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ON SOME METABELIAN 2-GROUPS AND APPLICATIONS I

BY

ABDELMALEK AZIZI (Oujda), ABDELKADER ZEKHNINI (Nador)
and MOHAMMED TAOUS (Errachidia)

Abstract. Let G be some metabelian 2-group satisfying the condition $G/G' \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. In this paper, we construct all the subgroups of G of index 2 or 4, we give the abelianization types of these subgroups and we compute the kernel of the transfer map. Then we apply these results to study the capitulation problem for the 2-ideal classes of some fields $\mathbf{k}$ satisfying the condition $\mathrm{Gal}(\mathbf{k}_2^{(2)}/\mathbf{k}) \simeq G$, where $\mathbf{k}_2^{(2)}$ is the second Hilbert 2-class field of $\mathbf{k}$.

1. Introduction. Let k be an algebraic number field and let $\mathrm{Cl}(k)$ denote its class group. Let $k^{(1)}$ be the Hilbert class field of k, that is, the maximal abelian unramified extension of k. Let $k^{(2)}$ be the Hilbert class field of $k^{(1)}$ and set $G = \mathrm{Gal}(k^{(2)}/k)$. Denote by F a finite extension of k and by H the subgroup of G which fixes F. Then we say that an ideal class of k *capitulates* in F if it is in the kernel of the homomorphism

$$j_{k\to F} : \mathrm{Cl}(k) \to \mathrm{Cl}(F)$$

induced by extension of ideals from k to F. An important problem in number theory is to explicitly determine the kernel of $j_{k\to F}$, which is usually called the *capitulation kernel*. As $j_{k\to F}$ corresponds, by the Artin reciprocity law, to the group-theoretical transfer (for details see [Mi])

$$\mathrm{V}_{G\to H} : G/G' \to H/H',$$

where G' (resp. H') is the derived group of G (resp. H), to determine $\ker j_{k\to F}$ is equivalent to determining $\ker V_{G\to H}$, which transforms the capitulation problem to a problem in group theory. That is why the capitulation problem is completely solved if $G/G' \simeq (2,2)$, since the groups G such that $G/G' \simeq (2,2)$ are determined and well classified (see [Ki, Mi]). If $G/G' \simeq (2,2^n)$ for some integer $n \geq 2$, then G is metacyclic or not; in

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the first case the capitulation problem is completely solved, whereas in the second case the problem is open (see [ATZ, BS]). If $G/G' \simeq (2,2,2)$, then the structure of G is unknown in most cases, so the capitulation problem is also open.

It is the purpose of this paper to provide answers to this problem in a particular case, continuing a project we started in [AZT4]; we give some group-theoretical results to solve the capitulation problem, in a particular case, if G satisfies the last condition. For this, we consider the following family of groups, defined for integers $n \geq 1$ and $m \geq 2$:

$$G_{m,n} = \langle \sigma, \tau, \rho : \rho^4 = \sigma^{2^m} = \tau^{2^{n+1}} = 1,\ \rho^2 = \varphi,\ [\tau,\sigma] = 1,\ [\rho,\sigma] = \sigma^2,\ [\rho,\tau] = \tau^2 \rangle, \tag{1.1}$$

where

$$\varphi = \sigma^{2^{m-1}} \text{ or } \tau^{2^n}\sigma^{2^{m-1}}.$$

In this paper, we construct all the subgroups of $G_{m,n}$ of index 2 or 4, we give the abelianization types of these subgroups and we compute the kernel of the transfer map $\mathrm{V}_{G\to H} : G_{m,n}/G'_{m,n} \to H/H'$ for any subgroup H of $G_{m,n}$, defined by the Artin map. Then we apply these results to study the capitulation of 2-ideal classes of some fields $\mathbf{k}$ satisfying $\mathrm{Gal}(\mathbf{k}_2^{(2)}/\mathbf{k}) \simeq G_{m,n}$, where $\mathbf{k}_2^{(2)}$ is the second Hilbert 2-class field of $\mathbf{k}$. Finally, we illustrate our results by examples which show that our group is realizable, i.e. there is a field $\mathbf{k}$ such that $\mathrm{Gal}(\mathbf{k}_2^{(2)}/\mathbf{k}) \simeq G_{m,n}$.

2. Main results.

Recall first that a group G is said to be *metabelian* if its derived group G' is abelian, and a subgroup H of G, not reduced to the unit, is called *maximal* if it is the unique subgroup of G distinct from G and containing H. Let x, y and z be elements of G. Set $x^y = y^{-1}xy$. Then we easily show that

$$[xy, z] = [x,z]^y[y,z] \quad \text{and} \quad [x, yz] = [x,z][x,y]^z.$$

Let $G_{m,n}$ be the group defined by (1.1). Since $[\tau,\sigma] = 1$, $[\rho,\sigma] = \sigma^2$ and $[\rho,\tau] = \tau^2$, we have $G'_{m,n} = \langle \sigma^2, \tau^2 \rangle$, which is abelian. Thus $G_{m,n}$ is metabelian and $G_{m,n}/G'_{m,n} \simeq (2,2,2)$, since $\rho^2 = \sigma^{2^{m-1}}$ or $\tau^{2^n}\sigma^{2^{m-1}}$. Hence $G_{m,n}$ admits seven subgroups of index 2, denoted $H_{i,2}$, and seven subgroups of index 4, denoted $H_{i,4}$, where $1 \leq i \leq 7$. These subgroups, their derived groups and their abelianizations are given in Tables 1–4 below. Set $a = \min(m-1, n)$ and $b = \max(m, n+1)$.

First case: $\rho^2 = \sigma^{2^{m-1}}$.

Table 1. Subgroups of $G_{m,n}$ of index 2

i	$H_{i,2}$	$H'_{i,2}$	$H_{i,2}/H'_{i,2}$
1	$\langle\sigma,\tau\rangle$	$\langle 1\rangle$	$(2^m, 2^{n+1})$
2	$\langle\sigma,\rho,\tau^2\rangle$	$\langle\sigma^2,\tau^4\rangle$	$(2,2,2)$
3	$\langle\tau,\rho,\sigma^2\rangle$	$\langle\tau^2,\sigma^4\rangle$	$(2,2,2)$
4	$\langle\sigma\tau,\rho,\sigma^2\rangle$	$\langle(\sigma\tau)^2,\sigma^4\rangle$	$(2,2,2)$
5	$\langle\sigma\rho,\sigma,\tau^2\rangle$	$\langle\tau^4,\sigma^2\rangle$	$(2,2,2)$
6	$\langle\tau\rho,\tau,\sigma^2\rangle$	$\langle\tau^2,\sigma^4\rangle$	$(2,2,2)$
7	$\langle\sigma\tau,\tau\rho,\sigma^2\rangle$	$\langle(\sigma\tau)^2,\sigma^4\rangle$	$(2,2,2)$

Table 2. Subgroups of $G_{m,n}$ of index 4

i	$H_{i,4}$	$H'_{i,4}$	$H_{i,4}/H'_{i,4}$
1	$\langle\sigma,\tau^2\rangle$	$\langle 1\rangle$	$(2^m, 2^n)$
2	$\langle\sigma^2,\tau\rangle$	$\langle 1\rangle$	$(2^{m-1}, 2^{n+1})$
3	$\langle\rho,\sigma^2,\tau^2\rangle$	$\langle\sigma^4,\tau^4\rangle$	$(2,2,2)$
4	$\langle\sigma\tau,\tau^2\rangle$	$\langle 1\rangle$	$(2^a, 2^b)$
5	$\langle\sigma\rho,\sigma^2,\tau^2\rangle$	$\langle\sigma^4,\tau^4\rangle$	$(2,2,2)$
6	$\langle\tau\rho,\sigma^2,\tau^2\rangle$	$\langle\sigma^4,\tau^4\rangle$	$(2,2,2)$
7	$\langle\sigma\tau\rho,\sigma^2,\tau^2\rangle$	$\langle\sigma^4,\tau^4\rangle$	$(2,2,2)$

Second case: $\rho^2 = \tau^{2^n}\sigma^{2^{m-1}}$.

