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ON THE UMD CONSTANT OF THE SPACE ℓ_1^N

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Abstract. Let $N \geq 2$ be a given integer. Suppose that $df = (df_n)_{n\geq 0}$ is a martingale difference sequence with values in ℓ_1^N and let $(\varepsilon_n)_{n\geq 0}$ be a deterministic sequence of signs. The paper contains the proof of the estimate

$$\mathbb{P}\Big(\sup_{n\geq 0}\Big\|\sum_{k=0}^n \varepsilon_k df_k\Big\|_{\ell_1^N} \geq 1\Big) \leq \frac{\ln N + \ln(3\ln N)}{1-(2\ln N)^{-1}} \sup_{n\geq 0} \mathbb{E}\Big\|\sum_{k=0}^n df_k\Big\|_{\ell_1^N}.$$

It is shown that this result is asymptotically sharp in the sense that the least constant C_N in the above estimate satisfies $\lim_{N\to\infty} C_N/\ln N = 1$. The novelty in the proof is the explicit verification of the ζ-convexity of the space ℓ_1^N.

1. Introduction. Suppose that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, equipped with a *filtration* $(\mathcal{F}_n)_{n\geq 0}$, i.e., a non-decreasing sequence of sub-σ-fields of $\mathcal{F}$. Assume further that $(\mathbb{B}, \|\cdot\|_{\mathbb{B}})$ is a Banach space and let $f = (f_n)_{n\geq 0}$ be an adapted martingale taking values in $\mathbb{B}$. Then we may define $df = (df_n)_{n\geq 0}$, the *difference sequence* of f, by the formulas $df_0 = f_0$ and $df_n = f_n - f_{n-1}$, $n \geq 1$. A Banach space $\mathbb{B}$ is said to be a *UMD space* (where UMD stands for Unconditional for Martingale Differences) if for some $1 < p < \infty$ (equivalently, for all $1 < p < \infty$) there is a finite constant $\beta = \beta_p$ with the following property: for any deterministic sequence $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$ with values in $\{-1, 1\}$ and any f as above,

$$\Big\|\sum_{k=0}^n \varepsilon_k df_k\Big\|_{L_p(\Omega;\mathbb{B})} \leq \beta_p \Big\|\sum_{k=0}^n df_k\Big\|_{L_p(\Omega;\mathbb{B})}, \qquad n = 0, 1, 2, \ldots. \tag{1.1}$$

For given p and $\mathbb{B}$, let $\beta_{p,\mathbb{B}}$ denote the smallest possible value of the constant β_p allowed above. Then, as shown by Burkholder [6], we have $\beta_{p,\mathbb{R}} = p^*-1$, where $p^* = \max\{p, p/(p-1)\}$. Actually, the same is true if $\mathbb{R}$ is replaced by any separable Hilbert space $\mathcal{H}$ (cf. [8]). By Fubini's theorem, this yields $\beta_{p,L_p(X;\mathcal{H})} = p^* - 1$ for $1 < p < \infty$, where $L_p(X;\mathcal{H})$ denotes the L_p-space of $\mathcal{H}$-valued functions on a given measurable space X. Thus, Hilbert spaces and L_p-spaces are UMD. Other examples include all finite-dimensional Banach spaces, reflexive Orlicz spaces, reflexive trace-class spaces and the reflexive

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non-commutative $L_p(M,\tau)$-spaces associated with a von Neumann algebra M having a faithful, normal, semifinite trace τ. But for these, the values of the corresponding constants $\beta_{p,\mathbb{B}}$ are not known. The negative examples include the spaces ℓ_1, ℓ_∞, $L_1(0,1)$ and $L^\infty(0,1)$. Actually, as Aldous proved in [1], any UMD space is superreflexive (but on the other hand, there are superreflexive spaces which are not UMD: see the work of Pisier [17]).

Many classical results from harmonic analysis on Hilbert spaces carry over to the UMD setting. For example, these spaces arise when one tries to extend the work of M. Riesz on the L_p-boundedness of the Hilbert transform, and that of Calderón and Zygmund on more general singular integral operators, to the case of functions with values in a Banach space. To be more specific, let $1 < p < \infty$ be a fixed number. It turns out that the (periodic) Hilbert transform is bounded as an operator on $L_p(\mathbb{T};\mathbb{B})$ if and only if $\mathbb{B}$ has the UMD property: this equivalence is due to Burkholder and McConnell (see [5]), who showed that UMD spaces are well-behaved for the Hilbert transform, and Bourgain [3], who established the reverse implication. This, by the use of Calderón–Zygmund's method of rotations, shows that UMD spaces form a natural context for the study of singular integrals with odd kernels. These spaces also provide the right setting for the study of evolution equations (cf. Coulhon and Lamberton [11]), the closedness of the sum of two closed operators (see Dore and Venni [12]), spectral theory (Berkson, Gillespie and Muhly [2]), multiplier theory (see Hytönen [13], McConnell [15]), and many other areas.

