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Adding a lot of Cohen reals by adding a few. II

by

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Abstract. We study pairs (V, V_1), $V \subseteq V_1$, of models of ZFC such that adding κ-many Cohen reals over V_1 adds λ-many Cohen reals over V for some $\lambda > \kappa$.

1. Introduction. We continue our previous study [GG]. We investigate pairs (V, V_1), $V \subseteq V_1$, of models of ZFC with the same ordinals, such that adding κ-many Cohen reals over V_1 adds λ-many Cohen reals over V for some $\lambda > \kappa$ [1]. We are mainly interested in when V and V_1 have the same cardinals and reals. We prove that for such models, adding κ-many Cohen reals over V_1 cannot produce more Cohen reals over V for κ below the first fixed point of the $\aleph$-function, but the situation at that point is different. We also reduce the large cardinal assumptions from [G1], [GG] to the optimal ones.

2. Adding many Cohen reals by adding a few: a general result. In this section we prove the following general result.

THEOREM 2.1. *Suppose $\kappa < \lambda$ are infinite (regular or singular) cardinals, and let V_1 be an extension of V. Suppose that in V_1:*

(a) *$\kappa < \lambda$ are still infinite cardinals* [2],
(b) *there exists an increasing sequence $\langle \kappa_n : n < \omega \rangle$ of regular cardinals, cofinal in κ, in particular $\mathrm{cf}(\kappa) = \omega$,*
(c) *there is an increasing (mod finite) sequence $\langle f_\alpha : \alpha < \lambda \rangle$ of functions in the product $\prod_{n<\omega}(\kappa_{n+1} \setminus \kappa_n)$,*

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(1) By "λ-many Cohen reals" we mean a generic object $\langle s_\alpha : \alpha < \lambda \rangle$ for the poset $\mathbb{C}(\lambda)$ of finite partial functions from $\lambda \times \omega$ to 2.

(2) λ can be a regular or a singular cardinal, but by (b), κ is necessarily a singular cardinal of cofinality ω.

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(d) *there is a splitting* $\langle S_\sigma : \sigma < \kappa\rangle$ *of* λ *into sets of size* λ *such that for every countable set* $I \in V$ *and every* $\sigma < \kappa$ *we have* $|I \cap S_\sigma| < \aleph_0$.

Then adding κ*-many Cohen reals over* V_1 *produces* λ*-many Cohen reals over* V.

REMARK 2.2. Condition (c) holds automatically for $\lambda = \kappa^+$; given any collection $\mathcal{F}$ of κ-many elements of $\prod_{n<\omega}(\kappa_{n+1} \setminus \kappa_n)$, there exists f such that for each $g \in \mathcal{F}$, $f(n) > g(n)$ for all large n (3). Thus we can define, by induction on $\alpha < \kappa^+$, an increasing (mod finite) sequence $\langle f_\alpha : \alpha < \kappa^+\rangle$ in $\prod_{n<\omega}(\kappa_{n+1} \setminus \kappa_n)$ (4).

Proof of Theorem 2.1. Force to add κ-many Cohen reals over V_1. Split them into $\langle r_{i,\sigma} : i, \sigma < \kappa\rangle$ and $\langle r'_\sigma : \sigma < \kappa\rangle$. Also, in V, split κ into κ-blocks B_σ, $\sigma < \kappa$, each of size κ, and let $\langle f_\alpha : \alpha < \lambda\rangle \in V_1$ be an increasing (mod finite) sequence in $\prod_{n<\omega}(\kappa_{n+1} \setminus \kappa_n)$. Let $\alpha < \lambda$. We define a real s_α as follows. Pick $\sigma < \kappa$ such that $\alpha \in S_\sigma$. Let $k_\alpha = \min\{k < \omega : r'_\sigma(k)\} = 1$ and set

$$\forall n < \omega, \qquad s_\alpha(n) = r_{f_\alpha(n+k_\alpha),\sigma}(0).$$

LEMMA 2.3. $\langle s_\alpha : \alpha < \lambda\rangle$ *is a sequence of* λ*-many Cohen reals over* V.

NOTATION 2.4. (a) For a forcing notion $\mathbb{P}$ and $p, q \in \mathbb{P}$, we let $p \le q$ mean p is stronger than q.

(b) For each set I, let $\mathbb{C}(I)$ be the Cohen forcing for adding I-many Cohen reals. Thus $\mathbb{C}(I) = \{p : p \text{ is a finite partial function from } I \times \omega \text{ to } 2\}$, ordered by $p \le q$ iff $p \supseteq q$.

