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REFLEXIVITY OF TOEPLITZ OPERATORS IN MULTIPLY CONNECTED REGIONS

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Abstract. Subspaces of Toeplitz operators on the Hardy spaces over a multiply connected region in the complex plane are investigated. A universal covering map of such a region and the group of automorphisms invariant with respect to the covering map connect the Hardy space on this multiply connected region with a certain subspace of the classical Hardy space on the disc. We also present some connections of Toeplitz operators on both spaces from the reflexivity point of view.

1. Introduction. Reflexivity and transitivity are two notions connected with invariant subspaces of operators. In [16] the reflexivity of the algebra of all analytic Toeplitz operators on the Hardy space on the unit disc was proved. Toeplitz operators on the Bergman space were investigated in this respect in [8]. The dichotomic behavior (transitivity or reflexivity) of subspaces of Toeplitz operators on the Hardy space on the unit disc was shown in [2]. Precise conditions for verifying the dichotomy were also given. This completely characterizes subspaces of Toeplitz operators from this point of view. The similar dichotomic behavior of subspaces of Toeplitz operators was proved for the Hardy space on the upper-half plane [12] and on simply connected regions [13]. The results were obtained by transferring the result from the unit disc proved in [2].

The problem of reflexivity and transitivity can also be posed for subspaces and subalgebras of Toeplitz operators on the Hardy spaces on bounded, open and multiply connected region in the complex plane whose boundary consists of a finite number of nonintersecting analytic Jordan curves. Using the universal covering map of such a region and the group of automorphisms invariant with respect to the covering map, one can connect the Hardy space on this multiply connected region with some subspace of the classical Hardy space on the disc. The connections of Toeplitz operators on both spaces from the reflexivity point of view are also investigated in this paper.

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2. Preliminaries. Let X_*, Y_* be Banach spaces and let X, Y be their dual spaces with dual actions $\langle \cdot, \cdot \rangle$. The *preannihilator* of $\mathcal{S} \subset X$ is the set

$$\mathcal{S}_\perp = \{x_* \in X_* : \langle x, x_* \rangle = 0 \text{ for all } x \in \mathcal{S}\}.$$

Recall that if $T\colon X \to Y$ is a weak* continuous, bounded linear transformation, then there exists (see [3, Proposition 2.5]) a bounded linear transformation $T_*\colon Y_* \to X_*$ such that

$$\langle x, T_* y_* \rangle = \langle Tx, y_* \rangle \quad \text{for all } x \in X,\ y_* \in Y_*. \tag{2.1}$$

For $1 \le p \le \infty$ let $L^p(X)$ denote the L^p space for a σ-finite measure space $(X, \mathcal{B}, \mu)$. It is well known (see for instance [4, p. 375]) that $L^\infty(X)$ is the dual space of $L^1(X)$ and an isometric isomorphism between $L^\infty(X)$ and $L^1(X)^*$ is given by

$$L\colon L^\infty(X) \ni \varphi \mapsto L_\varphi \in L^1(X)^*, \tag{2.2}$$

where

$$L_\varphi(f) = \langle \varphi, f \rangle := \int_X \varphi f \, d\mu \tag{2.3}$$

for $\varphi \in L^\infty(X)$ and $f \in L^1(X)$. This allows us to identify $L^\infty(X)_*$ with $L^1(X)$.

Let $\mathcal{H}$ be a complex Hilbert space with inner product denoted by $\langle \cdot, \cdot \rangle$. Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators acting on $\mathcal{H}$ and let $\mathcal{S}$ be a subspace of $\mathcal{B}(\mathcal{H})$. The *reflexive closure* of $\mathcal{S}$ is defined as

$$\operatorname{ref} \mathcal{S} := \{B \in \mathcal{B}(\mathcal{H}) : Bh \in [\mathcal{S}h] \text{ for all } h \in \mathcal{H}\},$$

where $[\mathcal{S}h]$ denotes the norm-closed linear span of $\mathcal{S}h = \{Sh : S \in \mathcal{S}\}$. The subspace $\mathcal{S}$ is said to be *reflexive* if $\operatorname{ref} \mathcal{S} = \mathcal{S}$, and *transitive* if $\operatorname{ref} \mathcal{S} = \mathcal{B}(\mathcal{H})$. The reflexive closure of a subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ is the algebra $\operatorname{Alg} \operatorname{Lat} \mathcal{A} = \{B \in \mathcal{B}(\mathcal{H}) : \operatorname{Lat} \mathcal{A} \subset \operatorname{Lat}\{B\}\}$, where $\operatorname{Lat} \mathcal{A}$ denotes the lattice of subspaces that are invariant for all operators from $\mathcal{A}$.

We denote by $\tau c(\mathcal{H})$ the space of trace class operators. It is well known that $\mathcal{B}(\mathcal{H})_* = \tau c(\mathcal{H})$ with the dual action

$$\langle A, T \rangle := \operatorname{tr}(AT), \qquad A \in \mathcal{B}(\mathcal{H}),\ T \in \tau c(\mathcal{H}).$$

Denote by $F_k(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ the set of all operators of rank at most k. Every rank-one operator can be written as $g \otimes h$ for some $g, h \in \mathcal{H}$, where $(g \otimes h)f = \langle f, h \rangle g$ for $f \in \mathcal{H}$.

3. Hardy spaces. We will denote by $\mathbb{D}$ the open unit disc and by $\mathbb{T}$ the unit circle in the complex plane $\mathbb{C}$. Let $L^p(\mathbb{T}) := L^p(\mathbb{T}, m)$, where m is the normalized Lebesgue measure on $\mathbb{T}$. For $1 \le p < \infty$ define the *Hardy space*

$H^p(\mathbb{D})$ to be the set of functions f analytic on $\mathbb{D}$ such that

$$\|f\|_{H^p(\mathbb{D})} := \sup_{0\le r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^p\,d\theta\right)^{1/p} < \infty.$$

The space $H^\infty(\mathbb{D})$ consists of all bounded and analytic functions on $\mathbb{D}$ with supremum norm. It is known that $H^p(\mathbb{D})$ can be seen as a certain closed (weak* closed for $p=\infty$) subspace of $L^p(\mathbb{T})$.

Throughout the paper, $\Omega\subset\mathbb{C}$ will denote a bounded, open and multiply connected region whose boundary consists of a finite number of nonintersecting analytic Jordan curves. Fix $a\in\Omega$ and let ω_a denote the harmonic measure on $\partial\Omega$ for the point a. By [10, Theorem 1.6.1] if $a,b\in\Omega$, then the harmonic measures ω_a and ω_b are boundedly mutually absolutely continuous. Hence without loss of generality we can assume that $L^p(\partial\Omega) := L^p(\partial\Omega,\omega_a)$. We will consider the Hardy space on Ω defined as follows.

Definition 3.1. For $1\le p<\infty$ define $H^p(\Omega)$ as the set of all analytic functions $F\colon\Omega\to\mathbb{C}$ such that there exists a function u harmonic on Ω and

$$|F(z)|^p\le u(z),\qquad z\in\Omega.$$

By $H^\infty(\Omega)$ we denote the space of all bounded analytic functions on Ω.

