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Multiplicatively dependent triples of Tribonacci numbers

by

Carlos Alexis Gómez Ruiz (Cali) and
Florian Luca (Johannesburg)

1. Introduction. The Fibonacci sequence $\mathbf{F} := \{F_n\}_{n\geq 0}$ is given by $F_0 = 0$, $F_1 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. Carmichael's Primitive Divisor Theorem (see [2]) says that if $n \geq 13$, then there is a prime factor p of F_n which does not divide F_m for any $1 \leq m \leq n-1$. In particular, if $n > m \geq 1$ and F_n and F_m are multiplicatively dependent, then $\max\{m,n\} \leq 12$. Further, a quick check shows that in fact the only indices $1 \leq m < n$ corresponding to multiplicatively dependent Fibonacci numbers F_m and F_n have either $m \in \{1,2\}$ (for which $F_1 = F_2 = 1$), or $(m,n) = (3,6)$. In the same spirit, in [7], we looked at multiplicatively dependent pairs of terms in the k-generalized Fibonacci sequence $\mathbf{F}^{(k)} := \{F_n^{(k)}\}_{n\geq -(k-2)}$ given by

$$F_i^{(k)} = 0 \quad \text{for } i = -(k-2), -(k-3), \ldots, 0, \qquad F_1^{(k)} = 1,$$
$$F_{n+k}^{(k)} = F_{n+k-1}^{(k)} + F_{n+k-2}^{(k)} + \cdots + F_n^{(k)} \quad \text{for all } n \geq -(k-2).$$

Although there is no version of Carmichael's theorem for the k-generalized Fibonacci sequence when $k > 2$, we showed that if $1 \leq m < n$ are such that $F_m^{(k)}$ and $F_n^{(k)}$ are multiplicatively dependent, then either $m \in \{1,2\}$ (and $F_1^{(k)} = F_2^{(k)} = 1$), or $n \leq k+1$. Furthermore, since $F_m^{(k)}$ is a power of 2 for all m in the interval $[1, k+1]$, it follows that for any $1 \leq m < n \leq k+1$, $F_m^{(k)}$ and $F_n^{(k)}$ are multiplicatively dependent.

In this paper, we look at the Tribonacci sequence $\mathbf{T} := \{T_n\}_{n\geq 0}$ given by $T_0 = 0$, $T_1 = T_2 = 1$, and

$$T_{n+3} = T_{n+2} + T_{n+1} + T_n \quad \text{for all } n \geq 0.$$

We study the multiplicatively dependent triples of positive integers belong-

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ing to **T**. That is, we look at the Diophantine equation

$$T_\ell^x T_m^y T_n^z = 1 \quad \text{with } 1 \le \ell < m < n \text{ and } x, y, z \text{ integers.} \tag{1.1}$$

We discard the situation when one or more of the indices ℓ, m, n is 1 or 2 since $T_1 = T_2 = 1$. We also assume that any two of T_ℓ, T_m, T_n are multiplicatively independent, since if two of them are multiplicatively dependent, then by the main result in [7] these numbers are in $\{2, 4\}$, and the third one is either in this set as well or it is not really involved in the actual multiplicative dependence relation (i.e., its exponent in (1.1) is 0).

We prove the following result.

Main Theorem. *The only triples of Tribonacci numbers which exceed* 1 *and are multiplicatively dependent, but any two are pairwise multiplicatively independent, are:*

$$T_{15} = T_4^3 T_5, \quad T_{15} = T_3^6 T_5, \quad T_7^4 = T_3^{12} T_9, \quad T_7^4 = T_4^6 T_9,$$
$$T_{13}^2 = T_{17} T_9, \quad T_{16}^2 = T_{15} T_{17}, \quad T_{12}^2 = T_{15} T_9.$$

2. Preliminaries

2.1. The Tribonacci sequence. The characteristic polynomial of the Tribonacci sequence is

$$\Psi(X) = X^3 - X^2 - X - 1.$$

It has a real root

$$\alpha = \tfrac{1}{3}\big(1 + (19 - 3\sqrt{33})^{1/3} + (19 + 3\sqrt{33})^{1/3}\big)$$

and two complex conjugate roots

$$\beta = \alpha^{-1/2} e^{i\theta} \quad \text{and} \quad \gamma = \alpha^{-1/2} e^{-i\theta} \quad \text{with } \theta \in (\pi/2, \pi). \tag{2.1}$$

A recent result of Dresden and Du [4] establishes a Binet-like formula for k-generalized Fibonacci numbers. For Tribonacci numbers it states that

$$T_n = d_\alpha \alpha^n + d_\beta \beta^n + d_\gamma \gamma^n, \tag{2.2}$$

where $d_X = (X - 1)/(X(4X - 6))$. We set

$$d_\beta = \rho e^{i\omega} \quad \text{and} \quad d_\gamma = \rho e^{-i\omega} \quad \text{with } \omega \in (0, \pi). \tag{2.3}$$

Dresden and Du also showed that the contribution of the complex roots β and γ with absolute value less than 1 to the right-hand side of (2.2) is very small; more precisely,

$$|T_n - d_\alpha \alpha^n| < 1/2 \quad \text{for all } n \ge 0. \tag{2.4}$$

These facts were already known to Spickerman [9].

Furthermore,

$$T_n - d_\alpha \alpha^n = 2\,\mathrm{Re}(d_\beta \beta^n) = 2\rho \cos(\omega + n\theta)/\alpha^{n/2}. \tag{2.5}$$

It is also well-known (see [1]) that

$$\alpha^{n-2} \le T_n \le \alpha^{n-1} \quad \text{for all } n \ge 1. \tag{2.6}$$

Let $\mathbb{L} := \mathbb{Q}(\alpha, \beta)$ be the splitting field of Ψ over $\mathbb{Q}$. Then $d_{\mathbb{L}} = [\mathbb{L} : \mathbb{Q}] = 6$. Furthermore, $[\mathbb{Q}(\alpha) : \mathbb{Q}] = 3$. The Galois group of $\mathbb{L}$ over $\mathbb{Q}$ is

$$G := \mathrm{Gal}(\mathbb{L}/\mathbb{Q}) \cong \{(1), (\alpha\beta), (\alpha\gamma), (\beta\gamma), (\alpha\beta\gamma), (\alpha\gamma\beta)\} \cong S_3.$$

Here, we identify the automorphisms of G with the permutations of the roots of Ψ. For instance, the permutation $(\alpha\beta)$ corresponds to the automorphism $\sigma : \alpha \mapsto \beta$, $\beta \mapsto \alpha$, $\gamma \mapsto \gamma$.

We conclude with a few results which play important roles in our work.

Theorem 1. *Let α, d_α be the algebraic numbers given by* (2.2). *If r, s are integers such that $\alpha^r d_\alpha^s \in \mathbb{Q}$, then $r = s = 0$.*

Proof. If $s \neq 0$, then conjugating the equality $\alpha^r d_\alpha^s = t$ with some $t \in \mathbb{Q}$ by the automorphisms $(\alpha\beta)$ and $(\alpha\gamma)$, we obtain

$$\beta^r d_\beta^s = \gamma^r d_\gamma^s.$$

Since $(\beta/\gamma)^r$ is a unit in $\mathbb{L}$, we conclude that d_γ/d_β is also a unit, in particular, an algebraic integer. However, this is impossible because the minimal polynomial of this number over $\mathbb{Z}$ is

$$11X^6 + 33X^5 + 64X^4 - 73X^3 + 64X^2 + 33X + 11.$$

Thus, $s = 0$ and $\alpha^r \in \mathbb{Q}$. Now, if $r \neq 0$, then $\alpha^{|r|} = t > 1$ and conjugating again by $(\alpha\beta)$, we obtain $|\beta|^{|r|} = t > 1$, which is false because $|\beta| < 1$. Hence, $r = s = 0$. ∎

Theorem 2. *Let $m > \ell \ge 3$. Then*

$$\gcd(T_\ell, T_m) < \alpha^{2m/3}.$$

Proof. If $m = 4$, then $\ell = 3$ and $2 = \gcd(T_3, T_4) < \alpha^{8/3}$. From now on, we assume $m \ge 5$. We set $D := \gcd(T_\ell, T_m)$. Let c be a positive constant to be determined. We first consider the case $\ell < cm$. Then

$$D \le T_\ell \le T_{\lfloor cm \rfloor} \le \alpha^{\lfloor cm \rfloor - 1} < \alpha^{cm}.$$

Now, we assume that $\ell \ge cm$. By performing calculations in the integer ring of $\mathbb{K} := \mathbb{Q}(\alpha)$, we see that D divides the algebraic integer

$$\alpha^{m-\ell} T_\ell - T_m = d_\beta \beta^\ell (\alpha^{m-\ell} - \beta^{m-\ell}) + d_\gamma \gamma^\ell (\alpha^{m-\ell} - \gamma^{m-\ell}).$$

For the above calculation we have used (2.2). Hence, by calculating norms from $\mathbb{K}$ to $\mathbb{Q}$, we conclude that D^3 divides

$$|\mathrm{N}_{\mathbb{K}/\mathbb{Q}}(\alpha^{m-\ell} T_\ell - T_m)| = |\alpha^{m-\ell} T_\ell - T_m|\, |\beta^{m-\ell} T_\ell - T_m|\, |\gamma^{m-\ell} T_\ell - T_m|.$$

Observe that

$$\begin{aligned}|\alpha^{m-\ell}T_\ell - T_m| &= |d_\beta \beta^\ell(\alpha^{m-\ell}-\beta^{m-\ell}) + d_\gamma\gamma^\ell(\alpha^{m-\ell}-\gamma^{m-\ell})| \\ &< \frac{2\rho}{\alpha^{\ell/2}}\left(\alpha^{m-\ell}+\frac{1}{\alpha^{(m-\ell)/2}}\right) < 2\rho\alpha^{(1-3c/2)m}+\frac{2\rho}{\alpha^{m/2}}.\end{aligned}$$

In the above, we have used the fact that $\ell \geq \max\{3, cm\}$ as well as (2.1) and (2.3). On the other hand,

$$|\beta^{m-\ell}T_\ell - T_m|\,|\gamma^{m-\ell}T_\ell - T_m| < (T_\ell + T_m)^2 < 4T_m^2 < 4\alpha^{2m-2}.$$

Thus,

$$D^3 \leq |\mathrm{N}_{\mathbb{K}/\mathbb{Q}}(\alpha^{m-\ell}T_\ell - T_m)| < 8\rho\alpha^{(3-3c/2)m-2}\left(1+\frac{1}{\alpha^{\frac{3}{2}(1-c)m}}\right).$$

Hence,

$$D \leq \frac{2}{\alpha^{2/3}}\left(\rho+\frac{\rho}{\alpha^{\frac{3}{2}(1-c)m}}\right)^{1/3}\alpha^{(1-c/2)m}.$$