Proof of Lemma 2.3. First note that $\langle\langle r_{i,\sigma} : i, \sigma < \kappa\rangle, \langle r'_\sigma : \sigma < \kappa\rangle\rangle$ is $\mathbb{C}(\kappa\times\kappa)\times\mathbb{C}(\kappa)$-generic over V_1. By the c.c.c. of $\mathbb{C}(\lambda)$ it suffices to show that for any countable set $I \subseteq \lambda$, $I \in V$, the sequence $\langle s_\alpha : \alpha \in I\rangle$ is $\mathbb{C}(I)$-generic over V. Thus it suffices to prove the following:

(∗) For every $(p, q) \in \mathbb{C}(\kappa \times \kappa) \times \mathbb{C}(\kappa)$ and every open dense subset $D \in V$ of $\mathbb{C}(I)$, there is $(\overline{p}, \overline{q}) \le (p, q)$ such that $(\overline{p}, \overline{q}) \Vdash \ulcorner\langle \underset{\sim}{s}_\alpha : \alpha \in I\rangle$ extends some element of $D\urcorner$.

Let (p, q) and D be as above, and for simplicity suppose that $p = q = \emptyset$. Let $b \in D$, and let $\alpha_1, \ldots, \alpha_m$ be an enumeration of the components of b, i.e., those α such that $(\alpha, n) \in \mathrm{dom}(b)$ for some n. Also let $\sigma_1, \ldots, \sigma_m < \kappa$ be such that $\alpha_i \in S_{\sigma_i}$, $i = 1, \ldots, m$. By (d) each $I \cap S_{\sigma_i}$ is finite, thus by (c) we can find $n^* < \omega$ such that for all $n \ge n^*$, $1 \le i \le m$ and $\alpha^*_1 < \alpha^*_2$ in

(3) To see this let $\mathcal{F} = \bigcup_{n<\omega} \mathcal{F}_n$, where $\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \cdots$ and $|\mathcal{F}_n| < \kappa_{n+1}$, and define f so that $\sup\{g(n) : g \in \mathcal{F}_n\} < f(n) \in \kappa_{n+1} \setminus \kappa_n$.

(4) Let f_0 be arbitrary. Given $\alpha < \kappa^+$, we can apply the above to find f_α so that $f_\alpha(n) > f_\beta(n)$ for all large n and all $\beta < \alpha$.

$I \cap S_{\sigma_i}$ we have $f_{\alpha_1^*}(n) < f_{\alpha_2^*}(n)$. Let

$$\overline{q} = \{\langle \sigma_i, n, 0\rangle : 1 \le i \le m,\ n < n^*\}.$$

Then $\overline{q} \in \mathbb{C}(\kappa)$ and $(\emptyset, \overline{q}) \Vdash \ulcorner k_{\alpha_i} \ge n^* \urcorner$ for all $1 \le i \le m$. Let

$$\overline{p} = \{\langle f_{\alpha_i}(n + k_{\alpha_i}), \sigma_i, 0, b(\alpha_i, n)\rangle : 1 \le i \le m,\ (\alpha_i, n) \in \mathrm{dom}(b)\}.$$

Then $\overline{p} \in \mathbb{C}(\kappa \times \kappa)$ is well-defined, and for $(\alpha_i, n) \in \mathrm{dom}(b), 1 \le i \le m$, we have

$$(\overline{p}, \overline{q}) \Vdash \ulcorner \underset{\sim}{s}_{\alpha_i}(n) = \underset{\sim}{r}_{f_{\alpha_i}(n+k_{\alpha_i}),\sigma_i}(0) = \overline{p}(f_{\alpha_i}(n + k_{\alpha_i}), \sigma_i, 0) = b(\alpha_i, n)\urcorner,$$

and hence

$$(\overline{p}, \overline{q}) \Vdash \ulcorner \langle \underset{\sim}{s}_\alpha : \alpha \in I\rangle \text{ extends } b\urcorner.$$

Thus $(*)$ is established and we are done. ∎

Theorem 2.1 follows from the above lemma. ∎

3. Getting results from optimal hypotheses

THEOREM 3.1. *Suppose GCH holds, and κ is a cardinal of countable cofinality and there are κ-many measurable cardinals below κ. Then there is a cardinal preserving not adding a real extension V_1 of V in which there is a splitting $\langle S_\sigma : \sigma < \kappa\rangle$ of κ^+ into sets of size κ^+ such that for every countable set $I \in V$ and every $\sigma < \kappa$, we have $|I \cap S_\sigma| < \aleph_0$.*

Proof. Let X be a set of measurable cardinals below κ of size κ which is discrete, i.e., contains none of its limit points, and for each $\xi \in X$ fix a normal measure U_ξ on ξ. For each $\xi \in X$ let $\mathbb{P}_\xi$ be the Prikry forcing associated with the measure U_ξ, and let $\mathbb{P}_X$ be the Magidor iteration of $\mathbb{P}_\xi$'s, $\xi \in X$ (cf. [G2], [Ma]). Since X is discrete, each condition in $\mathbb{P}_X$ can be seen as $p = \langle\langle s_\xi, A_\xi\rangle : \xi \in X\rangle$, where for $\xi \in X$, $\langle s_\xi, A_\xi\rangle \in \mathbb{P}_\xi$ and $\mathrm{supp}(p) = \{\xi \in X : s_\xi \ne \emptyset\}$ is finite. We may further suppose that for each $\xi \in X$ the Prikry sequence for ξ is contained in $(\sup(X \cap \xi), \xi)$. Let G be $\mathbb{P}_X$-generic over V. Note that G is uniquely determined by a sequence $(x_\xi : \xi \in X)$, where each x_ξ is an ω-sequence cofinal in ξ, V and $V[G]$ have the same cardinals, and GCH holds in $V[G]$.