Following [14] let us recall that $H^p(\Omega)$ is a Banach space with norm

$$\|F\|_{H^p(\Omega)} := \begin{cases}(u_F(a))^{1/p} & \text{for } 1\le p<\infty,\\ \sup\limits_{z\in\Omega}|F(z)| & \text{for } p=\infty,\end{cases}$$

where u_F is the least harmonic majorant of $|F|^p$. Furthermore, $H^2(\Omega)$ is a Hilbert space with reproducing kernel. Thanks to [14, Theorem 3.2], we can also think of $H^p(\Omega)$ as a certain closed subspace of $L^p(\partial\Omega)$. Recall that if $\Omega=\mathbb{D}$ and $a=0$ then the above definition of $H^p(\mathbb{D})$ is equivalent to the classical definition. We refer the reader to [9], [10] and [14] for further information on $H^p(\Omega)$.

4. Relationships between Hardy spaces $H^p(\Omega)$ and $H^p(\mathbb{D})$. Recall the definition of covering map and universal covering space.

Definition 4.1. Let X and Ω be topological spaces. A surjective continuous function $\tau\colon X\to\Omega$ is said to be a *covering map* if for every $a\in\Omega$ there is a neighbourhood U_a such that $\tau^{-1}(U_a)$ is a disjoint union of open sets of X each of which is mapped by τ homeomorphically onto U_a.

If $\tau\colon X\to\Omega$ is a covering map and X is simply connected, then we say that the pair (X,τ) is a *universal covering space* of Ω, and

$$\operatorname{Aut}(X,\tau) := \{T\colon X\to X : T \text{ a homeomorphism}, \ \tau\circ T=\tau\}$$

is the *group of automorphisms* of (X,τ).

Following [7] and [15] we give some examples of universal covering spaces.

EXAMPLE 4.2. Let $\Omega := \{z \in \mathbb{C} : 1 < |z| < e^{\pi}\}$. If

$$\tau(z) := \exp\left(i \log \frac{1+z}{1-z} + \frac{\pi}{2}\right), \qquad z \in \mathbb{D},$$

then $(\mathbb{D}, \tau)$ is a universal covering space of the annulus Ω.

EXAMPLE 4.3. Let $\operatorname{Arg} z$ denote the principal branch of the argument of $z \in \mathbb{C}$. Let $r_0 \in (0,1)$ and

$$\Omega_{r_0} := \{z \in \mathbb{C} : r_0 < \operatorname{Re} z < 1,\ \operatorname{Im} z \in \mathbb{R}\}.$$

The function $g\colon \mathbb{D} \to \Omega_{r_0}$ given by

$$g(z) := \sqrt{r_0}\, \exp\left(-\frac{1}{\pi} \log r_0 \operatorname{Arg}\left(\frac{1+z}{1-z}\right)\right) + \frac{i}{\pi} \log r_0 \log\left|\frac{1+z}{1-z}\right|, \qquad z \in \mathbb{D},$$

is an analytic bijection. Define $h(x+iy) := xe^{iy}$, where $x, y \in \mathbb{R}$.

If $\tau := h \circ g$, then $(\mathbb{D}, \tau)$ is a universal covering space of the annulus $\Omega = \{z \in \mathbb{C} : r_0 < |z| < 1\}$.

Now, we will assume that $\Omega \subset \mathbb{C}$ is an open, bounded and multiply connected region whose boundary consists of a finite number of nonintersecting analytic Jordan curves. By [7, Theorem XVI.5.1] there is a unique universal covering space $(\mathbb{D}, \tau)$ of Ω such that the covering map τ is analytic and $\tau(0) = a$, $\tau'(0) > 0$. Henceforth, we reserve the symbol τ for such a covering map. Note (see [1, Theorem 1.8]) that there is an open subset $\mathbb{T}_0$ of the unit circle $\mathbb{T}$ such that $m(\mathbb{T} \setminus \mathbb{T}_0) = 0$ and we can extend τ to an analytic surjection from $\mathbb{T}_0$ onto $\partial\Omega$.

Let us remark (see [7, Proposition 2.1] and [6, Theorem VI.2.5]) that the group of automorphisms of $(\mathbb{D}, \tau)$ can be described as follows:

$$\operatorname{Aut}(\mathbb{D}, \tau) = \{T_{b,\theta} : b \in \mathbb{D},\ \theta \in (-\pi, \pi],\ \tau \circ T_{b,\theta} = \tau\}, \tag{4.1}$$

where $T_{b,\theta}$ is the Schwarz map given by $T_{b,\theta}(z) := e^{i\theta} \frac{z-b}{1-\bar{b}z}$, $z \in \mathbb{D}$.

For the rest of the paper we write $\mathcal{G}$ for $\operatorname{Aut}(\mathbb{D}, \tau)$. We denote by $\widehat{\mathcal{G}}$ the *dual grup* of $\mathcal{G}$, i.e. $\widehat{\mathcal{G}} = \{\alpha\colon \mathcal{G} \to \mathbb{T} : \alpha \text{ a homomorphism}\}$. The elements of $\widehat{\mathcal{G}}$ are called *characters*.

For $1 \le p \le \infty$ and a character $\alpha \in \widehat{\mathcal{G}}$, we define

$$L^p_\alpha(\mathbb{T}) := \{f \in L^p(\mathbb{T}) : f \circ T = \alpha(T) f \text{ for all } T \in \mathcal{G}\}, \tag{4.2}$$

$$H^p_\alpha(\mathbb{D}) := \{f \in H^p(\mathbb{D}) : f \circ T = \alpha(T) f \text{ for all } T \in \mathcal{G}\}. \tag{4.3}$$

We say that a function $f \in H^p(\mathbb{D})$ is *modulus automorphic* if $|f \circ T| = |f|$ for all $T \in \mathcal{G}$. Note that if $f \in H^p_\alpha(\mathbb{D})$, then f is modulus automorphic. Conversely, if $f \in H^p(\mathbb{D})$ is modulus automorphic and $f \neq 0$, then $f \in H^p_\alpha(\mathbb{D})$, where the character α is given by $\alpha(T) := (f \circ T)/f$, $T \in \mathcal{G}$. Let $e \in \widehat{\mathcal{G}}$ denote the trivial character, i.e. $e(T) = 1$, $T \in \mathcal{G}$.

REMARK 4.4. If a region Ω is simply connected and $\tau\colon \mathbb{D} \to \Omega$ is the covering map then τ is the Riemann map. Moreover, $L^p_e(\mathbb{T}) = L^p(\mathbb{T})$ and $H^p_e(\mathbb{D}) = H^p(\mathbb{D})$.

From now on, to avoid trivial cases, by a multiply connected region we will mean a non-simply connected one. Let $T \in \mathcal{G}$ and $T(0) = b$. We know that $T^{-1} \in \mathcal{G}$ and T^{-1} is the conformal mapping that maps the unit disc $\mathbb{D}$ onto $\mathbb{D}$ and transforms the unit circle $\mathbb{T}$ to itself. We also know that $m_b := m \circ T^{-1}$ is the harmonic measure on $\mathbb{T}$ for the point b. If $1 \le p \le \infty$, then the operator (see [1, Proposition 1.3]) given by

$$V_p f := f \circ T^{-1}, \qquad f \in L^p(\mathbb{T}), \tag{4.4}$$

is an isometric isomorphism between $L^p(\mathbb{T})$ and $L^p(\mathbb{T}, m_b)$. Moreover, $V_p(H^p(\mathbb{D})) = H^p(\mathbb{D}, m_b)$.