In the early eighties, Burkholder provided a beautiful geometrical characterization of UMD spaces. To recall it, we need some more definitions. Suppose that $D \subseteq \mathbb{B}\times\mathbb{B}$ is a *biconvex set*, i.e., for any $z\in\mathbb{B}$, the sections $\{x\in\mathbb{B} : (x,z)\in D\}$ and $\{y\in\mathbb{B} : (z,y)\in D\}$ are convex subsets of $\mathbb{B}$. A function $\zeta : D\to\mathbb{R}$ is called *biconvex* if for any $z\in\mathbb{B}$ the functions $x\mapsto\zeta(x,z)$ and $y\mapsto\zeta(z,y)$ are convex. Let $\mathbb{K}=\mathbb{K}_{\mathbb{B}}$ be the unit ball of $\mathbb{B}$. Following Burkholder [4], we say that $\mathbb{B}$ is ζ-*convex* if there is a biconvex function ζ on $\mathbb{K}_{\mathbb{B}}\times\mathbb{K}_{\mathbb{B}}$ satisfying

$$\zeta(0,0) > 0, \tag{1.2}$$

$$\zeta(x,y) \le \|x+y\|_{\mathbb{B}} \qquad \text{if } \|x\|_{\mathbb{B}} = \|y\|_{\mathbb{B}} = 1. \tag{1.3}$$

Burkholder showed (see [4] and [7, Lemma 3.1]) that $\mathbb{B}$ is UMD if and only if it is ζ-convex.

Let us explain the interplay between the existence of such a function and the validity of (1.1). If there is ζ satisfying (1.2) and (1.3), then

$$\mathbb{P}\Big(\sup_n \Big\|\sum_{k=0}^n \varepsilon_k df_k\Big\|_{\mathbb{B}} \ge 1\Big) \le \frac{2}{\zeta(0,0)} \sup_n \Big\|\sum_{k=0}^n df_k\Big\|_{L_1(\Omega;\mathbb{B})}. \tag{1.4}$$

Now, using the classical good-lambda approach of Burkholder and Gundy

[10], one proves that for $1 < p < \infty$,

$$\Big\|\sum_{k=0}^{n} \varepsilon_k df_k\Big\|_{L_p(\Omega;\mathbb{B})} \le \frac{72}{\zeta(0,0)} \cdot \frac{(p+1)^2}{p-1} \Big\|\sum_{k=0}^{n} df_k\Big\|_{L_p(\Omega;\mathbb{B})} \tag{1.5}$$

for $n = 0, 1, 2, \ldots$. And conversely: Burkholder showed (see e.g. [7, Section 6]) that the validity of (1.1) for some given $1 < p < \infty$ implies the existence of a biconvex function ζ on the whole $\mathbb{B} \times \mathbb{B}$, which enjoys $\zeta(0,0) \ge (\beta_{p,\mathbb{B}})^{-1}$ and the property (1.3). This was done by providing an abstract, non-explicit formula for ζ.

For a general UMD space $\mathbb{B}$, the class of all biconvex functions ζ satisfying (1.2) and (1.3) is infinite. Indeed, if ζ satisfies (1.2) and (1.3), then any convex combination of ζ and the function $(x, y) \mapsto \|x + y\|_{\mathbb{B}}$ also has all the required properties. Nonetheless, one can distinguish a certain extremal element: it can be proved that there is the largest function in this class, namely $\zeta_{\mathbb{B}}(x, y) = \sup_\zeta \zeta(x, y)$ for all $x, y \in \mathbb{K}$ (see [4], [7]). This extremal object brings a lot of information on the size of optimal constants in the weak- and strong-type estimates above. More precisely, it can be shown that the constant $2/\zeta_{\mathbb{B}}(0, 0)$ in (1.4) is the best possible (cf. [4]). Furthermore, it follows from (1.5) and [7, Section 6] that

$$\frac{1}{\zeta_{\mathbb{B}}(0,0)} \le \beta_{p,\mathbb{B}} \le \frac{72}{\zeta_{\mathbb{B}}(0,0)} \cdot \frac{(p+1)^2}{p-1}.$$

Thus, for a given UMD space $\mathbb{B}$, it is of significant interest to find the explicit formula for $\zeta_{\mathbb{B}}$ or, at least, to identify the value $\zeta_{\mathbb{B}}(0, 0)$. This is a very difficult task, as it requires the understanding of the very delicate geometrical structures of $\mathbb{B}$.