Work in $V[G]$. We now force $\langle S_\sigma : \sigma < \kappa\rangle$ as follows. The set of conditions $\mathbb{P}$ consists of pairs $p = \langle \tau, \langle s_\sigma : \sigma < \kappa\rangle\rangle \in V[G]$ such that:

(1) $\tau < \kappa^+$,
(2) $\langle s_\sigma : \sigma < \kappa\rangle$ is a splitting of τ,
(3) for every countable set $I \in V$ and every $\sigma < \kappa$, $|I \cap s_\sigma| < \aleph_0$.

REMARK 3.2. (a) Given a condition $p \in \mathbb{P}$ as above, p decides an initial segment of S_σ, namely $S_\sigma \cap \tau$, to be s_σ. Condition (3) guarantees that each component in this initial segment has finite intersection with countable sets from the ground model.

(b) Let $t_0 = \bigcup_{\xi\in X} x_\xi$. By genericity arguments, it is easily seen that t_0 is a subset of κ of size κ such that for all countable sets $I \in V$, we have $|I \cap t_0| < \aleph_0$. For each $i < \kappa$ set $t_i = t_0 + i = \{\alpha + i : \alpha \in t_0\}$. Then again by genericity arguments, $|I \cap t_i| < \aleph_0$ for every countable set $I \in V$. Define s_i, $i < \kappa$, by recursion as $s_0 = t_0$ and $s_i = t_i \setminus \bigcup_{j<i} t_j$ for $i > 0$. Then $p = \langle \kappa, \langle s_\sigma : \sigma < \kappa\rangle\rangle \in \mathbb{P}$ (since again by genericity arguments, $\langle s_\sigma : \sigma < \kappa\rangle$ is a splitting of κ), and hence $\mathbb{P}$ is non-trivial.

We call τ the *height* of p and denote it by $\mathrm{ht}(p)$. For $p = \langle \tau, \langle s_\sigma : \sigma < \kappa\rangle\rangle$ and $q = \langle \nu, \langle t_\sigma : \sigma < \kappa\rangle\rangle$ in $\mathbb{P}$, we define $p \le q$ iff

- $\tau \ge \nu$,
- for every $\sigma < \kappa$, we have $s_\sigma \cap \nu = t_\sigma$, i.e., each s_σ end extends t_σ.

LEMMA 3.3.

(a) $\mathbb{P}$ *satisfies the* κ^{++}*-c.c,*
(b) $\mathbb{P}$ *is* $<\kappa$*-distributive.*

Proof. (a) is trivial, as $|\mathbb{P}| \le 2^\kappa = \kappa^+$. For (b), fix $\delta < \kappa$, δ regular, and let $p \in \mathbb{P}$ and $\underset{\sim}{g} \in V[G]^{\mathbb{P}}$ be such that

$$p \Vdash \ulcorner \underset{\sim}{g} : \delta \to \mathrm{On} \urcorner.$$

We find $q \le p$ which decides $\underset{\sim}{g}$. Fix in V a splitting of κ into δ-many sets of size κ, $\langle Z_i : i < \delta\rangle$ (5). Let θ be a large enough regular cardinal. Pick an increasing continuous sequence $\langle M_i : i \le \delta\rangle$ of elementary submodels of $\langle H(\theta), \in\rangle$ of size κ such that (6):

(1) $\langle M_i : i \le \delta\rangle \in V[G]$,
(2) $p, \mathbb{P}, \underset{\sim}{g}, \langle Z_i : i < \delta\rangle \in M_0$,
(3) if $i < \delta$ is a limit ordinal, then $\langle M_j : j \le i\rangle \in M_{i+1}$,
(4) $\mathrm{cf}(M_\delta \cap \kappa^+) = \delta$,
(5) if i is not a limit ordinal, then $\mathrm{cf}^V(M_{i+1} \cap \kappa^+) = \xi_i$ for a measurable ξ_i of V in X,
(6) $i < j \Rightarrow \xi_i < \xi_j$,
(7) $\langle M_i \cap V : i \le \delta\rangle \in V$.

(5) Note that this is possible, as $\delta < \kappa$ are cardinals in V. The splitting can also be chosen in $V[G]$.

(6) Condition (5) can be guaranteed using the fact that the set $K = \{\alpha < \kappa^+ : \mathrm{cf}^V(\alpha) \in X\}$ is a stationary subset of κ^+ in $V[G]$ (given M_i, build a suitable continuous increasing chain $\langle N_j : j < \kappa^+\rangle$ consisting of models of size κ. Then $\langle \sup(N_j \cap \kappa^+) : j < \kappa^+\rangle$ forms a club of κ^+, and M_{i+1} can be chosen to be one of those N_j so that $\sup(N_j \cap \kappa^+) \in K$). Condition (7) can be guaranteed by the fact that $\mathbb{P}_X$ satisfies the κ^+-c.c. and the models have size κ (use the fact that for any model N of size κ, there exists a model in V of the same size which contains $N \cap V$).