Now we give some properties of $L^p_e(\mathbb{T})$ and $H^p_e(\mathbb{D})$. First note that V_p is the identity operator on $L^p_e(\mathbb{T})$ and on $H^p_e(\mathbb{D})$.

PROPOSITION 4.5. *For $1 \le p < \infty$, $L^p_e(\mathbb{T})$ is a proper closed subspace of $L^p(\mathbb{T})$. Furthermore, $L^\infty_e(\mathbb{T})$ is a proper weak* closed subspace of $L^\infty(\mathbb{T})$.*

Proof. It follows immediately from [7, Corollary XVI.1.6] that $L^p_e(\mathbb{T}) \ne L^p(\mathbb{T})$. Concerning closedness, we will show for example that $L^\infty_e(\mathbb{T})$ is a weak* closed subspace of $L^\infty(\mathbb{T})$. Take a net $\{f_\alpha\} \subset L^\infty_e(\mathbb{T})$ with $f_\alpha \xrightarrow{*} f$. It is obvious that $f \in L^\infty(\mathbb{T})$ and $f_\alpha = f_\alpha \circ T$ for $T \in \mathcal{G}$. Thus we only need to show that $f \circ T = f$ for $T \in \mathcal{G}$. Since V_∞ is a weak* homeomorphism (see for example [13, Theorem 4.2]), we have $V_\infty f = f$. It follows that for all $T \in \mathcal{G}$,

$$f = f \circ T^{-1} \circ T = V_\infty f \circ T = f \circ T. \ \blacksquare$$

Similarly we obtain

PROPOSITION 4.6. *For $1 \le p < \infty$, $H^p_e(\mathbb{D})$ is a proper closed subspace of $H^p(\mathbb{D})$. Furthermore, $H^\infty_e(\mathbb{D})$ is a proper weak* closed subspace of $H^\infty(\mathbb{D})$.*

The following lemmas will be crucial in further considerations.

LEMMA 4.7. *If $P_{H^2(\mathbb{D})}\colon L^2(\mathbb{T}) \to H^2(\mathbb{D})$ is the orthogonal projection, then*

$$P_{H^2(\mathbb{D})}(L^2_e(\mathbb{T})) = H^2_e(\mathbb{D}). \tag{4.5}$$

Proof. We only need to show that $P_{H^2(\mathbb{D})}(L^2_e(\mathbb{T})) \subset H^2_e(\mathbb{D})$. Let $f \in L^2_e(\mathbb{T})$. Since $L^2_e(\mathbb{T}) \subset L^2(\mathbb{T})$, it follows that there are $g \in H^2(\mathbb{D})$ and $h \in L^2(\mathbb{T}) \ominus H^2(\mathbb{D})$ such that $f = g + h$. Then $f = f \circ T = g \circ T + h \circ T$ for $T \in \mathcal{G}$. Since $g \in H^2(\mathbb{D})$, we have $g \circ T \in H^2(\mathbb{D})$. Now, we will show that $h \circ T \in L^2(\mathbb{T}) \ominus H^2(\mathbb{D})$. Take $u \in H^2(\mathbb{D})$. As remarked earlier, V_2 defined by (4.4) is an isometry. Hence

$$\langle u, h\circ T\rangle = \langle V_2 u, V_2(h\circ T)\rangle = \langle u\circ T^{-1}, h\rangle = 0,$$

since $u\circ T^{-1}\in H^2(\mathbb{D})$. Thus $h\circ T\in L^2(\mathbb{T})\ominus H^2(\mathbb{D})$ and

$$g = P_{H^2(\mathbb{D})} f = P_{H^2(\mathbb{D})}(f\circ T) = g\circ T,$$

so $g\in H^2_e(\mathbb{D})$. ■

LEMMA 4.8. *If $\varphi\in H^\infty_e(\mathbb{D})$ and $g\in H^2_e(\mathbb{D})$, then $\varphi g\in H^2_e(\mathbb{D})$. Conversely, if $\varphi\in L^\infty_e(\mathbb{T})$ and $\varphi g\in H^2_e(\mathbb{D})$ for all $g\in H^2_e(\mathbb{D})$, then $\varphi\in H^\infty_e(\mathbb{D})$.*

Proof. The first assertion follows directly from [5, Proposition 24.8]. For the converse, note that $1\in H^2_e(\mathbb{D})$, since $H^2(\mathbb{D})\ni 1 = 1\circ T$ for all $T\in\mathcal{G}$. If $\varphi\in L^\infty_e(\mathbb{T})$ then $\varphi=\varphi 1\in H^2_e(\mathbb{D})$ and finally $\varphi\in H^\infty_e(\mathbb{D})$. ■

Notice that $H^p_e(\mathbb{D})$ can be identified with a subspace of $L^p_e(\mathbb{T})$. The following lemma gives a relation between $L^p(\partial\Omega)$ and $L^p_e(\mathbb{T})$ and also between $H^p(\Omega)$ and $H^p_e(\mathbb{D})$ (see [1, Lemmas 1.11 and 1.10]).

LEMMA 4.9. *Let $1\le p\le\infty$. The operator $U_p\colon L^p(\partial\Omega)\to L^p_e(\mathbb{T})$ given by*

$$U_p(F) := F\circ\tau,\qquad F\in L^p(\partial\Omega), \tag{4.6}$$

is an isometric isomorphism and

$$U_p(H^p(\Omega)) = H^p_e(\mathbb{D}). \tag{4.7}$$

The next lemma is an immediate consequence of Lemmas 4.8 and 4.9.

LEMMA 4.10. *If $\Phi\in H^\infty(\Omega)$ and $F\in H^2(\Omega)$, then $\Phi F\in H^2(\Omega)$. Conversely, if $\Phi\in L^\infty(\partial\Omega)$ and $\Phi F\in H^2(\Omega)$ for all $F\in H^2(\Omega)$, then $\Phi\in H^\infty(\Omega)$.*

COROLLARY 4.11. *$H^\infty(\Omega)$ is a subalgebra of $L^\infty(\partial\Omega)$.*

We will need a relationship between the harmonic measure ω_a on $\partial\Omega$ and the Lebesgue measure on $\mathbb{T}$ (see for instance [10]):

$$\int_{\partial\Omega} F\,d\omega_a = \int_{\mathbb{T}} (F\circ\tau)\,dm,\qquad F\in L^1(\partial\Omega). \tag{4.8}$$

The following theorem will be useful in several contexts.

THEOREM 4.12. *If $U_1\colon L^1(\partial\Omega)\to L^1_e(\mathbb{T})$ is given by $U_1F := F\circ\tau$, and $U_\infty\colon L^\infty(\partial\Omega)\to L^\infty_e(\mathbb{T})$ is given by $U_\infty\Phi := \Phi\circ\tau$, then*

(a) *$\langle\Phi, F\rangle = \langle U_\infty\Phi, U_1F\rangle$ for all $\Phi\in L^\infty(\partial\Omega)$ and $F\in L^1(\partial\Omega)$,*
(b) *the dual space of $L^1_e(\mathbb{T})$ is $L^\infty_e(\mathbb{T})$,*
(c) *U_∞ is a weak* homeomorphism.*

Proof. By (4.8), for any $\Phi \in L^\infty(\partial\Omega)$ and $F \in L^1(\partial\Omega)$, we obtain

$$\langle U_\infty\Phi, U_1F\rangle = \int_{\mathbb{T}}(U_\infty\Phi)(U_1F)\,dm = \int_{\mathbb{T}}(\Phi F)\circ\tau\,dm = \int_{\partial\Omega}\Phi F\,d\omega_a = \langle\Phi, F\rangle,$$

which proves (a).