So far, this problem has been successfully solved for Hilbert spaces only. More precisely, Burkholder [7] showed that

$$\zeta_{\mathbb{B}}(x, y) = [1 + 2\langle x, y\rangle_{\mathbb{B}} + \|x\|_{\mathbb{B}}^2 \|y\|_{\mathbb{B}}^2]^{1/2},$$

where $\langle\cdot,\cdot\rangle_{\mathbb{B}}$ denotes the scalar product in $\mathbb{B}$. For non-Hilbert spaces, essentially nothing is known. The only non-trivial result is the formula for a function ζ when $\mathbb{B} = L_p(X; \mathcal{H})$ is the space of p-integrable functions on a fixed measure space (X, μ) taking values in a certain Hilbert space $\mathcal{H}$, $1 < p < \infty$. In that case, one can take

$$\zeta(x, y) = \frac{2}{1 + (p^* - 1)^p} \Big[1 - \int_X U(x(s), y(s))\, d\mu(s)\Big],$$

where, for $a, b \in \mathcal{H}$,

$$U(a, b) = \alpha_p \Big\{ \Big\|\frac{a+b}{2}\Big\|_{\mathcal{H}} - (p^* - 1)\Big\|\frac{a-b}{2}\Big\|_{\mathcal{H}} \Big\} \Big\{ \Big\|\frac{a+b}{2}\Big\|_{\mathcal{H}} + \Big\|\frac{a-b}{2}\Big\|_{\mathcal{H}} \Big\}^{p-1}$$

and $\alpha_p = p(1-1/p^*)^{p-1}$. See [9] for details. However, this function is far from being optimal: we have

$$\zeta(0,0) = \frac{2}{1+(p^*-1)^p},$$

and the inequality (1.5) gives a constant of order $O(p^p)$ as $p \to \infty$, while the correct order is $O(p)$.

The main contribution of the present paper is to provide an explicit formula for the function ζ in the case when $\mathbb{B} = \ell_1^N = \ell_1^N(\mathcal{H})$, where $\mathcal{H}$ is a given separable Hilbert space and $N \geq 2$.

Theorem 1.1. *Let $N \geq 2$. There is a biconvex function $\zeta : \mathbb{K}_{\ell_1^N} \times \mathbb{K}_{\ell_1^N} \to \mathbb{R}$ which satisfies the conditions*

$$\zeta(0,0) = \frac{2}{\ln N + \ln(3\ln N)}\left(1 - \frac{1}{2\ln N}\right), \tag{1.6}$$

$$\zeta(x,y) \leq \|x+y\|_{\ell_1^N} \quad \text{if } \|x\|_{\ell_1^N} = \|y\|_{\ell_1^N} = 1. \tag{1.7}$$

The above function is close to $\zeta_{\ell_1^N}$ in the following sense. Observe that when $N \to \infty$, the value $\zeta(0,0)$ above behaves like $2/\ln N$ (in the sense that the ratio of these two quantities tends to 1). The order $1/\ln N$ and the factor 2 in the numerator are both optimal even when $\mathcal{H} = \mathbb{R}$, as the following statement indicates.

Theorem 1.2. *Let $\mathcal{H} = \mathbb{R}$. Then for any $N \geq 2$ we have*

$$\zeta_{\ell_1^N}(0,0) \leq \frac{2}{\ln(2N)}.$$

As a by-product, we obtain the following information on the size of the constants in the weak- and strong-type estimates discussed above.

Corollary 1.3.

(i) *For any $N \geq 2$,*

$$\mathbb{P}\Big(\sup_n \Big\|\sum_{k=0}^n \varepsilon_k df_k\Big\|_{\ell_1^N} \geq 1\Big) \leq \frac{\ln N + \ln(3\ln N)}{1-(2\ln N)^{-1}} \sup_n \Big\|\sum_{k=0}^n df_k\Big\|_{L_1(\Omega;\ell_1^N)}$$

and the least constant C_N here satisfies $\lim_{N\to\infty} C_N/\ln N = 1$.

(ii) *For any $1 < p < \infty$ and $n = 0,1,2,\ldots$,*

$$\Big\|\sum_{k=0}^n \varepsilon_k df_k\Big\|_{L_p(\Omega;\ell_1^N)} \leq \frac{36(p+1)^2}{p-1} \cdot \frac{\ln N + \ln(3\ln N)}{1-(2\ln N)^{-1}} \Big\|\sum_{k=0}^n df_k\Big\|_{L_p(\Omega;\ell_1^N)}.$$

Furthermore, β_{p,ℓ_1^N} is of order $O(\ln N)$ as $N \to \infty$.

While the behavior of the constants β_{p,ℓ_1^N} as $N \to \infty$ is well-known, the above precise information on the weak-type constants seems to be new. This result should be compared to a related "dual" result for ℓ_∞^N, obtained by the author in [16], in the context of a different geometrical characterization of UMD spaces obtained by Lee [14].

The remainder of this paper is organized as follows. The next section contains the construction of the function ζ of Theorem 1.1. The proof of Theorem 1.2 can be found in Section 3.