For each non-limit $i < \delta$, $M_{i+1} \cap V$ is in V by clause (7), and so by clause (5), $\mathrm{cf}^V(M_{i+1} \cap \kappa^+) = \xi_i$, where $\xi_i \in X$, so we can pick in V a sequence $\langle \eta^i_\alpha : \alpha < \xi_i \rangle$ cofinal in $M_{i+1} \cap \kappa^+$, where $\eta^i_\alpha > M_i \cap \kappa^+$ for all $\alpha < \xi_i$ (7).

Denote by ξ'_i the first element of the Prikry sequence of ξ_i. We define a descending sequence $p_i = \langle \tau_i, \langle s_{i,\sigma} : \sigma < \kappa \rangle \rangle$ of conditions by induction as follows:

Case $i = 0$. Set $p_0 = p$.

Case $i = j + 1$. Assume p_j is constructed so that $p_j \in M_j$ if j is not a limit ordinal, and $p_j \in M_{j+1}$ if j is a limit ordinal and p_j decides $\underset{\sim}{g} \restriction j$. Fix a bijection $f_j : Z_j \to (\mathrm{ht}(p_j), \eta^j_{\xi'_j})$ in M_{j+1}, and set (8)

$$p'_{j+1} = \langle \eta^j_{\xi'_j}, \langle s_{j,\sigma} \cup \{f_j(\sigma)\} : \sigma \in Z_j \rangle ^\frown \langle s_{j,\sigma} : \sigma \in \kappa \setminus Z_j \rangle \rangle.$$

Clearly $p'_{j+1} \in M_{j+1}$. Let $p_{j+1} \in M_{j+1}$ be an extension of p'_{j+1} which decides $\underset{\sim}{g}(j)$.

Case limit(i). Let $p_i = \langle \sup_{j<i} \mathrm{ht}(p_j), \langle \bigcup_{j<i} s_{j,\sigma} : \sigma < \kappa \rangle \rangle$. Let us show that this sequence is well-defined. Thus we need to show that for each $i \le \delta$, $p_i \in \mathbb{P}$. We prove this by induction on i. The successor case is trivial. Thus fix a limit ordinal $i \le \delta$. If $p_i \notin \mathbb{P}$, we can find a countable set $I \in V$, $I \subseteq \kappa^+$, and $\sigma < \kappa$ such that $I \cap s_{i,\sigma}$ is infinite. Define the sequence $\langle \alpha(j) : j < i \rangle$ as follows:

- if $I \cap (M_{j+1} \setminus M_j) \neq \emptyset$, then $\alpha(j) \in [\sup(X \cap \xi_j), \xi_j]$ is the least such that $\eta^j_{\alpha(j)} > \sup(I \cap (M_{j+1} \setminus M_j))$,
- $\alpha(j) = \sup(X \cap \xi_j)$ otherwise; note that in this case $\alpha(j) < \xi'_j$ (because the Prikry sequence for ξ was chosen in the interval $(\sup(X \cap \xi), \xi)$).

Clearly $\langle \alpha(j) : j < i \rangle \in V$.

Lemma 3.4. *The set $K = \{j < i : \xi'_j \le \alpha(j)\}$ is finite.*

Proof. Let $p \in \mathbb{P}_X$, $p = \langle \langle s_\xi, A_\xi \rangle : \xi \in X \rangle$. Extend p to $q = \langle \langle t_\xi, B_\xi \rangle : \xi \in X \rangle$ by setting

- $t_\xi = s_\xi$ and $B_\xi = A_\xi$ for $\xi \in \mathrm{supp}(p)$,
- $t_\xi = \emptyset$ and $B_\xi = A_\xi \setminus (\alpha(j) + 1)$ if $\xi = \xi_j$ (some $j < i$) and $\xi \notin \mathrm{supp}(p)$,
- $t_\xi = \emptyset$ and $B_\xi = A_\xi$ otherwise.

Then $q \le p$ and $q \Vdash \ulcorner \underset{\sim}{K} \subseteq \{j < i : \xi_j \in \mathrm{supp}(p)\} \urcorner$, so $q \Vdash \ulcorner \underset{\sim}{K}$ is finite$\urcorner$. ∎

(7) Note that $\sup(M_{i+1} \cap \kappa^+) = M_{i+1} \cap \kappa^+$. This is because if $\xi < \kappa^+$ and $\xi \in M_{i+1}$, then since $\kappa \cup \{\kappa\} \subseteq M_{i+1}$ and $M_{i+1} \models |\xi| = \kappa$, we have $\xi \subseteq M_{i+1}$. Also, as the sequence of M_i's is increasing continuous, we have $\sup(M_i \cap \kappa^+) = M_i \cap \kappa^+$ for limit ordinals i.