We choose $c = 2/3$, and use the fact that $m \geq 5$ to get

$$\frac{2}{\alpha^{2/3}}\left(\rho+\frac{\rho}{\alpha^{\frac{3}{2}(1-c)m}}\right)^{1/3} \leq \frac{2}{\alpha^{2/3}}(\rho+\frac{\rho}{\alpha^{5/2}})^{1/3} < 1,$$

and therefore conclude that $D < \alpha^{2m/3}$. ∎

Lemma 1. *There do not exist positive integers $a, b, c, \ell < m < n$ with $\max\{a, b, c\} < n$ such that*

$$a\frac{T_\ell}{\alpha^\ell} + c\frac{T_n}{\alpha^n} = b\frac{T_m}{\alpha^m}. \tag{2.7}$$

Proof. Multiply equation (2.7) by α^n and rearrange terms to get

$$cT_n + (-bT_m)\alpha^{n-m} + aT_\ell\alpha^{n-\ell} = 0.$$

Write $u = n - m$, $v = n - \ell$ and note that $1 \leq u < v$. Conjugating the above equation by any conjugation with $\alpha \mapsto \beta$, then with $\alpha \mapsto \gamma$, we find that $\mathbf{U} = (cT_n, -bT_m, aT_\ell)^T$ is a vector in the null-space of the matrix

$$A_{u,v} = \begin{pmatrix} 1 & \alpha^u & \alpha^v \\ 1 & \beta^u & \beta^v \\ 1 & \gamma^u & \gamma^v \end{pmatrix}. \tag{2.8}$$

By the main result of [6], we have $(u, v) = (3, 4), (13, 16), (13, 17), (16, 17)$ and in each case the matrix $A_{u,v}$ has rank 2. Thus, its null-space is onedimensional. A quick computation shows that the vectors

$$\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 56 \\ -9 \end{pmatrix}, \quad \begin{pmatrix} 2 \\ 103 \\ -9 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 103 \\ -56 \end{pmatrix} \tag{2.9}$$

are in the null-space of $A_{u,v}$ for $(u, v) = (3, 4), (13, 16), (13, 17), (16, 17)$, respectively. For example, one can check that each of the polynomials

$$\begin{aligned} X^4 - 2X^3 + 1, \qquad & 9X^{16} + 56X^{13} + 1, \\ 9X^{17} + 103X^{13} + 2, \qquad & -56X^{17} + 103X^{16} + 1 \end{aligned}$$

has $X^3 - X^2 - X - 1$ as a factor. Thus, $\mathbf{U}$ is parallel to one of the four vectors from (2.9). The last three are excluded because in $\mathbf{U}$ the first and last components have the same sign, whereas in the last three vectors in (2.9) the first and last components have different signs. Thus, the only possibility is the first one for which $\ell = n-4$, $m = n-3$ and $cT_n = aT_{n-4} = (b/2)T_{n-3}$. We get $T_n/T_{n-4} = a/c$. Hence, $T_n/\gcd(T_n, T_{n-4}) = a/\gcd(a, c) \le n$. Thus,

$$\alpha^{n-2} \le T_n \le n\gcd(T_{n-4}, T_n) < n\alpha^{2n/3},$$

giving $\alpha^n < (\alpha^2 n)^3$, so $n \le 20$. In the above argument, we have used Theorem 2. Now one prints T_n/T_{n-4} for all $n \in \{5, \dots, 20\}$ and checks that none of these fractions is of the form a/c with $\max\{a, c\} < n$, so there are no examples satisfying (2.7). ∎

2.2. Linear forms in logarithms. Let η be an algebraic number of degree d over $\mathbb{Q}$ with minimal primitive polynomial

$$f(X) := a_0 \prod_{i=1}^{d} (X - \eta^{(i)}) \in \mathbb{Z}[X],$$

where $a_0 > 0$. The *logarithmic height* of η is given by

$$h(\eta) := \frac{1}{d}\Big(\log a_0 + \sum_{i=1}^{d} \log\max\{|\eta^{(i)}|, 1\}\Big).$$

The following properties of the logarithmic height function $h(\cdot)$ will be used:

$$h(\eta\gamma^{\pm 1}) \le h(\eta) + h(\gamma) \quad \text{and} \quad h(\eta^s) = |s|h(\eta) \quad \text{for } s \in \mathbb{Z}.$$

Our main tool is a lower bound for a linear form in logarithms of algebraic numbers given by the following result of Matveev [8].

THEOREM 3 (Matveev's theorem). *Let $\mathbb{K}$ be a number field of degree D over $\mathbb{Q}$, $\eta_1, \dots, \eta_t$ nonzero elements of $\mathbb{K}$, and $b_1, \dots, b_t$ rational integers. Set*

$$\Lambda := \eta_1^{b_1} \cdots \eta_t^{b_t} - 1 \quad \textit{and} \quad B \ge \max\{|b_1|, \dots, |b_t|\}.$$

Let $A_i \geq \max\{Dh(\eta_i), |\log \eta_i|, 0.16\}$ *be real numbers for* $i = 1, \ldots, t$*. Then, assuming that* $\Lambda \neq 0$*, we have*

$$|\Lambda| > \exp\bigl(-3 \cdot 30^{t+4} \cdot (t+1)^{5.5} \cdot D^2(1+\log D)(1+\log(tB))A_1 \cdot A_t\bigr).$$

If in addition $\mathbb{K}$ *is real, then*

$$|\Lambda| > \exp\bigl(-1.4 \cdot 30^{t+3} \cdot t^{4.5} \cdot D^2(1+\log D)(1+\log B)A_1 \cdots A_t\bigr).$$

2.3. The reduction lemmas. In the course of our calculations, we get some upper bounds on our variables which are very large, so we need to reduce them. To this end, we use some results of the theory of continued fractions and geometry of numbers.

The following results, well-known in Diophantine approximation, will be used when dealing with homogeneous linear forms in two integer variables.

Lemma 2. *Let* M *be a positive integer, and let* $p_1/q_1, p_2/q_2, \ldots$ *be convergents of the continued fraction of the irrational* τ *such that* $M < q_{N+1}$ *for some* N*. Write* $a_M = \max\{a_t : t = 0, 1, \ldots, N+1\}$*. Then*

$$|m\tau - n| > \frac{1}{(a_M+2)m}$$

for all pairs (n, m) *of integers with* $0 < m < M$*.*

For nonhomogeneous linear forms in two integer variables, we will use a slight variation of a result due to Dujella and Pethő [5], which itself is a generalization of a result of Baker and Davenport. For a real number X, we write $\|X\| = \min\{|X - n| : n \in \mathbb{Z}\}$ for the distance from X to the nearest integer.

Lemma 3. *Let* M *be a positive integer, let* p/q *be a convergent of the continued fraction of the irrational* τ *such that* $q > 6M$*, and let* A, B, μ *be real numbers with* $A > 0$ *and* $B > 1$*. Let* $\epsilon := \|\mu q\| - M\|\tau q\|$*. If* $\epsilon > 0$*, then there is no solution to the inequality*

$$0 < |m\tau - n + \mu| < AB^{-k}$$

in positive integers m*,* n *and* k *with*

$$m \leq M \quad \text{and} \quad k \geq \frac{\log(Aq/\epsilon)}{\log B}.$$

On various occasions, we will need to find a lower bound for the absolute value of linear forms in three and four integer variables:

$$|x_1\tau_1 + \cdots + x_t\tau_t| \quad \text{with } |x_i| \leq X_i. \tag{2.10}$$

To this end, we set $X := \max\{X_i\}$, choose $C > (tX)^t$, and consider the integer lattice Ω generated by

$$b_j = \mathbf{e}_j + \lfloor C\tau_j \rceil \mathbf{e}_t \quad \text{for } 1 \leq j \leq t-1 \quad \text{and} \quad b_t = \lfloor C\tau_t \rceil \mathbf{e}_t.$$

We first calculate a reduced base $\{\mathbf{b}_1, \ldots, \mathbf{b}_t\}$ using the LLL-algorithm and afterwards its Gram–Schmidt associated basis $\{\mathbf{b}_1^*, \ldots, \mathbf{b}_t^*\}$. We further compute the following values:

$$c_1 = \max_{1\le i\le t} \frac{\|\mathbf{b}_1\|}{\|\mathbf{b}_i^*\|}, \quad m = \frac{\|\mathbf{b}_1\|}{c_1}, \quad Q = \sum_{i=1}^{t-1} X_i^2, \quad T = \sum_{i=1}^{t} X_i/2.$$

Finally, from the geometry of numbers we conclude that if $m^2 \ge T^2 + Q$, then

$$|x_1\tau_1 + \cdots + x_t\tau_t| > (\sqrt{m^2 - Q} - T)/C.$$

For more details, see [3, Chapter 2].

3. Proof of the Main Theorem

3.1. Bounds on exponents. We recall that our goal is to solve the Diophantine equation (1.1). Without loss of generality, we can assume that x, y and z are relatively prime. Furthermore, we suppose that at most one of T_ℓ, T_m and T_n is a power of two.

Let $P = \{p_1, \ldots, p_t\}$ be the set of all primes involved in the factorization of $T_\ell T_m T_n$. Thus

$$T_\ell = \prod_{p\in P} p^{\ell_p}, \quad T_m = \prod_{p\in P} p^{m_p}, \quad T_n = \prod_{p\in P} p^{n_p}. \tag{3.1}$$

As a consequence of inequality (2.6) and $\alpha < 2$ we have

$$\max_{p\in P}\{\ell_p, m_p, n_p\} \le n.$$

For a prime p and a nonzero integer m, we write $\upsilon_p(m)$ for the exact exponent of p in the factorization of m.

Lemma 4. *Let T_ℓ, T_m, T_n be Tribonacci numbers of indices at least 3 which are pairwise multiplicatively independent. If $T_\ell^x T_m^y T_n^z = 1$ and $\upsilon_p(T_t) \le k$ for $t \in \{\ell, m, n\}$, then exactly one of the numbers x, y, z has an opposite sign to the other two and*

$$\max\{|x|, |y|, |z|\} < k^2.$$

Proof. It is easy to note that exactly one of the numbers x, y, z has opposite sign to the other two. For the second assertion, we take the $\mathbb{Q}$-vector space

$$H := \langle \log T_\ell, \log T_m, \log T_n\rangle \subseteq \langle \log p : p \in P\rangle.$$

Then $\dim_{\mathbb{Q}} H = 2$. Indeed, since T_ℓ, T_m, T_n are multiplicatively dependent, we have $\dim_{\mathbb{Q}} H \le 2$. However, $\dim_{\mathbb{Q}} H = 1$ would contradict the hypothesis that any two of T_ℓ, T_m, T_n are multiplicatively independent.

