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LAZY 2-COCYCLES OVER MONOIDAL HOM-HOPF ALGEBRAS

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Abstract. We introduce the notion of a lazy 2-cocycle over a monoidal Hom-Hopf algebra and determine all lazy 2-cocycles for a class of monoidal Hom-Hopf algebras. We also study the extension of lazy 2-cocycles to a Radford Hom-biproduct.

1. Introduction. Let H be a Hopf algebra over a field $\Bbbk$. A left 2-cocycle $\sigma : H \otimes H \to \Bbbk$ is called *lazy* if

$$\sigma(h_1, g_1)h_2g_2 = h_1g_1\sigma(h_2, g_2)$$

for any $h, g \in H$ (see [11]). An important property used in Chen's study [6] of Hopf algebras is that all normalized and convolution invertible lazy 2-cocycles form a group denoted by $Z_L^2(H)$. Moreover, Schauenburg [23] defines the lazy 2-coboundary subgroup $B_L^2(H)$ of $Z_L^2(H)$ and the second lazy cohomology group $H_L^2(H) = Z_L^2(H)/B_L^2(H)$, generalizing Sweedler's second cohomology group of a cocommutative Hopf algebra. In connection with Brauer groups of Hopf algebras, bi-Galois groups, projective representations, lazy cocycles have been studied systematically in [3], [5], [11] and [21].

Motivated by certain problems in physics, various classes of nonassociative algebras such as Hom-Lie algebras, quasi-Hom-Lie algebras, Hom-Lie superalgebras etc. have been studied (see [2], [1] and [13]). With the same idea of modifying associativity-like conditions by endomorphisms, the concepts of Hom-algebras, Hom-colgebras, Hom-Hopf algebras etc. were introduced in [17], [18], [19] and [27]. In [4], the authors consider Hom-structures from the point of view of monoidal categories and introduce monoidal Hom-algebras, monoidal Hom-coalgebras etc. in a symmetric monoidal category, which are slightly different from the above Hom-algebras and Hom-coalgebras. Clearly, the notion of monoidal Hom-Hopf algebra is a generalization of the ordinary Hopf algebra. The theory of monoidal Hom-Hopf algebras was further developed by many scholars [7–10], [14–16].

The main purpose of this paper is to establish a theory of lazy 2-cocycles in the setting of monoidal Hom-Hopf algebras. The paper is organized as

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follows. In Section 2, we recall basic definitions and facts on monoidal Hom-Hopf algebras, Hom-modules, Hom-comodules, Hom-Yetter–Drinfeld modules, and Radford's Hom-biproducts. In Section 3, we introduce the notions of left 2-cocycle, right 2-cocycle and lazy 2-cocycle $\sigma : H \otimes H \to \Bbbk$ over a monoidal Hom-Hopf algebra H. Then we compute all lazy 2-cocycles over a class of monoidal Hom-Hopf algebras including a 3-dimensional monoidal Hom-Hopf algebra and Sweedler's 4-dimensional monoidal Hom-Hopf algebra [7]. The main result of that section is Theorem 3.5 asserting that all normalized and convolution invertible lazy 2-cocycles form a group. Then we define the second lazy cohomology group $H_L^2(H)$. Some properties of left 2-cocycles are also studied.

Sections 4 and 5 are devoted to the extension of lazy 2-cocycles to a Radford Hom-biproduct. Namely, let (H,α) be a monoidal Hom-Hopf algebra with a bijective antipode, and (B,β) be a Hopf algebra in the Hom-Yetter–Drinfeld category ${}^H_H\mathcal{HYD}$ (see [15] for details). In Section 4, we present a new construction $(B^{\times}_{\sharp}H, \beta\otimes\alpha)$ generalizing Radford's Hom-smash product and we obtain a lazy 2-cocycle over $(B^{\times}_{\sharp}H, \beta\otimes\alpha)$ from a lazy 2-cocycle over (H,α). In Section 5, we define a lazy 2-cocycle in the setting of Hom-Yetter–Drinfeld categories and study some of its properties similar to ones of Section 3. Moreover, we show that a lazy 2-cocycle over (B,β) induces a lazy 2-cocycle over $(B^{\times}_{\sharp}H, \beta\otimes\alpha)$.

Throughout this paper, $\Bbbk$ is a fixed field. Unless otherwise stated, all vector spaces, algebras, coalgebras, maps and unadorned tensor products are over $\Bbbk$. For a coalgebra C, we denote its comultiplication by $\Delta(c) = c_1 \otimes c_2$ for any $c \in C$; for a left C-comodule (M,ρ), we write its coaction $\rho(m) = m_{(-1)} \otimes m_{(0)}$ for any $m \in M$, where the summation symbols are omitted. Throughout this paper we freely use the Hopf algebra terminology introduced in [12], [20], [22], [25], [26].

2. Preliminaries. Let $\mathcal{M}_\Bbbk = (\mathcal{M}_\Bbbk, \otimes, \Bbbk, a, l, r)$ be the category of $\Bbbk$-modules. Following [4] we form a new monoidal category $\tilde{\mathcal{H}}(\mathcal{M}_\Bbbk) = (\mathcal{H}(\mathcal{M}_\Bbbk), \otimes, (\Bbbk, \mathrm{id}_\Bbbk), \tilde{a}, \tilde{l}, \tilde{r})$. The objects of $\mathcal{H}(\mathcal{M}_\Bbbk)$ are pairs (M,μ), where $M \in \mathcal{M}_\Bbbk$ and $\mu \in \mathrm{Aut}_\Bbbk(M)$. Any morphism $f : (M,\mu) \to (N,\nu)$ in $\mathcal{H}(\mathcal{M}_\Bbbk)$ is a $\Bbbk$-linear map from M to N such that $\nu \circ f = f \circ \mu$. For any $(M,\mu),(N,\nu) \in \mathcal{H}(\mathcal{M}_\Bbbk)$, the monoidal structure is given by

$$(M,\mu) \otimes (N,\nu) = (M \otimes N, \mu \otimes \nu),$$

and the unit is $(\Bbbk, \mathrm{id}_\Bbbk)$.

Generally speaking, all Hom-structures are objects in the monoidal category $\tilde{\mathcal{H}}(\mathcal{M}_\Bbbk) = (\mathcal{H}(\mathcal{M}_\Bbbk), \otimes, (\Bbbk, \mathrm{id}_\Bbbk), \tilde{a}, \tilde{l}, \tilde{r})$, where the associativity constraint $\tilde{a}$ is given by the formula

$$\tilde{a}_{M,N,L} = a_{M,N,L} \circ ((\mu \otimes \mathrm{id}) \otimes \lambda^{-1}) = (\mu \otimes (\mathrm{id} \otimes \lambda^{-1})) \circ a_{M,N,L},$$

and the unit constraints $\tilde{l}$ and $\tilde{r}$ are defined by

$$\tilde{l}_M = \mu \circ l_M = l_M \circ (\mathrm{id} \otimes \mu), \qquad \tilde{r}_M = \mu \circ r_M = r_M \circ (\mu \otimes \mathrm{id}),$$

for any $(M,\mu),(N,\nu),(L,\lambda) \in \mathcal{H}(\mathcal{M}_{\Bbbk})$. The category $\tilde{\mathcal{H}}(\mathcal{M}_{\Bbbk})$ is called the *Hom-category associated to the monoidal category* $\mathcal{M}_{\Bbbk}$.

Remark 2.1. We recall from [10, Section 5] that there is an exact functorial isomorphism

$$\phi : \tilde{\mathcal{H}}(\mathcal{M}_{\Bbbk}) \to \mathrm{Mod}(\Bbbk[t,t^{-1}])$$

between the monoidal category $\tilde{\mathcal{H}}(\mathcal{M}_{\Bbbk})$ defined above and the category $\mathrm{Mod}(\Bbbk[t,t^{-1}])$ of all modules over the $\Bbbk$-algebra $\Bbbk[t,t^{-1}]$ of all polynomials in one indeterminate t, with coefficients in $\Bbbk$, localized at the multiplicative system $\{1,t,t^2,\dots\}$. Therefore our monoidal category $\mathcal{H}(\mathcal{M}_{\Bbbk})$ is nothing else than the module category $\mathrm{Mod}(\Bbbk[t,t^{-1}])$. Consequently, the monoidal category $\tilde{\mathcal{H}}(\mathcal{M}_{\Bbbk})$ can be viewed as a full exact subcategory of the category $\mathrm{Rep}_{\Bbbk}\, Q$ of all $\Bbbk$-linear representations of the quiver Q with one vertex and one loop (see Sections 14.1–14.4 of the monograph [24]).

This interpretation of the category $\tilde{\mathcal{H}}(\mathcal{M}_{\Bbbk})$ in terms of quiver representations could probably simplify part of our study.

Now we recall from [4], [7] and [15] some definitions on Hom-structures.

Definition 2.2. (i) A *unital monoidal Hom-associative algebra* is an object (A,α) in the category $\tilde{\mathcal{H}}(\mathcal{M}_{\Bbbk})$ together with an element $1_A \in A$ and a linear map $m : A \otimes A \to A$, $a \otimes b \mapsto ab$, such that

$$\alpha(a)(bc) = (ab)\alpha(c), \qquad a1_A = \alpha(a) = 1_A a, \tag{2.1}$$

$$\alpha(ab) = \alpha(a)\alpha(b), \quad \alpha(1_A) = 1_A, \tag{2.2}$$

for all $a,b,c \in A$.

(ii) Let (A,α) and (A',α') be two monoidal Hom-algebras. A *Hom-algebra map* $f : (A,\alpha) \to (A',\alpha')$ is a linear map such that $f \circ \alpha = \alpha' \circ f$, $f(ab) = f(a)f(b)$ and $f(1_A) = 1_{A'}$.