(8) It is easily seen by induction on $j \le i$ that $\mathrm{ht}(p_j) < \eta^j_{\xi'_j}$: If $j = 0$ or j is a successor ordinal, then $p_j \in M_j$, so $\mathrm{ht}(p_j) \in M_j \cap \kappa^+ < \eta^j_{\xi'_j}$. If j is a limit ordinal, then $\mathrm{ht}(p_j) = \sup_{k<j} \mathrm{ht}(p_k) \le \sup_{k<j} M_k \cap \kappa^+ = M_j \cap \kappa^+ < \eta^j_{\xi'_j}$.

Take $i_0 < i$ large enough so that no point $\geq i_0$ is in K. Then for all $j \geq i_0$ we have $\xi'_j > \alpha(j)$, hence $\eta^j_{\xi'_j} > \sup(I \cap (M_{j+1}))$ ([9]).

CLAIM 3.5. *We have*

$$I \cap s_{i,\sigma} \subseteq I \cap (s_{i_0,\sigma} \cup \{f_{i_1}(\sigma)\})$$

where i_1 is the unique ordinal less than δ so that $\sigma \in Z_{i_1}$.

Proof. Assume towards a contradiction that the inclusion fails, and let $t \in I \cap s_{i,\sigma}$ be such that $t \notin I \cap (s_{i_0,\sigma} \cup \{f_{i_1}(\sigma)\})$. As i is a limit ordinal, $I \cap s_{i,\sigma} = I \cap \bigcup_{j<i} s_{j,\sigma}$. Let $j < i$ be the least such that $t \in s_{j+1,\sigma}$. Then as $t \in I \cap M_{j+1}$ and $j \geq i_0$ we have $t < \eta^j_{\xi'_j}$, so that by our definition of p'_{j+1}, t must be of the form $f_j(\sigma)$, where $\sigma \in Z_j$. But then $j = i_1$, and hence $t = f_{i_1}(\sigma)$. This is a contradiction, and the result follows. ∎

Thus, as $I \cap s_{i,\sigma}$ is infinite, so is $I \cap s_{i_0,\sigma}$, and this contradicts our inductive assumption.

It then follows that $q = p_\delta \in \mathbb{P}$, and it decides $\underset{\sim}{g}$. This finishes the proof of Lemma 3.3. ∎

Let H be $\mathbb{P}$-generic over $V[G]$ and set $V_1 = V[G][H]$. It follows from Lemma 3.3 that all cardinals $\leq \kappa$ and $\geq \kappa^{++}$ are preserved. Also note that κ^+ is preserved, as otherwise it would have cofinality less than κ, which is impossible by the $<\kappa$-distributivity of $\mathbb{P}$. Hence V_1 is forcing extension of $V[G]$, and hence of V, that is cardinal preserving and not adding reals. For $\sigma < \kappa$ set $S_\sigma = \bigcup_{\langle \tau, \langle s_\sigma : \sigma < \kappa \rangle \rangle \in H} s_\sigma$.

LEMMA 3.6. *The sequence $\langle S_\sigma : \sigma < \kappa \rangle$ is as required.*

Proof. For each $\tau < \kappa^+$, it is easily seen that the set of all conditions p such that $\mathrm{ht}(p) \geq \tau$ is dense, so $\langle S_\sigma : \sigma < \kappa \rangle$ is a partition of κ^+. Now suppose that $I \in V$ is a countable subset of κ^+. Find $p = \langle \tau, \langle s_\sigma : \sigma < \kappa \rangle \rangle \in H$ such that $\tau \supseteq I$. Then for all $\sigma < \kappa$, $S_\sigma \cap I = s_\sigma \cap I$, hence $|S_\sigma \cap I| = |s_\sigma \cap I| < \aleph_0$. ∎

Theorem 3.1 follows. ∎

REMARK 3.7. (a) The size of a set I in V can be changed from countable to any fixed $\eta < \kappa$. Given such η, we start with the Magidor iteration of Prikry forcings above η ([10]). The rest of the conclusions are the same.

([9]) This is trivial if $I \cap (M_{j+1} \setminus M_j) \neq \emptyset$, as then $\eta^j_{\xi'_j} > \eta^j_{\alpha(j)} > \sup(I \cap (M_{j+1} \setminus M_j)) = \sup(I \cap M_{j+1})$. If $I \cap (M_{j+1} \setminus M_j) = \emptyset$, then $\eta^j_{\xi'_j} > M_j \cap \kappa^+ = \sup(M_j \cap \kappa^+) \geq \sup(I \cap M_j)$ (as $I \subseteq \kappa^+$) and $\sup(I \cap M_{j+1}) = \sup(I \cap M_j)$ (since I has no points in $M_{j+1} \setminus M_j$), and hence again $\eta^j_{\xi'_j} > \sup(I \cap (M_{j+1}))$.