The Diophantine equation (1.1) can be represented in matrix form as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \begin{bmatrix} \ell_{p_1} & \cdots & \ell_{p_t} \\ m_{p_1} & \cdots & m_{p_t} \\ n_{p_1} & \cdots & n_{p_t} \end{bmatrix} \begin{bmatrix} \log p_1 \\ \vdots \\ \log p_t \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}^T \begin{bmatrix} \log T_\ell \\ \log T_m \\ \log T_n \end{bmatrix} = 0.$$

Now, as $\dim_{\mathbb{Q}} H = 2$, the $3 \times t$-matrix on the left-hand side has rank 2. So, there are $p_a, p_b \in P$ such that $u_{p_a} = [\ell_{p_a}, m_{p_a}, n_{p_a}]$ and $u_{p_b} = [\ell_{p_b}, m_{p_b}, n_{p_b}]$ are linearly independent. In particular, $[x, y, z]$ is parallel to the vector cross product

$$\begin{aligned} u_{p_a} \times u_{p_b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \ell_{p_a} & m_{p_a} & n_{p_a} \\ \ell_{p_b} & m_{p_b} & n_{p_b} \end{vmatrix} \\ &= \hat{i}(m_{p_a} n_{p_b} - n_{p_a} m_{p_b}) - \hat{j}(\ell_{p_a} n_{p_b} - n_{p_a} \ell_{p_b}) + \hat{k}(\ell_{p_a} m_{p_b} - m_{p_a} \ell_{p_b}), \end{aligned}$$

and since $\gcd(x, y, z) = 1$, it follows that x, y, z are divisors of the components of the vector $u_{p_a} \times u_{p_b}$ above. Since these components are each a difference of two nonnegative integers each of size at most k^2, we conclude that $\max\{|x|, |y|, |z|\} \le k^2$. ∎

From Lemma 4, we conclude that $\max\{|x|, |y|, |z|\} \le n^2$, and we may assume that among x, y, z there are two positive integers and one negative integer.

For the rest of this paper, we distinguish two cases:

$$d_\alpha^{x+y+z} \alpha^{\ell x + my + nz} \neq 1 \quad \text{and} \quad d_\alpha^{x+y+z} \alpha^{\ell x + my + nz} = 1.$$

4. The case $d_\alpha^{x+y+z} \alpha^{\ell x + my + nz} \neq 1$. For technical reasons, we assume that $n > 50$. Note that by (2.5), we can write

$$T_n = d_\alpha \alpha^n + e_n / \alpha^{n/2}, \quad \text{where} \quad e_n := 2\rho \cos(\omega + n\theta). \tag{4.1}$$

We have

$$T_n^z = d_\alpha^z \alpha^{nz} \left(1 + \frac{e_n}{d_\alpha \alpha^{3n/2}}\right)^z. \tag{4.2}$$

We look at the elements

$$(1 + r)^z \quad \text{and} \quad k := zr, \quad \text{where} \quad r := \frac{e_n}{d_\alpha \alpha^{3n/2}}. \tag{4.3}$$

Since $n > 50$, $e_n / d_\alpha < \alpha$ and $|z| \le n^2$, we have

$$|k| = |zr| < 2n^2 / \alpha^{3n/2} \quad \text{and in particular} \quad |k| < 3 \cdot 10^{-16}.$$

Now, if $z > 0$ and $r < 0$, then

$$1 > (1 + r)^z = \exp(z \log(1 - |r|)) \ge \exp(-2|k|) > 1 - 2|k|,$$

while if $z > 0$ and $r > 0$, then

$$1 < (1+r)^z = (1+|k|/z)^z < \exp|k| < 1+2|k|,$$

because $|r| < 1/2$ and $|k|$ is very small.

Thus, in either case assuming that $z > 0$ we have

$$T_n^z = d_\alpha^z \alpha^{nz}(1+\zeta_n) \quad \text{with} \quad |\zeta_n| < 3n^2/\alpha^{3n/2}. \tag{4.4}$$

Regarding T_ℓ^x and T_m^y, we assume that $m > \ell > 10\log n$ (later we will show that $\ell < m = O(\log n)$). By the respective choices of r and k, we use the same argument as above to conclude that

$$T_\ell^x = d_\alpha^x \alpha^{x\ell}(1+\zeta_\ell) \qquad \text{with} \quad |\zeta_\ell| < \frac{3n^2}{\alpha^{3\ell/2}}, \tag{4.5}$$

$$T_m^y = d_\alpha^y \alpha^{my}(1+\zeta_m) \quad \text{with} \quad |\zeta_m| < \frac{3n^2}{\alpha^{3m/2}}, \tag{4.6}$$

provided that x and y are positive.

Now, supposing $z < 0$ (the same conclusion is obtained in the other two cases when x or y is negative), we make use of (4.4)–(4.6) in the Diophantine equation (1.1) to obtain

$$d_\alpha^{x+y}\alpha^{\ell x+my}(1+\zeta_\ell)(1+\zeta_m) = d_\alpha^{|z|}\alpha^{n|z|}(1+\zeta_n).$$

Separating the dominant terms, we get

$$d_\alpha^{x+y}\alpha^{\ell x+my} - d_\alpha^{|z|}\alpha^{n|z|} = d_\alpha^{|z|}\alpha^{n|z|}\zeta_n - d_\alpha^{x+y}\alpha^{\ell x+my}(\zeta_\ell+\zeta_m+\zeta_\ell\zeta_m).$$

Dividing by $d_\alpha^{|z|}\alpha^{n|z|}$ and taking absolute value, we conclude that

$$\begin{aligned} |d_\alpha^{x+y+z}\alpha^{\ell x+my+nz} - 1| &< |\zeta_n| + \frac{d_\alpha^{x+y}\alpha^{\ell x+my}}{d_\alpha^{|z|}\alpha^{n|z|}}|\zeta_\ell+\zeta_m+\zeta_\ell\zeta_m| \\ &< \frac{9n^2}{\alpha^{3\ell/2}}. \end{aligned} \tag{4.7}$$

Above, we have used the inequalities

$$|\zeta_n| < \frac{0.5n^2}{\alpha^{3\ell/2}} \quad \text{and} \quad |\zeta_\ell+\zeta_m+\zeta_\ell\zeta_m| < \frac{4.25n^2}{\alpha^{3\ell/2}}$$

as well as

$$\frac{d_\alpha^{x+y}\alpha^{\ell x+my}}{d_\alpha^{|z|}\alpha^{n|z|}} = \frac{1+\zeta_n}{(1+\zeta_\ell)(1+\zeta_m)} < \frac{1+0.8}{(1-4\cdot 10^{-4})^2} < 2,$$

which follows from (4.4)–(4.6). In the above inequality, we have also used the fact that the function $f(n) = 3n^2/\alpha^{3n/2}$ is decreasing, and that $f(n) \le f(5) < 0.8$ for all $n \ge 5$, as well as that

$$\max\{|\zeta_\ell|, |\zeta_m|\} < \frac{3n^2}{\alpha^{15\log n}} = \frac{3}{n^{15\log\alpha-2}} < \frac{3}{5^{15\log\alpha-2}} < 4\cdot 10^{-4}.$$

On the left-hand side of inequality (4.7), we have a linear form in $t := 2$ logarithms, with $\eta_1 := d_\alpha$, $\eta_2 := \alpha$, $b_1 := x+y+z$, $b_2 := \ell x + my + nz$. So, $\Lambda_1 := d_\alpha^{x+y+z}\alpha^{\ell x+my+nz} - 1$ is nonzero by hypothesis, and from (4.7) we deduce that

$$|\Lambda_1| < \frac{9n^2}{\alpha^{3\ell/2}}. \tag{4.8}$$

The field $\mathbb{K} := \mathbb{Q}(\alpha)$ contains η_1, η_2 and has $D = [\mathbb{K} : \mathbb{Q}] = 3$. Since the minimal polynomial of d_α is $44X^3 - 2X - 1$, and d_α and its conjugates d_β and d_γ are all inside the unit disk, we can take $A_1 := \log 44$. Further, by the properties of the roots of Ψ, we take $A_2 := 0.7 > \log\alpha$. Since $\mathbb{K}$ is real, Theorem 3 gives the following lower bound for $|\Lambda_1|$:

$$\exp\bigl(-1.4 \times 30^5 \times 2^{4.5}3^2(1+\log 3)(1+\log(2n^3))(\log\alpha)(0.7)\bigr),$$

which is smaller than $9n^2/\alpha^{3\ell/2}$ by (4.8). Taking logarithms on both sides and performing the corresponding calculations, we get

$$\ell < 1.5 \cdot 10^{11} \log n, \tag{4.9}$$

where we have used $1 + \log(2n^3) < 4.1 \log n$ for all $n \geq 5$.

We go back to equation (1.1). Replacing T_m^y, T_n^z according to (4.4) and (4.6), by the same arguments used to derive (4.7) we obtain

$$|d_\alpha^{y+z}\alpha^{my+nz}T_\ell^x - 1| < 5n^2/\alpha^{3m/2}. \tag{4.10}$$

Again we use the real version of Matveev's theorem, with $t := 3$,

$$\eta_1 := d_\alpha, \quad \eta_2 := \alpha, \quad \eta_3 := T_\ell,$$
$$b_1 := y+z, \quad b_2 := my+nz, \quad b_3 := x.$$

So, $\Lambda_2 := d_\alpha^{y+z}\alpha^{my+nz}T_\ell^x - 1$ and

$$|\Lambda_2| < 5n^2/\alpha^{3m/2}. \tag{4.11}$$

We can take again $\mathbb{K} := \mathbb{Q}(\alpha)$, $D := 3$, $A_1 := \log 44, A_2 := 0.7$ and $B := 2n^3$. For A_3, we note that $T_\ell \leq \alpha^{\ell-1} < 2^\ell$, so we can take $A_3 := 0.7\ell$. We are ready to use Theorem 1 since $\Lambda_2 \neq 0$: indeed, otherwise by Lemma 1 we would obtain $y+z = my+nz = 0$. Since $m \neq n$, we get $y = z = 0$. So, $T_\ell^x = 1$ and thus $x = 0$. However, this contradicts our hypothesis.

Combining the conclusion of Theorem 3 with inequality (4.11), we get, after taking logarithms, the following upper bound for m:

$$\frac{3\log\alpha}{2}m - \log(5n^2) < 1.4 \cdot 30^6 \cdot 3^{4.5} \cdot 3^2(1+\log 3)$$
$$\times(1+\log(2n^3))(\log 44)(0.7)(0.7\ell).$$

Using again the fact that $1 + \log(2n^3) < 4.1 \log n$ for all $n \geq 5$ and that

$\ell < 1.5 \cdot 10^{11} \log n$, we get

$$m < 2.6 \cdot 10^{24} \log^2 n. \tag{4.12}$$

Returning once again to (1.1), we replace T_n^z according to (4.4). We now obtain

$$|\Lambda_3| := |d_\alpha^z \alpha^{nz} T_\ell^x T_m^y - 1| < 4n^2/\alpha^{3n/2}. \tag{4.13}$$

A new application of Theorem 3 (real case) with the data

$$t := 3, \quad \eta_1 := d_\alpha, \quad \eta_2 := \alpha, \quad \eta_3 := T_\ell, \quad \eta_4 := T_m,$$
$$b_1 := z, \quad b_2 := nz, \quad b_3 := x, \quad b_4 := y,$$

where we take

$$\mathbb{K} = \mathbb{Q}(\alpha), \quad D = 3, \quad A_1 = \log 44, \quad A_2 = 0.7, \quad A_3 = 0.7\ell, \quad A_4 = 0.7m$$

and $B = n^3$, leads to

$$n < 1.2 \cdot 10^{15} \ell m \log n.$$

The fact that $\Lambda_3 \neq 0$ is an immediate application of Lemma 1. Inserting (4.9) and (4.12) in the above inequality, we get $n < 4.5 \cdot 10^{50} \log^4 n$, which leads to $n < 3.3 \cdot 10^{58}$. From (4.9) and (4.12), we deduce that $\ell < 1.5 \cdot 10^{13}$ and $m < 4 \cdot 10^{28}$.