Note that the first part of (2.1) can be rewritten as

$$a(b\alpha^{-1}(c)) = (\alpha^{-1}(a)b)c. \tag{2.3}$$

In the language of Hopf algebras, m is called the *Hom-multiplication*, α is the *twisting automorphism*, and 1_A is the *unit*.

Definition 2.3. (i) A *counital monoidal Hom-coassociative coalgebra* is an object (C,γ) in the category $\tilde{\mathcal{H}}(\mathcal{M}_{\Bbbk})$ together with linear maps $\Delta : C \to C \otimes C$, $c \mapsto c_1 \otimes c_2$, and $\varepsilon : C \to \Bbbk$ such that

$$\gamma^{-1}(c_1) \otimes \Delta(c_2) = \Delta(c_1) \otimes \gamma^{-1}(c_2), \quad c_1\varepsilon(c_2) = \varepsilon(c_1)c_2 = \gamma^{-1}(c), \tag{2.4}$$

$$\Delta(\gamma(c)) = \gamma(c_1) \otimes \gamma(c_2), \qquad \varepsilon\gamma(c) = \varepsilon(c), \tag{2.5}$$

for all $c \in C$.

(ii) Let (C,γ) and (C',γ') be two monoidal Hom-coalgebras. A *Hom-coalgebra map* $f:(C,\gamma)\to(C',\gamma')$ is a linear map such that $f\circ\gamma=\gamma'\circ f$, $\Delta_{C'}\circ f=(f\otimes f)\circ\Delta_C$ and $\varepsilon_{C'}\circ f=\varepsilon_C$.

Note that the first part of (2.4) is equivalent to

$$c_1\otimes c_{21}\otimes\gamma(c_{22})=\gamma(c_{11})\otimes c_{12}\otimes c_2. \tag{2.6}$$

Definition 2.4. (i) A *monoidal Hom-bialgebra* $H=(H,\alpha,m,1_H,\Delta,\varepsilon)$ is a bialgebra in the category $\tilde{\mathcal{H}}(\mathcal{M}_{\Bbbk})$, which means that $(H,\alpha,m,1_H)$ is a monoidal Hom-algebra and $(H,\alpha,\Delta,\varepsilon)$ is a monoidal Hom-coalgebra such that Δ and ε are Hom-algebra maps, that is, for any $h,g\in H$,

$$\begin{aligned}\Delta(hg)&=\Delta(h)\Delta(g), &\quad \Delta(1_H)&=1_H\otimes 1_H,\\ \varepsilon(hg)&=\varepsilon(h)\varepsilon(g), &\quad \varepsilon(1_H)&=1_{\Bbbk}.\end{aligned}$$

(ii) A monoidal Hom-bialgebra (H,α) is called a *monoidal Hom-Hopf algebra* if there exists a linear map (called the *antipode*) $S:H\to H$ in $\tilde{\mathcal{H}}(\mathcal{M}_{\Bbbk})$ (i.e., $S\circ\alpha=\alpha\circ S$), which is the convolution inverse of the identity map (i.e., $S(h_1)h_2=\varepsilon(h)1_H=h_1S(h_2)$ for any $h\in H$).

As in the case of Hopf algebras, the antipode of a monoidal Hom-Hopf algebra is a morphism of Hom-anti-algebras and Hom-anti-coalgebras.

Definition 2.5. (i) Let (A,α) be a monoidal Hom-algebra. A *left* (A,α)-*Hom-module* is an object (M,μ) in $\tilde{\mathcal{H}}(\mathcal{M}_{\Bbbk})$ together with a linear map $\varphi:A\otimes M\to M,\ a\otimes m\mapsto am$, such that

$$\alpha(a)(bm)=(ab)\mu(m),\qquad 1_Am=\mu(m),\qquad \mu(am)=\alpha(a)\mu(m),$$

for all $a,b\in A$ and $m\in M$.

(ii) If (M,μ) and (N,ν) are two left (A,α)-Hom-modules, then a linear map $f:M\to N$ is called a *left A-module map* if for any $a\in A$ and $m\in M$ we have $f(am)=af(m)$ and $f\circ\mu=\nu\circ f$.

Definition 2.6. (i) Let (C,γ) be a monoidal Hom-coalgebra. A *left* (C,γ)-*Hom-comodule* is an object (M,μ) in $\tilde{\mathcal{H}}(\mathcal{M}_{\Bbbk})$ together with a linear map $\rho_M:M\to C\otimes M,\ m\mapsto m_{(-1)}\otimes m_{(0)}$, such that

$$\begin{aligned}&\Delta(m_{(-1)})\otimes\mu^{-1}(m_{(0)})=\gamma^{-1}(m_{(-1)})\otimes\rho_M(m_{(0)}),\qquad \varepsilon(m_{(-1)})m_{(0)}=\mu^{-1}(m),\\ &\rho_M(\mu(m))=\gamma(m_{(-1)})\otimes\mu(m_{(0)}),\end{aligned}$$

for all $m\in M$.

(ii) If (M,μ) and (N,ν) are two left (C,γ)-Hom-comodules, then a linear map $g:M\to N$ is called a *left C-comodule map* if $g\circ\mu=\nu\circ g$ and $\rho_N(g(m))=(\mathrm{id}\otimes g)\rho_M(m)$ for any $m\in M$.

Definition 2.7. Let (H,α) be a monoidal Hom-bialgebra and (B,β) be a monoidal Hom-algebra.

(i) (B,β) is called a *left* (H,α)*-Hom-module algebra* if (B,β) is a left (H,α)-Hom-module with the action $\cdot$ and satisfies

$$h\cdot(ab)=(h_1\cdot a)(h_2\cdot b),\qquad h\cdot 1_B=\varepsilon(h)1_B,$$

for any $a,b\in B$ and $h\in H$.

(ii) (B,β) is called a *left* (H,α)*-Hom-comodule algebra* if (B,β) is a left (H,α)-Hom-comodule with the coaction ρ and satisfies

$$\rho(ab)=a_{(-1)}b_{(-1)}\otimes a_{(0)}b_{(0)},\qquad \rho(1_B)=1_H\otimes 1_B,$$

for any $a,b\in B$.

Definition 2.8. Let (H,α) be a monoidal Hom-bialgebra and (C,γ) be a monoidal Hom-coalgebra.

(i) (C,γ) is called a *left* (H,α)*-Hom-module coalgebra* if (C,γ) is a left (H,α)-Hom-module with the action $\cdot$ and satisfies

$$\Delta(h\cdot c)=h_1\cdot c_1\otimes h_2\cdot c_2,\qquad \varepsilon_C(h\cdot c)=\varepsilon_H(h)\varepsilon_C(c),$$

for any $c\in C$ and $h\in H$;

(ii) (C,γ) is called a *left* (H,α)*-Hom-comodule coalgebra* if (C,γ) is a left (H,α)-Hom-comodule with the coaction ρ and satisfies

$$c_{(-1)}\otimes\Delta(c_{(0)})=c_{1(-1)}c_{2(-1)}\otimes c_{1(0)}\otimes c_{2(0)},\qquad c_{(-1)}\varepsilon(c_{(0)})=\varepsilon(c)1_H,$$

for any $c\in C$ and $h\in H$.

Definition 2.9. Let (H,α) be a monoidal Hom-bialgebra and (B,β) be a left (H,α)-Hom-module algebra. The *Hom-smash product* $(B\,\sharp\,H,\beta\,\sharp\,\alpha)$ of (B,β) and (H,α) is defined as follows, for all $a,b\in B,h,g\in H$:

(i) $B\,\sharp\,H=B\otimes H$, when we view them as $\Bbbk$-vector spaces,
(ii) Hom-multiplication is given by

$$(a\,\sharp\,h)(b\,\sharp\,g)=a(h_1\cdot\beta^{-1}(b))\,\sharp\,\alpha(h_2)g.$$

Note that $(B\,\sharp\,H,\beta\,\sharp\,\alpha)$ is a monoidal Hom-algebra with unit $1_B\,\sharp\,1_H$.

Definition 2.10. Let (H,α) be a monoidal Hom-bialgebra and (B,β) be a left (H,α)-Hom-comodule coalgebra. Their *Hom-smash coproduct* $(B\times H,\beta\times\alpha)$ is defined as follows, for all $b\in B,h\in H$:

(i) $B\times H=B\otimes H$, when we view them as $\Bbbk$-vector spaces,
(ii) Hom-comultiplication is given by

$$\Delta(b\times h)=(b_1\times b_{2(-1)}\alpha^{-1}(h_1))\otimes(\beta(b_{2(0)})\times h_2).$$

Note that $(B\times H,\beta\times\alpha)$ is a monoidal Hom-coalgebra with counit $\varepsilon_B\times\varepsilon_H$.

Let (H,α) be a monoidal Hom-bialgebra and (B,β) be a left (H,α)-Hom-module algebra and a left (H,α)-Hom-comodule coalgebra. Denote the Hom-smash product $(B\sharp H,\beta\sharp\alpha)$ and the Hom-coproduct $(B\times H,\beta\times\alpha)$ by $(B_\sharp^\times H,\beta\otimes\alpha)$. In [15], the authors proved that $(B_\sharp^\times H,\beta\otimes\alpha)$ is a monoidal Hom-bialgebra if and only if the following conditions hold:

(i) ε_B is an algebra map and $\Delta_B(1_B)=1_B\otimes 1_B$,
(ii) (B,β) is a left (H,α)-Hom-module coalgebra,
(iii) (B,β) is a left (H,α)-Hom-comodule algebra,
(iv) $\Delta_B(ab)=a_1(a_{2(-1)}\cdot\beta^{-1}(b_1))\otimes\beta(a_{2(0)})b_2$,
(v) $(h_1\cdot\beta^{-1}(b))_{(-1)}h_2\otimes\beta((h_1\cdot\beta^{-1}(b))_{(0)})=h_1b_{(-1)}\otimes h_2\cdot b_{(0)}$, for all $a,b\in B$ and $h\in H$.