([10]) The reason for starting the iteration above η is to add no subsets of η. This will guarantee that if t_0 is defined as in Remark 3.2(b), then t_0 has finite intersection with sets from V of size η. Using this fact we can show as before that there is a splitting of κ

(b) It is possible to add a one element Prikry sequence to each $\xi \in X$ ([11]). Then V_1 will be a cofinality preserving generic extension of V.

The next corollary follows from Theorem 3.1 and Remark 2.2.

COROLLARY 3.8. *Suppose that GCH holds in V, κ is a cardinal of countable cofinality and there, are κ-many measurable cardinals below κ. Then there is an extension V_1 of V that is cardinal preserving, not adding a real, and such that adding κ-many Cohen reals over V_1 produces κ^+-many Cohen reals over V.*

THEOREM 3.9. *Assume that there is no sharp for a strong cardinal. Suppose $V_1 \subseteq V_2$ have the same cardinals, same reals, and there is an infinite set of ordinals S in V_2 which does not contain an infinite subset which is in V_1. Then either*

(1) *S is countable, and then there is a measurable cardinal $\le \sup(S)$ in $\mathcal{K}$, or*
(2) *S is uncountable, and then there is $\delta \le \sup(S)$ which is a limit of $|S|$-many or δ-many measurable cardinals of $\mathcal{K}$.*

Proof. Given a model V, let $\mathcal{K}(V)$ denote the core model of V below the strong cardinal. Note that $\mathcal{K}(V_1) = \mathcal{K}(V_2)$, since the models V_1 and V_2 agree about cardinals. We denote this common core model by $\mathcal{K}$.

Let us first assume that S is countable. Suppose otherwise, i.e., there are no measurable cardinals $\le \sup(S)$ in $\mathcal{K}$. Then by the Covering Theorem (see [Mi]) there is $Y \in \mathcal{K}$, $|Y| = \aleph_1$ which covers S. Fix some $f : \aleph_1 \leftrightarrow Y$ in V_1. Consider $Z = (f^{-1})''S$. Then Z also does not contain an infinite subset which is in V_1. But Z is countable, hence there is $\eta < \omega_1$ with $Z \subseteq \eta$. Let $g : \omega \leftrightarrow \eta$ in V_1. Consider $X = (g^{-1})''Z$. Then X also does not contain an infinite subset which is in V_1. But this is impossible since V_1 and V_2 have the same reals (and hence X itself is in V_1), a contradiction.

Let us deal now with the uncountable case. Suppose otherwise, i.e., there is no $\delta \le \sup(S)$ which is a limit of $|S|$-many or δ-many measurable cardinals of $\mathcal{K}$. Pick a counterexample S with $\sup(S)$ as small as possible. Denote $\sup(S)$ by δ. By minimality, δ is a cardinal. Also, the measurable cardinals of $\mathcal{K}$ are unbounded in δ. For otherwise, let ξ be their supremum. Pick $S' \subseteq S$ of size ξ. By the Covering Theorem, S' can be covered by a set in $\mathcal{K}$ of size

into κ sets, each of them having finite intersection with ground model sets of size η. This makes the second step of the above forcing construction well-behaved.

([11]) Conditions in the forcing are of the form $\langle p_\xi : \xi \in X \rangle$, where for each $\xi \in X$, p_ξ is either of the form A_ξ for some $A_\xi \in U_\xi$, or α_ξ for some $\alpha_\xi < \xi$. We also require that there are only finitely many p_ξ's of the form α_ξ. When extending a condition, we either allow A_ξ to become thinner, or replace it by some ordinal $\alpha_\xi \in A_\xi$.

$\xi < \delta$, and then we get a contradiction to the minimality of δ, as witnessed by ξ and S' ([12]).

Clearly, δ must be a singular cardinal, and by the above, δ is a limit of measurable cardinals in $\mathcal{K}$. Fix a cofinal sequence $\langle \delta_i : i < \mathrm{cf}(\delta)\rangle$. Denote by η the cardinality of the set $\{\alpha < \delta : \alpha \text{ is a measurable cardinal in } \mathcal{K}\}$. By the assumption, $|S| > \eta \geq \mathrm{cf}(\delta)$. But then there is $i^* < \mathrm{cf}(\delta)$ such that $S \cap \delta_{i^*}$ has size $> \eta$. This is impossible by the minimality of δ, a contradiction. ∎

The conclusions of the theorem are optimal. A Prikry sequence witnesses this in the countable case, and the Magidor iteration of Prikry forcing witnesses this in the uncountable case.

Theorem 3.10. *Suppose that $V_1 \supseteq V$ are such that:*

(a) *V_1 and V have the same cardinals and reals,*
(b) *$\kappa < \lambda$ are infinite cardinals of V_1,*
(c) *there is no splitting $\langle S_\sigma : \sigma < \kappa\rangle$ of λ in V_1 as in Theorem* 2.1(d).

Then adding κ-many Cohen reals over V_1 cannot produce λ-many Cohen reals over V.