In summary, we have proved the following result.

LEMMA 5. *Let (ℓ, m, n, x, y, z) be a solution of* (1.1) *with $3 \leq \ell < m < n$ such that $d_\alpha^{x+y+z} \alpha^{mx+ny+\ell z} \neq 1$. Then* $\max\{|x|, |y|, |z|\} \leq n^2$ *and*

$$\ell < 1.5 \cdot 10^{13}, \quad m < 4 \cdot 10^{28}, \quad n < 1.6 \cdot 10^{59}.$$

The rest of this section is dedicated to reducing the bounds given in this lemma. For this purpose, we return to Λ_1, Λ_2 and Λ_3.

First of all, we consider

$$\Gamma_1 := (x + y + z) \log(d_\alpha) + (\ell x + my + nz) \log \alpha.$$

Then $e^{\Gamma_1} - 1 = \Lambda_1$. Assuming that $\ell > 310$, we have $|\Lambda_1| < 1/2$ (given that $n < 1.6 \cdot 10^{59}$), so $e^{|\Gamma_1|} < 3/2$ and

$$|\Gamma_1| < e^{|\Gamma_1|} |e^{\Gamma_1} - 1| < 13.5 n^2/\alpha^{3\ell/2}. \tag{4.14}$$

From the above inequality, we note that $|\Gamma_1| < 1$. Thus, without loss of generality, we can suppose that $x + y + z$ and $\ell x + my + nz$ are positive.

Dividing both sides of (4.14) by $(x + y + z) \log \alpha$, we obtain

$$\left| \frac{\log(d_\alpha^{-1})}{\log \alpha} - \frac{\ell x + my + nz}{x + y + z} \right| < \frac{23n^2}{\alpha^{3\ell/2}(x + y + z)}. \tag{4.15}$$

We set $\tau := \log(d_\alpha^{-1})/\log \alpha$, and compute a few initial terms of its continued fraction $[a_0, a_1, a_2, \ldots]$ and its convergents $p_1/q_1, p_2/q_2, \ldots$. Then we find an integer t such that $q_t > 5.2 \cdot 10^{118} > 2n^2 > x + y + z$ and take $a_M := \max\{a_i : 0 \leq i \leq t\}$. Computationally, we confirm that $q_{231} > 5.2 \cdot 10^{118}$

and $a_M = 174$. Thus, combining (4.15) with the conclusion of Lemma 2, we get

$$\alpha^{3\ell/2} < 4.1 \cdot 10^3 (x+y+z)n^2 < 8.2 \cdot 10^3 n^4.$$

Using the fact that $n < 1.6 \cdot 10^{59}$, we conclude that $\ell \le 610$.

We now go back to the inequality for Λ_2, where we set

$$\Gamma_2 := x \log T_\ell + (y+z) \log d_\alpha + (my+nz) \log \alpha.$$

It is easy to see from (4.11) that

$$|\Gamma_2| < 8n^2/\alpha^{3m/2}. \tag{4.16}$$

For each $\ell \in [3, 610]$ we estimate $|\Gamma_2|$ from below via the procedure described in Section 2.3 (LLL-algorithm). First of all, note that $\Gamma_2 \neq 0$ because $\Lambda_2 \neq 0$.

As in (2.10), we set $t := 3$,

$$\tau_1 := \log T_\ell, \quad \tau_2 := \log d_\alpha, \quad \tau_3 := \log \alpha,$$
$$x_1 := x, \quad x_2 := y+z, \quad x_3 := my+nz.$$

Further, we take $X := 2 \cdot (1.6 \cdot 10^{59})^3$ as an upper bound for $|x|$, $|y+z|$ and $|my+nz|$, and $C := (3X)^3$. A computer search then reveals that $|\Gamma_2| > 2.3 \cdot 10^{-360}$. Combining this with (4.16), we conclude that $m \le 1210$.

Returning to the application of Matveev's theorem for Λ_3, we use the latest bounds for ℓ and m, instead of (4.9) and (4.12), to obtain $n < 4.4 \cdot 10^{22}$. We return to Γ_1 and Γ_2 with this new bound on n and suppose that $m > \ell > 120$. So, $|\Gamma_1|, |\Gamma_2| < 1/2$, and (4.14) and (4.16) are satisfied. In our new reduction of the bound of ℓ, we find that $q_{108} > 4 \cdot 10^{45} > 2n^2 > x+y+z$ and $a_M = 49$. This time we obtain $\ell \le 240$. Regarding m, we redefine $X := 2 \cdot (4 \cdot 10^{22})^3$. By the LLL-algorithm, we obtain $|\gamma_2| > 1.7 \cdot 10^{-140}$, from which we conclude that $m \le 470$.

Now, with $\ell \in [3, 240]$, $m \in [\ell+1, 470]$ and $n \in [m+1, 4.4 \cdot 10^{22}]$, we go back to Λ_3. Taking

$$\Gamma_3 := x \log T_\ell + y \log T_m + z \log d_\alpha + nz \log \alpha,$$

we get $e^{\Gamma_3} - 1 = \Lambda_3$ and $|\Gamma_3| < 6n^2/\alpha^{3n/2}$ (here we have used $n > 50$).

We use the LLL-algorithm with $X := (4.4 \cdot 10^{22})^3$ (a current upper bound on $|x|, |y|, |z|, |nz|$) to find a lower bound of $|\Gamma_3|$. Computationally we confirm that $|\Gamma_3| > 10^{-412}$. Thus, $n \le 1050$. Once again, we reduce the bounds on ℓ and m using $|\Gamma_1|$ and $|\Gamma_2|$, respectively (now it is enough to assume that $m > \ell > 25$). This time we obtain $\ell \le 40$ and $m \le 72$. Finally, applying the LLL-algorithm to $|\Gamma_3|$ with $\ell \in [3, 40]$ and $m \in [\ell+1, 72]$, we obtain $n \le 130$.

A thorough inspection, through the analysis of the primitive prime factors of T_ℓ, T_m and T_n (here, we say that $p \mid T_n$ is *primitive* if $p \nmid T_k$ for all

$1 \le k \le n-1$) with $\ell \in [3,40]$, $m \in [\ell+1, 72]$ and $n \in [m+1, 130]$, reveals that the only solutions of (1.1) in this case are

$$T_{15} = T_4^3 T_5, \quad T_{15} = T_3^6 T_5, \quad T_7^4 = T_3^{12} T_9, \quad T_7^4 = T_4^6 T_9.$$

5. The case $d_\alpha^{x+y+z} \alpha^{\ell x + my + nz} = 1$. This case is a lot more challenging. By Theorem 1, we conclude

$$x + y + z = 0, \qquad \ell x + my + nz = 0. \tag{5.1}$$

So, $x+y = -z$ and $(n-\ell)x + (n-m)y = 0$. Solving this system with respect to x and y while treating z as a parameter, we get, by Cramer's rule,

$$x = \frac{\begin{vmatrix} -z & 1 \\ 0 & n-m \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ n-\ell & n-m \end{vmatrix}} = \frac{z(n-m)}{m-\ell}, \tag{5.2}$$

$$y = \frac{\begin{vmatrix} 1 & -z \\ n-\ell & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ n-\ell & n-m \end{vmatrix}} = \frac{z(n-\ell)}{\ell-m}. \tag{5.3}$$

Taking into account that $n-\ell$, $n-m$ and $m-\ell$ are all positive, we deduce that x and z have the same sign, so are positive, while y is negative. Even more, from (5.1) we get $|y| = x + z$. Thus, (1.1) becomes

$$T_\ell^x T_n^z = T_m^{x+z}. \tag{5.4}$$

On the other hand, as $\gcd(x,z) = \gcd(y,z) = 1$, from (5.2) and (5.3) we get

$$z \,|\, m-\ell, \quad x \,|\, n-m, \quad y \,|\, n-\ell. \tag{5.5}$$

Thus,

$$\max\{x, |y|, z\} < n. \tag{5.6}$$

We now go back to (2.2), (4.1) and (4.2) in order to derive new expressions for T_ℓ^x, T_n^z and T_m^{x+z} with two dominant terms. As in the previous section, we begin by assuming that $m > \ell > 10 \log n$. We analyze

$$(1 + e_\ell d_\alpha^{-1} \alpha^{-3\ell/2})^x$$

by using the binomial theorem. We write

$$s_\ell := (1 + e_\ell d_\alpha^{-1} \alpha^{-3\ell/2})^x - 1 - x e_\ell d_\alpha^{-1} \alpha^{-3\ell/2} \quad \text{and} \quad \kappa := 2\rho/d_\alpha,$$

so

$$|s_\ell| \le \sum_{j=2}^{x} \binom{x}{j} \left(\frac{\kappa}{\alpha^{3\ell/2}}\right)^j < \frac{\kappa^2 x^2}{\alpha^{3\ell}} \sum_{j=0}^{\infty} \left(\frac{\kappa x}{\alpha^{3\ell/2}}\right)^j$$
$$< \frac{\kappa^2 n^2}{\alpha^{3\ell}} \sum_{j=0}^{\infty} \left(\frac{\kappa n}{\alpha^{3\ell/2}}\right)^j < \frac{1.1 \cdot \kappa^2 n^2}{\alpha^{3\ell}},$$

where we have used the inequalities

$$\binom{x}{j} < x^j \le n^j \quad \text{and} \quad \frac{\kappa n}{\alpha^{3\ell/2}} < \frac{\kappa}{n^{15\log\alpha-1}} \le \frac{\kappa}{5^{15\log\alpha-1}} < 4\cdot 10^{-6}.$$

In summary, we have shown that

$$(5.7)\quad T_\ell^x = d_\alpha^x \alpha^{\ell x}\left(1+\frac{\kappa x}{\alpha^{3\ell/2}}\cos(\omega+\ell\theta)+s_\ell\right),\quad |s_\ell| < \frac{1.1\cdot\kappa^2 n^2}{\alpha^{3\ell}}.$$

In the same way, we obtain

$$(5.8)\quad T_n^z = d_\alpha^z \alpha^{nz}\left(1+\frac{\kappa z}{\alpha^{3n/2}}\cos(\omega+n\theta)+s_n\right),\qquad |s_n| < \frac{1.1\cdot\kappa^2 n^2}{\alpha^{3n}},$$

$$(5.9)\quad T_m^{|y|} = d_\alpha^{|y|} \alpha^{m|y|}\left(1+\frac{\kappa |y|}{\alpha^{3m/2}}\cos(\omega+m\theta)+s_m\right),\quad |s_m| < \frac{1.1\cdot\kappa^2 n^2}{\alpha^{3m}}.$$