Table 3. Subgroups of $G_{m,n}$ of index 2

i	Conditions	$H_{i,2}$	$H'_{i,2}$	$H_{i,2}/H'_{i,2}$
1		$\langle\sigma,\tau\rangle$	$\langle 1\rangle$	$(2^m, 2^{n+1})$
2	$n=1$	$\langle\sigma,\rho\rangle$	$\langle\sigma^2\rangle$	$(2,4)$
	$n\geq 2$	$\langle\sigma,\rho,\tau^2\rangle$	$\langle\sigma^2,\tau^4\rangle$	$(2,2,2)$
3	$m=2$	$\langle\tau,\rho\rangle$	$\langle\tau^2\rangle$	$(2,4)$
	$m\geq 3$	$\langle\tau,\rho,\sigma^2\rangle$	$\langle\tau^2,\sigma^4\rangle$	$(2,2,2)$
4	$n=1$ and $m\geq 3$	$\langle\sigma\tau,\rho\rangle$	$\langle(\sigma\tau)^2\rangle$	$(2,4)$
	$n\geq 2$ and $m=2$	$\langle\sigma\tau,\rho\rangle$	$\langle(\sigma\tau)^2\rangle$	$(2,4)$
	$n=1$ and $m=2$	$\langle\sigma\tau,\rho,\sigma^2\rangle$	$\langle(\sigma\tau)^2\rangle$	$(2,2,2)$
	$n\geq 2$ and $m\geq 3$	$\langle\sigma\tau,\rho,\sigma^2\rangle$	$\langle(\sigma\tau)^2,\sigma^4\rangle$	$(2,2,2)$
5	$m=2$	$\langle\sigma\rho,\tau\rangle$	$\langle\tau^2\rangle$	$(2,4)$
	$m\geq 3$	$\langle\sigma\rho,\tau,\sigma^2\rangle$	$\langle\tau^2,\sigma^4\rangle$	$(2,2,2)$
6	$n=1$	$\langle\sigma,\tau\rho\rangle$	$\langle\sigma^2\rangle$	$(2,4)$
	$n\geq 2$	$\langle\sigma,\tau\rho,\tau^2\rangle$	$\langle\sigma^2,\tau^4\rangle$	$(2,2,2)$
7	$n=1$ and $m\geq 3$	$\langle\sigma\tau,\tau\rho\rangle$	$\langle(\sigma\tau)^2\rangle$	$(2,4)$
	$n\geq 2$ and $m=2$	$\langle\sigma\tau,\tau\rho\rangle$	$\langle(\sigma\tau)^2\rangle$	$(2,4)$
	$n=1$ and $m=2$	$\langle\sigma\tau,\tau\rho,\sigma^2\rangle$	$\langle(\sigma\tau)^2\rangle$	$(2,2,2)$
	$n\geq 2$ and $m\geq 3$	$\langle\sigma\tau,\tau\rho,\sigma^2\rangle$	$\langle(\sigma\tau)^2,\sigma^4\rangle$	$(2,2,2)$

Table 4. Subgroups of $G_{m,n}$ of index 4

i	Conditions	$H_{i,4}$	$H'_{i,4}$	$H_{i,4}/H'_{i,4}$
1		$\langle\sigma,\tau^2\rangle$	$\langle 1\rangle$	$(2^m,2^n)$
2		$\langle\sigma^2,\tau\rangle$	$\langle 1\rangle$	$(2^{m-1},2^{n+1})$
3	$n=1$ and $m\geq 3$	$\langle\sigma^2,\rho\rangle$	$\langle\sigma^4\rangle$	$(2,4)$
	$m=2$ and $n\geq 2$	$\langle\tau^2,\rho\rangle$	$\langle\tau^4\rangle$	$(2,4)$
	$n=1$ and $m=2$	$\langle\sigma^2,\rho\rangle=\langle\tau^2,\rho\rangle$	$\langle 1\rangle$	$(2,4)$
	$n\geq 2$ and $m\geq 3$	$\langle\rho,\tau^2,\sigma^2\rangle$	$\langle\tau^4,\sigma^4\rangle$	$(2,2,2)$
4		$\langle\sigma\tau,\tau^2\rangle$	$\langle 1\rangle$	$(2^a,2^b)$
5	$n=1$ and $m\geq 3$	$\langle\sigma^2,\sigma\rho\rangle$	$\langle\sigma^4\rangle$	$(2,4)$
	$m=2$ and $n\geq 2$	$\langle\tau^2,\sigma\rho\rangle$	$\langle\tau^4\rangle$	$(2,4)$
	$n=1$ and $m=2$	$\langle\sigma^2,\sigma\rho\rangle=\langle\tau^2,\sigma\rho\rangle$	$\langle 1\rangle$	$(2,4)$
	$n\geq 2$ and $m\geq 3$	$\langle\sigma\rho,\tau^2,\sigma^2\rangle$	$\langle\tau^4,\sigma^4\rangle$	$(2,2,2)$
6	$n=1$ and $m\geq 3$	$\langle\sigma^2,\tau\rho\rangle$	$\langle\sigma^4\rangle$	$(2,4)$
	$m=2$ and $n\geq 2$	$\langle\tau^2,\tau\rho\rangle$	$\langle\tau^4\rangle$	$(2,4)$
	$n=1$ and $m=2$	$\langle\sigma^2,\tau\rho\rangle=\langle\tau^2,\tau\rho\rangle$	$\langle 1\rangle$	$(2,4)$
	$n\geq 2$ and $m\geq 3$	$\langle\tau\rho,\tau^2,\sigma^2\rangle$	$\langle\tau^4,\sigma^4\rangle$	$(2,2,2)$
7	$n=1$ and $m\geq 3$	$\langle\sigma^2,\sigma\tau\rho\rangle$	$\langle\sigma^4\rangle$	$(2,4)$
	$m=2$ and $n\geq 2$	$\langle\tau^2,\sigma\tau\rho\rangle$	$\langle\tau^4\rangle$	$(2,4)$
	$n=1$ and $m=2$	$\langle\sigma^2,\sigma\tau\rho\rangle=\langle\tau^2,\sigma\tau\rho\rangle$	$\langle 1\rangle$	$(2,4)$
	$n\geq 2$ and $m\geq 3$	$\langle\sigma\tau\rho,\tau^2,\sigma^2\rangle$	$\langle\tau^4,\sigma^4\rangle$	$(2,2,2)$

To check the tables entries, we need the following lemma.

LEMMA 2.1. *Let $G_{m,n}=\langle\sigma,\tau,\rho\rangle$ denote the group defined above. Then:*

(1) $\rho^{-1}\sigma\rho=\sigma^{-1}$.
(2) $\rho^{-1}\tau\rho=\tau^{-1}$.
(3) ρ^2 *commutes with* σ *and* τ.
(4) $(\sigma\tau\rho)^2=(\sigma\rho)^2=(\tau\rho)^2=\rho^2$.
(5) *For all* $r\in\mathbb{N}$, $[\rho,\tau^{2^r}]=\tau^{2^{r+1}}$ *and* $[\rho,\sigma^{2^r}]=\sigma^{2^{r+1}}$.

Proof. (1) and (2) are obvious, since $[\rho,\sigma]=\sigma^2$ and $[\rho,\tau]=\tau^2$.

(3) As $\rho^2=\sigma^{2^{m-1}}$ or $\tau^{2^n}\sigma^{2^{m-1}}$, we have $\rho^2\in\langle\tau,\sigma\rangle$, which is an abelian group, because $[\tau,\sigma]=1$. Hence the result.

(4) $(\tau\rho)^2=\tau\rho\tau\rho=\tau\rho^2\rho^{-1}\tau\rho=\tau\rho^2\tau^{-1}=\rho^2$. To prove the other results, we proceed similarly.

(5) Since $[\rho,\tau]=\tau^2$, we have $[\rho,\tau^2]=\tau^4$. By induction, we show that $[\rho,\tau^{2^r}]=\tau^{2^{r+1}}$ for all $r\in\mathbb{N}$. Similarly, we prove that $[\rho,\sigma^{2^r}]=\sigma^{2^{r+1}}$. ∎

Let us now prove some entries of the tables, using Lemma 2.1.

First case: $\rho^2 = \sigma^{2^{m-1}}$. For $H_{1,2} = \langle\sigma, \tau, G'_{m,n}\rangle = \langle\sigma, \tau\rangle$, we have $H'_{1,2} = \langle 1\rangle$, since $[\sigma, \tau] = 1$. As $\sigma^{2^m} = \tau^{2^{n+1}} = 1$, we obtain $H_{1,2}/H'_{1,2} \simeq (2^m, 2^{n+1}) = (2^2, 2^{n+2})$.