Note that if $(B_\sharp^\times H,\beta\otimes\alpha)$ is a monoidal Hom-bialgebra as above, it is called a *Radford Hom-biproduct*. In this case, the pair $((H,\alpha),(B,\beta))$ is called an *admissible pair*. Moreover, if (H,α) is a monoidal Hom-Hopf algebra with antipode S_H and $S_B:B\to B$ in $\tilde{\mathcal{H}}(\mathcal{M}_\Bbbk)$ (i.e., $S_B\circ\beta=\beta\circ S_B$) is a convolution inverse of id_B, then $(B_\sharp^\times H,\beta\otimes\alpha)$ is a monoidal Hom-Hopf algebra with antipode S given by

$$S(b\times h)=\big(1_B\times S_H(\alpha^{-1}(b_{(-1)})\alpha^{-2}(h))\big)\big(S_B(b_{(0)})\times 1_H\big)$$

for all $b\in B$ and $h\in H$.

DEFINITION 2.11. Let (H,α) be a monoidal Hom-Hopf algebra. A *left-left (H,α)-Hom-Yetter–Drinfeld module* is an object (M,β) in $\tilde{\mathcal{H}}(\mathcal{M}_\Bbbk)$ such that (M,β) is a left (H,α)-Hom-module (with notation $h\otimes m\mapsto h\cdot m$) and a left (H,α)-Hom-comodule (with notation $m\mapsto m_{(-1)}\otimes m_{(0)}$) satisfying the following compatibility condition:

$$h_1m_{(-1)}\otimes h_2\cdot m_{(0)}=(h_1\cdot\beta^{-1}(m))_{(-1)}h_2\otimes\beta((h_1\cdot\beta^{-1}(m))_{(0)}),$$

which is equivalent to the equation

$$\rho(h\cdot m)=(h_{11}\alpha^{-1}(m_{(-1)}))S(h_2)\otimes(\alpha(h_{12})\cdot m_{(0)}),$$

for all $h\in H$, and $m\in M$.

Let ${}^H_H\mathcal{HYD}$ be the category of all left-left (H,α)-Hom-Yetter–Drinfeld modules and left H-linear left H-colinear maps. If the antipode of (H,α) is bijective, then the category $({}^H_H\mathcal{HYD},\otimes,(\Bbbk,\mathrm{id}),a,l,r,c)$ is a braided monoidal category, where for any $(M,\mu),(N,\nu)\in\mathcal{H}(\mathcal{M}_\Bbbk)$, the monoidal structure is given by $(M,\mu)\otimes(N,\nu)=(M\otimes N,\mu\otimes\nu)$, $((M\otimes N,\mu\otimes\nu)\in{}^H_H\mathcal{HYD}$ in the usual way), the unit is $(\Bbbk,\mathrm{id})$, $((\Bbbk,\mathrm{id})\in{}^H_H\mathcal{HYD}$ in the usual way), the associativity and unit constraints are given by

$$a_{U,V,W}:(U\otimes V)\otimes W\to U\otimes(V\otimes W),\quad (u\otimes v)\otimes w\mapsto\beta(u)\otimes(v\otimes\tau^{-1}(w)),$$
$$l_V:\Bbbk\otimes V\to V,\qquad k\otimes v\mapsto k\gamma(v),$$
$$r_V:V\otimes\Bbbk\to V,\qquad v\otimes k\mapsto k\gamma(v),$$

and the braiding is given by

$$c_{U,V} : U \otimes V \to V \otimes U, \quad u \otimes v \mapsto u_{(-1)} \cdot \gamma^{-1}(v) \otimes \beta(u_{(0)}),$$

for any $(U,\beta), (V,\gamma), (W,\tau) \in {}^H_H\mathcal{HYD}$ and $u \in U$, $v \in V$, $w \in W$, $k \in \Bbbk$.

Recall from [15, Proposition 4.7] that if (H,α) is a monoidal Hom-Hopf algebra and (B,β) is a Hopf algebra in ${}^H_H\mathcal{HYD}$, then $(B{}^{\times}_{\sharp}H, \beta \otimes \alpha)$ is a monoidal Hom-Hopf algebra.

3. Lazy 2-cocycles over monoidal Hom-Hopf algebras. In this section, we always let (H,α) denote a monoidal Hom-Hopf algebra and $\sigma : H \otimes H \to \Bbbk$ be a $\Bbbk$-linear *α-invariant* map, i.e., $\sigma \circ (\alpha \otimes \alpha) = \sigma$.

DEFINITION 3.1. Let $\sigma : H \otimes H \to \Bbbk$ be a $\Bbbk$-linear α-invariant map.

(i) σ is called a *left 2-cocycle* if $\sigma(h_1, g_1)\sigma(h_2g_2, l) = \sigma(g_1, l_1)\sigma(h, g_2l_2)$;
(ii) σ is called a *right 2-cocycle* if $\sigma(h_1g_1, l)\sigma(h_2, g_2) = \sigma(h, g_1l_1)\sigma(g_2, l_2)$;
(iii) σ is called *lazy* if $\sigma(h_1, g_1)h_2g_2 = h_1g_1\sigma(h_2, g_2)$;
(iv) σ is called *normalized* if $\sigma(h, 1) = \sigma(1, h) = \varepsilon(h)$,

for any $h, g, l \in H$.

REMARK 3.2. (i) If $\sigma : H \otimes H \to \Bbbk$ is a convolution invertible left 2-cocycle, then σ^{-1} is a right 2-cocycle;

(ii) If $\sigma : H \otimes H \to \Bbbk$ is a lazy left 2-cocycle, then it is also a right 2-cocycle and in this case, we call σ a *lazy 2-cocycle.*

EXAMPLE 3.3. Let $(H = \Bbbk\{1, g, g^2\}, \alpha)$ be a 3-dimensional monoidal Hom-Hopf algebra, where the Hom-multiplication is given by

H	1	g	g^2
1	1	g^2	g
g	g^2	g	1
g^2	g	1	g^2

the Hom-comultiplication is given by

$$\Delta(1) = 1 \otimes 1, \quad \Delta(g) = g^2 \otimes g^2, \quad \Delta(g^2) = g \otimes g,$$

the counit is given by

$$\varepsilon(1) = \varepsilon(g) = \varepsilon(g^2) = 1,$$

the antipode is given by

$$S(1) = 1, \quad S(g) = g^2, \quad S(g^2) = g,$$

and $\alpha \in \mathrm{Aut}_{\Bbbk}(H)$ is given by

$$\alpha(1) = 1, \quad \alpha(g) = g^2, \quad \alpha(g^2) = g.$$

It is easy to see that any $\Bbbk$-linear α-invariant map $\sigma : H \otimes H \to \Bbbk$ is of the form

σ	1	g	g^2
1	k_1	k_2	k_2
g	k_3	k_4	k_5
g^2	k_3	k_5	k_4

for some $k_i \in \Bbbk$, $i = 1, 2, 3, 4, 5$.

Since (H, α) is cocommutative, any left 2-cocycle is lazy. A computation shows that any lazy 2-cocycle σ must be equal to $k\sigma_i$ for some $k \in \Bbbk$ and $i \in \{1, 2, 3, 4, 5, 6\}$, where

σ_1	1	g	g^2
1	1	1	1
g	1	1	1
g^2	1	1	1

σ_2	1	g	g^2
1	1	1	1
g	1	0	0
g^2	1	0	0

σ_3	1	g	g^2
1	1	1	1
g	0	0	0
g^2	0	0	0

σ_4	1	g	g^2
1	1	0	0
g	1	0	0
g^2	1	0	0

σ_5	1	g	g^2
1	1	0	0
g	0	0	0
g^2	0	0	0

σ_6	1	g	g^2
1	0	0	0
g	0	0	1
g^2	0	1	0

EXAMPLE 3.4. Recall from [7, Example 3.5] that $(H_4 = \Bbbk\{1, g, x, y\}, \alpha)$ is a 4-dimensional monoidal Hom-Hopf algebra (usually called *Sweedler's 4-dimensional monoidal Hom-Hopf algebra*), where the Hom-multiplication is given by

H_4	1	g	x	y
1	1	g	cx	cy
g	g	1	cy	cx
x	cx	$-cy$	0	0
y	cy	$-cx$	0	0

the Hom-comultiplication is given by

$$\Delta(1) = 1 \otimes 1, \qquad \Delta(g) = g \otimes g,$$

$$\Delta(x) = \frac{1}{c}(x \otimes 1 + g \otimes x), \qquad \Delta(y) = \frac{1}{c}(y \otimes g + 1 \otimes y),$$

the counit is given by

$$\varepsilon(1) = \varepsilon(g) = 1, \qquad \varepsilon(x) = \varepsilon(y) = 0,$$

the antipode is given by

$$S(1) = 1, \qquad S(g) = g, \qquad S(x) = -y, \qquad S(y) = x,$$

and $\alpha \in \mathrm{Aut}_{\Bbbk}(H_4)$ is given by

$$\alpha(1) = 1, \quad \alpha(g) = g, \quad \alpha(x) = cx, \quad \alpha(y) = cy,$$

for any $0 \neq c \in \Bbbk$.

We will find all lazy 2-cocycles of (H_4, α). When $c = 1$, (H_4, α) is just the ordinary Sweedler's 4-dimensional Hopf algebra and any lazy 2-cocycle of (H_4, α) is of the form

σ	1	g	x	y
1	1	1	0	0
g	1	1	0	0
x	0	0	$t/2$	$-t/2$
y	0	0	$t/2$	$-t/2$

for some $t \in \Bbbk$ (see [3, Example 2.1]).

When $c = -1$, any lazy 2-cocycle of (H_4, α) is of the form

σ	1	g	x	y
1	0	0	0	0
g	0	0	0	0
x	0	0	k_1	k_2
y	0	0	k_3	k_4

or

σ	1	g	x	y
1	k	k	0	0
g	k	k	0	0
x	0	0	t	$-t$
y	0	0	t	$-t$

for any $k_1, k_2, k_3, k_4, k, t \in \Bbbk$, and $k \neq 0$.