2. A biconvex function for ℓ_1^N. From now on, $\mathcal{H}$ will be a fixed separable Hilbert space, with a norm $|\cdot|$ and a scalar product denoted by $\langle\cdot,\cdot\rangle$. Let $a > 0$ be a fixed parameter. The first step of the construction of ζ is to introduce an auxiliary special function $z = z^a : \mathcal{H}\times\mathcal{H} \to \mathbb{R}$. If $|x+y| + |x-y| \le 2/a$, define

$$z(x,y) = \frac{a\langle x,y\rangle}{2} - \frac{1}{2a}.$$

On the other hand, if $|x+y|+|x-y| > 2/a$, set

$$z(x,y) = \frac{|x+y|}{2}\ln\left[\frac{a}{2}(|x+y|+|x-y|)\right] - \frac{|x-y|}{2}.$$

It is easy to see that the function z is continuous (simply use the identity $\langle x,y\rangle = (|x+y|^2 - |x-y|^2)/4$). Let us study further crucial properties of this function.

LEMMA 2.1. *The function z is biconvex on $\mathcal{H}\times\mathcal{H}$.*

Proof. Observe that z satisfies the symmetry property $z(x,y) = z(y,x)$ for all $x, y \in \mathcal{H}$. Consequently, it is enough to establish the convexity with respect to the first variable. So, fix $x, y, h \in \mathcal{H}$ and consider the function $G = G_{x,y,h} : \mathbb{R} \to \mathbb{R}$ given by

$$G(t) = z(x+th, y).$$

We must show that G is convex.

By continuity of z, we may assume that $|x+y+th|$ and $|x-y+th|$ are non-zero for all $t \in \mathbb{R}$ (indeed, if this is not the case, it suffices to add to x a small vector orthogonal to the subspace spanned by y and h). Then, as we shall prove now, G is of class C^1. This is evident if we have $|x+y+th| + |x+y-th| \ge 2/a$ for all $t \in \mathbb{R}$. On the other hand, if there is $t \in \mathbb{R}$ for which $|x+y+th| + |x-y+th| < 2/a$, then there exist $t_-, t_+ \in \mathbb{R}$, $t_- < t_+$, such that $|x+y+t_\pm h| + |x-y+t_\pm h| = 2/a$. Now we verify directly that

$$\frac{d}{dt}\left[\frac{a\langle x+th,y\rangle}{2} - \frac{1}{2a}\right]\bigg|_{t=t_\pm} = \frac{a\langle h,y\rangle}{2}$$

and

$$\begin{aligned}
&\frac{d}{dt}\left\{\frac{|x+y+th|}{2}\ln\left[\frac{a}{2}(|x+y+th|+|x-y+th|)\right]-\frac{|x-y+th|}{2}\right\}\Bigg|_{t=t_\pm}\\
&\qquad=\frac{|x+y+t_\pm h|}{2}\cdot\frac{a}{2}\left(\frac{\langle x+y+t_\pm h,h\rangle}{|x+y+t_\pm h|}+\frac{\langle x-y+t_\pm h,h\rangle}{|x-y+t_\pm h|}\right)\\
&\qquad\quad-\frac{\langle x-y+t_\pm h,h\rangle}{|x-y+t_\pm h|}\\
&\qquad=\frac{a\langle h,y\rangle}{2}.
\end{aligned}$$

This yields the smoothness of G.

So, to show the desired convexity, it is enough to check that $G''(t)\geq 0$ provided $|x+y+th|+|x-y+th|\neq 2/a$ (clearly, then the second derivative exists). Since G has the translation property $G_{x,y,h}(t+s)=G_{x+th,y,h}(s)$, it suffices to prove the inequality in question for $t=0$. If $|x+y|+|x-y|<2/a$, then $G''(0)=0$; on the other hand, if $|x+y|+|x-y|>2/a$, some tedious calculations show that $G''(0)=\mathrm{I}+\mathrm{II}$, where

$$\begin{aligned}
\mathrm{I}&=\frac{1}{2}\left(\frac{|h|^2|x+y|^2-\langle h,x+y\rangle^2}{|x+y|^3}\right)\ln\left[\frac{a}{2}(|x+y|+|x-y|)\right],\\
\mathrm{II}&=\frac{|x-y|}{2(|x+y|+|x-y|)^2}\left[\frac{\langle x+y,h\rangle}{|x+y|}+\frac{\langle x-y,h\rangle}{|x-y|}\right]^2.
\end{aligned}$$

Of course, both I and II are non-negative, and hence so is $G''(0)$. ∎

In our further considerations, we will also make use of the following majorization.