To see (b), let

$$\widetilde{L}\colon L_e^\infty(\mathbb{T}) \ni \varphi \mapsto \widetilde{L}_\varphi \in L_e^1(\mathbb{T})^*, \tag{4.9}$$

where $\widetilde{L}_\varphi$ is the restriction of L_φ given by (2.3) to $L_e^1(\mathbb{T})$. Since L, defined by (2.2), is an isometry, so is $\widetilde{L}$. We show that $\widetilde{L}(L_e^\infty(\mathbb{T})) = L_e^1(\mathbb{T})^*$. Let $\Lambda \in L_e^1(\mathbb{T})^*$. By the definition of the dual operator U_1^* we have $U_1^*(\Lambda) = \Lambda\circ U_1 \in L^1(\partial\Omega)^*$. Using (2.2) and (2.3) we find that there is $\Phi \in L^\infty(\partial\Omega)$ such that $(\Lambda\circ U_1)F = \langle\Phi, F\rangle$ for all $F \in L^1(\partial\Omega)$. Let $\varphi := U_\infty\Phi$ and $F := U_1^{-1}f$ for $f \in L_e^1(\mathbb{T})$. Now, (a) gives

$$\Lambda(f) = (\Lambda\circ U_1)(F) = \langle\Phi, F\rangle = \langle U_\infty\Phi, U_1F\rangle = \langle\varphi, f\rangle,$$

which proves the surjectivity of $\widetilde{L}$ and completes the proof of (b).

Thanks to (a) we see that U_∞ is weak* continuous by [3, Proposition 2.5]. Hence, by [3, Theorem 2.7], we obtain (c). ∎

5. Toeplitz operators on $H^2(\mathbb{D})$ with $L_e^\infty(\mathbb{T})$ symbols. Recall that for every $\varphi \in L^\infty(\mathbb{T})$, the *Toeplitz operator* on $H^2(\mathbb{D})$ with symbol φ is the operator T_φ defined by $T_\varphi f := P_{H^2(\mathbb{D})}(\varphi f)$ for $f \in H^2(\mathbb{D})$, where $P_{H^2(\mathbb{D})}$ is the orthogonal projection of $L^2(\mathbb{T})$ onto $H^2(\mathbb{D})$. If φ is an element of $H^\infty(\mathbb{D})$, then T_φ is called an *analytic Toeplitz operator*. We denote by $\mathcal{T}(\mathbb{D})$ the space of all Toeplitz operators and by $\mathcal{A}(\mathbb{D})$ the algebra of all analytic Toeplitz operators on $H^2(\mathbb{D})$. The mapping $\xi\colon L^\infty(\mathbb{T}) \to \mathcal{B}(H^2(\mathbb{D}))$ given by $\xi(\varphi) := T_\varphi$ is said to be the *symbol map* of Toeplitz operators on $H^2(\mathbb{D})$. It is known (see [2, Corollary 2.3]) that ξ is a weak* homeomorphism of $L^\infty(\mathbb{T})$ onto $\mathcal{T}(\mathbb{D})$.

In [16] it was shown that the algebra $\mathcal{A}(\mathbb{D})$ is reflexive and in [2] it was proved that $\mathcal{T}(\mathbb{D})$ is transitive. Moreover, from [2, Theorem 1.1′], we know that every weak* closed subspace of $\mathcal{T}(\mathbb{D})$ is either reflexive or transitive and a precise condition for verifying this dichotomy is given.

We will need the following definition.

Definition 5.1. A weak* closed subspace $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$ has *property* $\mathbb{A}_1$ (resp. *property* $\mathbb{A}_1(1)$) if for every weak* continuous functional ϕ on $\mathcal{S}$ (and for every $\varepsilon > 0$, respectively) there are vectors $g, h \in \mathcal{H}$ such that $\phi(A) = \langle Ag, h\rangle$ for all $A \in \mathcal{S}$ (and, in addition, $\|g\|\,\|h\| < (1+\varepsilon)\|\phi\|$).

A reflexive subspace $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$ is said to be *hereditarily reflexive* if every weak* closed subspace of $\mathcal{S}$ is reflexive. In [11] it was pointed out that a reflexive subspace $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$ is hereditarily reflexive if and only if $\mathcal{S}$ has property $\mathbb{A}_1$. Hence $\mathcal{A}(\mathbb{D})$ is hereditarily reflexive, because it has property $\mathbb{A}_1(1)$ (see [5, Proposition 60.5]).

We now consider some properties of Toeplitz operators on $H^2(\mathbb{D})$ with symbols in $L_e^\infty(\mathbb{T})$. Let

$$\mathcal{T}_e(\mathbb{D}) := \xi(L_e^\infty(\mathbb{T})). \tag{5.1}$$

Then $\mathcal{T}_e(\mathbb{D})$ is a weak* closed subspace of $\mathcal{T}(\mathbb{D})$ by Proposition 4.5. Now we will prove that $\mathcal{T}_e(\mathbb{D})$ is reflexive.

THEOREM 5.2.

(a) *The subspace $\mathcal{T}_e(\mathbb{D})$ is reflexive.*
(b) *If $\mathcal{B}$ is a proper weak* closed subspace of $\mathcal{T}_e(\mathbb{D})$, then $\mathcal{B}$ is reflexive. Moreover, there is a function $f \in L_e^1(\mathbb{T})$ such that $\log|f| \in L_e^1(\mathbb{T})$ and $\int_{\mathbb{T}} \varphi f\, dm = 0$ for all $T_\varphi \in \mathcal{B}$.*

Proof. Proposition 4.6 says that $H_e^2(\mathbb{D}) \neq H^2(\mathbb{D})$. Take $g \in H_e^2(\mathbb{D})$ and $h \in H^2(\mathbb{D}) \ominus H_e^2(\mathbb{D})$. Then, by (4.5), for $T_\varphi \in \mathcal{T}_e(\mathbb{D})$ we get

$$\langle T_\varphi, g \otimes h\rangle = \langle T_\varphi g, h\rangle = \langle P_{H^2(\mathbb{D})}(\varphi g), h\rangle = 0.$$

Thus the rank-one operator $g \otimes h$ is in $\mathcal{T}_e(\mathbb{D})_\perp$, so $\mathcal{T}_e(\mathbb{D})$ is not transitive. Hence, by [2, Theorem 1.1′], it is reflexive.