When $c^2 \neq 1$, any lazy 2-cocycle of (H_4, α) is of the form

σ	1	g	x	y
1	k	k	0	0
g	k	k	0	0
x	0	0	0	0
y	0	0	0	0

for any $k \in \Bbbk$.

Notation. (i) The set of normalized and convolution invertible $\Bbbk$-linear α-invariant maps $\sigma : H \otimes H \to \Bbbk$ is denoted by $\mathrm{Reg}^2(H, \alpha)$; it is a group under convolution product.

(ii) The set of lazy elements of $\mathrm{Reg}^2(H, \alpha)$, denoted by $\mathrm{Reg}^2_L(H, \alpha)$, is a subgroup of $\mathrm{Reg}^2(H, \alpha)$.

(iii) We denote by $Z^2(H, \alpha)$ the set of left 2-cocycles on (H, α) and by $Z^2_L(H, \alpha)$ the set $Z^2(H, \alpha) \cap \mathrm{Reg}^2_L(H, \alpha)$ of normalized and convolution invertible lazy 2-cocycles.

It is well known that $Z^2(H, \alpha)$ is in general not closed under convolution. Next we show that one of the main features of lazy 2-cocycles is that $Z^2_L(H, \alpha)$ is closed under the convolution product.

THEOREM 3.5. *The subset $Z^2_L(H,\alpha)$ of $Z^2(H,\alpha)$ is a group under the convolution product.*

Proof. One easily shows that $\sigma \in Z^2_L(H,\alpha)$ implies $\sigma^{-1} \in Z^2_L(H,\alpha)$. It remains to show that $\sigma * \tau \in Z^2_L(H,\alpha)$ for any $\sigma, \tau \in Z^2_L(H,\alpha)$, i.e.,

$$(\sigma * \tau)(h_1, g_1)(\sigma * \tau)(h_2 g_2, l) = (\sigma * \tau)(g_1, l_1)(\sigma * \tau)(h, g_2 l_2)$$

for any $h, g, l \in H$. Indeed, we have

$$\begin{aligned}
(\sigma * \tau)(h_1, g_1)(\sigma * \tau)(h_2 g_2, l) &= \sigma(h_{11}, g_{11})\tau(h_{12}, g_{12})\sigma(h_{21}g_{21}, l_1)\tau(h_{22}g_{22}, l_2)\\
&= \sigma(h_1, g_1)\sigma(\alpha(h_{212})\alpha(g_{212}), l_1)\tau(\alpha(h_{211}), \alpha(g_{211}))\tau(h_{22}g_{22}, l_2)\\
&= \sigma(h_1, g_1)\sigma(\alpha(h_{211})\alpha(g_{211}), l_1)\tau(h_{212}, g_{212})\tau(h_{22}g_{22}, l_2)\\
&= \sigma(h_1, g_1)\sigma(h_{21}g_{21}, l_1)\tau(h_{221}, g_{221})\tau(h_{222}g_{222}, \alpha^{-1}(l_2))\\
&= \sigma(h_1, g_1)\sigma(h_{21}g_{21}, l_1)\tau(g_{221}, \alpha^{-1}(l_{21}))\tau(h_{22}, g_{222}\alpha^{-1}(l_{22}))\\
&= \sigma(h_{11}, g_{11})\sigma(h_{12}g_{12}, l_1)\tau(g_{21}, l_{21})\tau(h_2, g_{22}l_{22})\\
&= \sigma(g_{11}, l_{11})\sigma(h_1, g_{12}l_{12})\tau(g_{21}, l_{21})\tau(h_2, g_{22}l_{22})\\
&= \sigma(g_1, l_1)\sigma(h_1, \alpha(g_{211})\alpha(l_{211}))\tau(g_{212}, l_{212})\tau(h_2, g_{22}l_{22})\\
&= \sigma(g_1, l_1)\tau(g_{211}, l_{211})\sigma(h_1, \alpha(g_{212})\alpha(l_{212}))\tau(h_2, g_{22}l_{22})\\
&= \sigma(g_1, l_1)\tau(g_{21}, l_{21})\sigma(h_1, \alpha(g_{221})\alpha(l_{221}))\tau(h_2, \alpha(g_{222})\alpha(l_{222}))\\
&= \sigma(g_1, l_1)\tau(g_{21}, l_{21})(\sigma * \tau)(h, \alpha(g_{22})\alpha(l_{22}))\\
&= \sigma(g_{11}, l_{11})\tau(g_{12}, l_{12})(\sigma * \tau)(h, g_2 l_2) = (\sigma * \tau)(g_1, l_1)(\sigma * \tau)(h, g_2 l_2). \quad \blacksquare
\end{aligned}$$

EXAMPLE 3.6. If (H,α) is a monoidal Hom-Hopf algebra of Example 3.3, then one easily shows that $Z^2_L(H,\alpha) = \{\sigma_1\}$ in the notation of Example 3.3.

EXAMPLE 3.7. Let (H_4,α) be a monoidal Hom-Hopf algebra of Example 3.4. Then Example 3.5 yields:

(i) for $c = 1$, the elements in the group $Z^2_L(H_4,\alpha)$ are of the form

σ	1	g	x	y
1	1	1	0	0
g	1	1	0	0
x	0	0	$\lambda/2$	$-\lambda/2$
y	0	0	$\lambda/2$	$-\lambda/2$

with $\lambda \in \Bbbk$;

(ii) for $c = -1$, the elements in the group $Z^2_L(H_4,\alpha)$ are of the form

σ	1	g	x	y
1	1	1	0	0
g	1	1	0	0
x	0	0	μ	$-\mu$
y	0	0	μ	$-\mu$

with $\mu \in \Bbbk$;

(iii) for $c^2 \neq 1$, the group $Z_L^2(H_4, \alpha)$ has a unique element σ of the form

σ	1	g	x	y
1	1	1	0	0
g	1	1	0	0
x	0	0	0	0
y	0	0	0	0

Next we define the second lazy cohomology group of (H, α).

Definition 3.8. Let $\gamma : H \to \mathbb{k}$ be a $\mathbb{k}$-linear α-invariant map, i.e., $\gamma \circ \alpha = \gamma$.

(i) We say that γ is *normalized* if $\gamma(1_H) = 1_{\mathbb{k}}$.
(ii) We say that γ is *lazy* if $\gamma(h_1)h_2 = h_1\gamma(h_2)$ for any $h \in H$.

Theorem 3.9. (i) *The set of normalized and convolution invertible $\mathbb{k}$-linear α-invariant maps $\gamma : H \to \mathbb{k}$, denoted by* $\mathrm{Reg}^1(H, \alpha)$*, is obviously a group under the convolution product.*

(ii) *The set of lazy elements of* $\mathrm{Reg}^1(H, \alpha)$*, denoted by* $\mathrm{Reg}_L^1(H, \alpha)$*, is a central subgroup of* $\mathrm{Reg}^1(H, \alpha)$*.*

Lemma 3.10. *For any $\gamma \in \mathrm{Reg}^1(H, \alpha)$, the map $D^1(\gamma) : H \otimes H \to \mathbb{k}$ defined by*

$$D^1(\gamma)(h, g) = \gamma(h_1)\gamma(g_1)\gamma^{-1}(h_2 g_2)$$

for any $h, g \in H$ is a normalized and convolution invertible left 2-cocycle. Moreover, if γ is lazy, then so is $D^1(\gamma)$.

Proof. Clearly, $D^1(\gamma)$ is $\mathbb{k}$-linear, α-invariant and normalized. We check that $D^1(\gamma)$ is a left 2-cocycle. Indeed, for any $h, g, l \in H$, we have

$$\begin{aligned}
&D^1(\gamma)(h_1, g_1)D^1(\gamma)(h_2g_2, l) \\
&\qquad = \gamma(h_{11})\gamma(g_{11})\gamma^{-1}(h_{12}g_{12})\gamma(h_{21}g_{21})\gamma(l_1)\gamma^{-1}((h_{22}g_{22})l_2) \\
&\qquad = \gamma(h_1)\gamma(g_1)\gamma^{-1}(h_{211}g_{211})\gamma(h_{212}g_{212})\gamma(l_1)\gamma^{-1}((h_{22}g_{22})l_2) \\
&\qquad = \gamma(g_1)\gamma(l_1)\gamma(h_1)\gamma^{-1}(h_2\alpha^{-1}(g_2l_2)) \\
&\qquad = \gamma(g_1)\gamma(l_1)\gamma(h_1)\gamma^{-1}(g_{211}l_{211})\gamma(g_{212}l_{212})\gamma^{-1}(h_2\alpha(g_{22}l_{22})) \\
&\qquad = \gamma(g_{11})\gamma(l_{11})\gamma^{-1}(g_{12}l_{12})\gamma(h_1)\gamma(g_{21}l_{21})\gamma^{-1}(h_2(g_{22}l_{22})) \\
&\qquad = D^1(\gamma)(g_1, l_1)D^1(\gamma)(h, g_2l_2).
\end{aligned}$$

Hence $D^1(\gamma)$ is a left 2-cocycle. Next we prove that $D^1(\gamma)$ is convolution invertible. Define a map $T^1(\gamma) : H \otimes H \to \mathbb{k}$ as

$$T^1(\gamma)(h, g) = \gamma(h_1g_1)\gamma^{-1}(h_2)\gamma^{-1}(g_2)$$

for any $h, g \in H$. We show that $D^1(\gamma) * T^1(\gamma) = T^1(\gamma) * D^1(\gamma) = \varepsilon_{H\otimes H}$. Indeed, we have

$$\begin{aligned}
&(D^1(\gamma) * T^1(\gamma))(h, g)\\
&\qquad = \gamma(h_{11})\gamma(g_{11})\gamma^{-1}(h_{12}g_{12})\gamma(h_{21}g_{21})\gamma^{-1}(h_{22})\gamma^{-1}(g_{22})\\
&\qquad = \gamma(h_1)\gamma(g_1)\gamma^{-1}(h_{21}g_{21})\gamma(h_{221}g_{221})\gamma^{-1}(h_{222})\gamma^{-1}(g_{222})\\
&\qquad = \gamma(h_1)\gamma(g_1)\gamma^{-1}(h_{211}g_{211})\gamma(h_{212}g_{212})\gamma^{-1}(h_{22})\gamma^{-1}(g_{22})\\
&\qquad = \varepsilon(h)\varepsilon(g).
\end{aligned}$$