For $H_{2,2} = \langle\sigma, \rho, G'_{m,n}\rangle = \langle\sigma, \rho, \tau^2, \sigma^2\rangle = \langle\sigma, \rho, \tau^2\rangle$. Therefore, by Lemma 2.1, we get $H'_{2,2} = \langle\sigma^2, \tau^4\rangle$, thus $H_{2,2}/H'_{2,2} \simeq (2, 2, 2)$.

For $H_{1,4} = \langle\sigma, G'_{m,n}\rangle = \langle\sigma, \sigma^2, \tau^2\rangle = \langle\sigma, \tau^2\rangle$, we have $H'_{1,4} = \langle 1\rangle$, since $[\sigma, \tau] = 1$. As $\sigma^{2^m} = \tau^{2^{n+1}} = 1$, we obtain $H_{1,4}/H'_{1,4} \simeq (2^m, 2^n)$.

For $H_{2,4} = \langle\tau, G'_{m,n}\rangle = \langle\tau, \tau^2, \sigma^2\rangle = \langle\tau, \sigma^2\rangle$, we have $H'_{2,4} = \langle 1\rangle$, hence $H_{2,4}/H'_{2,4} \simeq (2^{m-1}, 2^{n+1})$.

The other entries of Tables 1 and 2 are checked similarly.

Second case: $\rho^2 = \tau^{2^n}\sigma^{2^{m-1}}$. We have $H_{2,2} = \langle\sigma, \rho, G'_{m,n}\rangle = \langle\sigma, \rho, \tau^2\rangle$. Since $\rho^2 = \tau^{2^n}\sigma^{2^{m-1}}$, if $n = 1$ then $\rho^2 = \tau^2\sigma^{2^{m-1}}$, and thus $\rho^2\sigma^{-2^{m-1}} = \tau^2$. Hence $H_{2,2} = \langle\sigma, \rho\rangle$. If $n \geq 2$, then $H_{2,2} = \langle\sigma, \rho, \tau^2\rangle$. Therefore, by Lemma 2.1, we get

$$H'_{2,2} = \begin{cases} \langle\sigma^2\rangle & \text{if } n = 1, \\ \langle\sigma^2, \tau^4\rangle & \text{if } n \geq 2. \end{cases}$$

As $\rho^4 = 1$ and $\rho^2 = \tau^{2^n}\sigma^{2^{m-1}}$, we obtain

$$H_{2,2}/H'_{2,2} \simeq \begin{cases} (2, 4) & \text{if } n = 1, \\ (2, 2, 2) & \text{if } n \geq 2. \end{cases}$$

We have $H_{4,2} = \langle\sigma\tau, \rho, G'_{m,n}\rangle = \langle\sigma\tau, \rho, \tau^2\rangle = \langle\sigma\tau, \rho, \sigma^2\rangle$. Since $\rho^2 = \tau^{2^n}\sigma^{2^{m-1}}$, we obtain:

- If $n = 1$ and $m \geq 3$, then $\rho^2 = \tau^2\sigma^{2^{m-1}}$ and $\tau^4 = 1$, thus $\rho^2 = \tau^2(\sigma\tau)^{2^{m-1}}$, which implies $\rho^2(\sigma\tau)^{-2^{m-1}} = \tau^2$. Hence $H_{4,2} = \langle\sigma\tau, \rho\rangle$, and Lemma 2.1 yields $H'_{4,2} = \langle(\sigma\tau)^2\rangle$. Thus $H_{4,2}/H'_{4,2} \simeq (2, 4)$ since $\rho^4 = 1$.
- If $n \geq 2$ and $m = 2$, then $\rho^2 = \tau^{2^n}\sigma^2$ and $\sigma^4 = 1$, thus $\rho^2 = (\sigma\tau)^{2^n}\sigma^2$, which implies $\rho^2(\sigma\tau)^{-2^n} = \sigma^2$. Hence $H_{4,2} = \langle\sigma\tau, \rho\rangle$, and Lemma 2.1 yields $H'_{4,2} = \langle(\sigma\tau)^2\rangle$. Thus $H_{4,2}/H'_{4,2} \simeq (2, 4)$ since $\rho^4 = 1$.
- If $n = 1$ and $m = 2$, then $\rho^2 = \tau^2\sigma^2 = (\sigma\tau)^2$ and $\sigma^4 = \tau^4 = 1$, hence $H_{4,2} = \langle\sigma\tau, \rho, \sigma^2\rangle$, and Lemma 2.1 yields $H'_{4,2} = \langle(\sigma\tau)^2\rangle$. Thus $H_{4,2}/H'_{4,2} \simeq (2, 2, 2)$.
- If $n \geq 2$ and $m \geq 3$, then $H_{4,2} = \langle\sigma\tau, \rho, \sigma^2\rangle$, and Lemma 2.1 yields $H'_{4,2} = \langle(\sigma\tau)^2, \sigma^4\rangle$. Thus $H_{4,2}/H'_{4,2} \simeq (2, 2, 2)$.

For $H_{1,4} = \langle\sigma, G'_{m,n}\rangle = \langle\sigma, \sigma^2, \tau^2\rangle = \langle\sigma, \tau^2\rangle$, we have $H'_{1,4} = \langle 1\rangle$, since $[\sigma, \tau] = 1$. As $\sigma^{2^m} = \tau^{2^{n+1}} = 1$, it follows that $H_{1,4}/H'_{1,4} \simeq (2^m, 2^n)$.

For $H_{3,4} = \langle\rho, G'_{m,n}\rangle = \langle\rho, \tau^2, \sigma^2\rangle$, we have:

- If $n = 1$ and $m \geq 3$, then $\rho^2 = \tau^2\sigma^{2^{m-1}}$ and $\tau^4 = 1$, which implies $\rho^2\sigma^{-2^{m-1}} = \tau^2$. Hence $H_{3,4} = \langle \rho, \sigma^2 \rangle$, and Lemma 2.1 yields $H'_{3,4} = \langle \sigma^4 \rangle$. Thus $H_{3,4}/H'_{3,4} \simeq (2,4)$ since $\rho^4 = 1$.
- If $n \geq 2$ and $m = 2$, then $\rho^2 = \tau^{2^n}\sigma^2$ and $\sigma^4 = 1$, which implies $\rho^2\tau^{-2^n} = \sigma^2$. Hence $H_{3,4} = \langle \tau^2, \rho \rangle$, and Lemma 2.1 yields $H'_{3,4} = \langle \tau^4 \rangle$. Thus $H_{3,4}/H'_{3,4} \simeq (2,4)$ since $\rho^4 = 1$.
- If $n = 1$ and $m = 2$, then $\rho^2 = \tau^2\sigma^2$ and $\sigma^4 = \tau^4 = 1$, hence $H_{3,4} = \langle \rho, \sigma^2 \rangle = \langle \rho, \tau^2 \rangle$, and Lemma 2.1 yields $H'_{3,4} = \langle 1 \rangle$. Thus $H_{3,4}/H'_{3,4} \simeq (2,4)$.
- If $n \geq 2$ and $m \geq 3$, then $H_{3,4} = \langle \rho, \sigma^2, \tau^2 \rangle$, and Lemma 2.1 yields $H'_{3,4} = \langle \tau^4, \sigma^4 \rangle$. Thus $H_{3,4}/H'_{3,4} \simeq (2,2,2)$.

The other entries of Tables 3 and 4 are checked similarly. ■

PROPOSITION 2.2. *Let $G_{m,n}$ be the group defined by* (1.1). *Then:*

(1) *The order of $G_{m,n}$ is 2^{m+n+2} and that of $G'_{m,n}$ is 2^{m+n-1}.*
(2) *The coclass of $G_{m,n}$ is $\min(m, n+1)+1$ and its nilpotency class is $\max(m, n+1)$.*
(3) *The center, $Z(G)$, of G is of type* $(2,2)$.