Proof. Suppose not. Let $\langle r_\alpha : \alpha < \lambda\rangle$ be a sequence of λ-many Cohen reals over V added after forcing with $\mathbb{C}(\kappa)$ over V_1. Let G be $\mathbb{C}(\kappa)$-generic over V_1. For each $p \in \mathbb{C}(\kappa)$ set

$$C_p = \{\alpha < \lambda : p \text{ decides } \underset{\sim}{r}_\alpha(0)\}.$$

Then by genericity $\lambda = \bigcup_{p\in G} C_p$. Fix an enumeration $\langle p_\xi : \xi < \kappa\rangle$ of G, and define a splitting $\langle S_\sigma : \sigma < \kappa\rangle$ of λ in $V_1[G]$ by setting $S_\sigma = C_{p_\sigma} \setminus \bigcup_{\xi<\sigma} C_{p_\xi}$. By (a), (c) we can find a countable $I \in V$ and $\sigma < \kappa$ such that $I \subseteq S_\sigma$ ([13]). Suppose for simplicity that $\forall \alpha \in S_\sigma$, $p_\sigma \Vdash \ulcorner \underset{\sim}{r}_\alpha(0) = 0\urcorner$. Let $q \in \mathbb{C}(\kappa)$ be such that

$$q \Vdash^V \ulcorner I \in V \text{ is countable and } \forall \alpha \in I,\ \underset{\sim}{r}_\alpha(0) = 0\urcorner.$$

Pick $\langle 0, \alpha\rangle \in \omega \times I$ such that $\langle 0,\alpha\rangle \notin \mathrm{supp}(q)$. Let $\bar{q} = q \cup \{\langle\langle 0,\alpha\rangle, 1\rangle\}$. Then $\bar{q} \in \mathbb{C}(\kappa)$, $\bar{q} \leq q$ and $\bar{q} \Vdash \ulcorner \underset{\sim}{r}_\alpha(0) = 1\urcorner$, which is a contradiction. ∎

The following corollary answers a question from [G1].

Corollary 3.11. *The following are equiconsistent:*

(a) *There exists a pair (V_1, V_2), $V_1 \subseteq V_2$, of models of set theory with the same cardinals and reals, and a cardinal κ of cofinality ω (in V_2)*

([12]) We then have $\sup(S') = \xi < \delta$, and S' is a counterexample to our assumption of smaller supremum.

([13]) In fact, by (c) there exist a countable $I \in V$ and some $\sigma < \kappa$ such that $I \cap S_\sigma$ is infinite. By (a), V and V_1 have the same reals, and hence $I \cap S_\sigma \in V$. So by replacing I with $I \cap S_\sigma$, if necessary, we can assume that $I \subseteq S_\sigma$.

such that adding κ-many Cohen reals over V_2 adds more than κ-many Cohen reals over V_1.

(b) *There exists a cardinal δ which is a limit of δ-many measurable cardinals.*

Proof. Assume (a) holds for some pair (V_1, V_2) of models of set theory, $V_1 \subseteq V_2$, which have the same cardinals and reals. If there is a sharp for a strong cardinal, then clearly in $\mathcal{K}$, the core model for a strong cardinal, there is a cardinal δ which is a limit of δ-many measurable cardinals ([14]). So assume there is no sharp for a strong cardinal. Then by Theorem 3.10 there exists a splitting $\langle S_\sigma : \sigma < \kappa \rangle$ of κ^+ in V_2 such that for every countable set $I \in V_1$ and $\sigma < \kappa$, $I \cap S_\sigma$ is finite. Take S to be one of the sets S_σ which has size κ^+. So by Theorem 3.9, we get the consistency of (b) ([15]).

Conversely, if (b) is consistent, then by Corollary 3.8 the consistency of (a) follows ([16]). ∎

4. Below the first fixed point of the $\aleph$-function

THEOREM 4.1. *Suppose that $V_1 \supseteq V$ are such that V_1 and V have the same cardinals and reals. Suppose $\aleph_\delta$ is smaller than the first fixed point of the $\aleph$-function, $X \subseteq \aleph_\delta$, $X \in V_1$ and $|X| \geq \delta^+$ (in V_1). Then X has a countable subset which is in V.*

Proof. We proceed by induction on δ that are smaller than the first fixed point of the $\aleph$-function.

CASE 1: $\delta = 0$. Then $X \in V$ by the fact that V_1 and V have the same reals.

CASE 2: $\delta = \delta' + 1$. We have $\delta' < \aleph_{\delta'}$, hence $\delta^+ < \aleph_\delta$, thus we may suppose that $|X| \leq \aleph_{\delta'}$. Let $\eta = \sup(X) < \aleph_\delta$. Pick $f_\eta : \aleph_{\delta'} \leftrightarrow \eta$, $f_\eta \in V$. Set $Y = f_\eta^{-1''} X$. Then $Y \subseteq \aleph_{\delta'}$, $\delta' < \aleph_{\delta'}$ and $|Y| \geq \delta^+ = \delta'^+$. Hence by induction there is a countable set $B \in V$ such that $B \subseteq Y$. Let $A = f_\eta'' B$. Then $A \in V$ is a countable subset of X.