Similarly, we get $T^1(\gamma) * D^1(\gamma) = \varepsilon_{H\otimes H}$. If γ is lazy, it is easy to see that $D^1(\gamma)$ is lazy. ∎

Proposition 3.11. *The map $D^1(\alpha)$ defined in Lemma* 3.10 *induces a group morphism $\mathrm{Reg}_L^1(H, \alpha) \to Z_L^2(H, \alpha)$; its image, denoted by $B_L^2(H, \alpha)$, is contained in the center of $Z_L^2(H, \alpha)$.*

Proof. By Lemma 3.10, we have $D^1(\gamma) \in Z_L^2(H, \alpha)$ for any $\gamma \in \mathrm{Reg}_L^1(H, \alpha)$. Next we check that $D^1(\gamma * \gamma') = D^1(\gamma) * D^1(\gamma')$ for any $\gamma, \gamma' \in \mathrm{Reg}_L^1(H, \alpha)$, and $D^1(\varepsilon) = \varepsilon_{H\otimes H}$. Indeed, for any $h, g \in H$, we have

$$\begin{aligned}
D^1(\gamma * \gamma')(h, g) &= \gamma(h_{11})\gamma'(h_{12})\gamma(g_{11})\gamma'(g_{12})\gamma'^{-1}(h_{21}g_{21})\gamma^{-1}(h_{22}g_{22})\\
&= \gamma(h_1)\gamma(g_1)\gamma^{-1}(h_{22}g_{22})\gamma'(h_{211})\gamma'(g_{211})\gamma'^{-1}(h_{212}g_{212})\\
&= \gamma(h_1)\gamma(g_1)\gamma^{-1}(h_{22}g_{22})D^1(\gamma')(h_{21}, g_{21})\\
&= \gamma(h_1)\gamma(g_1)\gamma^{-1}(h_{21}g_{21})D^1(\gamma')(h_{22}, g_{22})\\
&= \gamma(h_{11})\gamma(g_{11})\gamma^{-1}(h_{12}g_{12})D^1(\gamma')(h_2, g_2)\\
&= (D^1(\gamma) * D^1(\gamma'))(h, g),
\end{aligned}$$

and $D^1(\varepsilon)(h, g) = \varepsilon(h_1)\varepsilon(g_1)\varepsilon(h_2g_2) = \varepsilon(h)\varepsilon(g)$.

Finally, we show that $B_L^2(H, \alpha)$ is contained in the center of $Z_L^2(H, \alpha)$, i.e., $\sigma * D^1(\gamma) = D^1(\gamma) * \sigma$ for any $\gamma \in \mathrm{Reg}_L^1(H, \alpha)$ and $\sigma \in Z_L^2(H, \alpha)$. Indeed, for any $h, g \in H$, we have

$$\begin{aligned}
(\sigma * D^1(\gamma))(h, g) &= \sigma(h_1, g_1)\gamma(h_{21})\gamma(g_{21})\gamma^{-1}(h_{22}g_{22})\\
&= \sigma(h_1, g_1)\gamma(h_{22})\gamma(g_{22})\gamma^{-1}(h_{21}g_{21})\\
&= \sigma(h_{11}, g_{11})\gamma(h_2)\gamma(g_2)\gamma^{-1}(h_{12}g_{12})\\
&= \sigma(h_{12}, g_{12})\gamma(h_2)\gamma(g_2)\gamma^{-1}(h_{11}g_{11})\\
&= \sigma(h_{21}, g_{21})\gamma(h_{22})\gamma(g_{22})\gamma^{-1}(h_1g_1)\\
&= \sigma(h_{22}, g_{22})\gamma(h_{21})\gamma(g_{21})\gamma^{-1}(h_1g_1)\\
&= \sigma(h_2, g_2)\gamma(h_{12})\gamma(g_{12})\gamma^{-1}(h_{11}g_{11})\\
&= \gamma(h_{11})\gamma(g_{11})\gamma^{-1}(h_{12}g_{12})\sigma(h_2, g_2) = (D^1(\gamma) * \sigma)(h, g).
\end{aligned}$$ ∎

Definition 3.12. Let (H,α) be a monoidal Hom-Hopf algebra.

(i) The elements of $B_L^2(H,\alpha)$ are called *lazy 2-coboundaries.*

(ii) The quotient group
$$H_L^2(H,\alpha) := Z_L^2(H,\alpha)/B_L^2(H,\alpha)$$
is called *the second lazy cohomology group of* (H,α).

Finally, we list some properties of left (right) 2-cocycles.

Proposition 3.13. *If we define a Hom-multiplication $\cdot_\sigma$ on (H,α) by $h\cdot_\sigma g = \sigma(h_1,g_1)\alpha(h_2g_2)$ for any $h,g\in H$, then $({}_\sigma H,\alpha) = (H,\cdot_\sigma,1_H,\alpha)$ is a monoidal Hom-associative algebra if and only if σ is a normalized left 2-cocycle.*

Proof. For any $h\in H$, it is easy to see that $h\cdot_\sigma 1_H = \alpha(h)$ if and only if $\sigma(h,1_H)=\varepsilon(h)$ and $1_H\cdot_\sigma h = \alpha(h)$ if and only if $\sigma(1_H,h)=\varepsilon(h)$. For any $h,g,l\in H$, we have
$$\begin{aligned}\alpha(h)\cdot_\sigma(g\cdot_\sigma l) &= \sigma(g_1,l_1)\sigma(\alpha(h_1),\alpha(g_{21})\alpha(l_{21}))\alpha^2(h_2)(\alpha^2(g_{22})\alpha^2(l_{22}))\\ &= \sigma(g_{11},l_{11})\sigma(h_1,g_{12}l_{12})\alpha^2(h_2)(\alpha(g_2)\alpha(l_2)),\end{aligned}$$
and
$$\begin{aligned}(h\cdot_\sigma g)\cdot_\sigma \alpha(l) &= \sigma(h_1,g_1)\sigma(\alpha(h_{21})\alpha(g_{21}),\alpha(l_1))(\alpha^2(h_{22})\alpha^2(g_{22}))\alpha^2(l_2)\\ &= \sigma(h_{11},g_{11})\sigma(h_{12}g_{12},l_1)\alpha^2(h_2)(\alpha(g_2)\alpha(l_2)).\end{aligned}$$
Hence, if $\cdot_\sigma$ is Hom-associative, we get
$$\begin{aligned}&\sigma(g_{11},l_{11})\sigma(h_1,g_{12}l_{12})\alpha^2(h_2)(\alpha(g_2)\alpha(l_2))\\ &\qquad\qquad = \sigma(h_{11},g_{11})\sigma(h_{12}g_{12},l_1)\alpha^2(h_2)(\alpha(g_2)\alpha(l_2)).\end{aligned}$$
Applying ε to both sides, we obtain
$$\sigma(g_1,l_1)\sigma(h,g_2l_2) = \sigma(h_1,g_1)\sigma(h_2g_2,l),$$
which means σ is a left 2-cocycle.

Conversely, if σ is a left 2-cocycle, it is straightforward to deduce that $\alpha(h)\cdot_\sigma(g\cdot_\sigma l) = (h\cdot_\sigma g)\cdot_\sigma\alpha(l)$, i.e., $\cdot_\sigma$ is Hom-associative. ∎

Proposition 3.14. *Let $\sigma : H\otimes H\to \Bbbk$ be a normalized left 2-cocycle. Then $({}_\sigma H,\alpha)$ is a right (H,α)-Hom-comodule algebra via Δ_H.*

Proof. From the above proposition, we know that $({}_\sigma H,\alpha)$ is a monoidal Hom-associative algebra. Clearly, it is a right (H,α)-Hom-comodule via Δ_H. We just need to show that $\Delta_H(h\cdot_\sigma g) = h_1\cdot_\sigma g_1\otimes h_2g_2$. Indeed,
$$\begin{aligned}\Delta_H(h\cdot_\sigma g) &= \sigma(h_1,g_1)\alpha(h_{21})\alpha(g_{21})\otimes\alpha(h_{22})\alpha(g_{22})\\ &= \sigma(h_{11},g_{11})\alpha(h_{12})\alpha(g_{12})\otimes h_2g_2 = h_1\cdot_\sigma g_1\otimes h_2g_2. \ ∎\end{aligned}$$

By applying the arguments in the proofs of Propositions 3.13 and 3.14, we get the following three propositions.

PROPOSITION 3.15. *If we define a Hom-multiplication* $\cdot_\sigma$ *on* (H,α) *by* $h\cdot_\sigma g=\alpha(h_1g_1)\sigma(h_2,g_2)$ *for any* $h,g\in H$, *then* $(H_\sigma,\alpha)=(H,\cdot_\sigma,1_H,\alpha)$ *is a monoidal Hom-associative algebra if and only if* σ *is a normalized right* 2*-cocycle.*

PROPOSITION 3.16. *Let* σ *be a normalized right* 2*-cocycle. Then* (H_σ,α) *is a left* (H,α)*-Hom-comodule algebra via* Δ_H.

PROPOSITION 3.17. *Let* σ *be a normalized lazy* 2*-cocycle. Then* $({}_\sigma H,\alpha)=(H_\sigma,\alpha)$, *and we denote it by* $H(\sigma)$. *It is an* (H,α)*-Hom-bicomodule algebra via* Δ_H.