LEMMA 2.2. *If x,y belong to the unit ball of $\mathcal{H}$ and $a\geq\sqrt{e}/3$, then*

$$z(x,-x)\leq -|x|, \tag{2.1}$$

$$z(x,2y+x)\leq \ln(3a)\cdot|x+y|-|y|. \tag{2.2}$$

Proof. The estimate (2.1) is evident: if $|x|\leq 1/a$, then the inequality is equivalent to $(|x|-a^{-1})^2\geq 0$; if $|x|>1/a$, then $z(x,-x)=-|x|$. To show (2.2), suppose first that $|x+y|+|y|>1/a$. Then the majorization can be rewritten in the form

$$\ln[a(|x+y|+|y|)]\leq\ln(3a),$$

which follows directly from the assumption $|x|,|y|\leq 1$. On the other hand, if $|x+y|+|y|\leq 1/a$, then we must prove that

$$\frac{a}{2}(|x+y|^2-|y|^2)-\frac{1}{2a}\leq\ln(3a)\cdot|x+y|-|y|,$$

or equivalently,

$$|x+y|\left(\ln(3a)-\frac{a|x+y|}{2}\right)+\frac{a}{2}(|y|-a^{-1})^2\geq 0. \tag{2.3}$$

However, $|x+y|\leq 1/a$; furthermore, $a\geq\sqrt{e}/3$, as we have assumed in the statement of the lemma. Therefore,

$$\ln(3a)-\frac{a|x+y|}{2}\geq \ln\sqrt{e}-\frac{1}{2}=0.$$

So, the first summand on the left-hand side of (2.3) is non-negative; clearly the second summand also has this property. ■

We are ready to introduce the formula for a function ζ corresponding to the UMD space $\ell_1^N=\ell_1^N(\mathcal{H})$. Actually, we will provide a whole family of special functions. Recall that $\mathbb{K}_{\ell_1^N}$ denotes the unit ball of ℓ_1^N. For a fixed $a\geq\sqrt{e}/3$, let $\zeta=\zeta^a:\mathbb{K}_{\ell_1^N}\times\mathbb{K}_{\ell_1^N}\to\mathbb{R}$ be given by

$$\zeta(x,y)=\frac{2}{\ln(3a)}\Big(1+\sum_{j=1}^N z(x_j,y_j)\Big),$$

where $x_1,\dots,x_N,y_1,\dots,y_N\in\mathcal{H}$ are the coordinates of the vectors $x,y\in\ell_1^N$.

THEOREM 2.3. *For any $a>N/2$, the function $\zeta=\zeta^a$ is biconvex, and*

$$\zeta(0,0)=\frac{2}{\ln(3a)}\left(1-\frac{N}{2a}\right), \tag{2.4}$$

$$\zeta(x,y)\leq\|x+y\|_{\ell_1^N}\quad \textit{provided } \|x\|_{\ell_1^N}=\|y\|_{\ell_1^N}=1. \tag{2.5}$$

Proof. The biconvexity of ζ follows at once from Lemma 2.1. The equality (2.4) is also clear. To show (2.5), note that the condition $\|x\|_{\ell_1^N}=\|y\|_{\ell_1^N}=1$ implies that for each $j=1,\dots,N$, the coordinates x_j, y_j belong to the unit ball of $\mathcal{H}$. Furthermore, since $N\geq 2$, we have $N/2\geq 1>\sqrt{e}/3$. Consequently, we are allowed to apply (2.1) and (2.2) to x_j and y_j, and obtain

$$z(x_j,-x_j)\leq -|x_j|,\qquad z(x_j,x_j+2y_j)\leq\ln(3a)\cdot|x_j+y_j|-|y_j|.$$

Summing over $j=1,\dots,N$, we get

$$1+\sum_{j=1}^N z(x_j,-x_j)\leq 1-\|x\|_{\ell_1^N}=0$$

and

$$1+\sum_{j=1}^N z(x_j,x_j+2y_j)\leq 1+\ln(3a)\|x+y\|_{\ell_1^N}-\|y\|_{\ell_1^N}=\ln(3a)\|x+y\|_{\ell_1^N}.$$

These two estimates combined with the biconvexity of z imply

$$\begin{aligned}\zeta(x,y) &= \frac{2}{\ln(3a)}\Big(1+\sum_{j=1}^{N} z(x_j,y_j)\Big)\\ &\leq \frac{1}{2}\cdot\frac{2}{\ln(3a)}\Big(1+\sum_{j=1}^{N} z(x_j,-x_j)\Big)+\frac{1}{2}\cdot\frac{2}{\ln(3a)}\Big(1+\sum_{j=1}^{N} z(x_j,x_j+2y_j)\Big)\\ &\leq \|x+y\|_{\ell_1^N}. \ \blacksquare\end{aligned}$$

To establish Theorem 1.1, it suffices to set $a = N\ln N$. Up to a numerical factor, this choice maximizes the right-hand side of (2.4) over all admissible values of the parameter a.