In the same way we prove that $\mathcal{B}$ is reflexive. Then, by condition (2) of [2, Theorem 1.1′], there is $f \in L^1(\mathbb{T})$ such that $\log|f| \in L^1(\mathbb{T})$ and $\int_{\mathbb{T}} \varphi f\, dm = 0$ for all $T_\varphi \in \mathcal{B}$. Since $\mathcal{B}$ is a subspace of $\mathcal{T}_e(\mathbb{D})$, it follows that $\xi^{-1}(\mathcal{B}) \subset L_e^\infty(\mathbb{T})$. Thus $\xi^{-1}(\mathcal{B})_\perp \subset L_e^1(\mathbb{T})$, because $L_e^1(\mathbb{T})^* = L_e^\infty(\mathbb{T})$. This implies that $f \in L_e^1(\mathbb{T})$ and finally $\log|f| \in L_e^1(\mathbb{T})$. ∎

The next corollary is an easy consequence of the previous theorem and [11].

COROLLARY 5.3. *The subspace $\mathcal{T}_e(\mathbb{D})$ has property $\mathbb{A}_1$.*

Using Theorem 5.2 we get examples of reflexive subspaces consisting of Toeplitz operators on $H^2(\mathbb{D})$.

PROPOSITION 5.4. *Let $\Omega \subset \mathbb{C}$ be a bounded, open and multiply connected region whose boundary consists of a finite number of nonintersecting analytic Jordan curves. If $(\mathbb{D}, \tau)$ is a universal covering space of Ω, then the subspace*

$$\mathcal{B}_\Omega := \{T_{\Phi\circ\tau} : \Phi \in L^\infty(\partial\Omega)\} \subset \mathcal{T}(\mathbb{D})$$

is reflexive.

Proof. This follows immediately from Lemma 4.9 and Theorem 5.2. ∎

Combining the above proposition with Examples 4.3 and 4.2 we obtain

EXAMPLE 5.5. If $\Omega_1 := \{z \in \mathbb{C} : 1 < |z| < e^\pi\}$, then the subspace

$$\mathcal{B}_{\Omega_1} := \Big\{T_{\Phi\circ\tau} : \tau(z) = \exp\big(i \log \tfrac{1+z}{1-z} + \tfrac{\pi}{2}\big),\ z \in \mathbb{D},\ \Phi \in L^\infty(\partial\Omega_1)\Big\}$$

is reflexive.

EXAMPLE 5.6. If $\Omega_2 := \{z \in \mathbb{C} : 1/2 < |z| < 1\}$, then the subspace $\mathcal{B}_{\Omega_2} := \big\{T_{\Phi\circ\tau} : \tau(z) = \exp\big(\frac{i}{\pi}\log\frac{1}{2}\operatorname{Log}\frac{1+z}{1-z} + \frac{1}{2}\log\frac{1}{2}\big),\ z \in \mathbb{D},\ \Phi \in L^\infty(\partial\Omega_2)\big\}$ is reflexive.

6. Toeplitz operators on $H^2(\Omega)$. Recall that Ω denotes a bounded, open and multiply connected region whose boundary consists of a finite number of nonintersecting analytic Jordan curves.

DEFINITION 6.1. Let $\Phi \in L^\infty(\partial\Omega)$. The operator $M_\Phi\colon L^2(\partial\Omega) \to L^2(\partial\Omega)$ defined by $M_\Phi F := \Phi F$ is called a *multiplication operator* on $L^2(\partial\Omega)$.

A standard reference concerning Toeplitz operators on the Hardy space on Ω is [1].

DEFINITION 6.2. For every $\Phi \in L^\infty(\partial\Omega)$, the *Toeplitz operator* on $H^2(\Omega)$ with symbol Φ is the operator T_Φ defined by

$$T_\Phi := P_{H^2(\Omega)} M_{\Phi|_{H^2(\Omega)}},$$

where $P_{H^2(\Omega)}$ is the orthogonal projection of $L^2(\partial\Omega)$ onto $H^2(\Omega)$. If Φ is in $H^\infty(\Omega)$, then T_Φ is called an *analytic Toeplitz operator*.

Using Lemma 4.10 it follows that if T_Φ is an analytic Toeplitz operator, then $T_\Phi F = \Phi F$ for all $F \in H^2(\Omega)$. Notice also that if $\Phi \in L^\infty(\partial\Omega)$ and $F, G \in H^2(\Omega)$, then

$$\langle T_\Phi F, G\rangle = \langle P_{H^2(\Omega)}(\Phi F), G\rangle = \langle \Phi F, G\rangle = \int_{\partial\Omega} \Phi F \overline{G}\, d\omega_a. \tag{6.1}$$

The next lemma is an easy application of the above equality.

LEMMA 6.3. *If $\Phi \in L^\infty(\partial\Omega)$ and $G \in H^\infty(\Omega)$, then*

(a) $T_\Phi^* = T_{\overline{\Phi}}$,
(b) $T_\Phi T_G = T_{\Phi G}$ *and* $T_{\overline{G}} T_\Phi = T_{\overline{G}\Phi}$.

Let $\mathcal{T}(\Omega)$ stand, as usual, for the space of all Toeplitz operators and let $\mathcal{A}(\Omega)$ be the space of all analytic Toeplitz operators on $H^2(\Omega)$. Observe that the previous lemma and Lemma 4.10 imply that $\mathcal{A}(\Omega)$ is an algebra.

DEFINITION 6.4. The *symbol map* of Toeplitz operators on $H^2(\Omega)$ is the function $\eta\colon L^\infty(\partial\Omega) \to \mathcal{T}(\Omega) \subset \mathcal{B}(H^2(\Omega))$ defined by $\eta(\Phi) := T_\Phi$.

LEMMA 6.5. *If η is the symbol map of Toeplitz operators on $H^2(\Omega)$, then*

(a) *η is a weak* homeomorphism from $L^\infty(\partial\Omega)$ onto $\mathcal{T}(\Omega)$,*
(b) *$\mathcal{T}(\Omega)$ is a weak* closed subspace of $\mathcal{B}(H^2(\Omega))$,*
(c) $\mathcal{T}(\Omega)_* = \tau c(H^2(\Omega))/\mathcal{T}(\Omega)_\perp$,
(d) *$\eta_*\colon \mathcal{T}(\Omega)_* \to L^1(\partial\Omega)$ is a weak homeomorphism and*

$$\langle T_\Phi, \eta_*^{-1}(F)\rangle = \langle \Phi, F\rangle \quad \text{for } \Phi \in L^\infty(\partial\Omega),\ F \in L^1(\partial\Omega). \tag{6.2}$$

Proof. From [1, Theorem 2.11] we know that η is an isometry. Let $\{\Phi_\alpha\} \subset L^\infty(\partial\Omega)$ be a net with $\Phi_\alpha \xrightarrow{*} \Phi \in L^\infty(\partial\Omega)$. Let $T \in \tau c(H^2(\Omega))$. By [5, Theorem 18.13] we have $T = \sum_n F_n \otimes G_n$, where $\{F_n\}$ and $\{G_n\}$ are square summable sequences of vectors from $H^2(\Omega)$. Applying [5, Proposition 20.4] we see that $\sum_n F_n \overline{G}_n \in L^1(\partial\Omega)$. Hence, by (6.1),

$$\begin{aligned}\operatorname{tr}(T_{\Phi_\alpha}T) &= \sum_n \langle T_{\Phi_\alpha}F_n, G_n\rangle = \int_{\partial\Omega} \Phi_\alpha \sum_n F_n\overline{G}_n\, d\omega_a \\ &\xrightarrow{\alpha} \int_{\partial\Omega} \Phi \sum_n F_n\overline{G}_n\, d\omega_a = \sum_n \langle T_\Phi F_n, G_n\rangle = \operatorname{tr}(T_\Phi T).\end{aligned}$$

Therefore η is weak* continuous, which implies (a) and (b) by [3, Theorem 2.7].