4. Extending (lazy) 2-cocycles to a Radford biproduct, I. We begin this section with the following construction.

PROPOSITION 4.1. *Let* (H,α) *be a monoidal Hom-bialgebra,* (B,β) *a left* (H,α)*-Hom-module algebra and* (A,γ) *a left* (H,α)*-Hom-comodule algebra. Then on the space* $B\otimes A$ *we have a Hom-associative algebra structure, denoted by* $(B\ltimes A,\beta\otimes\gamma)$, *with unit* $1_B\ltimes 1_A$ *and Hom-multiplication*

$$(b\ltimes a)(b'\ltimes a')=b(a_{(-1)}\cdot\beta^{-1}(b'))\ltimes\gamma(a_{(0)})a'$$

for any $b,b'\in B$ *and* $a,a'\in A$.

Proof. We can easily see that $1_B\ltimes 1_A$ is the unit. Next we just show the Hom-associativity of the Hom-multiplication, i.e.,

$$(\beta\otimes\gamma)(b\ltimes a)((b'\ltimes a')(b''\ltimes a''))=((b\ltimes a)(b'\ltimes a'))(\beta\otimes\gamma)(b''\ltimes a'')$$

for any $b,b',b''\in B$ and $a,a',a''\in A$. In fact, we have

$$\begin{aligned}
&(\beta\otimes\gamma)(b\ltimes a)((b'\ltimes a')(b''\ltimes a''))\\
&\quad=\beta(b)\big(\gamma(a)_{(-1)}\cdot\beta^{-1}(b'(a'_{(-1)}\cdot\beta^{-1}(b'')))\big)\ltimes\gamma(\gamma(a)_{(0)})(\gamma(a'_{(0)})a'')\\
&\quad=\beta(b)\big((\alpha(a_{(-1)1})\cdot\beta^{-1}(b'))(\alpha(a_{(-1)2})\cdot\beta^{-1}(a'_{(-1)}\cdot\beta^{-1}(b'')))\big)\\
&\qquad\ltimes\gamma^2(a_{(0)})(\gamma(a'_{(0)})a'')\\
&\quad=(b(a_{(-1)}\cdot\beta^{-1}(b')))\beta\big(\alpha(a_{(0)(-1)})\cdot(\alpha^{-1}(a'_{(-1)})\cdot\beta^{-2}(b''))\big)\\
&\qquad\ltimes(\gamma^2(a_{(0)(0)})\gamma(a'_{(0)}))\gamma(a'')\\
&\quad=(b(a_{(-1)}\cdot\beta^{-1}(b')))(\alpha(a_{(0)(-1)})a'_{(-1)}\cdot b'')\ltimes(\gamma^2(a_{(0)(0)})\gamma(a'_{(0)}))\gamma(a'')\\
&\quad=(b(a_{(-1)}\cdot\beta^{-1}(b')))(\gamma(a_{(0)})_{(-1)}a'_{(-1)}\cdot b'')\ltimes\gamma(\gamma(a_{(0)})_{(0)}a'_{(0)})\gamma(a'')\\
&\quad=\big(b(a_{(-1)}\cdot\beta^{-1}(b'))\ltimes\gamma(a_{(0)})a'\big)(\beta(b'')\ltimes\gamma(a''))\\
&\quad=((b\ltimes a)(b'\ltimes a'))(\beta\otimes\gamma)(b''\ltimes a''). \ \blacksquare
\end{aligned}$$

PROPOSITION 4.2. *If $((H,\alpha),(B,\beta))$ is an admissible pair and (A,γ) is a left (H,α)-Hom-comodule algebra, then $(B\ltimes A,\beta\otimes\gamma)$ becomes a left $(B^{\times}_{\sharp}H,\beta\otimes\alpha)$-Hom-comodule algebra with coaction*

$$\begin{aligned}&\lambda: B\ltimes A\to(B^{\times}_{\sharp}H)\otimes(B\ltimes A),\\&\lambda(b\ltimes a)=(b_1\times b_{2(-1)}\alpha^{-1}(a_{(-1)}))\otimes(\beta(b_{2(0)})\ltimes a_{(0)}),\end{aligned}$$

for any $b\in B$ and $a\in A$.

Proof. We first prove that $((B\ltimes A,\beta\otimes\gamma),\lambda)$ is a left $(B^{\times}_{\sharp}H,\beta\otimes\alpha)$-Hom-comodule. For this, we have the following computations:

$$\begin{aligned}(\varepsilon\otimes\mathrm{id})\lambda(b\ltimes a)&=\varepsilon(b_1\times b_{2(-1)}\alpha^{-1}(a_{(-1)}))(\beta(b_{2(0)})\ltimes a_{(0)})\\&=(\beta^{-1}\otimes\gamma^{-1})(b\ltimes a),\\\lambda(\beta\otimes\gamma)(b\ltimes a)&=(\beta(b_1)\times\beta(b)_{2(-1)}\alpha^{-1}(\gamma(a)_{(-1)}))\otimes(\beta(\beta(b)_{2(0)})\ltimes\gamma(a)_{(0)})\\&=(\beta(b_1)\times\alpha(b_{2(-1)})a_{(-1)})\otimes(\beta^2(b_{2(0)})\ltimes\gamma(a_{(0)}))\\&=(\beta\otimes\alpha\otimes\beta\otimes\gamma)\lambda(b\ltimes a),\end{aligned}$$

and

$$\begin{aligned}&((\beta\otimes\alpha)^{-1}\otimes\lambda)\lambda(b\ltimes a)\\&=\big(\beta^{-1}(b_1)\times\alpha^{-1}(b_{2(-1)})\alpha^{-2}(a_{(-1)})\big)\\&\quad\otimes\big((\beta(b_{2(0)1})\times\beta(b_{2(0)2})_{(-1)}\alpha^{-1}(a_{(0)(-1)}))\otimes(\beta(\beta(b_{2(0)2})_{(0)})\ltimes a_{(0)(0)})\big)\\&=\big(\beta^{-1}(b_1)\times(\alpha^{-1}(b_{21(-1)})\alpha^{-1}(b_{22(-1)}))\alpha^{-1}(a_{(-1)1})\big)\\&\quad\otimes\big((\beta(b_{21(0)})\times\alpha(b_{22(0)(-1)})\alpha^{-1}(a_{(-1)2}))\otimes(\beta^2(b_{22(0)(0)})\ltimes\gamma^{-1}(a_{(0)}))\big)\\&=\big(b_{11}\times(\alpha^{-1}(b_{12(-1)})\beta^{-1}(b_2)_{(-1)1})\alpha^{-1}(a_{(-1)1})\big)\\&\quad\otimes\big((\beta(b_{12(0)})\times\alpha(\beta^{-1}(b_2)_{(-1)2})\alpha^{-1}(a_{(-1)2}))\otimes(\beta(\beta^{-1}(b_2)_{(0)})\ltimes\gamma^{-1}(a_{(0)}))\big)\\&=\big(b_{11}\times b_{12(-1)}(\alpha^{-1}(b_{2(-1)1})\alpha^{-2}(a_{(-1)1}))\big)\otimes\big((\beta(b_{12(0)})\times b_{2(-1)2}\alpha^{-1}(a_{(-1)2}))\\&\quad\otimes(b_{2(0)}\ltimes\gamma^{-1}(a_{(0)}))\big)\\&=(\Delta_{B^{\times}_{\sharp}H}\otimes(\beta\otimes\gamma)^{-1})\lambda(b\ltimes a),\end{aligned}$$

for any $b\in B$ and $a\in A$. We proceed to show that λ is a Hom-algebra map. Clearly, $\lambda(1_B\ltimes 1_A)=(1_B\times 1_H)\otimes(1_B\ltimes 1_A)$. For any $b,b'\in B$ and $a,a'\in A$, we have

$$\begin{aligned}&\lambda((b\ltimes a)(b'\ltimes a'))\\&\quad=\big((b(a_{(-1)}\cdot\beta^{-1}(b')))_1\times(b(a_{(-1)}\cdot\beta^{-1}(b')))_{2(-1)}\alpha^{-1}((\gamma(a_{(0)})a')_{(-1)})\big)\\&\qquad\otimes\big(\beta((b(a_{(-1)}\cdot\beta^{-1}(b')))_{2(0)})\ltimes(\gamma(a_{(0)})a')_{(0)}\big)\end{aligned}$$