3. An upper bound for $\zeta_{\ell_1^N}(0,0)$. Now we turn to the proof of Theorem 1.2. In the light of the discussion in the introductory section, it suffices to provide an efficient *lower* bound for the best constant C_N in the estimate

$$\mathbb{P}\Big(\sup_n \Big\|\sum_{k=0}^{n} \varepsilon_k df_k\Big\|_{\ell_1^N} \geq 1\Big) \leq C_N \sup_n \Big\|\sum_{k=0}^{n} df_k\Big\|_{L_1(\Omega;\ell_1^N)}, \tag{3.1}$$

i.e., we need to construct appropriate examples. Let N, K be positive integers and set $\delta = (N-1)/(2NK)$. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a given non-atomic probability space. Consider the sequence $(\xi_j)_{j=1}^{2K+1}$ of independent, real-valued random variables with the distribution uniquely determined by the following requirements:

(i) We have

$$\mathbb{P}\big(\xi_1 = -(2N)^{-1}\big) = \mathbb{P}\big(\xi_1 = (2N)^{-1}\big) = 1/2.$$

(ii) For $n = 2,3,\ldots,2K$,

$$\mathbb{P}\big(\xi_n = -N^{-1}-(n-2)\delta\big) = 1-\mathbb{P}(\xi_n=\delta) = \frac{\delta}{N^{-1}+(n-1)\delta}.$$

(iii) We have

$$\mathbb{P}(\xi_{2K+1} = -1+\delta) = 1-\mathbb{P}(\xi_{2K+1}=1+\delta) = \frac{1+\delta}{2}.$$

Observe that the variables ξ_n have mean zero.

Let ε be a Rademacher variable, independent of $(\xi_n)_{n=1}^{2K+1}$. Introduce $\tau = \inf\{n : \xi_n \leq 0 \text{ or } n = 2K+1\}$; then τ is a stopping time with respect to the natural filtration of the sequence $(\xi_n)_{n=1}^{2K+1}$, so by Doob's optional sampling theorem, the process

$$f_n = \varepsilon\big((2N)^{-1}+\xi_1+\cdots+\xi_{\tau\wedge n}\big), \qquad n = 0,1,\ldots,2K+1,$$

is a mean-zero martingale. Let $g=(g_n)_{n=0}^{2K+1}$ be the transform of f by the deterministic sequence $v=((-1)^n)_{n=0}^{2K+1}$, that is,

$$g_n=\sum_{k=0}^{n}(-1)^k df_k=\frac{\varepsilon}{2N}+\sum_{k=1}^{n}(-1)^k\varepsilon\xi_k,\qquad n=0,1,\dots,2K+1.$$

To gain some intuition about the pair (f,g), let us look at the pattern of its behavior. Because of the random sign ε, we see that the variable (f_0,g_0) takes the values $\left(\pm\frac{1}{2N},\pm\frac{1}{2N}\right)$ (each with probability 1/2). Suppose that (f_0,g_0) is equal to $\left(\frac{1}{2N},\frac{1}{2N}\right)$ (if it equals $\left(-\frac{1}{2N},-\frac{1}{2N}\right)$, the movement is symmetric with respect to $(0,0)$). Then (f_1,g_1) moves along the line of slope -1 and jumps either to $(0,1/N)$ or to $(1/N,0)$. If the first possibility occurs, then $\tau=1$ (that is, $(\xi_n)_{n=0}^{2K+1}$ experiences its first negative jump) and the evolution of (f,g) stops (that is, $f_1=\dots=f_{2K+1}$, $g_1=\dots=g_{2K+1}$). If $(f_1,g_1)=(1/N,0)$, then (f,g) starts moving along the line of slope 1, and goes to $(0,-1/N)$ or to $(1/N+\delta,\delta)$. In the first case, we see that $\tau=2$ and (f,g) stops. If $(f_2,g_2)=(1/N+\delta,\delta)$, then the pair continues its evolution and moves along the line of slope -1, jumping to $(0,1/N+2\delta)$ or to $(1/N+2\delta,0)$. In the first case the pair stops, in the second its evolution continues, according to the above pattern. The procedure (almost) finishes after $2K$ steps: by this time, (f,g) either has already landed on the y-axis, or gets to the point $(1/N+(2K-1)\delta,\delta)=(1-\delta,\delta)$; in the latter case, the pair makes its final, $2K+1$-st move, either jumping to $(0,1)$ or to $(2,-1)$.

