \otimes \mathrm{id}_V \otimes \mathrm{id}_W) \circ (\mathrm{id}_V \otimes \mathrm{id}_W \otimes S)(v \otimes w \otimes v' \otimes w' \otimes a) \\
&\quad = (\mu_A \otimes \mathrm{id}_V \otimes \mathrm{id}_W) \circ (\mathrm{id}_A \otimes \theta) \circ (S \otimes \mathrm{id}_V \otimes \mathrm{id}_W) \\
&\qquad (v \otimes w \otimes (a_P)_R \otimes v'_R \otimes w'_P) \\
&\quad = (\mu_A \otimes \mathrm{id}_V \otimes \mathrm{id}_W) \circ (\mathrm{id}_A \otimes \theta)\big((((a_P)_R)_p)_r \otimes v_r \otimes w_p \otimes v'_R \otimes w'_P\big) \\
&\quad = (((a_P)_R)_p)_r \sigma_1(v_r, (v'_R)_Q)\nu_1((w_p)_Q, w'_P)_{\mathcal{R}} \\
&\qquad \otimes \sigma_2(v_r, (v'_R)_Q)_{\mathcal{R}} \otimes \nu_2((w_p)_Q, w'_P) \\
&\quad \overset{(3.3)}{=} (((a_P)_p)_R)_r \sigma_1(v_r, (v'_Q)_R)\nu_1((w_Q)_p, w'_P)_{\mathcal{R}} \\
&\qquad \otimes \sigma_2(v_r, (v'_Q)_R)_{\mathcal{R}} \otimes \nu_2((w_Q)_p, w'_P)
\end{aligned}$$

$$
\begin{aligned}
&\overset{(2.8)}{=} \sigma_1(v, v'_Q)((a_P)_p)_R\nu_1((w_Q)_p, w'_P)_{\mathcal{R}} \otimes (\sigma_2(v, v'_Q)_R)_{\mathcal{R}} \otimes \nu_2((w_Q)_p, w'_P)\\
&\overset{(2.6)}{=} \sigma_1(v, v'_Q)[(a_P)_p\nu_1((w_Q)_p, w'_P)]_R \otimes \sigma_2(v, v'_Q)_R \otimes \nu_2((w_Q)_p, w'_P)\\
&\overset{(2.8)}{=} \sigma_1(v, v'_Q)[\nu_1(w_Q, w')a_P]_R \otimes \sigma_2(v, v'_Q)_R \otimes \nu_2(w_Q, w')_P\\
&\overset{(2.6)}{=} \sigma_1(v, v'_Q)\nu_1(w_Q, w')_R(a_P)_r \otimes (\sigma_2(v, v'_Q)_R)_r \otimes \nu_2(w_Q, w')_P\\
&= (\mu_A \otimes \mathrm{id}_V \otimes \mathrm{id}_W) \circ (\mathrm{id}_A \otimes S)(\sigma_1(v, v'_Q)\nu_1(w_Q, w')_R\\
&\quad \otimes \sigma_2(v, v'_Q)_R \otimes \nu_2(w_Q, w') \otimes a)\\
&= (\mu_A \otimes \mathrm{id}_V \otimes \mathrm{id}_W) \circ (\mathrm{id}_A \otimes S) \circ (\theta \otimes \mathrm{id}_A)(v \otimes w \otimes v' \otimes w' \otimes a), \quad \text{q.e.d.}
\end{aligned}
$$

So $A \otimes_{S,\theta} (V \otimes W)$ is indeed a crossed product. With a similar computation one can prove that $(A \otimes_{R,\sigma} V) \otimes_{T,\eta} W$ is a crossed product; the only thing left to prove is that the multiplications of $A \otimes_{S,\theta} (V \otimes W)$ and $(A \otimes_{R,\sigma} V) \otimes_{T,\eta} W$ coincide. A straightforward computation shows that the multiplication of $A \otimes_{S,\theta} (V \otimes W)$ is given by the formula

$$
\begin{aligned}
(a \otimes v \otimes w)(a' \otimes v' \otimes w') = {} & a(a'_P)_{\mathcal{R}}\sigma_1(v_{\mathcal{R}}, v'_Q)\nu_1((w_P)_Q, w')_r\\
& \otimes \sigma_2(v_{\mathcal{R}}, v'_Q)_r \otimes \nu_2((w_P)_Q, w').
\end{aligned}
$$