Proof. (1) Since $\sigma^{2^m} = \tau^{2^{n+1}} = 1$, we have $\langle \sigma, \tau \rangle \simeq (2^m, 2^{n+1})$. Moreover, as $\rho^2 = \sigma^{2^{m-1}}$ or $\tau^{2^n}\sigma^{2^{m-1}}$, we obtain $\langle \sigma, \tau, \rho \rangle \simeq (2^m, 2^{n+1}, 2)$. Thus $|G_{m,n}| = 2^{m+n+2}$. Similarly, we prove that $|G'_{m,n}| = 2^{m+n-1}$, since $G'_{m,n} = \langle \sigma^2, \tau^2 \rangle$.

(2) The lower central series of $G_{m,n}$ is defined inductively by $\gamma_1(G_{m,n}) = G_{m,n}$ and $\gamma_{i+1}(G_{m,n}) = [\gamma_i(G_{m,n}), G_{m,n}]$, that is, the subgroup of $G_{m,n}$ generated by the set $\{[a,b] = a^{-1}b^{-1}ab : a \in \gamma_i(G_{m,n}),\, b \in G_{m,n}\}$, so the coclass of $G_{m,n}$ is defined to be $cc(G_{m,n}) = h - c$, where $|G_{m,n}| = 2^h$ and $c = c(G_{m,n})$ is the nilpotency class of $G_{m,n}$. We easily get

$$\begin{aligned} \gamma_1(G_{m,n}) &= G_{m,n}, \\ \gamma_2(G_{m,n}) &= G'_{m,n} = \langle \sigma^2, \tau^2 \rangle, \\ \gamma_3(G_{m,n}) &= [G'_{m,n}, G_{m,n}] = \langle \sigma^4, \tau^4 \rangle. \end{aligned}$$

Then Lemma 2.1(5) yields $\gamma_{j+1}(G_{m,n}) = [\gamma_j(G_{m,n}), G_{m,n}] = \langle \sigma^{2^j}, \tau^{2^j} \rangle$. Hence, if we set $v = \max(m, n+1)$, then $\gamma_{v+1}(G_{m,n}) = \langle \sigma^{2^v}, \tau^{2^v} \rangle = \langle 1 \rangle$ and $\gamma_v(G_{m,n}) = \langle \sigma^{2^{v-1}}, \tau^{2^{v-1}} \rangle \neq \langle 1 \rangle$. As $|G_{m,n}| = 2^{m+n+2}$, it follows that $c(G_{m,n}) = \max(m, n+1)$ and

$$cc(G_{m,n}) = m + n + 2 - \max(m, n+1) = \min(m, n+1) + 1.$$

(3) We use [Is, Lemma 12.12, p. 204] which states that if G is a p-group and A is a normal abelian subgroup of G such that G/A is cyclic, then $A/A \cap Z(G) \simeq G'$. Let $A = H_{1,2}$, so A is abelian and $[G : A] = 2$, thus

$Z(G) \subset A$ and $A/Z(G) \simeq G'$. Hence $|G| = |A|[G:A] = 2|G'|\,|Z(G)|$, thus $|Z(G)| = \frac{1}{2}|G/G'| = 4$. On the other hand, by Lemma 2.1 we have $[\rho, \sigma^{2^{m-1}}] = \sigma^{2^m} = 1$ and $[\rho, \tau^{2^n}] = \sigma^{2^{n+1}} = 1$, so $\langle \sigma^{2^{m-1}}, \tau^{2^n} \rangle \subset Z(G)$. As $|\langle \sigma^{2^{m-1}}, \tau^{2^n} \rangle| = 4$, we conclude that $\langle \sigma^{2^{m-1}}, \tau^{2^n} \rangle = Z(G) \simeq (2,2)$. ■

We continue with the following results.

PROPOSITION 2.3 ([Mi]). *Let H be a normal subgroup of a group G. For $g \in G$, write $f = [\langle g \rangle.H : H]$ and let $\{x_1, \dots, x_t\}$ be a set of representatives of $G/\langle g \rangle H$. Then the transfer map $\mathrm{V}_{G\to H} : G/G' \to H/H'$ is given by*

$$\mathrm{V}_{G\to H}(gG') = \prod_{i=1}^{t} x_i^{-1} g^f x_i.H'. \tag{2.1}$$

The following corollaries can be proved easily.

COROLLARY 2.4. *Let H be a subgroup of $G_{m,n}$ of index 2. If $G_{m,n}/H = \{1, zH\}$, then*

$$\mathrm{V}_{G\to H}(gG'_{m,n}) = \begin{cases} gz^{-1}gz.H' = g^2[g,z].H' & \text{if } g \in H, \\ g^2.H' & \text{if } g \notin H. \end{cases}$$

COROLLARY 2.5. *Let H be a normal subgroup of $G_{m,n}$ of index 4. If $G_{m,n}/H = \{1, zH, z^2H, z^3H\}$, then*

$$\mathrm{V}_{G\to H}(gG'_{m,n}) = \begin{cases} gz^{-1}gz^{-1}gz^{-1}gz^3.H' & \text{if } g \in H, \\ g^4.H' & \text{if } gH = zH, \\ g^2z^{-1}g^2z.H' & \text{if } g \notin H \text{ and } gH \neq zH. \end{cases}$$

COROLLARY 2.6. *Let H be a normal subgroup of $G_{m,n}$ of index 4. If $G_{m,n}/H = \{1, z_1H, z_2H, z_3H\}$ with $z_3 = z_1z_2$, then*

$$\mathrm{V}_{G\to H}(gG'_{m,n}) = \begin{cases} gz_1^{-1}gz_1z_2^{-1}gz_1^{-1}gz_1z_2.H' & \text{if } g \in H, \\ g^2z_i^{-1}g^2z_i.H' & \text{if } gH = z_jH \text{ with } i \neq j. \end{cases}$$

In what follows, we denote by $\ker \mathrm{V}_H$ the kernel of the transfer map $\mathrm{V}_{G\to H} : G_{m,n}/G'_{m,n} \to H/H'$, where H is a subgroup of $G_{m,n}$.

THEOREM 2.7. *Keep the previous notation.*

(I) *If $\rho^2 = \sigma^{2^{m-1}}$, then*

$$\ker \mathrm{V}_{H_{1,2}} = \langle \sigma G'_{m,n}, \tau G'_{m,n} \rangle,$$
$$\ker \mathrm{V}_{H_{2,2}} = \langle \sigma G'_{m,n}, \tau\rho G'_{m,n} \rangle,$$
$$\ker \mathrm{V}_{H_{3,2}} = \begin{cases} \langle \tau G'_{m,n}, \rho G'_{m,n} \rangle & \text{if } m = 2, \\ \langle \tau G'_{m,n}, \sigma\rho G'_{m,n} \rangle & \text{if } m \geq 3, \end{cases}$$
$$\ker \mathrm{V}_{H_{4,2}} = \begin{cases} \langle \sigma\tau G'_{m,n}, \rho G'_{m,n} \rangle & \text{if } m = 2, \\ \langle \sigma\tau G'_{m,n}, \sigma\rho G'_{m,n} \rangle & \text{if } m \geq 3, \end{cases}$$

$$\ker \mathrm{V}_{H_{5,2}} = \begin{cases} \langle \tau G'_{m,n}, \sigma\rho G'_{m,n}\rangle & \textit{if } m=2, \\ \langle \tau G'_{m,n}, \rho G'_{m,n}\rangle & \textit{if } m\geq 3, \end{cases}$$
$$\ker \mathrm{V}_{H_{6,2}} = \langle \sigma G'_{m,n}, \rho G'_{m,n}\rangle,$$
$$\ker \mathrm{V}_{H_{7,2}} = \begin{cases} \langle \sigma\tau G'_{m,n}, \tau\rho G'_{m,n}\rangle & \textit{if } m=2, \\ \langle \sigma\tau G'_{m,n}, \rho G'_{m,n}\rangle & \textit{if } m\geq 3. \end{cases}$$