From the above description, we immediately extract several useful properties of the sequence (f,g). First, the martingales f, g are *simple*, i.e., they are finite and for each n the variables f_n and g_n take only a finite number of values. Secondly, we see that the martingale f does not change its sign (more precisely, $\operatorname{sgn} f_n=\operatorname{sgn}\varepsilon$), and hence $\|f\|_{L_1(\Omega;\mathbb{R})}=\sup_n\|f_n\|_{L_1(\Omega;\mathbb{R})}=\mathbb{E}|f_0|=(2N)^{-1}$. Finally, we easily compute the distribution of the variable $|g_{2K+1}|$. From the above discussion, it is clear that it takes values in the set $\{N^{-1},N^{-1}+2\delta,N^{-1}+4\delta,\dots,1\}$. So, if $N=1$, then $|g_{2K+1}|=1$ almost surely. On the other hand, if $N\ge 2$, then we see that

$$\mathbb{P}(|g_{2K+1}|=N^{-1}+2n\delta)=\mathbb{P}(\tau=2n+1\text{ or }\tau=2n+2)$$

for $n=0,1,\dots,K-1$, and

$$\mathbb{P}(|g_{2K+1}|=N^{-1}+2K\delta)=\mathbb{P}(\tau=2K+1).$$

These probabilities are easy to compute. We have

$$\begin{aligned}\mathbb{P}(\tau=2n+1)&=\mathbb{P}(\xi_1>0,\dots,\xi_{2n}>0,\xi_{2n+1}\le 0)\\&=\mathbb{P}(\xi_1>0)\dots\mathbb{P}(\xi_{2n}>0)\mathbb{P}(\xi_{2n+1}\le 0).\end{aligned}$$

Using the above information on the distribution of $(\xi_n)_{n=0}^{2K+1}$, we get

$$\mathbb{P}(\tau = 2n+1) = \begin{cases} 1/2 & \text{if } n = 0, \\ \dfrac{(2N)^{-1}}{N^{-1} + (2n-1)\delta} \cdot \dfrac{\delta}{N^{-1} + 2n\delta} & \text{if } n = 1, \ldots, K-1, \\ \dfrac{(2N)^{-1}}{N^{-1} + (2K-1)\delta} & \text{if } n = K. \end{cases}$$

Similarly, one derives that for $n = 1, \ldots, K-1$,

$$\mathbb{P}(\tau = 2n+2) = \frac{(2N)^{-1}}{N^{-1} + 2n\delta} \cdot \frac{\delta}{N^{-1} + (2n+1)\delta}.$$

Now we can construct the ℓ_1^N-valued extremal martingales. The definition is inductive:

THEOREM 3.1. *Let $(\zeta_N)_{N\geq 1}$ be a sequence defined by the recursion*

$$\zeta_1 = \frac{1}{2}, \qquad \zeta_N = \frac{1}{2N} + \left(1 - \frac{1}{N} - \frac{\ln N}{2N}\right)\zeta_{N-1}.$$

For any positive integer N and any positive number η there is an ℓ_1^N-valued, mean-zero simple martingale F satisfying $\|F\|_{L_1(\Omega;\ell_1^N)} \leq \zeta_N + \eta$ such that its transform G by the deterministic sequence $((-1)^n)_{n\geq 0}$ satisfies $\mathbb{P}(\sup_n \|G_n\|_{\ell_1^N} \geq 1) = 1$.

Proof. For $N = 1$, we use the above example with $K = 1$: then $\|F\|_{L_1(\Omega;\mathbb{R})} = 1/2$ and $\mathbb{P}(|G_3| \geq 1) = 1$, so the required conditions are satisfied. Now suppose that $N \geq 2$ and that the assertion of the theorem holds for $N-1$. For a given $\eta > 0$, let $\tilde{F}$ be the ℓ_1^{N-1}-valued martingale given by the inductive assumption and let $f = (f_n)_{n=0}^{2K+1}$ be a martingale as in the above construction. We define F as follows: for $n = 0, 1, \ldots, 2K+1$ we set $F_n = (f_n, 0, \ldots, 0)$ ($N-1$ zeros). To define F_n for $n > 2K+1$, pick an arbitrary atom A of the σ-algebra generated by $f_1, \ldots, f_{2K+1}$, satisfying $\mathbb{P}(A) > 0$. On this atom the random variable g_{2K+1} is constant, say $g_{2K+1} = c$ (from the above analysis, we know that $c \in \{\pm N^{-1}, \pm(N^{-1} + 2\delta), \ldots, \pm 1\}$). If $|c| = 1$, then we set $F_n = F_{2K+1}$ for $n > 2K+1$; if $|c| < 1$, then we define F by saying that the distribution of the $(N-1)$-dimensional vector $(F_n^2, \ldots, F_n^N)_{n\geq 2K+2}$ is the same as the distribution of $(1-|c|)\tilde{F}$. The reason for choosing the scaling factor $1-|c|$ is that then the transform G of the martingale F we have just constructed (the transforming sequence is $((-1)^n)_{n\geq 0}$, as usual) has the following property: On each atom A as above,

$$\sup_n |G_n^1| = |g_{2K+1}| = |c|, \qquad \sup_n \|(G_n^2, \ldots, G_n^N)\|_{\ell_1^{N-1}} \geq 1 - |c|$$

with probability 1 (here we use the inductive assumption), and therefore we have $\mathbb{P}(\sup_n \|G_n\|_{\ell_1^N} \geq 1) = 1$, as desired.