We now compute the multiplication of $(A \otimes_{R,\sigma} V) \otimes_{T,\eta} W$:

$$
\begin{aligned}
&(a \otimes v \otimes w)(a' \otimes v' \otimes w')\\
&\quad = (a \otimes v)(a' \otimes v')_T\eta_1(w_T, w') \otimes \eta_2(w_T, w')\\
&\quad = (a \otimes v)(a'_P \otimes v'_Q)\eta_1((w_P)_Q, w') \otimes \eta_2((w_P)_Q, w')\\
&\quad = (a \otimes v)(a'_P \otimes v'_Q)\big(\nu_1((w_P)_Q, w') \otimes 1_V\big) \otimes \nu_2((w_P)_Q, w')\\
&\quad = (a \otimes v)\big(a'_P\nu_1((w_P)_Q, w')_R \otimes (v'_Q)_R\big) \otimes \nu_2((w_P)_Q, w')\\
&\quad = a[a'_P\nu_1((w_P)_Q, w')_R]_r\sigma_1(v_r, (v'_Q)_R) \otimes \sigma_2(v_r, (v'_Q)_R) \otimes \nu_2((w_P)_Q, w')\\
&\quad \overset{(2.6)}{=} a(a'_P)_{\mathcal{R}}\big(\nu_1((w_P)_Q, w')_R\big)_r\sigma_1((v_{\mathcal{R}})_r, (v'_Q)_R)\\
&\qquad \otimes \sigma_2((v_{\mathcal{R}})_r, (v'_Q)_R) \otimes \nu_2((w_P)_Q, w')\\
&\quad \overset{(2.8)}{=} a(a'_P)_{\mathcal{R}}\sigma_1(v_{\mathcal{R}}, v'_Q)\nu_1((w_P)_Q, w')_r \otimes \sigma_2(v_{\mathcal{R}}, v'_Q)_r \otimes \nu_2((w_P)_Q, w'),
\end{aligned}
$$

and we can see that the two multiplications coincide. ■

Example 3.2. We recall from [JLPV] what was called there an *iterated twisted tensor product* of algebras. Let A, B, C be associative unital algebras, $R_1 : B \otimes A \to A \otimes B$, $R_2 : C \otimes B \to B \otimes C$, $R_3 : C \otimes A \to A \otimes C$ twisting maps satisfying the braid equation

$$
(\mathrm{id}_A \otimes R_2) \circ (R_3 \otimes \mathrm{id}_B) \circ (\mathrm{id}_C \otimes R_1) = (R_1 \otimes \mathrm{id}_C) \circ (\mathrm{id}_B \otimes R_3) \circ (R_2 \otimes \mathrm{id}_A).
$$

Then we have an algebra structure on $A \otimes B \otimes C$ (called the iterated twisted tensor product) with unit $1_A \otimes 1_B \otimes 1_C$ and multiplication

$$(a \otimes b \otimes c)(a' \otimes b' \otimes c') = a(a'_{R_3})_{R_1} \otimes b_{R_1} b'_{R_2} \otimes (c_{R_3})_{R_2} c'.$$

We define $V = B$, $W = C$, $R = R_1$, $P = R_3$, $Q = R_2$ and the linear maps

$$\sigma : V \otimes V \to A \otimes V, \qquad \sigma(b \otimes b') = 1_A \otimes bb', \quad \forall b, b' \in V,$$
$$\nu : W \otimes W \to A \otimes W, \qquad \nu(c \otimes c') = 1_A \otimes cc', \quad \forall c, c' \in W.$$

Then, for the crossed products $A \otimes_{R,\sigma} V = A \otimes_{R_1} B$, $A \otimes_{P,\nu} W = A \otimes_{R_3} C$ and the map Q, one can check that the hypotheses of Theorem 3.1 are satisfied and the crossed products $A \otimes_{S,\theta} (V \otimes W) \equiv (A \otimes_{R,\sigma} V) \otimes_{T,\eta} W$ (notation as in that theorem) coincide with the iterated twisted tensor product.

EXAMPLE 3.3. Let $A \otimes_{R,\sigma} V$ be a crossed product and W an (associative unital) algebra. Define the linear maps

$$P : W \otimes A \to A \otimes W, \qquad P(w \otimes a) = a \otimes w, \qquad \forall a \in A,\ w \in W,$$
$$\nu : W \otimes W \to A \otimes W, \qquad \nu(w \otimes w') = 1_A \otimes ww', \quad \forall w, w' \in W,$$

so we have the crossed product $A \otimes_{P,\nu} W$ which is just the ordinary tensor product of algebras $A \otimes W$. Define the linear map $Q : W \otimes V \to V \otimes W$, $Q(w \otimes v) = v \otimes w$ for all $v \in V$, $w \in W$. Then one can easily check that the hypotheses of Theorem 3.1 are satisfied.

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