(II) *If* $\rho^2 = \tau^{2^n}\sigma^{2^{m-1}}$, *then*

$$\ker \mathrm{V}_{H_{1,2}} = \langle \sigma G'_{m,n}, \tau G'_{m,n}\rangle,$$
$$\ker \mathrm{V}_{H_{2,2}} = \begin{cases} \langle \sigma G'_{m,n}, \rho G'_{m,n}\rangle & \textit{if } n=1, \\ \langle \sigma G'_{m,n}, \tau\rho G'_{m,n}\rangle & \textit{if } n\geq 2, \end{cases}$$
$$\ker \mathrm{V}_{H_{3,2}} = \begin{cases} \langle \tau G'_{m,n}, \rho G'_{m,n}\rangle & \textit{if } m=2, \\ \langle \tau G'_{m,n}, \sigma\rho G'_{m,n}\rangle & \textit{if } m\geq 3, \end{cases}$$
$$\ker \mathrm{V}_{H_{4,2}} = \begin{cases} \langle \sigma\tau G'_{m,n}, \rho G'_{m,n}\rangle & \textit{if } \begin{cases} n=1 \textit{ and } m\geq 3 \textit{ or} \\ n\geq 2 \textit{ and } m=2, \end{cases} \\ \langle \sigma\tau G'_{m,n}, \sigma\rho G'_{m,n}\rangle & \textit{if } \begin{cases} n=1 \textit{ and } m=2 \textit{ or} \\ n\geq 2 \textit{ and } m\geq 3, \end{cases} \end{cases}$$
$$\ker \mathrm{V}_{H_{5,2}} = \begin{cases} \langle \tau G'_{m,n}, \sigma\rho G'_{m,n}\rangle & \textit{if } m=2, \\ \langle \tau G'_{m,n}, \rho G'_{m,n}\rangle & \textit{if } m\geq 3, \end{cases}$$
$$\ker \mathrm{V}_{H_{6,2}} = \begin{cases} \langle \sigma G'_{m,n}, \tau\rho G'_{m,n}\rangle & \textit{if } n=1, \\ \langle \sigma G'_{m,n}, \rho G'_{m,n}\rangle & \textit{if } n\geq 2, \end{cases}$$
$$\ker \mathrm{V}_{H_{7,2}} = \begin{cases} \langle \sigma\tau G'_{m,n}, \tau\rho G'_{m,n}\rangle & \textit{if } \begin{cases} n=1 \textit{ and } m\geq 3 \textit{ or} \\ n\geq 2 \textit{ and } m=2, \end{cases} \\ \langle \sigma\tau G'_{m,n}, \rho G'_{m,n}\rangle & \textit{if } \begin{cases} n=1 \textit{ and } m=2 \textit{ or} \\ n\geq 2 \textit{ and } m\geq 3. \end{cases} \end{cases}$$

(III) *For all* $1\leq i\leq 7$, $\ker \mathrm{V}_{H_{i,4}} = G_{m,n}/G'_{m,n}$.

Proof. We prove only some assertions, the others are shown similarly.

(I) Assume $\rho^2 = \sigma^{2^{m-1}}$. We know, from Table 1, that $H_{1,2} = \langle \sigma, \tau\rangle$; then $G_{m,n}/H_{1,2} = \{1, \rho H_{1,2}\}$ and $H'_{1,2} = \langle 1\rangle$. Hence, by Corollary 2.4 and Lemma 2.1, we get

$$\mathrm{V}_{G_{m,n}\to H_{1,2}}(\sigma G'_{m,n}) = \sigma^2[\sigma,\rho]H'_{1,2} = \sigma^2\sigma^{-2}H'_{1,2} = H'_{1,2},$$
$$\mathrm{V}_{G_{m,n}\to H_{1,2}}(\tau G'_{m,n}) = \tau^2[\tau,\rho]H'_{1,2} = \tau^2\tau^{-2}H'_{1,2} = H'_{1,2},$$
$$\mathrm{V}_{G_{m,n}\to H_{1,2}}(\rho G'_{m,n}) = \rho^2 H'_{1,2} \neq H'_{1,2}.$$

Therefore $\ker \mathrm{V}_{H_{1,2}} = \langle \sigma G'_{m,n}, \tau G'_{m,n}\rangle$.

Similarly, from Table 1, we get $H_{3,2} = \langle \tau, \rho, \sigma^2\rangle$, so $G_{m,n}/H_{3,2} = \{1, \sigma H_{3,2}\}$ and $H'_{3,2} = \langle \sigma^4, \tau^2\rangle$. Hence, by Corollary 2.4 and Lemma 2.1, we get

$$\mathrm{V}_{G_{m,n}\to H_{3,2}}(\sigma G'_{m,n}) = \sigma^2 H'_{3,2} \neq H'_{3,2},$$

$$\begin{aligned}
\mathrm{V}_{G_{m,n}\to H_{3,2}}(\tau G'_{m,n}) &= \tau^2[\tau,\sigma]H'_{3,2} = \tau^2 H'_{3,2} = H'_{3,2},\\
\mathrm{V}_{G_{m,n}\to H_{3,2}}(\rho G'_{m,n}) &= \rho^2[\rho,\sigma]H'_{3,2} = \rho^2\sigma^2 H'_{3,2} = \sigma^2\sigma^{2^{m-1}} H'_{3,2},
\end{aligned}$$

since $\rho^2 = \sigma^{2^{m-1}}$. If $m = 2$, then $\sigma^2\sigma^{2^{m-1}}H'_{3,2} = \sigma^2\sigma^2 H'_{3,2} = H'_{3,2}$; and if $m \geq 3$, then $\sigma^2\sigma^{2^{m-1}}H'_{3,2} = \sigma^2 H'_{3,2} \neq H'_{3,2}$. Moreover,

$$\mathrm{V}_{G_{m,n}\to H_{3,2}}(\sigma\rho G'_{m,n}) = \rho^2 H'_{3,2} = \sigma^{2^{m-1}} H'_{3,2},$$

since $\rho^2 = \sigma^{2^{m-1}}$. If $m = 2$, then $\sigma^{2^{m-1}}H'_{3,2} = \sigma^2 H'_{3,2} \neq H'_{3,2}$; and if $m \geq 3$, then $\sigma^{2^{m-1}}H'_{3,2} = (\sigma^4)^{2^{m-3}}H'_{3,2} = H'_{3,2}$.

Therefore

$$\ker \mathrm{V}_{H_{3,2}} = \begin{cases} \langle \tau G'_{m,n}, \rho G'_{m,n}\rangle & \text{if } m = 2,\\ \langle \tau G'_{m,n}, \sigma\rho G'_{m,n}\rangle & \text{if } m \geq 3.\end{cases}$$

(II) Assume now $\rho^2 = \tau^{2^n}\sigma^{2^{m-1}}$. We know, from Table 3, that

$$H_{2,2} = \begin{cases} \langle \sigma, \rho\rangle & \text{if } n = 1,\\ \langle \sigma, \rho, \tau^2\rangle & \text{if } n \geq 2.\end{cases}$$

Then

$$H'_{2,2} = \begin{cases} \langle \sigma^2\rangle & \text{if } n = 1,\\ \langle \sigma^2, \tau^4\rangle & \text{if } n \geq 2,\end{cases} \quad \text{and} \quad G_{m,n}/H_{2,2} = \{1, \tau H_{1,2}\}.$$

Hence, by Corollary 2.4 and Lemma 2.1, we get

$$\begin{aligned}
\mathrm{V}_{G_{m,n}\to H_{2,2}}(\sigma G'_{m,n}) &= \sigma^2[\sigma,\tau]H'_{2,2} = \sigma^2 H'_{2,2} = H'_{2,2},\\
\mathrm{V}_{G_{m,n}\to H_{2,2}}(\tau G'_{m,n}) &= \tau^2 H'_{2,2} \neq H'_{2,2},\\
\mathrm{V}_{G_{m,n}\to H_{2,2}}(\rho G'_{m,n}) &= \rho^2[\rho,\tau]H'_{2,2}\\
&= \tau^{2^n}\sigma^{2^{m-1}}\tau^2 H'_{2,2}\\
&= \begin{cases} \tau^2\sigma^{2^{m-1}}\tau^2 H'_{2,2} & \text{if } n = 1,\\ \tau^{2^n}\sigma^{2^{m-1}}\tau^2 H'_{2,2} & \text{if } n \geq 2,\end{cases}\\
&= \begin{cases} \tau^4\sigma^{2^{m-1}} H'_{2,2} & \text{if } n = 1,\\ \tau^2(\tau^4)^{2^{n-2}}\sigma^{2^{m-1}} H'_{2,2} & \text{if } n \geq 2,\end{cases}\\
&= \begin{cases} H'_{2,2} & \text{if } n = 1, \text{ since } \tau^4 = 1,\\ \tau^2 H'_{2,2} \neq H'_{2,2} & \text{if } n \geq 2,\end{cases}\\
\mathrm{V}_{G_{m,n}\to H_{2,2}}(\tau\rho G'_{m,n}) &= \rho^2 H'_{2,2}\\
&= \tau^{2^n}\sigma^{2^{m-1}} H'_{2,2}\\
&= \begin{cases} \tau^2\sigma^{2^{m-1}} H'_{2,2} & \text{if } n = 1,\\ \tau^{2^n}\sigma^{2^{m-1}} H'_{2,2} & \text{if } n \geq 2,\end{cases}
\end{aligned}$$

$$= \begin{cases} \tau^2 H'_{2,2} \neq H'_{2,2} & \text{if } n = 1, \\ H'_{2,2} & \text{if } n \geq 2. \end{cases}$$