Inserting (5.7)–(5.9) in (5.4), and using $d_\alpha^{x+y+z}\alpha^{\ell x+my+nz} = 1$ to simplify the resulting expressions, we obtain

$$\left(1+\frac{\kappa x}{\alpha^{3\ell/2}}\cos(\omega+\ell\theta)+s_\ell\right)\cdot\left(1+\frac{\kappa z}{\alpha^{3n/2}}\cos(\omega+n\theta)+s_n\right) = 1+\frac{\kappa(x+z)}{\alpha^{3m/2}}\cos(\omega+m\theta)+s_m.$$

Expanding the left-hand side and performing some calculations, we arrive at

$$(5.10)\qquad \frac{\kappa x}{\alpha^{3\ell/2}}\cos(\omega+\ell\theta) = \frac{\kappa(x+z)}{\alpha^{3m/2}}\cos(\omega+m\theta) - \frac{\kappa z}{\alpha^{3n/2}}\cos(\omega+n\theta) - \frac{\kappa^2 xz}{\alpha^{3(n+\ell)/2}}\cos(\omega+\ell\theta)\cos(\omega+n\theta) - \frac{\kappa x}{\alpha^{3\ell/2}}\cos(\omega+\ell\theta)s_n - \frac{\kappa z}{\alpha^{3n/2}}\cos(\omega+n\theta)s_\ell + s_m - s_\ell - s_n - s_\ell s_n.$$

Multiplying (5.10) by $\alpha^{3\ell/2}/\kappa x$ and taking absolute values, we get

$$|\cos(\omega+\ell\theta)| < \frac{3n^2}{\alpha^{3\min\{\ell,m-\ell\}/2}}.$$

But

$$2|\cos(\omega+\ell\theta)| = |1+e^{2i(\omega+\ell\theta)}| = \left|1-\left(-\frac{d_\beta}{d_\gamma}\right)\left(\frac{\beta}{\gamma}\right)^\ell\right|.$$

Thus,

$$(5.11)\qquad \left|1-\left(-\frac{d_\beta}{d_\gamma}\right)\left(\frac{\beta}{\gamma}\right)^\ell\right| < \frac{6n^2}{\alpha^{3\min\{\ell,m-\ell\}/2}}.$$

In order to find an upper bound for ℓ and m in terms of $\log n$, we use the complex version of Theorem 3 with the parameters

$$t := 2, \quad \eta_1 := -d_\beta/d_\gamma, \quad \eta_2 := \beta/\gamma, \quad b_1 := 1, \quad b_2 := \ell.$$

Thus, $\Lambda_4 := 1 - (-d_\beta/d_\gamma)(\beta/\gamma)^\ell$, and from (5.11) we obtain

$$|\Lambda_4| < \frac{6n^2}{\alpha^{3\min\{\ell, m-\ell\}/2}}. \tag{5.12}$$

The number field $\mathbb{K} = \mathbb{Q}(\alpha, \beta)$ contains η_1, η_2 and has degree $D = 6$ over $\mathbb{Q}$. A simple check shows that the minimal polynomials of η_1 and η_2 are

$$\prod_{\sigma \in G}\left(X + \sigma\left(\frac{d_\beta}{d_\gamma}\right)\right) = 11X^6 - 33X^5 + 64X^4 + 73X^3 + 64X^2 - 33X + 11,$$
$$\prod_{\sigma \in G}(X - \sigma(\beta/\gamma)) = X^6 + 4X^5 + 11X^4 + 12X^3 + 11X^2 + 4X + 1,$$

respectively, where G is the Galois group $\mathrm{Gal}(\mathbb{K}/\mathbb{Q})$. Furthermore, the conjugates of η_1 and η_2 satisfy

$$\left|\frac{d_\beta}{d_\gamma}\right| = \left|\frac{d_\gamma}{d_\beta}\right| = 1, \quad \left|\frac{d_\beta}{d_\alpha}\right| = \left|\frac{d_\gamma}{d_\alpha}\right| = 0.773\ldots, \quad \left|\frac{d_\alpha}{d_\beta}\right| = \left|\frac{d_\alpha}{d_\gamma}\right| = 1.293\ldots$$
$$\left|\frac{\beta}{\gamma}\right| = \left|\frac{\gamma}{\beta}\right| = 1, \quad \left|\frac{\beta}{\alpha}\right| = \left|\frac{\gamma}{\alpha}\right| = 0.4008\ldots, \quad \left|\frac{\alpha}{\beta}\right| = \left|\frac{\alpha}{\gamma}\right| = 2.494\ldots.$$

Hence,

$$h(\eta_1) = \frac{1}{6}\left(\log 11 + 2\log\left|\frac{d_\alpha}{d_\beta}\right|\right) < 0.5, \quad h(\eta_2) = \frac{1}{3}\log\left|\frac{\alpha}{\beta}\right| < 0.31.$$

So, we can take $A_1 := 3$ and $A_2 := 2$, given that $|\log \eta_1| < 2$ and $|\log \eta_2| < 2$. Finally, $\Lambda_4 \neq 0$, because β/γ is an algebraic integer while d_γ/d_β is not. We set $B := n$.

By Theorem 3, we obtain

$$\begin{aligned} |\Lambda_4| &> \exp\bigl(-3 \cdot 30^6 \cdot 3^{5.5} \cdot 6^2 \cdot (1 + \log 6) \cdot (1 + \log(2n)) \cdot 3 \cdot 2\bigr) \\ &> \exp(-1.7 \cdot 10^{15} \log n). \end{aligned} \tag{5.13}$$

Here we have used $1 + \log(2n) < 3\log n$, valid for all $n \geq 5$.

Combining (5.12) and (5.13), we arrive at

$$\begin{aligned} \min\{\ell, m-\ell\} &< \frac{2\log 6}{3\log\alpha} + \frac{4\log n}{3\log\alpha} + \frac{3.4 \cdot 10^{15}}{3\log\alpha}\log n \\ &< 2 \cdot 10^{15}\log n. \end{aligned} \tag{5.14}$$

We now analyze two cases according to whether ℓ or $m - \ell$ is smaller.

Case 1. $\ell \leq m - \ell$. By (5.14), we have

$$\ell < 2 \cdot 10^{15}\log n. \tag{5.15}$$

As in the above section, we go back to the Diophantine equation (5.4) and replace T_m^y, T_n^z according to (4.6) and (4.4), respectively. This time, we get an inequality analogous to (4.10):

$$|\Lambda_2'| := |d_\alpha^{-x}\alpha^{-\ell x}T_\ell^x - 1| < 5n/\alpha^{3m/2}. \tag{5.16}$$

Here, we have used analogues of (4.4) and (4.6),

$$|\zeta_m| < 3n/\alpha^{3m/2} \quad \text{and} \quad |\zeta_n| < 3n/\alpha^{3n/2},$$

which hold because $\max\{|y|, z\} < n$ (see (5.6)).

We next apply Matveev's theorem again with $t := 1$ and

$$\eta_1 := d_\alpha^{-1}\alpha^{-\ell}T_\ell \quad \text{and} \quad b_1 := x.$$

Note that $\eta_1 \in \mathbb{K} := \mathbb{Q}(\alpha)$ and $[\mathbb{K} : \mathbb{Q}] = 3$. Using the properties of logarithmic height (see Section 2.2), we get

$$h(\eta_1) \le h(d_\alpha) + \ell h(\alpha) + h(T_\ell) < h(d_\alpha) + 2\ell h(\alpha) < 2\ell,$$

where we have used (2.6) and the fact that $h(d_\alpha) < 1.3$ (since the minimal polynomial of d_α is $44X^3 - 2X - 1$). Thus, we take $A_1 := 6\ell$ and $B := n$. It is easy to see that Λ_2' is nonzero: otherwise $x = 0$, which is not true.

Theorem 3 gives the following lower bound for $|\Lambda_2'|$:

$$\exp\big(-1.4 \cdot 30^4 \cdot 3^2(1 + \log 3)(1 + \log n)(6\ell)\big).$$

Combining (5.15), (5.16) and the above bound, we conclude that

$$m < 5.7 \cdot 10^{23} \log^2 n. \tag{5.17}$$

We go back to (5.4) and replace only T_n^z by using (4.4), to obtain

$$|\Lambda_3'| := |d_\alpha^{-z}\alpha^{-nz}T_\ell^{-x}T_m^{x+z} - 1| < 3n/\alpha^{3n/2}. \tag{5.18}$$

Clearly, $\Lambda_3' \ne 0$, because otherwise $z = 0$, which is not true.

With appropriate choices of $\mathbb{K}, D, \eta_i, b_i, A_i, B$, we obtain from Matveev's theorem (real case) the following lower bound on $\log|\Lambda_3'|$:

$$-1.4 \cdot 30^7 \cdot 4^{4.5} \cdot 3^2(1 + \log 3)(1 + 2\log n)(\log 44)(0.7)(0.7\ell)(0.7m). \tag{5.19}$$

But, from (5.18), we have the upper bound

$$\log|\Lambda_3'| < \log(3n) - (1.5\log\alpha)n.$$

Hence, using (5.15) and (5.17), we get $n < 10^{54}\log^4 n$. This last inequality leads to the following absolute bounds on ℓ, m and n.

Lemma 6. *Let (ℓ, m, n, x, y, z) be a solution of (5.4) with $3 \le \ell < m < n$ and $\ell < m - \ell$ and $d_\alpha^{x+y+z}\alpha^{\ell x+my+nz} = 1$. Then $\max\{x, |y|, z\} < n$ and*

$$\ell < 3.2 \cdot 10^{17}, \quad m < 8 \cdot 10^{32}, \quad n < 4.2 \cdot 10^{62}.$$

We now reduce the bounds given in this lemma. We begin by assuming that $\ell > 480$. From (5.11), we get

$$|\sin(\omega + \ell\theta - \pi/2)| = |\cos(\omega + \ell\theta)| < (3n^2)\alpha^{-3\ell/2} < 2\alpha^{-\ell/2}.$$

Setting $t := \lfloor(\omega + \ell\theta - \pi/2)/\pi\rceil$, where $\lfloor y\rceil$ is the nearest integer to the real number y, we obtain $-\pi/2 \le \omega + \ell\theta - \pi/2 - t\pi \le \pi/2$. Hence,

$$2\alpha^{-\ell/2} > |\sin(\omega + \ell\theta - \pi/2)| = |\sin(\omega + \ell\theta - \pi/2 - t\pi)| \ge \left|\frac{2\omega}{\pi} + \frac{2\theta}{\pi}\ell - 2t - 1\right|, \tag{5.20}$$

where we have used the inequality

$$|\sin y| = \sin|y| \ge \frac{2}{\pi}|y| \quad \text{for all } -\pi/2 \le y \le \pi/2.$$

We conclude from (5.20) that

$$\left|\frac{\theta}{\pi}\ell - t + \left(\frac{\omega}{\pi} - \frac{1}{2}\right)\right| < \alpha^{-\ell/2}. \tag{5.21}$$

We note that

$$\frac{\theta}{\pi}\ell - t + \left(\frac{\omega}{\pi} - \frac{1}{2}\right)$$

is nonzero. We set

$$\tau := \frac{\theta}{\pi}, \qquad \mu := \frac{\omega}{\pi} - \frac{1}{2}, \qquad A := 1, \qquad B := \alpha^{1/2}.$$

Inequality (5.21) can be rewritten as

$$0 < |\tau\ell - t + \mu| < AB^{-\ell}. \tag{5.22}$$

The fact that $\mathbf{T}$ is nondegenerate ensures that τ is an irrational number (otherwise the ratio β/γ is a root of unity, which is not the case). Lastly, we take $M := 3.2 \cdot 10^{17}$ which is an upper bound on ℓ by the inequalities in Lemma 6, and apply Lemma 3 to (5.22). With the help of Mathematica, we find that $q_{38} > 6M$ and $\epsilon = 0.39065\ldots$. Thus, the maximum value of $\lfloor\log(Aq/\epsilon)/\log B\rfloor$ is 142, which is an upper bound on ℓ, according to Lemma 3. However, we assumed that $\ell > 480$. This contradiction shows that $\ell \le 480$.