Condition (c) is obvious from (b). Thanks to (a) and (2.1), for any $\Phi \in L^\infty(\partial\Omega)$ and $F \in L^1(\partial\Omega)$ we get

$$\langle T_\Phi, \eta_*^{-1}(F)\rangle = \langle \eta(\Phi), \eta_*^{-1}(F)\rangle = \langle \Phi, \eta_*\eta_*^{-1}(F)\rangle = \langle \Phi, F\rangle. \ \blacksquare$$

7. The algebra $\mathcal{B}(H_e^2(\mathbb{D}))$ and Toeplitz operators. Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces and let $\mathcal{S}_1 \subset \mathcal{B}(\mathcal{H}_1)$, $\mathcal{S}_2 \subset \mathcal{B}(\mathcal{H}_2)$ be some subspaces. We say that $\mathcal{S}_1$ and $\mathcal{S}_2$ are *spatially isomorphic* (*or unitarily equivalent*) if there is an isomorphism $U\colon \mathcal{H}_1 \to \mathcal{H}_2$ such that $U\mathcal{S}_1U^{-1} = \mathcal{S}_2$. For the following lemma, see for example [13, Lemma 2.1].

LEMMA 7.1. *If an operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ is an isometric isomorphism and if $\widetilde{U}$ is defined by $\widetilde{U}(A) := UAU^{-1}$, $A \in \mathcal{B}(\mathcal{H}_1)$, then*

(a) $\widetilde{U}\colon \mathcal{B}(\mathcal{H}_1) \to \mathcal{B}(\mathcal{H}_2)$ *is a weak* homeomorphism,*
(b) $\operatorname{ref}(\widetilde{U}(\mathcal{S})) = \widetilde{U}(\operatorname{ref}\mathcal{S})$ *for* $\mathcal{S} \subset \mathcal{B}(\mathcal{H}_1)$,
(c) $\mathcal{S} \subset \mathcal{B}(\mathcal{H}_1)$ *is reflexive* (*respectively, transitive*) *if and only if* $\widetilde{U}(\mathcal{S})$ *is reflexive* (*respectively, transitive*).

Notice that the spaces $\mathcal{T}(\Omega)$ and $\mathcal{T}_e(\mathbb{D})$ are not spatially isomorphic, since $H^2(\Omega)$ and $H^2(\mathbb{D})$ are not isomorphic (see Lemma 4.9). On the other hand, the next proposition says that $\mathcal{T}(\Omega)$ and $\mathcal{T}_e(\mathbb{D})$ are isomorphic.

PROPOSITION 7.2. *Let $\xi : L^\infty(\mathbb{T}) \to \mathcal{T}(\mathbb{D})$, $\xi(\varphi) := T_\varphi$, and $\eta : L^\infty(\partial\Omega) \to \mathcal{T}(\Omega)$, $\eta(\Phi) := T_\Phi$, be the symbol maps of Toeplitz operators on $H^2(\mathbb{D})$ and on $H^2(\Omega)$, respectively. Let $U_\infty\colon L^\infty(\partial\Omega) \to L_e^\infty(\mathbb{T})$ be given by $U_\infty\Phi := \Phi\circ\tau$. Then there exists a weak* homeomorphism U_e from $\mathcal{T}(\Omega)$ onto $\mathcal{T}_e(\mathbb{D})$ such that the following diagram commutes:*

$$\begin{array}{ccc} L^\infty(\partial\Omega) & \xrightarrow{\ \eta\ } & \mathcal{T}(\Omega) \\ {\scriptstyle U_\infty}\downarrow & & \downarrow{\scriptstyle U_e} \\ L_e^\infty(\mathbb{T}) & \xrightarrow{\ \xi\ } & \mathcal{T}_e(\mathbb{D}) \end{array} \tag{7.1}$$

Proof. Define
$$U_e := \xi \circ U_\infty \circ \eta^{-1}$$
and apply Theorem 4.12, equality (5.1) and Lemma 6.5. ∎

Let us consider a new set of operators on $H^2_e(\mathbb{D})$. Take $\varphi \in L^\infty_e(\mathbb{T})$ and a Toeplitz operator $T_\varphi \in \mathcal{T}_e(\mathbb{D})$. Define $\widetilde{T}_\varphi$ to be the restriction of T_φ to $H^2_e(\mathbb{D})$. By (4.5) we see that $\widetilde{T}_\varphi \colon H^2_e(\mathbb{D}) \to H^2_e(\mathbb{D})$ and

$$\widetilde{T}_\varphi f = P_{H^2(\mathbb{D})}(\varphi f), \qquad f \in H^2_e(\mathbb{D}). \tag{7.2}$$

For $\varphi \in H^\infty_e(\mathbb{D})$, applying Lemma 4.8, we obtain

$$\widetilde{T}_\varphi f = \varphi f, \qquad f \in H^2_e(\mathbb{D}). \tag{7.3}$$

Set

$$\widetilde{\mathcal{T}}_e(\mathbb{D}) := \{\widetilde{T}_\varphi : \varphi \in L^\infty_e(\mathbb{T})\} \subset \mathcal{B}(H^2_e(\mathbb{D})), \tag{7.4}$$

$$\widetilde{\mathcal{A}}_e(\mathbb{D}) := \{\widetilde{T}_\varphi : \varphi \in H^\infty_e(\mathbb{D})\} \subset \widetilde{\mathcal{T}}_e(\mathbb{D}). \tag{7.5}$$

We now introduce multiplication operators on $L^2_e(\mathbb{T})$. First, note that if $\varphi \in L^\infty_e(\mathbb{T})$ and $f \in L^2_e(\mathbb{T})$, then $\varphi f \in L^2_e(\mathbb{T})$.

Definition 7.3. Let $\varphi \in L^\infty_e(\mathbb{T})$. The operator $\widetilde{M}_\varphi \colon L^2_e(\mathbb{T}) \to L^2_e(\mathbb{T})$ defined by $\widetilde{M}_\varphi f := \varphi f$ is called a *multiplication operator* on $L^2_e(\mathbb{T})$.

Remark 7.4. If $\varphi \in L^\infty_e(\mathbb{T})$ and $f \in H^2_e(\mathbb{D})$, then $\varphi f \in L^2_e(\mathbb{D})$. Moreover,

$$\widetilde{T}_\varphi = P_{H^2(\mathbb{D})} \widetilde{M}_\varphi|_{H^2_e(\mathbb{D})}. \tag{7.6}$$

Lemma 4.9 says that the operator $U_2 \colon L^2(\partial\Omega) \to L^2_e(\mathbb{T})$ given by $U_2 F := F \circ \tau$ is an isomorphism. Hence we get a relation between multiplication operators on $L^2(\partial\Omega)$ and $L^2_e(\mathbb{T})$.

Lemma 7.5. *If $\Phi \in L^\infty(\partial\Omega)$ and M_Φ is the multiplication operator by Φ on $L^2(\partial\Omega)$, then*

$$U_2 M_\Phi U_2^{-1} = \widetilde{M}_{\Phi\circ\tau} \tag{7.7}$$

is the multiplication operator by $\Phi \circ \tau$ on $L^2_e(\mathbb{T})$.