Therefore

$$\ker \mathrm{V}_{H_{2,2}} = \begin{cases} \langle \sigma G'_{m,n}, \rho G'_{m,n} \rangle & \text{if } n = 1, \\ \langle \sigma G'_{m,n}, \tau\rho G'_{m,n} \rangle & \text{if } n \geq 2. \end{cases}$$

Similarly, from Table 4,

$$H_{4,2} = \begin{cases} \langle \sigma\tau, \rho \rangle & \text{if } n = 1 \text{ and } m \geq 3, \\ \langle \sigma\tau, \rho \rangle & \text{if } n \geq 2 \text{ and } m = 2, \\ \langle \sigma\tau, \rho, \sigma^2 \rangle & \text{if } n = 1 \text{ and } m = 2, \\ \langle \sigma\tau, \rho, \sigma^2 \rangle & \text{if } n \geq 2 \text{ and } m \geq 3. \end{cases}$$

Hence

$$H'_{4,2} = \begin{cases} \langle (\sigma\tau)^2 \rangle & \text{if } n = 1 \text{ and } m \geq 3, \\ \langle (\sigma\tau)^2 \rangle & \text{if } n \geq 2 \text{ and } m = 2, \\ \langle (\sigma\tau)^2 \rangle & \text{if } n = 1 \text{ and } m = 2, \text{ since } \sigma^4 = 1, \\ \langle (\sigma\tau)^2, \sigma^4 \rangle & \text{if } n \geq 2 \text{ and } m \geq 3. \end{cases}$$

On the other hand, $G_{m,n}/H_{4,2} = \{1, \sigma H_{4,2}\} = \{1, \tau H_{4,2}\}$. Thus by Corollary 2.4 and Lemma 2.1, we get

$$\mathrm{V}_{G_{m,n}\to H_{4,2}}(\sigma G'_{m,n}) = \sigma^2 H'_{4,2} \neq H'_{3,2},$$
$$\mathrm{V}_{G_{m,n}\to H_{4,2}}(\tau G'_{m,n}) = \tau^2 H'_{4,2} \neq H'_{4,2},$$
$$\mathrm{V}_{G_{m,n}\to H_{4,2}}(\sigma\tau G'_{m,n}) = (\sigma\tau)^2[\sigma\tau, \sigma] H'_{4,2} = H'_{4,2},$$

$$\mathrm{V}_{G_{m,n}\to H_{4,2}}(\rho G'_{m,n}) = \rho^2[\rho, \sigma] H'_{4,2} = \rho^2\sigma^2 H'_{4,2} = \sigma^2\tau^{2^n}\sigma^{2^{m-1}} H'_{4,2}$$

$$= \begin{cases} \sigma^2\tau^2(\sigma\tau)^{2^{m-1}} H'_{4,2} & \text{if } n = 1 \text{ and } m \geq 3, \text{ since } \tau^4 = 1, \\ \sigma^4(\sigma\tau)^{2n} H'_{4,2} & \text{if } n \geq 2 \text{ and } m = 2, \text{ since } \sigma^4 = 1, \\ \sigma^2\tau^2\sigma^2 H'_{4,2} & \text{if } n = 1 \text{ and } m = 2, \\ \sigma^2(\sigma\tau)^{2^n}\sigma^{-2^n}\sigma^{2^{m-1}} H'_{4,2} & \text{if } n \geq 2 \text{ and } m \geq 3, \end{cases}$$

$$= \begin{cases} H'_{4,2} & \text{if } n = 1 \text{ and } m \geq 3, \text{ since } \tau^4 = 1, \\ H'_{4,2} & \text{if } n \geq 2 \text{ and } m = 2, \text{ since } \sigma^4 = 1, \\ \tau^2 H'_{4,2} \neq H'_{4,2} & \text{if } n = 1 \text{ and } m = 2, \\ \sigma^2 H'_{4,2} \neq H'_{4,2} & \text{if } n \geq 2 \text{ and } m \geq 3, \end{cases}$$

$$\mathrm{V}_{G_{m,n}\to H_{4,2}}(\sigma\rho G'_{m,n}) = (\sigma\rho)^2 H'_{4,2} = \rho^2 H'_{4,2} = \tau^{2^n}\sigma^{2^{m-1}} H'_{4,2}$$

$$= \begin{cases} \tau^2(\sigma\tau)^{2^{m-1}} H'_{4,2} & \text{if } n = 1 \text{ and } m \geq 3, \text{ since } \tau^4 = 1, \\ \sigma^2(\sigma\tau)^{2n} H'_{4,2} & \text{if } n \geq 2 \text{ and } m = 2, \text{ since } \sigma^4 = 1, \\ \sigma^2\tau^2 H'_{4,2} & \text{if } n = 1 \text{ and } m = 2, \\ (\sigma\tau)^{2^n}\sigma^{-2^n}\sigma^{2^{m-1}} H'_{4,2} & \text{if } n \geq 2 \text{ and } m \geq 3, \end{cases}$$

$$= \begin{cases} \tau^2 H'_{4,2} \neq H'_{4,2} & \text{if } n=1 \text{ and } m\geq 3, \text{ since } \tau^4=1,\\ \sigma^2 H'_{4,2} \neq H'_{4,2} & \text{if } n\geq 2 \text{ and } m=2, \text{ since } \sigma^4=1,\\ H'_{4,2} & \text{if } n=1 \text{ and } m=2,\\ H'_{4,2} & \text{if } n\geq 2 \text{ and } m\geq 3.\end{cases}$$

Therefore

$$\ker \mathrm{V}_{H_{4,2}} = \begin{cases} \langle \sigma\tau G'_{m,n}, \rho G'_{m,n}\rangle & \text{if } \begin{cases} n=1 \text{ and } m\geq 3 \text{ or}\\ n\geq 2 \text{ and } m=2,\end{cases}\\ \langle \sigma\tau G'_{m,n}, \sigma\rho G'_{m,n}\rangle & \text{if } \begin{cases} n=1 \text{ and } m=2 \text{ or}\\ n\geq 2 \text{ and } m\geq 3.\end{cases}\end{cases}$$

(III) We know, from Table 2, that $H_{1,4} = \langle\sigma, \tau^2\rangle$, so $G_{m,n}/H_{1,4} = \{1, \tau H_{1,4}, \rho H_{1,4}, \tau\rho H_{1,4}\}$ and $H'_{1,4} = \langle 1\rangle$. Hence Corollary 2.6 and Lemma 2.1 yield

$$\begin{aligned} \mathrm{V}_{G_{m,n}\to H_{1,4}}(\sigma G'_{m,n}) &= \sigma\tau^{-1}\sigma\tau\rho^{-1}\sigma\tau^{-1}\sigma\tau\rho H'_{1,4} = H'_{1,4},\\ \mathrm{V}_{G_{m,n}\to H_{1,4}}(\tau G'_{m,n}) &= \tau^2\rho^{-1}\tau^2\rho H'_{1,4} = H'_{1,4},\\ \mathrm{V}_{G_{m,n}\to H_{1,4}}(\rho G'_{m,n}) &= \rho^2\tau^{-1}\rho^{-2}\tau H'_{1,4} = H'_{1,4}.\end{aligned}$$

Therefore $\ker \mathrm{V}_{H_{1,4}} = \langle \sigma G'_{m,n}, \tau G'_{m,n}, \rho G'_{m,n}\rangle = G_{m,n}/G'_{m,n}$. ■