We now go back to (5.16) and note that

$$3.3 \cdot 10^{-191} < \min_{\ell\in[3,480]} |d_\alpha^{-1}\alpha^{-\ell}T_\ell - 1| \le |(d_\alpha^{-1}\alpha^{-\ell}T_\ell)^x - 1| < \frac{5n}{\alpha^{3m/2}}. \tag{5.23}$$

This leads to $m \le 640$.

Returning to the application of Matveev's theorem in Λ_3', we use the latest bounds for ℓ and m, instead of (5.15) and (5.17), to obtain $n < 1.4 \cdot 10^{22}$. Using this new bound on n, we return to our application of continued fractions in (5.21). We now assume that $\ell > 180$ and take $M := 480$ (the current upper bound of ℓ). The same arguments used before lead to $\ell \le 180$. Redoing the calculations for $\ell \in [3, 180]$, we obtain $2.6 \cdot 10^{-72}$ as a lower bound on the right-hand side of (5.23). Thus, $m \le 240$. Once again, returning to Λ_3', we obtain $n < 3.6 \cdot 10^{19}$.

With the new bounds, namely

$$\ell \in [3, 180], \qquad m \in [\ell + 1, 240], \qquad n \in [m + 1, 3.6 \cdot 10^{19}],$$

we implement the LLL-algorithm on Λ_3'. We write

$$\Gamma_3' := x \log T_\ell + y \log T_m + z \log d_\alpha + nz \log \alpha. \tag{5.24}$$

Since $|\Lambda_3'| < 3n/\alpha^{3n/2} < 1/2$ for all $n > 50$, we conclude that $|\Gamma_3'| < 5n/\alpha^{3n/2}$. We note that $\max\{x, |y|, z, nz\} < n^2$, so we set $X := (3.6 \cdot 10^{19})^2$. Computationally, we verify the lower bound

$$1.1 \cdot 10^{-190} < |\Gamma_3'| < 5n/\alpha^{3n/2}.$$

Hence, $n \le 500$. Once again, we return to the argument using continued fractions (5.21), where we now assume that $\ell > 30$ and take $M := 180$. This time we have $q_9 > 6M$ and $\ell \le 60$. In (5.23), we now have

$$\min_{\ell \in [3,60]} |d_\alpha^{-1} \alpha^{-\ell} T_\ell - 1| > 2 \cdot 10^{-24}.$$

Using the above inequality instead of the left-hand side of (5.23), and the fact that $n \le 500$, we obtain $m \le 70$. We now repeat the LLL-algorithm with $\ell \in [3, 60]$ and $m \in [\ell + 1, 70]$, where we take $X := 500^2$. We get $|\Gamma_3'| > 1.2 \cdot 10^{-74}$, and so $n \le 200$. We finish our reduction here, since the current bound is acceptable for a computational search.

Case 2: $m - \ell \le \ell$. By (5.14), we have

$$m - \ell < 2 \cdot 10^{15} \log n. \tag{5.25}$$

Subcase 2.1: *In* (3.1) *we have* $m_p \ge 5\ell_p/6$ *for all* $p \in P$. Then, by Theorem 2,

$$\alpha^{5(\ell-2)/6} < T_\ell^{5/6} = \Big(\prod p^{\ell_p}\Big)^{5/6} \le \gcd(T_\ell, T_m) < \alpha^{2m/3}.$$

Thus, $5(\ell - 2)/6 < 2m/3$ and so

$$\ell < \tfrac{4}{5} m + 2. \tag{5.26}$$

Combining (5.25) and (5.26), we deduce

$$\ell < 9 \cdot 10^{15} \log n \quad \text{and} \quad m < 2 \cdot 10^{16} \log n. \tag{5.27}$$

We now return, as before, to (5.18) and (5.19), where we use (5.27) to obtain $n < 8.4 \cdot 10^{45} \log^3 n$. This inequality and (5.27) allow us to deduce the following result.

LEMMA 7. *Let* (ℓ, m, n, x, y, z) *be a solution of* (5.4) *with* $3 \le \ell < m < n$ *and* $d_\alpha^{x+y+z}\alpha^{\ell x+my+nz} = 1$. *Assume that* $m-\ell < \ell$ *and* $\upsilon_p(T_m) \ge 5\upsilon_p(T_\ell)/6$ *for all* $p \in P$. *Then* $\max\{x, |y|, z\} < n$ *and*

$$\ell < 1.1 \cdot 10^{18}, \qquad m < 1.3 \cdot 10^{18}, \qquad n < 1.5 \cdot 10^{52}.$$

We next start reducing the bounds on ℓ, m and n. Returning to (5.12), we have $|\cos(\omega+\ell\theta)| < 3n^2/\alpha^{3(m-\ell)/2}$. First assume $m-\ell > 390$. Repeating the arguments concerning continued fractions, we deduce an inequality similar to (5.21) but with $m-\ell$ instead of ℓ:

$$\left|\frac{\theta}{\pi}\ell - t + \left(\frac{\omega}{\pi} - \frac{1}{2}\right)\right| < \alpha^{-(m-\ell)/2}. \tag{5.28}$$

Setting $M := 1.1 \cdot 10^{18}$ (current bound on ℓ), we confirm with Mathematica that $q_{41} > 6M$ and $\epsilon = 0.0141\ldots$. Thus, $m-\ell \le 164$, which contradicts our assumption that $m-\ell > 390$. From now on, we assume that $m-\ell \le 390$. Therefore, by (5.26), we get

$$\ell \le 1570 \quad \text{and} \quad m \le 1950.$$

With these bounds, we go to the lower bound of $\log|\Lambda_3'|$ given in (5.19) and replace ℓ and m, to obtain $n < 1.4 \cdot 10^{23}$. Restarting our reduction cycle through the continued fractions argument, inequality (5.26) and the linear form in logarithms Λ_3', we conclude that $\ell \le 710$, $m \le 885$ and $n < 2.8 \cdot 10^{22}$.

We implement the LLL-algorithm with $\ell \in [3, 710]$ and $m \in [\ell+1, 885]$ on Γ_3' in (5.24). We now set $X := (2.8 \cdot 10^{22})^2$ (current bound on $\max\{x, |y|, z, nz\}$). We verify with Mathematica the lower bound

$$10^{-360} < |\Gamma_3'| < 5n/\alpha^{3n/2}.$$

Hence, $n \le 920$. Once again, we return to the argument using continued fractions (5.21), where we now assume that $m-\ell > 30$. We take $M := 710$. This time we have $m-\ell \le 60$. Thus, $\ell \le 250$. For $\ell \in [3, 250]$, we calculate an inequality similar to (5.23):

$$\begin{aligned} 1.2 \cdot 10^{-100} &< \min_{\ell \in [3,250]} |d_\alpha^{-1}\alpha^{-\ell}T_\ell - 1| \\ &\le |(d_\alpha^{-1}\alpha^{-\ell}T_\ell)^x - 1| < 5n/\alpha^{3(m-\ell)/2}. \end{aligned} \tag{5.29}$$

This leads to $m \le 260$. Finally, applying the LLL-algorithm algorithm on Γ_3' leads to the conclusion that $n \le 280$. The bounds $\ell < m < n \le 280$ are low enough to perform a computer search.

SUBCASE 2.2: *In* (3.1), *we have* $m_p < 5\ell_p/6$ *for some* $p \in P$. From the Diophantine equation (5.4), we deduce that

$$\ell_p x = \upsilon_p(T_\ell^x) \le \upsilon_p(T_m^{x+z}) = m_p(x+z) < \tfrac{5}{6}\ell_p(x+z).$$

Thus, $x < 5z$. Combining this with (5.2), and inequalities (5.5) and (5.25), we conclude that

$$x < 10^{16}\log n, \quad z < 2\cdot 10^{15}\log n, \tag{5.30}$$

$$m-\ell < 2\cdot 10^{15}\log n, \quad n-m < 10^{16}\log n, \quad n-\ell < 2\cdot 10^{16}\log n. \tag{5.31}$$

We go one last time to (5.4), and replace each term according to (5.7)–(5.9), where we now use the identity

$$\frac{\kappa}{\alpha^{3t/2}}\cos(\omega+t\theta) = \frac{d_\beta}{d_\alpha}\left(\frac{\beta}{\alpha}\right)^t + \frac{d_\gamma}{d_\alpha}\left(\frac{\gamma}{\alpha}\right)^t$$

for each $t=\ell,m,n$. Thus, the Diophantine equation $T_\ell^x T_n^z = T_m^{x+z}$ is reduced to

$$\begin{aligned}\left(1+x\left(\frac{d_\beta}{d_\alpha}\left(\frac{\beta}{\alpha}\right)^\ell + \frac{d_\gamma}{d_\alpha}\left(\frac{\gamma}{\alpha}\right)^\ell\right)+s_\ell\right)&\\ \times\left(1+z\left(\frac{d_\beta}{d_\alpha}\left(\frac{\beta}{\alpha}\right)^n + \frac{d_\gamma}{d_\alpha}\left(\frac{\gamma}{\alpha}\right)^n\right)+s_n\right)&\\ =1+(x+z)\left(\frac{d_\beta}{d_\alpha}\left(\frac{\beta}{\alpha}\right)^m + \frac{d_\gamma}{d_\alpha}\left(\frac{\gamma}{\alpha}\right)^m\right)+s_m&.\end{aligned}$$