Proof. Let $F \in L^2(\partial\Omega)$. Then

$$\begin{aligned} \widetilde{M}_{\Phi\circ\tau} U_2(F) &= \widetilde{M}_{\Phi\circ\tau}(F \circ \tau) = (\Phi \circ \tau)(F \circ \tau) = (\Phi F) \circ \tau \\ &= U_2(\Phi F) = U_2 M_\Phi (F). \end{aligned}$$ ∎

Applying Lemma 4.9 again we have $U_2(H^2(\Omega)) = H^2_e(\mathbb{D})$. It is easy to check that $U_2(L^2(\partial\Omega) \ominus H^2(\Omega)) = L^2_e(\mathbb{T}) \ominus H^2_e(\mathbb{D})$ and

$$U_2 P_{H^2(\Omega)} = P_{H^2(\mathbb{D})} U_2. \tag{7.8}$$

The following theorem gives a spatial isomorphism between $\mathcal{T}(\Omega)$ and $\widetilde{\mathcal{T}}_e(\mathbb{D})$.

THEOREM 7.6. *Let $\eta : L^\infty(\partial\Omega) \to \mathcal{T}(\Omega)$, $\eta(\Phi) := T_\Phi$, be the symbol map of Toeplitz operators on $H^2(\Omega)$. If $U_2\colon H^2(\Omega) \to H^2_e(\mathbb{D})$ is given by $U_2F := F \circ \tau$ and*

$$\widetilde{U}_e(B) := U_2BU_2^{-1}, \qquad B \in \mathcal{B}(H^2(\Omega)), \tag{7.9}$$

then

(a) *$U_2T_\Phi U_2^{-1} = \widetilde{T}_{\Phi\circ\tau}$ for all $\Phi \in L^\infty(\partial\Omega)$,*
(b) *$U_2(\mathcal{T}(\Omega))U_2^{-1} = \widetilde{\mathcal{T}}_e(\mathbb{D})$ and $U_2(\mathcal{A}(\Omega))U_2^{-1} = \widetilde{\mathcal{A}}_e(\mathbb{D})$,*
(c) *$\widetilde{U}_e : \mathcal{B}(H^2(\Omega)) \to \mathcal{B}(H^2_e(\mathbb{D}))$ is a weak* homeomorphism,*
(d) *if $\widetilde{\xi}(\varphi) := \widetilde{T}_\varphi$, $\varphi \in L^\infty_e(\mathbb{T})$, then the following diagram commutes:*

$$\begin{array}{ccc} L^\infty(\partial\Omega) & \xrightarrow{\ \eta\ } & \mathcal{T}(\Omega) \\ {\scriptstyle U_\infty}\downarrow & & \downarrow{\scriptstyle \widetilde{U}_e} \\ L^\infty_e(\mathbb{T}) & \xrightarrow{\ \widetilde{\xi}\ } & \widetilde{\mathcal{T}}_e(\mathbb{D}) \end{array} \tag{7.10}$$

(e) *$\widetilde{\xi}\colon L^\infty_e(\mathbb{T}) \to \widetilde{\mathcal{T}}_e(\mathbb{D})$ is a weak* homeomorphism.*

Proof. Let $\Phi \in L^\infty(\partial\Omega)$. Using (7.7), (7.8) and (7.6) we get

$$\begin{aligned} U_2T_\Phi &= U_2P_{H^2(\Omega)}M_\Phi|_{H^2(\Omega)} = P_{H^2(\mathbb{D})}U_2M_\Phi|_{H^2(\Omega)} \\ &= P_{H^2(\mathbb{D})}\widetilde{M}_{\Phi\circ\tau}U_2|_{H^2(\Omega)} = \widetilde{T}_{\Phi\circ\tau}U_2|_{H^2(\Omega)}. \end{aligned}$$

Thus condition (a) holds. Condition (b) is a consequence of (a) and Lemma 4.9. Now (c) follows from (b) by Lemma 7.1. Moreover,

$$(\widetilde{U}_e \circ \eta)(\Phi) = \widetilde{U}_e(T_\Phi) = \widetilde{T}_{\Phi\circ\tau} = \widetilde{\xi}(\Phi \circ \tau) = (\widetilde{\xi} \circ U_\infty)(\Phi),$$

which implies (d).

Theorem 4.12 and Lemma 6.5 show that $U_\infty\colon L^\infty(\partial\Omega) \to L^\infty_e(\mathbb{T})$ and $\eta\colon L^\infty(\partial\Omega) \to \mathcal{T}(\Omega)$ are weak* homeomorphisms. Hence conditions (b)–(d) imply (e). ■

8. Reflexivity and transitivity results. In this section we concentrate on the reflexivity and transitivity of Toeplitz operators on $H^2(\Omega)$.

THEOREM 8.1. *The subspace $\mathcal{T}(\Omega)$ is transitive.*

Proof. Let $F, G \in H^2(\Omega)$. Assume that a rank-one operator $F \otimes G$ is in $\mathcal{T}(\Omega)_\perp$. Then, for all $\Phi \in L^\infty(\partial\Omega)$, according to (6.1), we have

$$0 = \langle T_\Phi, F \otimes G\rangle = \langle T_\Phi F, G\rangle = \int_{\partial\Omega} \Phi F\overline{G}\, d\omega_a.$$

Since $F\overline{G} \in L^1(\partial\Omega)$, it follows that $F\overline{G} = 0$. But $F, G \in H^2(\Omega)$, therefore $F = 0$ or $G = 0$ by [1, Corollary 1.19]. Thus, we have shown that $\mathcal{T}(\Omega)_\perp \cap F_1(H^2(\Omega)) = \{0\}$, which implies that $\mathcal{T}(\Omega)$ is transitive. ■

The theorem below is a generalization of the classical reflexivity result for the algebra of analytic Toeplitz operators on the disc (see [16]).

THEOREM 8.2. *The algebra* $\mathcal{A}(\Omega)$ *is reflexive.*

This seems to be common knowledge. We present the proof for completeness.

Proof. The Hardy space $H^2(\Omega)$ has the reproducing property. Thus $\langle F, k_\lambda\rangle = F(\lambda)$ for all $\lambda \in \Omega$ and $F \in H^2(\Omega)$, where k_λ is the reproducing kernel. It is easily seen that

$$\mathbb{C}k_\lambda \in \operatorname{Lat}\mathcal{A}(\Omega)^*. \tag{8.1}$$

It has to be shown that $\operatorname{Alg}\operatorname{Lat}\mathcal{A}(\Omega) \subset \mathcal{A}(\Omega)$. Assume that $\lambda \in \Omega$ and $B \in \operatorname{Alg}\operatorname{Lat}\mathcal{A}(\Omega)$. Then $\operatorname{Lat}\mathcal{A}(\Omega)^* \subset \operatorname{Lat} B^*$. By (8.1) we see that $\mathbb{C}k_\lambda \in \operatorname{Lat} B^*$. It follows that there exists $\Phi(\lambda)$ such that $B^*k_\lambda = \overline{\Phi(\lambda)}k_\lambda$. Further, note that $\langle B(1), k_\lambda\rangle = \langle 1, B^*k_\lambda\rangle = \Phi(\lambda)$. Thus $\Phi = B(1) \in H^2(\Omega)$, which implies that Φ is analytic. On the other hand, for every $\lambda \in \Omega$ we have $\overline{\Phi(\lambda)} \in \sigma_p(B^*) \subset \sigma(B^*) = \overline{\sigma(B)}$. Therefore $\Phi(\lambda) \in \sigma(B)$, and hence $|\Phi(\lambda)| \le \|B\|$. This means that $\Phi \in H^\infty(\Omega)$. Take $F \in H^2(\Omega)$ and $\lambda \in \Omega$. Then

$$(BF)(\lambda) = \langle BF, k_\lambda\rangle = \langle F, B^*k_\lambda\rangle = \langle F, \overline{\Phi(\lambda)}k_\lambda\rangle = \Phi(\lambda)F(\lambda).$$

Thus B is an analytic Toeplitz operator with symbol Φ. ∎

The next corollary is a direct consequence of the preceding theorems and Lemma 7.1.