3. Applications. Let $\mathbf{k}$ be a number field and $\mathrm{C}_{\mathbf{k},2}$ be its 2-class group, that is, the 2-Sylow subgroup of the ideal class group $\mathrm{C}_{\mathbf{k}}$ of $\mathbf{k}$, in the wide sense. Let $\mathbf{k}_2^{(1)}$ be the Hilbert 2-class field of $\mathbf{k}$ in the wide sense. Then the Hilbert 2-class field tower of $\mathbf{k}$ is defined inductively by $\mathbf{k}_2^{(0)} = \mathbf{k}$ and $\mathbf{k}_2^{(n+1)} = (\mathbf{k}_2^{(n)})^{(1)}$, where n is a positive integer. Let $\mathbb{M}$ be an unramified extension of $\mathbf{k}$ and $\mathrm{C}_{\mathbb{M}}$ be the subgroup of $\mathrm{C}_{\mathbf{k}}$ associated to $\mathbb{M}$ by class field theory. Denote by $j_{\mathbf{k}\to\mathbb{M}} : \mathrm{C}_{\mathbf{k}} \to \mathrm{C}_{\mathbb{M}}$ the homomorphism that associates to the class of an ideal $\mathcal{A}$ of $\mathbf{k}$ the class of the ideal generated by $\mathcal{A}$ in $\mathbb{M}$, and by $\mathcal{N}_{\mathbb{M}/\mathbf{k}}$ the norm of the extension $\mathbb{M}/\mathbf{k}$.

Throughout this section, assume that $\mathrm{Gal}(\mathbf{k}_2^{(2)}/\mathbf{k}) \simeq G_{m,n}$. Hence, according to class field theory, $\mathrm{C}_{\mathbf{k},2} \simeq G_{m,n}/G'_{m,n} \simeq (2,2,2)$, thus $\mathrm{C}_{\mathbf{k},2} = \langle \mathfrak{a}, \mathfrak{b}, \mathfrak{c}\rangle \simeq \langle \sigma G'_{m,n}, \tau G'_{m,n}, \rho G'_{m,n}\rangle$, where $(\mathfrak{a}, \mathbf{k}_2^{(2)}/\mathbf{k}) = \sigma G'_{m,n}$, $(\mathfrak{b}, \mathbf{k}_2^{(2)}/\mathbf{k}) = \tau G'_{m,n}$ and $(\mathfrak{c}, \mathbf{k}_2^{(2)}/\mathbf{k}) = \rho G'_{m,n}$, with $(\cdot, \mathbf{k}_2^{(2)}/\mathbf{k})$ denoting the Artin symbol in $\mathbf{k}_2^{(2)}/\mathbf{k}$.

It is well known that each subgroup $H_{i,j}$, where $1 \leq i \leq 7$ and $j = 2$ or 4, of $\mathrm{C}_{\mathbf{k},2}$ is associated, by class field theory, to a unique unramified extension $\mathbf{K}_{i,j}$ of $\mathbf{k}_2^{(1)}$ such that $H_{i,j}/H'_{i,j} \simeq \mathrm{C}_{\mathbf{K}_{i,j},2}$.

Our goal is to study the capitulation problem of the 2-ideal classes of $\mathbf{k}$ in its unramified quadratic and biquadratic extensions $\mathbf{K}_{i,2}$ and $\mathbf{K}_{i,4}$. By class field theory, $\ker j_{\mathbf{k}\to\mathbb{M}}$ is determined by the kernel of the transfer map $\mathrm{V}_{G\to H} : G/G' \to H/H'$, where $G = \mathrm{Gal}(\mathbf{k}_2^{(2)}/\mathbf{k})$ and $H = \mathrm{Gal}(\mathbb{M}_2^{(2)}/\mathbb{M})$.

THEOREM 3.1. *Keep the previous notation.*

(1) *If* $\rho^2 = \sigma^{2^{m-1}}$, *then*

$$\begin{aligned}
\ker j_{\mathbf{k}\to\mathbf{K}_{1,2}} &= \langle \mathfrak{a}, \mathfrak{b}\rangle,\\
\ker j_{\mathbf{k}\to\mathbf{K}_{2,2}} &= \langle \mathfrak{a}, \mathfrak{b}\mathfrak{c}\rangle,\\
\ker j_{\mathbf{k}\to\mathbf{K}_{3,2}} &= \begin{cases} \langle \mathfrak{b}, \mathfrak{c}\rangle & \text{if } m = 2,\\ \langle \mathfrak{b}, \mathfrak{a}\mathfrak{c}\rangle & \text{if } m \geq 3,\end{cases}\\
\ker j_{\mathbf{k}\to\mathbf{K}_{4,2}} &= \begin{cases} \langle \mathfrak{a}\mathfrak{b}, \mathfrak{c}\rangle & \text{if } m = 2,\\ \langle \mathfrak{a}\mathfrak{b}, \mathfrak{a}\mathfrak{c}\rangle & \text{if } m \geq 3,\end{cases}\\
\ker j_{\mathbf{k}\to\mathbf{K}_{5,2}} &= \begin{cases} \langle \mathfrak{b}, \mathfrak{a}\mathfrak{c}\rangle & \text{if } m = 2,\\ \langle \mathfrak{b}, \mathfrak{c}\rangle & \text{if } m \geq 3,\end{cases}\\
\ker j_{\mathbf{k}\to\mathbf{K}_{6,2}} &= \langle \mathfrak{a}, \mathfrak{c}\rangle,\\
\ker j_{\mathbf{k}\to\mathbf{K}_{7,2}} &= \begin{cases} \langle \mathfrak{a}\mathfrak{b}, \mathfrak{b}\mathfrak{c}\rangle & \text{if } m = 2,\\ \langle \mathfrak{a}\mathfrak{b}, \mathfrak{c}\rangle & \text{if } m \geq 3.\end{cases}
\end{aligned}$$

(2) *If* $\rho^2 = \tau^{2^n}\sigma^{2^{m-1}}$, *then*

$$\begin{aligned}
\ker j_{\mathbf{k}\to\mathbf{K}_{1,2}} &= \langle \mathfrak{a}, \mathfrak{b}\rangle,\\
\ker j_{\mathbf{k}\to\mathbf{K}_{2,2}} &= \begin{cases} \langle \mathfrak{a}, \mathfrak{c}\rangle & \text{if } n = 1,\\ \langle \mathfrak{a}, \mathfrak{b}\mathfrak{c}\rangle & \text{if } n \geq 2,\end{cases}\\
\ker j_{\mathbf{k}\to\mathbf{K}_{3,2}} &= \begin{cases} \langle \mathfrak{b}, \mathfrak{c}\rangle & \text{if } m = 2,\\ \langle \mathfrak{b}, \mathfrak{a}\mathfrak{c}\rangle & \text{if } m \geq 3,\end{cases}\\
\ker j_{\mathbf{k}\to\mathbf{K}_{4,2}} &= \begin{cases} \langle \mathfrak{a}\mathfrak{b}, \mathfrak{c}\rangle & \text{if } \begin{cases} n = 1 \text{ and } m \geq 3 \text{ or}\\ n \geq 2 \text{ and } m = 2,\end{cases}\\ \langle \mathfrak{a}\mathfrak{b}, \mathfrak{a}\mathfrak{c}\rangle & \text{if } \begin{cases} n = 1 \text{ and } m = 2 \text{ or}\\ n \geq 2 \text{ and } m \geq 3,\end{cases}\end{cases}\\
\ker j_{\mathbf{k}\to\mathbf{K}_{5,2}} &= \begin{cases} \langle \mathfrak{b}, \mathfrak{a}\mathfrak{c}\rangle & \text{if } m = 2,\\ \langle \mathfrak{b}, \mathfrak{c}\rangle & \text{if } m \geq 3,\end{cases}\\
\ker j_{\mathbf{k}\to\mathbf{K}_{6,2}} &= \begin{cases} \langle \mathfrak{a}, \mathfrak{b}\mathfrak{c}\rangle & \text{if } n = 1,\\ \langle \mathfrak{a}, \mathfrak{c}\rangle & \text{if } n \geq 2,\end{cases}\\
\ker j_{\mathbf{k}\to\mathbf{K}_{7,2}} &= \begin{cases} \langle \mathfrak{a}\mathfrak{b}, \mathfrak{b}\mathfrak{c}\rangle & \text{if } \begin{cases} n = 1 \text{ and } m \geq 3 \text{ or}\\ n \geq 2 \text{ and } m = 2,\end{cases}\\ \langle \mathfrak{a}\mathfrak{b}, \mathfrak{c}\rangle & \text{if } \begin{cases} n = 1 \text{ and } m = 2 \text{ or}\\ n \geq 2 \text{ and } m \geq 3.\end{cases}\end{cases}
\end{aligned}$$

(3) *For all* $1 \leq i \leq 7$, $\ker j_{\mathbf{k}\to\mathbf{K}_{i,4}} = \mathrm{C}_{\mathbf{k},2}$.