Multiplying, simplifying and rearranging terms, we get

$$\begin{aligned}x\left(\frac{d_\beta}{d_\alpha}\left(\frac{\beta}{\alpha}\right)^\ell + \frac{d_\gamma}{d_\alpha}\left(\frac{\gamma}{\alpha}\right)^\ell\right)+z\left(\frac{d_\beta}{d_\alpha}\left(\frac{\beta}{\alpha}\right)^n + \frac{d_\gamma}{d_\alpha}\left(\frac{\gamma}{\alpha}\right)^n\right)&\\ -(x+z)\left(\frac{d_\beta}{d_\alpha}\left(\frac{\beta}{\alpha}\right)^m + \frac{d_\gamma}{d_\alpha}\left(\frac{\gamma}{\alpha}\right)^m\right)&\\ = s_m - s_\ell - s_n - s_\ell s_n - x\left(\frac{d_\beta}{d_\alpha}\left(\frac{\beta}{\alpha}\right)^\ell + \frac{d_\gamma}{d_\alpha}\left(\frac{\gamma}{\alpha}\right)^\ell\right)s_n&\\ - z\left(\frac{d_\beta}{d_\alpha}\left(\frac{\beta}{\alpha}\right)^n + \frac{d_\gamma}{d_\alpha}\left(\frac{\gamma}{\alpha}\right)^n\right)s_\ell&\\ - xz\frac{d_\beta}{d_\alpha}\frac{d_\gamma}{d_\alpha}\left(\left(\frac{\beta}{\alpha}\right)^n\left(\frac{\gamma}{\alpha}\right)^\ell + \left(\frac{\beta}{\alpha}\right)^\ell\left(\frac{\gamma}{\alpha}\right)^n\right)&\\ - xz\left(\left(\frac{d_\beta}{d_\alpha}\right)^2\left(\frac{\beta}{\alpha}\right)^{\ell+n} + \left(\frac{d_\gamma}{d_\alpha}\right)^2\left(\frac{\gamma}{\alpha}\right)^{\ell+n}\right)&.\end{aligned} \tag{5.32}$$

We work on the left-hand side of (5.32). We start reorganizing the terms:

$$\begin{aligned}\frac{d_\beta}{d_\alpha}\left(\frac{\beta}{\alpha}\right)^\ell\left[x+z\left(\frac{\beta}{\alpha}\right)^{n-\ell}-(x+z)\left(\frac{\beta}{\alpha}\right)^{m-\ell}\right]&\\ +\frac{d_\gamma}{d_\alpha}\left(\frac{\gamma}{\alpha}\right)^\ell\left[x+z\left(\frac{\gamma}{\alpha}\right)^{n-\ell}-(x+z)\left(\frac{\gamma}{\alpha}\right)^{m-\ell}\right]&.\end{aligned} \tag{5.33}$$

The second term in (5.33) is nonzero. Indeed, otherwise

$$x=(x+z)\left(\frac{\gamma}{\alpha}\right)^{m-\ell}-z\left(\frac{\gamma}{\alpha}\right)^{n-\ell}.$$

Taking absolute value and using

$$|\gamma/\alpha|=1/\alpha^{3/2},\qquad m<n,\qquad x<5z,\qquad z\le m-\ell,$$

we get

$$1\le x<\frac{x+z}{\alpha^{3(m-\ell)/2}}+\frac{z}{\alpha^{3(n-\ell)/2}}<\frac{7(m-\ell)}{\alpha^{3(m-\ell)/2}}.$$

The last inequality holds only for $m-\ell\le 3$. Thus,

$$x\in[1,15],\quad z\in[1,3],\quad m-\ell\in[1,3],\quad n-\ell\in[2,18].$$

However, a computational check reveals that

$$\left|x+z\left(\frac{\gamma}{\alpha}\right)^{n-\ell}-(x+z)\left(\frac{\gamma}{\alpha}\right)^{m-\ell}\right|>1$$

for x, z, $m-\ell$ and $n-\ell$ in the above range.

We now show that every expression on the left-hand side of (5.32) is nonzero. First of all, we note by (2.2) that for $t=\ell,m,n$,

$$\frac{d_\beta}{d_\alpha}\left(\frac{\beta}{\alpha}\right)^t+\frac{d_\gamma}{d_\alpha}\left(\frac{\gamma}{\alpha}\right)^t=\frac{T_t}{d_\alpha\alpha^t}-1.$$

Hence, if the left-hand side of (5.32) is zero, then

$$x\left(\frac{T_\ell}{d_\alpha\alpha^\ell}-1\right)+z\left(\frac{T_n}{d_\alpha\alpha^n}-1\right)=(x+z)\left(\frac{T_m}{d_\alpha\alpha^m}-1\right).$$

Thus,

$$x\frac{T_\ell}{d_\alpha\alpha^\ell}+z\frac{T_n}{d_\alpha\alpha^n}=(x+z)\frac{T_m}{d_\alpha\alpha^m}.$$

However, this is not possible by Lemma 1.

Factoring the second term in (5.33), we get

$$\begin{aligned}(5.34)\qquad &\frac{d_\gamma}{d_\alpha}\left(\frac{\gamma}{\alpha}\right)^\ell\left[x+z\left(\frac{\gamma}{\alpha}\right)^{n-\ell}-(x+z)\left(\frac{\gamma}{\alpha}\right)^{m-\ell}\right]\\ &\qquad\times\left[\frac{d_\beta}{d_\gamma}\left(\frac{\beta}{\gamma}\right)^\ell\frac{x+z\left(\frac{\beta}{\alpha}\right)^{n-\ell}-(x+z)\left(\frac{\beta}{\alpha}\right)^{m-\ell}}{x+z\left(\frac{\gamma}{\alpha}\right)^{n-\ell}-(x+z)\left(\frac{\gamma}{\alpha}\right)^{m-\ell}}+1\right].\end{aligned}$$

Below we work on the right-hand side of (5.32). We consider the following facts:

$$\left|\frac{d_\beta}{d_\gamma}\right|=1,\qquad\left|\frac{d_\beta}{d_\alpha}\right|=\left|\frac{d_\gamma}{d_\alpha}\right|<1,\qquad\left|\frac{\beta}{\alpha}\right|=\left|\frac{\gamma}{\alpha}\right|=\frac{1}{\alpha^{3/2}}.$$

Furthermore, in order to use (5.30), we note that, more generally, we can get slightly better inequalities than (5.7)–(5.9):

$$|s_\ell| < \frac{2.7x^2}{\alpha^{3\ell}}, \quad |s_n| < \frac{2.7z^2}{\alpha^{3n}}, \quad |s_m| < \frac{2.7(x+z)^2}{\alpha^{3m}},$$

where we have used $1.1 \cdot \kappa^2 < 2.7$.

We have shown that the absolute value of the right-hand side of (5.32) is less than

$$\frac{2.7x^2}{\alpha^{3\ell}} + \frac{2.7z^2}{\alpha^{3n}} + \frac{2.7(x+z)^2}{\alpha^{3m}} + \frac{7x^2z^2}{\alpha^{3(\ell+n)}} + \frac{4.8xz^2}{\alpha^{(3\ell/2)+3n}} + \frac{4.8x^2z}{\alpha^{(3n/2)+3\ell}} + \frac{4xz}{\alpha^{(3(n+\ell)/2)}}. \tag{5.35}$$

We set

$$\chi := x + z\left(\frac{\gamma}{\alpha}\right)^{n-\ell} - (x+z)\left(\frac{\gamma}{\alpha}\right)^{m-\ell}. \tag{5.36}$$

Keeping in mind that the absolute value of (5.34) is less than the expression in (5.35), we multiply by $(d_\alpha/\rho)\alpha^{3\ell/2}$ to obtain

$$|\chi| \cdot \left|\left(\frac{\beta}{\gamma}\right)^{\ell} \frac{d_\beta}{d_\gamma} \frac{\bar{\chi}}{\chi} + 1\right| < \frac{12x^2z^2}{\alpha^{3\ell/2}}. \tag{5.37}$$

We now give lower bounds for each absolute value.

Since γ/α is an algebraic integer in $\mathbb{L} := \mathbb{Q}(\alpha, \beta)$, we have $\chi \in \mathcal{O}_{\mathbb{L}}$. Thus, $\mathrm{N}_{\mathbb{L}/\mathbb{Q}}(\chi) \geq 1$. But

$$\mathrm{N}_{\mathbb{L}/\mathbb{Q}}(\chi) = \prod_{\sigma \in G} |\sigma(\chi)|,$$

where $G = \mathrm{Gal}(\mathbb{L}/\mathbb{Q})$. Hence,

$$|\chi| > \prod_{\substack{\sigma \in G \\ \sigma \neq (1)}} |\sigma(\chi)|^{-1}. \tag{5.38}$$

Now,

$$|\sigma(\chi)| \leq x + z\left|\sigma\left(\frac{\gamma}{\alpha}\right)\right|^{n-\ell} + (x+z)\left|\sigma\left(\frac{\gamma}{\alpha}\right)\right|^{m-\ell}.$$

We note that $|\sigma(\gamma/\alpha)| < \alpha^{3/2}$ for all $\sigma \in G$. Thus

$$\begin{aligned} |\sigma(\chi)| &\leq x + z\alpha^{3(n-\ell)/2} + (x+z)\alpha^{3(m-\ell)/2} \\ &\leq n(1 + \alpha^{3(n-\ell)/2} + 2\alpha^{3(m-\ell)/2}) \\ &< n\alpha^{2(n-\ell)} < n\alpha^{4\cdot 10^{16}\log n} = \exp(2.44 \cdot 10^{16} \log n), \end{aligned} \tag{5.39}$$

where we have used (5.31). Hence, returning to (5.38), we get

$$|\chi| > \exp(-1.3 \cdot 10^{17} \log n). \tag{5.40}$$

We now set

$$\Lambda_5 := 1 - \left(\frac{\beta}{\gamma}\right)^{\ell}\left(-\frac{d_\beta}{d_\gamma}\right)\frac{\bar{\chi}}{\chi}.$$

We use one last time Matveev's theorem (complex case), with the parameters $t := 2$ and

$$\eta_1 := \frac{\beta}{\gamma}, \quad \eta_2 := \left(-\frac{d_\beta}{d_\gamma}\right)\frac{\bar{\chi}}{\chi}, \quad b_1 := \ell, \quad b_2 := 1.$$

As before, we take $\mathbb{K} := \mathbb{Q}(\alpha, \beta)$, $D = 6$ and $B := \ell$. In addition, recall that from the application of Theorem 3 to Λ_4, we have $h(\beta/\alpha) < 0.31$ and $h(-d_\beta/d_\gamma) < 0.5$. Then, by the properties of logarithmic height,

$$h(\eta_2) \le h(-d_\beta/d_\gamma) + 2h(\chi).$$

We assume that d is the degree of χ over $\mathbb{Q}$ and use (5.39) to conclude that

$$\begin{aligned} h(\chi) &= \frac{1}{d}\sum_{\sigma\in G}\log(\max\{1, |\sigma(\chi)|\}) \\ &\le \log\Big(\max_{\sigma\in G}\{1, |\sigma(\chi)|\}\Big) < 2.44\cdot 10^{16}\log n. \end{aligned}$$

Hence, $h(\eta_2) < 5\cdot 10^{16}\log n$.