Let us now look at the first norm of F. From the construction and the induction hypothesis, we see that

$$\begin{aligned}
&\|F\|_{L_1(\Omega;\ell_1^N)} \\
&\quad = \mathbb{E}|f_{2K+1}| + \|(F^2, F^3, \ldots, F^N)\|_{L_1(\Omega;\ell_1^{N-1})} \\
&\quad \le (2N)^{-1} + \sum_{n=0}^{K} (1 - N^{-1} - 2n\delta)\|\tilde{F}\|_{L_1(\Omega;\ell_1^{N-1})}\mathbb{P}(|g_{2K+1}| = N^{-1} + 2n\delta).
\end{aligned}$$

We have computed the above probabilities in our earlier considerations. If we plug them in, we see that the above expression becomes an appropriate Riemann sum: if K is chosen sufficiently large, we can make the right-hand side arbitrarily close to

$$\begin{aligned}
\frac{1}{2N} + \|\tilde{F}\|_{L_1(\Omega;\ell_1^{N-1})}\biggl[\frac{1}{2}\biggl(1 - \frac{1}{N}\biggr) + \int_{1/N}^{1} \frac{(2N)^{-1}}{x^2}(1-x)\,dx\biggr]& \\
= \frac{1}{2N} + \|\tilde{F}\|_{L_1(\Omega;\ell_1^{N-1})}\biggl[1 - \frac{1}{N} - \frac{\ln N}{2N}\biggr]&.
\end{aligned}$$

It remains to recall that $\|\tilde{F}\|_{L_1(\Omega;\ell_1^{N-1})} \le \zeta_{N-1} + \eta$, where η was an arbitrary positive number. Thus we get the recursion defining $(\zeta_N)_{N\ge1}$, and hence if K and η are chosen appropriately, the norm $\|F\|_{L_1(\Omega;\ell_1^N)}$ can be as close to ζ_N as we wish. This proves Theorem 3.1. ∎

Thus, the above example shows that the optimal constant C_N in the weak-type inequality (3.1) satisfies $C_N \ge \zeta_N^{-1}$ and hence $\zeta_{\ell_1^N}(0,0) \le 2\zeta_N$. So, to get the assertion of Theorem 1.2, it is enough to establish the following statement.

LEMMA 3.2. *The sequence $(\zeta_N)_{N\ge1}$ satisfies $\zeta_N \ln(2N) \le 1$.*

Proof. We have $\zeta_1 \ln 2 = (\ln 2)/2 \le 1$, $\zeta_2 \ln 4 = (1/2 - (\ln 2)/8)\ln 4 \le (\ln 4)/2 \le 1$ and

$$\zeta_3 \ln 6 = \biggl[\frac{1}{6} + \biggl(\frac{2}{3} - \frac{\ln 3}{6}\biggr)\biggl(\frac{1}{2} - \frac{\ln 2}{8}\biggr)\biggr]\ln 6 \le \biggl(\frac{1}{6} + \frac{1}{3}\biggr)\ln 6 \le 1.$$

For $N \ge 4$, we use induction; assuming that $\zeta_{N-1}\ln(2N-2) \le 1$, we compute

$$\zeta_N = \frac{1}{2N} + \biggl(1 - \frac{1}{N} - \frac{\ln N}{2N}\biggr)\zeta_{N-1} \le \frac{1}{2N} + \biggl(1 - \frac{1}{N} - \frac{\ln N}{2N}\biggr)\frac{1}{\ln(2N-2)}.$$

Hence, it is enough to show that the latter expression does not exceed $1/\ln(2N)$. After some straightforward manipulations, this amounts to

$$\frac{1}{2} \le \frac{N \ln \frac{N-1}{N} + \ln(2N) + \frac{1}{2}\ln N \cdot \ln(2N)}{\ln(2N)\ln(2N-2)},$$

or

$$\frac{1}{2} \le \frac{N \ln \frac{N-1}{N} + (1 - \ln\sqrt{2}) \ln(2N) + \frac{1}{2} \ln(2N) \ln(2N)}{\ln(2N) \ln(2N-2)}.$$

Clearly, we will be done if we prove that $N \ln \frac{N-1}{N} + (1 - \ln\sqrt{2}) \ln 2N \ge 0$ for $N \ge 4$. But this is easy: the left-hand side is an increasing function of N, and for $N = 4$ it is equal to $0.208\ldots > 0$, as computer simulations show. The proof of Lemma 3.2, and hence of Theorem 1.2, is complete. ∎

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