$$
\begin{aligned}
&= \big(b_1(b_{2(-1)} \cdot \beta^{-1}(a_{(-1)1} \cdot \beta^{-1}(b'_1))) \\
&\qquad\qquad \times (\beta(b_{2(0)})(a_{(-1)2} \cdot \beta^{-1}(b'_2)))_{(-1)}\alpha^{-1}(\alpha(a_{(0)(-1)})a'_{(-1)})\big) \\
&\quad \otimes \big(\beta((\beta(b_{2(0)})(a_{(-1)2} \cdot \beta^{-1}(b'_2)))_{(0)}) \ltimes \gamma(a_{(0)(0)})a'_{(0)}\big) \\
&= \big(b_1(\alpha^{-1}(b_{2(-1)})\alpha^{-1}(a_{(-1)1}) \cdot \beta^{-1}(b'_1)) \\
&\qquad\qquad \times \alpha^2(b_{2(0)(-1)})(\alpha^{-1}((a_{(-1)2} \cdot \beta^{-1}(b'_2))_{(-1)}a_{(0)(-1)})\alpha^{-1}(a'_{(-1)}))\big) \\
&\quad \otimes \big(\beta^2(b_{2(0)(0)})\beta((a_{(-1)2} \cdot \beta^{-1}(b'_2))_{(0)}) \ltimes \gamma(a_{(0)(0)})a'_{(0)}\big) \\
&= \big(b_1(\alpha^{-1}(b_{2(-1)})\alpha^{-1}(a_{(-1)1}) \cdot \beta^{-1}(b'_1)) \\
&\qquad\qquad \times \alpha^2(b_{2(0)(-1)})(\alpha^{-1}((\alpha(a_{(-1)21}) \cdot \beta^{-1}(b'_2))_{(-1)}\alpha(a_{(-1)22}))\alpha^{-1}(a'_{(-1)}))\big) \\
&\quad \otimes \big(\beta^2(b_{2(0)(0)})\beta((\alpha(a_{(-1)21}) \cdot \beta^{-1}(b'_2))_{(0)}) \ltimes a_{(0)}a'_{(0)}\big) \\
&= \big(b_1(\alpha^{-1}(b_{2(-1)})\alpha^{-1}(a_{(-1)1}) \cdot \beta^{-1}(b'_1)) \\
&\qquad\qquad \times \alpha^2(b_{2(0)(-1)})(\alpha^{-1}(\alpha(a_{(-1)21})b'_{2(-1)})\alpha^{-1}(a'_{(-1)}))\big) \\
&\quad \otimes (\beta^2(b_{2(0)(0)})(\alpha(a_{(-1)22}) \cdot b'_{2(0)}) \ltimes a_{(0)}a'_{(0)}) \\
&= \big(b_1(b_{2(-1)1}a_{(-1)11} \cdot \beta^{-1}(b'_1)) \\
&\qquad\qquad \times \alpha^2(b_{2(-1)2})(\alpha^{-1}(\alpha(a_{(-1)12})b'_{2(-1)})\alpha^{-1}(a'_{(-1)}))\big) \\
&\quad \otimes (\beta(b_{2(0)})(a_{(-1)2} \cdot b'_{2(0)}) \ltimes a_{(0)}a'_{(0)}) \\
&= \big(b_1(b_{2(-1)1}a_{(-1)11} \cdot \beta^{-1}(b'_1)) \times \alpha(b_{2(-1)2}a_{(-1)12})(b'_{2(-1)}\alpha^{-1}(a'_{(-1)}))\big) \\
&\quad \otimes (\beta(b_{2(0)})(a_{(-1)2} \cdot b'_{2(0)}) \ltimes a_{(0)}a'_{(0)}) \\
&= \big(b_1((b_{2(-1)}\alpha^{-1}(a_{(-1)}))_1 \cdot \beta^{-1}(b'_1)) \\
&\qquad\qquad \times \alpha((b_{2(-1)}\alpha^{-1}(a_{(-1)}))_2)(b'_{2(-1)}\alpha^{-1}(a'_{(-1)}))\big) \\
&\quad \otimes (\beta(b_{2(0)})(a_{(0)(-1)} \cdot b'_{2(0)}) \ltimes \gamma(a_{(0)(0)})a'_{(0)}) \\
&= (b_1 \times b_{2(-1)}\alpha^{-1}(a_{(-1)}))(b'_1 \times b'_{2(-1)}\alpha^{-1}(a'_{(-1)})) \\
&\quad \otimes (\beta(b_{2(0)}) \ltimes a_{(0)})(\beta(b'_{2(0)}) \ltimes a'_{(0)}) \\
&= \lambda(b \ltimes a)\lambda(b' \ltimes a'),
\end{aligned}
$$

Hence, λ is a Hom-algebra map, and the proof is finished. ∎

Now we can obtain the main result of this section.

THEOREM 4.3. *Let $((H,\alpha),(B,\beta))$ be an admissible pair and let $\sigma : H \otimes H \to \Bbbk$ be a normalized and convolution invertible right 2-cocycle. Define a map*

$$
\tilde{\sigma} : (B^{\times}_{\sharp} H) \otimes (\tilde{\sigma} : B^{\times}_{\sharp} H) \to \Bbbk, \qquad \tilde{\sigma}(b \times h, b' \times h') = \varepsilon_B(b)\varepsilon_B(b')\sigma(h,h'),
$$

for any $b, b' \in B$ and $h, h' \in H$. Then $\tilde{\sigma}$ is a normalized and convolution *invertible right* 2*-cocycle on* $B^{\times}_{\sharp}H$*, and we have*

$$((B^{\times}_{\sharp}H)_{\tilde{\sigma}}, \beta\otimes\alpha) = (B \ltimes H_\sigma, \beta\otimes\alpha)$$

as left $(B^{\times}_{\sharp}H, \beta\otimes\alpha)$*-Hom-comodule algebras. Moreover,* $\tilde{\sigma}$ *is unique with this property.*

Proof. Clearly, $\tilde{\sigma}$ is $\beta\otimes\alpha$-invariant, normalized and convolution invertible. Next we show that it is a right 2-cocycle. By Propositions 3.15 and 3.16, we know that (H_σ, α) is a left (H, α)-Hom-comodule algebra via Δ_H. So by Proposition 4.1, $B \ltimes H_\sigma$ is a Hom-associative algebra. For any $b, b' \in B$ and $h, h' \in H$, we have

$$\begin{aligned}
&(b\times h)\cdot_{\tilde{\sigma}}(b'\times h')\\
&= (\beta\otimes\alpha)\big((b_1\times b_{2(-1)}\alpha^{-1}(h_1))(b'_1\times b'_{2(-1)}\alpha^{-1}(h'_1))\big)\\
&\quad \tilde{\sigma}(\beta(b_{2(0)})\times h_2, \beta(b'_{2(0)})\times h'_2)\\
&= (\beta\otimes\alpha)(b_1((b_{2(-1)}\alpha^{-1}(h_1))_1\cdot\beta^{-1}(b'_1))\times\alpha((b_{2(-1)}\alpha^{-1}(h_1))_2)b'_{2(-1)}\alpha^{-1}(h'_1))\\
&\quad \varepsilon_B(b_{2(0)})\varepsilon_B(b'_{2(0)})\sigma(h_2, h'_2)\\
&= \beta(b_1)\beta(1_H\alpha^{-1}(h_{11})\cdot\beta^{-1}(b'_1))\\
&\quad \times\alpha^2(1_H\alpha^{-1}(h_{12}))(1_Hh'_1)\varepsilon_B(b_2)\varepsilon_B(b'_2)\sigma(h_2, h'_2)\\
&= b(\alpha(h_{11})\cdot\beta^{-1}(b'))\times\alpha^2(h_{12})\alpha(h'_1)\sigma(h_2, h'_2)\\
&= b(h_1\cdot\beta^{-1}(b'))\times\alpha(\alpha(h_{21})h'_1)\sigma(\alpha(h_{22}), h'_2) = (b\ltimes h)(b'\ltimes h'),
\end{aligned}$$

which means the Hom-multiplication on $(B^{\times}_{\sharp}H)_{\tilde{\sigma}}$ coincides with the one on $B \ltimes H_\sigma$. So by Proposition 3.15, $\tilde{\sigma}$ is a right 2-cocycle and we have $((B^{\times}_{\sharp}H)_{\tilde{\sigma}}, \beta\otimes\alpha) = (B\ltimes H_\sigma, \beta\otimes\alpha)$ as Hom-associative algebras. It is obvious that they also coincide as left $(B^{\times}_{\sharp}H, \beta\otimes\alpha)$-Hom-comodules.

Finally, we show the uniqueness of $\tilde{\sigma}$. Since the Hom-multiplications on $((B^{\times}_{\sharp}H)_{\tilde{\sigma}}, \beta\otimes\alpha)$ and $(B\ltimes H_\sigma, \beta\otimes\alpha)$ coincide, we apply $\varepsilon_B\otimes\varepsilon_H$ to conclude that $\tilde{\sigma}(b\times h, b'\times h') = \varepsilon_B(b)\varepsilon_B(b')\sigma(h, h')$. ∎

5. Extending (lazy) 2-cocycles to a Radford biproduct, II. In this section, we always let (H, α) denote a monoidal Hom-Hopf algebra with a bijective antipode and (B, β) be a Hopf algebra in ${}^H_H\mathcal{HYD}$.

Let $\sigma : B\otimes B\to \Bbbk$ be a morphism in ${}^H_H\mathcal{HYD}$, that is,

$$\begin{aligned}
\sigma(\beta(b), \beta(b')) &= \sigma(b, b'),\\
\sigma(h_1\cdot b, h_2\cdot b') &= \varepsilon(h)\sigma(b, b'),\\
b_{(-1)}b'_{(-1)}\sigma(b_{(0)}, b'_{(0)}) &= \sigma(b, b')1_H,
\end{aligned}$$

for any $b, b' \in B$.

Let (B,β) be a Hopf algebra in ${}^H_H\mathcal{HYD}$. Then the Hom-coalgebra structure of $(B\otimes B,\beta\otimes\beta)$ in ${}^H_H\mathcal{HYD}$ is given by

$$\Delta_{B\otimes B}(b\otimes b')=(b_1\otimes b_{2(-1)}\cdot\beta^{-1}(b'_1))\otimes(\beta(b_{2(0)})\otimes b'_2)$$

for any $b,b'\in B$.

So, if $\sigma,\tau: B\otimes B\to \Bbbk$ are morphisms in ${}^H_H\mathcal{HYD}$, their convolution product in ${}^H_H\mathcal{HYD}$ is given by

$$(\sigma*\tau)(b,b')=\sigma(b_1,b_{2(-1)}\cdot\beta^{-1}(b'_1))\tau(\beta(b_{2(0)}),b'_2)$$

for any $b,b'\in B$.