On the other hand,

$$|\log\eta_2| \le |\log(-d_\beta/d_\gamma)| + 2|\log\chi| < 2 + 2|\log\chi|. \tag{5.41}$$

Furthermore,

$$\begin{aligned} |\log\chi| &\le \log x + \left|\log\left(1 - \left(\left(1+\frac{z}{x}\right)\left(\frac{\gamma}{\alpha}\right)^{m-\ell} - \frac{z}{x}\left(\frac{\gamma}{\alpha}\right)^{n-\ell}\right)\right)\right| \\ &\le \log x + \sum_{k=1}^{\infty}\left|\left(1+\frac{z}{x}\right)\left(\frac{\gamma}{\alpha}\right)^{m-\ell} - \frac{z}{x}\left(\frac{\gamma}{\alpha}\right)^{n-\ell}\right|^k \\ &< \log\log n + 70. \end{aligned}$$

In the above inequality, we have used the fact that $x < 10^{16}\log n$ (by (5.30)) and

$$\begin{aligned} &\left|\left(1+\frac{z}{x}\right)\left(\frac{\gamma}{\alpha}\right)^{m-\ell} - \frac{z}{x}\left(\frac{\gamma}{\alpha}\right)^{n-\ell}\right| \\ &\quad\le \left|\frac{\gamma}{\alpha}\right|^{m-\ell}\left(1+\frac{z}{x}\left(1+\left|\frac{\gamma}{\alpha}\right|^{n-m}\right)\right) \\ &\quad= \frac{1}{\alpha^{3(m-\ell)/2}}\left(1+\frac{z}{x}\left(1+\frac{1}{\alpha^{3(n-m)/2}}\right)\right) < \frac{2}{\alpha^{3/2}} + \frac{1}{\alpha^3} < 0.963. \end{aligned}$$

By (5.41), we conclude that $\log(\eta_2) < 2\log\log n + 150$. So, we can take $A_1 := 2$ and $A_2 := 3\cdot 10^{17}\log n$.

Applying Theorem 3 (complex case) with the above information, we obtain the following lower bound for $|\Lambda_5|$:

$$\left|\left(\frac{\beta}{\gamma}\right)^{\ell}\frac{d_\beta}{d_\gamma}\frac{\overline{\chi}}{\chi}+1\right| > \exp(-1.7\cdot 10^{32}\log n\log \ell). \tag{5.42}$$

Combining (5.37), (5.40) and (5.42), we get

$$\exp(-2\cdot 10^{32}\log n\log \ell) < |\chi|\cdot\left|\left(\frac{\beta}{\gamma}\right)^{\ell}\frac{d_\beta}{d_\gamma}\frac{\bar{\chi}}{\chi}+1\right| < \frac{12x^2z^2}{\alpha^{3\ell/2}}. \tag{5.43}$$

We now take logarithms on both sides, and consider the bounds on x and z given in (5.30), to obtain

$$\frac{\ell}{\log \ell} < 2.2\cdot 10^{32}\log n. \tag{5.44}$$

We use the fact that

$$\left(A>3 \text{ and } \frac{t}{\log t}<A\right) \;\Rightarrow\; t<2A\log A. \tag{5.45}$$

Taking $A := 2.2\cdot 10^{32}\log n$, we deduce from (5.44) and (5.45) that

$$\begin{aligned}\ell &< 2(2.2\cdot 10^{32}\log n)\log(2.2\cdot 10^{32}\log n)\\ &< 4.4\cdot 10^{32}(\log n)(75+\log\log n).\end{aligned}$$

Thus, by (5.31), we get

$$n < 2\cdot 10^{16}\log n+\ell < 2\cdot 10^{16}\log n + 4.4\cdot 10^{32}(\log n)(75+\log\log n),$$

which leads to $n<3\cdot 10^{36}$ and later to $\ell < 3\cdot 10^{36}$.

Repeating the arguments concerning continued fractions, we return to the linear form associated to Λ_4 (given in (5.28)), where we assume again that $m-\ell>390$. Applying Lemma 3 with $M := 3\cdot 10^{36}$ (current bound on ℓ), we obtain $q_{77} > 6M$, $\epsilon = 0.2423\ldots$ and $m-\ell\le 293$, which was confirmed with Mathematica. Since we have assumed in fact that $m-\ell>390$, we conclude that $m-\ell\le 390$.

We use the facts that $n-\ell=(n-m)+(m-\ell)$ and $x(m-\ell)=z(n-m)$ (by (5.2)) to derive the following result.

Lemma 8. *Let (ℓ,m,n,x,y,z) be a solution of (5.4) with $3\le \ell<m<n$ and $d_\alpha^{x+y+z}\alpha^{\ell x+my+nz}=1$. Assume further that $m-\ell<\ell$ and $\upsilon_p(T_m) < 5\upsilon_p(T_\ell)/6$ for some $p\in P$. Then $\max\{x,|y|,z\}<n$ and*

$$\ell<n<3\cdot 10^{36},\qquad m-\ell\le 390,\qquad n-\ell\le 6(m-\ell)\le 2340.$$

As before, our next step is to reduce the above bounds. To this end, we return to inequality (5.37), which we rewrite as

$$
\begin{aligned}
(5.46)\qquad |\Lambda_5| &= \left|\left(\frac{\beta}{\gamma}\right)^{\ell}\left(-\frac{d_\beta}{d_\gamma}\right)\frac{\bar{\chi}}{\chi}-1\right| < \frac{12x^2z}{|\tilde{\chi}|}\alpha^{-3\ell/2}\\
&< \frac{3\cdot 10^2(m-\ell)^3}{|\tilde{\chi}|}\alpha^{-3\ell/2},
\end{aligned}
$$

where $\tilde{\chi}$ corresponds to the simplification of z in χ (see (5.36)). Moreover,

$$
\begin{aligned}
\tilde{\chi} &= \frac{x}{z}+\left(\frac{\gamma}{\alpha}\right)^{n-\ell}-\left(\frac{x}{z}+1\right)\left(\frac{\gamma}{\alpha}\right)^{m-\ell}\\
&= \left(\frac{n-\ell}{m-\ell}-1\right)+\left(\frac{\gamma}{\alpha}\right)^{n-\ell}-\left(\frac{n-\ell}{m-\ell}+1\right)\left(\frac{\gamma}{\alpha}\right)^{m-\ell}.
\end{aligned}
$$

The previous calculations lead us to note that the upper bound in inequality (5.46) is only determined by the values

$$
\ell < 3\cdot 10^{36},\quad 1\le m-\ell\le 390,\quad m-\ell\le n-\ell\le 6(m-\ell)\le 2340.
$$

Before continuing, we note that

$$
\min_{\substack{1\le m-\ell\le 390\\ m-\ell<n-\ell\le 2340}} |\tilde{\chi}| > 2.5\cdot 10^{-3}.
$$

Thus, assuming that $\ell > 40$, we conclude from (5.46) that $|\Lambda_5| < 1/2$.

Now, taking $\log w = \log|w| + i\arg w$ with $-\pi < \arg w \le \pi$ (the logarithm of the complex number w), we get

$$
\log(1+w) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{w^n}{n}\quad \text{for } w\in\mathbb{C} \text{ with } |w|<1.
$$

From the above formula, one easily shows that $|\log(1+w)| \le 2|w|$ if $|w| \le 1/2$. Hence, with $w = \Lambda_5$, and recalling that the complex logarithm is additive modulo $2\pi i$, we deduce from (5.46) that

$$
(5.47)\qquad \left|\ell\log\frac{\beta}{\gamma}+\log\left(\frac{d_\beta}{d_\gamma}\frac{\tilde{\chi}}{\chi}\right)-2\pi k i\right| < \frac{6\cdot 10^2(m-\ell)^3}{|\tilde{\chi}|}\alpha^{-3\ell/2}
$$

for some $k\in\mathbb{Z}$. We note that β/γ and $(-\delta_\beta/d_\gamma)\tilde{\chi}/\chi$ are complex numbers of absolute value 1. Moreover,

$$
\frac{\beta}{\gamma} = e^{2\theta i},\quad -\frac{d_\alpha}{d_\gamma}\frac{\tilde{\chi}}{\chi} = e^{(2\delta+2\omega+\pi)i},
$$

where θ, δ and ω are the arguments of β/γ, δ_β/d_γ and $\tilde{\chi}/\chi$, respectively.

We see from inequality (5.47) that

$$
|2\theta\ell i + (2\delta+2\omega+\pi)i - 2\pi k\, i| < \frac{6\cdot 10^2(m-\ell)^3}{|\tilde{\chi}|}\alpha^{-3\ell/2}.
$$

Dividing both sides by $2\pi i$, we get

$$\left|\frac{\theta}{\pi}\ell - k + \left(\frac{\delta+\omega}{\pi} + \frac{1}{2}\right)\right| < \frac{3\cdot 10^2(m-\ell)^3}{\pi|\tilde{\chi}|}\alpha^{-3\ell/2}. \tag{5.48}$$

We note that the left-hand side is nonzero, and the fact that $\mathbf{T}$ is nondegenerate ensures that θ/π is an irrational number.

A new implementation of Lemma 3 in (5.48) for $m-\ell \in [3, 390]$ and $n-\ell \in [m-\ell, 6(m-\ell)]$, with

$$\gamma := \frac{\theta}{\pi}, \qquad \mu := \frac{\delta+\omega}{\pi} + \frac{1}{2}$$

and

$$A := \frac{3\cdot 10^2(m-\ell)^3}{\pi|\tilde{\chi}|}, \qquad B := \alpha^{3/2}, \qquad M := 3\cdot 10^{36},$$

yields $\ell \le 130$. Then, by Lemma 8, we get $n-\ell < 2340$, and so $n < 2470$.

We return one more time to (5.28), where we now assume that $m-\ell > 40$, and take $M := 130$. We conclude that $m-\ell \le 40$. Finally, we return to (5.46), where we now assume that $\ell > 20$, to conclude that $|\Lambda_5| < 1/2$ given that $n < 2340$. We apply Lemma 3 in (5.48) with $m-\ell \in [3, 40]$ and $n-\ell \in [m-\ell, 6(m-\ell)]$ and with $M := 130$ (current bound on ℓ). A quick calculation with Mathematica reveals that $\ell \le 40$, so $n \le 6(m-\ell)+\ell \le 280$.

Summarizing all the cases, we have

$$\ell < m < n \le 280.$$

Using the primitive prime factors of T_ℓ, T_m and T_n, we check that the only solutions of (5.4), corresponding to $x+z=|y|$ (see (5.1)), are

$$T_{13}^2 = T_{17}T_9, \qquad T_{16}^2 = T_{15}T_{17}, \qquad T_{12}^2 = T_{15}T_9.$$

This completes the proof of the Main Theorem.

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Carlos Alexis Gómez Ruiz
Departamento de Matemáticas
Universidad del Valle
Calle 13 No. 100-00
25360 Cali, Colombia
E-mail: carlos.a.gomez@correounivalle.edu.co

Florian Luca
School of Mathematics
University of the Witwatersrand
Private Bag X3, Wits 2050
Johannesburg, South Africa
E-mail: Florian.Luca@wits.ac.za

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