COROLLARY 8.3. *The subspace* $\widetilde{\mathcal{T}}_e(\mathbb{D})$ *is transitive and the algebra* $\widetilde{\mathcal{A}}_e(\mathbb{D})$ *is reflexive.*

Theorem 7.6 implies that $\widetilde{\mathcal{A}}_e(\mathbb{D})$ is a weak* closed subspace of $\mathcal{B}(H^2_e(\mathbb{D}))$. It is known that $\mathcal{A}(\mathbb{D}) \subset \mathcal{B}(H^2(\mathbb{D}))$ has property $\mathbb{A}_1(1)$ (see for instance [5, Proposition 60.5]). We obtain the following result.

THEOREM 8.4. *The algebra* $\widetilde{\mathcal{A}}_e(\mathbb{D}) \subset \mathcal{B}(H^2_e(\mathbb{D}))$ *has property* $\mathbb{A}_1(1)$.

Proof. Suppose that $\Lambda\colon \widetilde{\mathcal{A}}_e(\mathbb{D}) \to \mathbb{C}$ is a weak* continuous functional and $\|\Lambda\| = 1$. Let $\varepsilon > 0$. Let $\widetilde{\xi}$ be as in Theorem 7.6. Then $\Lambda \circ \widetilde{\xi}$ is a weak* continuous functional on $H^\infty_e(\mathbb{D})$. Since $L^1_e(\mathbb{T})^* = L^\infty_e(\mathbb{T})$ and $H^\infty_e(\mathbb{D})$ is a weak* closed subspace of $L^\infty_e(\mathbb{T})$, it follows that $(L^1_e(\mathbb{T})/H^\infty_e(\mathbb{D})_\perp)^*$ and $H^\infty_e(\mathbb{D})$ are isometrically isomorphic. Hence there exists $f \in L^1_e(\mathbb{T})$ such that, for $\varphi \in H^\infty_e(\mathbb{D})$, we have $(\Lambda \circ \widetilde{\xi}\,)\varphi = \langle \varphi, [f\,]\rangle$ and $\|f\|_{L^1(\mathbb{T})} < 1 + \varepsilon$. Thus

$$\Lambda(\widetilde{T}_\varphi) = (\Lambda \circ \widetilde{\xi}\,)(\varphi) = \langle \varphi, [f\,]\rangle = \int_{\mathbb{T}} \varphi f\, dm. \tag{8.2}$$

First note that the function $(|f| + \varepsilon)^{1/2}$ is bounded from below. In addition, $(|f| + \varepsilon)^{1/2} \in L^2_e(\mathbb{T})$, because $f \in L^1_e(\mathbb{T})$. By [5, Corollary 25.12], there is an outer function $g \in H^2(\mathbb{D})$ satisfying $|g| = (\,|f| + \varepsilon)^{1/2}$ m-a.e. on $\mathbb{T}$. If $T \in \mathcal{G}$, then

$$|g \circ T| = (|f \circ T| + \varepsilon)^{1/2} = (|f| + \varepsilon)^{1/2} = |g|.$$

Thus g is modulus automorphic. Hence there is a character $\alpha \in \widehat{\mathcal{G}}$ such that $g \in H^2_\alpha(\mathbb{D})$. Define $\beta := 1/\alpha \in \widehat{\mathcal{G}}$. By [1, Theorem 1.14] there is an inner function in $H^\infty_\beta(\mathbb{D})$, say ψ. Setting $g_1 := g\psi$ we get $g_1 \in H^2_e(\mathbb{D})$ and $|g_1| = |g|$.

Define $h := |f|/|g| = |f|/|g_1|$. Since f and g are modulus automorphic, $h \circ T = h$ for any $T \in \mathcal{G}$. Moreover, $h = [\,|f|/(|f| + \varepsilon)\,]^{1/2} \cdot |f|^{1/2}$, hence $h \in L^2_e(\mathbb{T})$. On the other hand, since $|f| = h|g_1|$, we have $f = uhg_1$, where u is a measurable function of modulus 1. Moreover, $u \circ T = u$ for all $T \in \mathcal{G}$, since $f \in L^1_e(\mathbb{T}), h \in L^2_e(\mathbb{T})$, and $g_1 \in H^2_e(\mathbb{D})$. Let $h_1 := P_{H^2(\mathbb{D})}(\overline{u}h)$. Since $\overline{u}h \in L^2_e(\mathbb{T})$, we have $h_1 \in H^2_e(\mathbb{D})$ by (4.5). Putting uhg_1 in place of f in (8.2), we get, for $\varphi \in H^\infty_e(\mathbb{D})$,

$$\begin{aligned} \Lambda(\widetilde{T}_\varphi) &= \int_{\mathbb{T}} \varphi g_1 uh\, dm = \langle \varphi g_1, \overline{u}h \rangle = \langle P_{H^2(\mathbb{D})}(\varphi g_1), \overline{u}h \rangle \\ &= \langle \varphi g_1, P_{H^2(\mathbb{D})}(\overline{u}h) \rangle = \langle \varphi g_1, h_1 \rangle = \langle \widetilde{T}_\varphi g_1, h_1 \rangle. \end{aligned}$$

So $\widetilde{\mathcal{A}}_e(\mathbb{D})$ has property $\mathbb{A}_1$.

To finish the proof, note that $\|g_1\|^2_{H^2(\mathbb{D})} < 1 + 2\varepsilon$ and $\|h_1\|^2_{H^2(\mathbb{D})} < 1 + \varepsilon$. Hence $\|g_1\|_{H^2(\mathbb{D})} \|h_1\|_{H^2(\mathbb{D})} < 1 + 4\varepsilon$, which completes the proof. ∎

The theorem above and Theorems 7.6 and 8.2 lead to

THEOREM 8.5. *The algebra $\mathcal{A}(\Omega)$ has property $\mathbb{A}_1(1)$ and every weak* closed subspace of $\mathcal{A}(\Omega)$ is reflexive.*

We end up with the following open question.

PROBLEM 8.6. *Let Ω be a multiply connected region in the complex plane. Is every weak* closed subspace of $\mathcal{T}(\Omega)$ either reflexive or transitive?*

As mentioned in the introduction, the above is true if $\Omega = \mathbb{D}$ (see [2]), if Ω is a simply connected region (see [13]), or if Ω is the upper half-plane (see [12]).

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