Definition 5.1. Let (H,α) be a monoidal Hom-Hopf algebra with a bijective antipode, (B,β) be a Hopf algebra in ${}^H_H\mathcal{HYD}$ and $\sigma: B\otimes B\to\Bbbk$ be a morphism in ${}^H_H\mathcal{HYD}$. For any $a,b,c\in B$,

(i) σ is called a *left 2-cocycle in* ${}^H_H\mathcal{HYD}$ if

$$\begin{aligned}\sigma(a_1,a_{2(-1)}\cdot\beta^{-1}(b_1))&\sigma(\beta(a_{2(0)})b_2,c)\\&=\sigma(b_1,b_{2(-1)}\cdot\beta^{-1}(c_1))\sigma(a,\beta(b_{2(0)})c_2);\end{aligned}$$

(ii) σ is called *lazy in* ${}^H_H\mathcal{HYD}$ if

$$\begin{aligned}\sigma(a_1,a_{2(-1)}\cdot\beta^{-1}(b_1))&\beta(a_{2(0)})b_2\\&=\sigma(\beta(a_{2(0)}),b_2)a_1(a_{2(-1)}\cdot\beta^{-1}(b_1));\end{aligned}$$

(iii) σ is called *normalized* if $\sigma(b,1)=\sigma(1,b)=\varepsilon(b)$.

Proposition 5.2. *If we define a Hom-multiplication $\cdot_\sigma$ on (B,β) by*

$$b\cdot_\sigma b'=\sigma(b_1,b_{2(-1)}\cdot\beta^{-1}(b'_1))\beta(\beta(b_{2(0)})b'_2)$$

for any $b,b'\in B$, then

(a) *$({}_\sigma B,\beta)=(B,\cdot_\sigma,1_B,\beta)$ is a monoidal Hom-associative algebra if and only if σ is a normalized left 2-cocycle in ${}^H_H\mathcal{HYD}$.*
(b) *$({}_\sigma B,\beta)$ is a left (H,α)-Hom-module algebra with the same action as (B,β).*

Proof. (a) Use the same idea as in the proof of Proposition 3.13.

(b) We check that $({}_\sigma B,\beta)$ is a left (H,α)-Hom-module algebra. Clearly, $h\cdot 1_B=\varepsilon(h)1_B$ for any $h\in H$. Next we show the identity $h\cdot(b\cdot_\sigma b')=(h_1\cdot b)\cdot_\sigma(h_2\cdot b')$ for any $h\in H$ and $b,b'\in B$. Indeed, we have

$$\begin{aligned}&(h_1\cdot b)\cdot_\sigma(h_2\cdot b')\\&\quad=\sigma\big(h_{11}\cdot b_1,(h_{12}\cdot b_2)_{(-1)}\cdot\beta^{-1}(h_{21}\cdot b'_1)\big)\beta\big(\beta((h_{12}\cdot b_2)_{(0)})(h_{22}\cdot b'_2)\big)\end{aligned}$$

$$= \sigma\big(h_{11} \cdot b_1, (h_{1211}\alpha^{-1}(b_{2(-1)}))S(h_{122}) \cdot \beta^{-1}(h_{21} \cdot b'_1)\big)$$
$$\beta\big(\beta(\alpha(h_{1212}) \cdot b_{2(0)})(h_{22} \cdot b'_2)\big)$$
$$= \sigma\big(h_{11} \cdot b_1, (\alpha^{-1}(h_{121})\alpha^{-1}(b_{2(-1)}))S(\alpha(h_{1222})) \cdot \beta^{-1}(h_{21} \cdot b'_1)\big)$$
$$\beta\big(\beta(\alpha(h_{1221}) \cdot b_{2(0)})(h_{22} \cdot b'_2)\big)$$
$$= \sigma\big(\alpha(h_{111}) \cdot b_1, \alpha(h_{112}) \cdot (\alpha^{-1}(b_{2(-1)})S(\alpha^{-1}(h_{122})) \cdot \beta^{-2}(h_{21} \cdot b'_1))\big)$$
$$\beta(\beta(h_{121} \cdot b_{2(0)})(h_{22} \cdot b'_2))$$
$$= \sigma\big(b_1, \alpha^{-1}(b_{2(-1)})S(\alpha^{-2}(h_{12})) \cdot (\alpha^{-2}(h_{21}) \cdot \beta^{-2}(b'_1))\big)\beta(h_{11} \cdot \beta(b_{2(0)}))$$
$$\beta(h_{22} \cdot b'_2)$$
$$= \sigma\big(b_1, \alpha^{-1}(b_{2(-1)})(\alpha^{-3}(S(h_{12}))\alpha^{-3}(h_{21})) \cdot \beta^{-1}(b'_1)\big)(\alpha(h_{11}) \cdot \beta^2(b_{2(0)}))$$
$$(\alpha(h_{22}) \cdot \beta(b'_2))$$
$$= \sigma\big(b_1, \alpha^{-1}(b_{2(-1)})(\alpha^{-3}(S(h_{21}))\alpha^{-2}(h_{221})) \cdot \beta^{-1}(b'_1)\big)(h_1 \cdot \beta^2(b_{2(0)}))$$
$$(\alpha^2(h_{222}) \cdot \beta(b'_2))$$
$$= \sigma\big(b_1, \alpha^{-1}(b_{2(-1)})(\alpha^{-3}(S(\alpha(h_{211})))\alpha^{-2}(h_{212})) \cdot \beta^{-1}(b'_1)\big)(h_1 \cdot \beta^2(b_{2(0)}))$$
$$(\alpha(h_{22}) \cdot \beta(b'_2))$$
$$= \sigma\big(b_1, b_{2(-1)} \cdot \beta^{-1}(b'_1)\big)(h_1 \cdot \beta^2(b_{2(0)}))(h_2 \cdot \beta(b'_2)) = h \cdot (b \cdot_\sigma b'). \blacksquare$$

THEOREM 5.3. *Let (H, α) be a monoidal Hom-Hopf algebra with a bijective antipode and (B, β) be a Hopf algebra in ${}^H_H\mathcal{HYD}$. If $\sigma : B \otimes B \to \Bbbk$ is a normalized left 2-cocycle in ${}^H_H\mathcal{HYD}$, and*

$$\bar\sigma : (B \overset{\times}{\sharp} H) \otimes (B \overset{\times}{\sharp} H) \to \Bbbk, \qquad \bar\sigma(b \times h, b' \times h') = \sigma(b, \alpha^{-1}(h) \cdot \beta^{-1}(b'))\varepsilon(h'),$$

for any $b, b' \in B$ and $h, h' \in H$, then $\bar\sigma$ is a normalized left 2-cocycle on $B \overset{\times}{\sharp} H$, and we have $({}_{\bar\sigma}(B \overset{\times}{\sharp} H), \beta \otimes \alpha) = ({}_\sigma B \sharp H, \beta \otimes \alpha)$ as monoidal Hom-algebras. Moreover, $\bar\sigma$ is unique with these properties.

Proof. It is easy to see that $\bar\sigma$ is normalized. We will show that the Hom-multiplications on $({}_\sigma B \sharp H, \beta \otimes \alpha)$ and $({}_{\bar\sigma}(B \overset{\times}{\sharp} H), \beta \otimes \alpha)$ coincide. Indeed,

$$(b \sharp h)(b' \sharp h')$$
$$= \sigma\big(b_1, b_{2(-1)} \cdot \beta^{-1}((h_1 \cdot \beta^{-1}(b'))_1)\big)\beta(\beta(b_{2(0)})(h_1 \cdot \beta^{-1}(b'))_2) \sharp \alpha(h_2)h'$$
$$= \sigma\big(b_1, b_{2(-1)} \cdot (\alpha^{-1}(h_{11}) \cdot \beta^{-2}(b'_1))\big)(\beta^2(b_{2(0)})(\alpha(h_{12}) \cdot b'_2)) \sharp \alpha(h_2)h'$$
$$= \sigma\big(b_1, \alpha^{-1}(b_{2(-1)})\alpha^{-2}(h_1) \cdot \beta^{-1}(b'_1)\big)(\beta^2(b_{2(0)})(\alpha(h_{21}) \cdot b'_2)) \sharp \alpha^2(h_{22})h'$$
$$= \sigma\big(b_1, \alpha^{-1}(b_{2(-1)})\alpha^{-2}(h_1) \cdot \beta^{-1}(b'_1)\big)(\beta \otimes \alpha)\big((\beta(b_{2(0)}) \times h_2)(b'_2 \times \alpha^{-1}(h'))\big)$$
$$= \sigma\big(b_1, \alpha^{-1}(b_{2(-1)})\alpha^{-2}(h_1) \cdot \beta^{-1}(b'_1)\big)\varepsilon(b'_{2(-1)}\alpha^{-1}(h'_1))$$
$$(\beta \otimes \alpha)((\beta(b_{2(0)}) \times h_2)(\beta(b'_{2(0)}) \times h'_2))$$

$$
\begin{aligned}
&= \bar{\sigma}\big(b_1 \times b_{2(-1)}\alpha^{-1}(h_1), b'_1 \times b'_{2(-1)}\alpha^{-1}(h'_1)\big) \\
&\quad (\beta \otimes \alpha)((\beta(b_{2(0)}) \times h_2)(\beta(b'_{2(0)}) \times h'_2)) \\
&= (b \times h) \cdot_{\bar{\sigma}} (b' \times h'),
\end{aligned}
$$

for any $b, b' \in {}_\sigma B$ and $h, h' \in H$. So from Proposition 5.2 and 3.13, we deduce that $\bar{\sigma}$ is a left 2-cocycle on $(B{}^{\times}_{\sharp}H, \beta \otimes \alpha)$ and certainly $({}_{\bar{\sigma}}(B{}^{\times}_{\sharp}H), \beta \otimes \alpha) = ({}_\sigma B \sharp H, \beta \otimes \alpha)$ as monoidal Hom-algebras.

Moreover, if $({}_{\bar{\sigma}}(B{}^{\times}_{\sharp}H), \beta \otimes \alpha) = ({}_\sigma B \sharp H, \beta \otimes \alpha)$ as monoidal Hom-algebras, the uniqueness of $\bar{\sigma}$ follows easily by applying $\varepsilon_B \otimes \varepsilon_H$ to the Hom-multiplications on $({}_{\bar{\sigma}}(B{}^{\times}_{\sharp}H), \beta \otimes \alpha)$ and $({}_\sigma B \sharp H, \beta \otimes \alpha)$. ∎

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