(4) *The* 2*-class group of* $\mathbf{k}_2^{(1)}$ *is of type* $(2^{m-1}, 2^n)$.

(5) *The Hilbert* 2*-class field tower of* $\mathbf{k}$ *stops at* $\mathbf{k}_2^{(2)}$.

Proof. (1) According to Theorem 2.7, since $\ker \mathrm{V}_{H_{1,2}} = \langle \sigma G'_{m,n}, \tau G'_{m,n}\rangle$, we have $\ker j_{\mathbf{k}\to\mathbf{K}_{1,2}} = \langle \mathfrak{a}, \mathfrak{b}\rangle$. Similarly, as

$$\ker \mathrm{V}_{H_{3,2}} = \begin{cases} \langle \tau G'_{m,n}, \rho G'_{m,n}\rangle & \text{if } m = 2,\\ \langle \tau G'_{m,n}, \sigma\rho G'_{m,n}\rangle & \text{if } m \geq 3,\end{cases}$$

we have

$$\ker j_{\mathbf{k}\to\mathbf{K}_{3,2}} = \begin{cases} \langle \mathfrak{b}, \mathfrak{c}\rangle & \text{if } m = 2,\\ \langle \mathfrak{b}, \mathfrak{ac}\rangle & \text{if } m \geq 3.\end{cases}$$

The other assertions are proved similarly.

(4) It is well known that $\mathrm{C}_{\mathbf{k}_2^{(1)},2} \simeq G'_{m,n}$, where $\mathrm{C}_{\mathbf{k}_2^{(1)},2}$ is the 2-class group of $\mathbf{k}_2^{(1)}$. As $G'_{m,n} = \langle \sigma^2, \tau^2\rangle \simeq (2^{m-1}, 2^n)$ since $\sigma^{2^m} = \tau^{2^{n+1}} = 1$, the result is proved.

(5) $H_{1,4}$, $H_{2,4}$ and $H_{4,4}$ are the three subgroups of index 2 of the group $H_{1,2}$, hence $\mathbf{K}_{1,4}$, $\mathbf{K}_{2,4}$ and $\mathbf{K}_{4,4}$ are the three unramified quadratic extensions of $\mathbf{K}_{1,2}$. On the other hand, the 2-class groups of these fields are of rank 2, since, by class field theory, $\mathrm{C}_{\mathbf{K}_{i,j},2} \simeq H_{i,j}/H'_{i,j}$ with $i = 1$, 2 or 4 and $j = 2$ or 4. Thus Tables 1, 2, 3 and 4 imply that $\mathrm{C}_{\mathbf{K}_{1,2},2} \simeq (2^m, 2^{n+1})$ and $\mathrm{C}_{\mathbf{K}_{1,4},2} \simeq (2^m, 2^n)$. Hence $h_2(\mathbf{K}_{1,4}) = h_2(\mathbf{K}_{1,2})/2$, where $h_2(K)$ denotes the 2-class number of the field K. Therefore, we can apply [BLS, Proposition 7], which says that $\mathbf{K}_{1,2}$ has an abelian 2-class field tower if and only if it has a quadratic unramified extension $\mathbf{K}_{1,4}/\mathbf{K}_{1,2}$ such that $h_2(\mathbf{K}_{1,4}) = h_2(\mathbf{K}_{1,2})/2$. Thus $\mathbf{K}_{1,2}$ has abelian 2-class field tower which terminates at the first stage; this implies that the 2-class field tower of $\mathbf{k}$ terminates at $\mathbf{k}_2^{(2)}$, since $\mathbf{k} \subset \mathbf{K}_{1,2}$. Moreover, we know from Proposition 2.2 that $|G_{m,n}| = 2^{n+m+2}$ and $|G'_{m,n}| = 2^{n+m-1}$, hence $\mathbf{k}_2^{(1)} \neq \mathbf{k}_2^{(2)}$. ■

4. Example. Let $p_1 \equiv p_2 \equiv 5 \pmod 8$ be different primes. Denote by $\mathbf{k}$ the imaginary bicyclic biquadratic field $\mathbb{Q}(\sqrt{d}, i)$, where $d = 2p_1p_2$. Let $\mathbf{k}_2^{(1)}$ be the Hilbert 2-class field of $\mathbf{k}$, $\mathbf{k}_2^{(2)}$ its second Hilbert 2-class field, and G the Galois group of $\mathbf{k}_2^{(2)}/\mathbf{k}$. According to [AT], $\mathbf{k}$ has an elementary abelian 2-class group $\mathbf{C}_{\mathbf{k},2}$ of rank 3, that is, of type $(2, 2, 2)$. Set $\mathbf{K} = \mathbf{k}(\sqrt{2}) = \mathbb{Q}(\sqrt{2}, \sqrt{p_1p_2}, \sqrt{-1})$, and let q denote the unit index of $\mathbf{K}^+ = \mathbb{Q}(\sqrt{2}, \sqrt{p_1p_2})$. Denote by $h_2(-p_1p_2)$ (resp. $h_2(p_1p_2)$) the 2-class number of $\mathbb{Q}(\sqrt{-p_1p_2})$ (resp. $\mathbb{Q}(\sqrt{p_1p_2})$). Then, from [Ka], we have $h_2(-p_1p_2) = 2^{m+1}$ with $m \geq 2$, and $h_2(p_1p_2) = 2^n$ with $n \geq 1$. Assume that $q = 1$. Then, by [AZT3, Lemma 6], $m \geq 2$ and $n \geq 1$, and by [AZT3, Theorem 2], $G \simeq G_{m,n}$. The following result is proved in [AZT3], and we give it here to illustrate the results shown above. For more details, see [AZT3].

THEOREM 4.1. *Let $p_1 \equiv p_2 \equiv 5 \pmod 8$ be two different primes. Set $\mathbf{k} = \mathbb{Q}(\sqrt{2p_1p_2}, i)$. Then $\mathbf{k}$ has fourteen unramified extensions within its first Hilbert 2-class field $\mathbf{k}_2^{(1)}$ (see [AZT1]). Denote by $\mathbf{C}_{\mathbf{k},2}$ the 2-class group of $\mathbf{k}$. Then the following assertions hold:*

(1) *Exactly four elements of $\mathbf{C}_{\mathbf{k},2}$ capitulate in each unramified quadratic extension of $\mathbf{k}$.*
(2) *All the 2-classes of $\mathbf{k}$ capitulate in each unramified biquadratic extension of $\mathbf{k}$.*
(3) *The Hilbert 2-class field tower of $\mathbf{k}$ stops at $\mathbf{k}_2^{(2)}$ (see [AZT2]).*
(4) $\mathbf{C}_{\mathbf{k}_2^{(1)},2} \simeq (2^n, 2^{m-1})$.
(5) *The coclass of G is 3 and its nilpotency class is $n+2$.*
(6) *The 2-class groups of the unramified quadratic extensions of $\mathbf{k}$ are of types $(2,4)$, $(2,2,2)$ or $(2^m, 2^{n+1})$.*
(7) *The 2-class groups of the unramified biquadratic extensions of $\mathbf{k}$ are of types $(2,4)$, $(2,2,2)$, $(2^m, 2^n)$, $(2^{m-1}, 2^{n+1})$ or $(2^{\min(m-1,n)}, 2^{\max(m,n+1)})$.*

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Abdelmalek Azizi
Department of Mathematics
Faculty of Sciences
Mohammed First University
Oujda, Morocco
E-mail: abdelmalekazizi@yahoo.fr

Mohammed Taous
Department of Mathematics
Faculty of Sciences and Technology
Moulay Ismail University
Errachidia, Morocco
E-mail: taousm@hotmail.com

Abdelkader Zekhnini
Department of Mathematics
Polydisciplinary Faculty of Nador
Mohammed First University
Nador, Morocco
E-mail: zekha1@yahoo.fr

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