CASE 3: limit(δ). Let $\langle \delta_\xi : \xi < \mathrm{cf}(\delta) \rangle$ be increasing and cofinal in δ. Pick $\xi < \mathrm{cf}(\delta)$ such that $|X \cap \aleph_{\delta_\xi}| \geq \delta^+$. By induction there is a countable set $A \in V$ such that $A \subseteq X \cap \aleph_{\delta_\xi} \subseteq X$. ∎

([14]) In fact there are many such cardinals δ.

([15]) Note that necessarily case (2) of Theorem 3.9 happens.

([16]) If $\mathrm{cf}(\delta) > \omega$, then we can find $\delta^* < \delta$ of cofinality ω which is a limit of δ^*-many measurable cardinals, so that Corollary 3.8 can be applied. To see such a δ^* exists, define an increasing sequence δ_n, $n < \omega$, of cardinals below δ, so that for any n, there are at least δ_n-many measurable cardinals below δ_{n+1}, and let $\delta^* = \sup_n \delta_n$.

The following corollary gives a negative answer to another question from [G1].

Corollary 4.2. *Suppose* V_1, V *and* δ *are as in Theorem* 4.1. *Then adding* $\aleph_\delta$*-many Cohen reals over* V_1 *cannot produce* $\aleph_{\delta+1}$*-many Cohen reals over* V.

Proof. Towards a contradiction suppose that adding $\aleph_\delta$-many Cohen reals over V_1 produces $\aleph_{\delta+1}$-many Cohen reals over V. Then by Theorem 3.10, there exists $X \subseteq \aleph_{\delta+1}$, $X \in V_1$, such that $|X| = \aleph_{\delta+1}$ ($\geq \delta^+$) and X does not contain any countable subset from V ([17]), which contradicts Theorem 4.1. ∎

5. At the first fixed point of the ℵ-function. The next theorem shows that Theorem 4.1 does not extend to the first fixed point of the ℵ-function.

Theorem 5.1. *Suppose GCH holds and* κ *is the least singular cardinal of cofinality* ω *which is a limit of* κ*-many measurable cardinals. Then there is a pair* $(V[G], V[H])$ *of generic extensions of* V *with* $V[G] \subseteq V[H]$ *such that:*

(a) $V[G]$ *and* $V[H]$ *have the same cardinals and reals,*
(b) κ *is the first fixed point of the ℵ-function in* $V[G]$ (*and hence in* $V[H]$),
(c) *in* $V[H]$ *there exists a splitting* $\langle S_\sigma : \sigma < \kappa \rangle$ *of* κ *into sets of size* κ *such that for every countable* $I \in V[G]$ *and* $\sigma < \kappa$, $|I \cap S_\sigma| < \aleph_0$.

Proof. We first make a simple observation.

Claim 5.2. *Suppose there is* $S \subseteq \kappa$ *of size* κ *in* $V[H] \supseteq V[G]$ *such that for every countable* $A \in V[G]$, $|A \cap S| < \aleph_0$. *Then there is a splitting* $\langle S_\sigma : \sigma < \kappa \rangle$ *of* κ *as in* (c).

Proof. Let $\langle \alpha_i : i < \kappa \rangle$ be an increasing enumeration of S. We may further suppose that $\alpha_0 = 0$, and each α_i, $i > 0$, is measurable ([18]) in V and is not a limit point of S ([19]). Note that for all $i < \kappa$, we have

([17]) In fact, there exists a splitting $\langle S_\sigma : \sigma < \aleph_\delta \rangle$ of $\aleph_{\delta+1}$ in V_1, consisting of sets of size $\aleph_{\delta+1}$, such that each S_σ has finite intersection with any countable set from V. The set X can be chosen to be any of S_σ's.

([18]) It suffices for each α_i to be inaccessible in V.

([19]) Let $f \in V$ be such that $f : \kappa \to X$ is a bijection, where X is a discrete set of measurable cardinals of V below κ of size κ. Then if $S \subseteq \kappa$ satisfies the claim, so does $f[S]$, hence we can suppose all non-zero elements of S are measurable in V, and are not a limit point of S.

$\sup_{j<i} \alpha_j < \alpha_i \setminus \sup_{j<i} \alpha_j$. Now set

$$S_0 = S,$$
$$S_\sigma = \{\alpha_l + \sigma : i \le l < \kappa\} \quad \text{for } 0 < \sigma \in \Big[\sup_{j<i} \alpha_j, \alpha_i\Big).$$

Then $\langle S_\sigma : \sigma < \kappa \rangle$ is as required (note that for $\sigma > 0$, $S_\sigma \subseteq S + \sigma = \{\alpha + \sigma : \alpha \in S\}$, and clearly $S + \sigma$, and hence S_σ, has finite intersection with countable sets from $V[G]$). ∎

Thus it is enough to find a pair $(V[G], V[H])$ of generic extensions of V satisfying (a) and (b) with $V[G] \subseteq V[H]$ such that in $V[H]$ there is $S \subseteq \kappa$ of size κ, composed of inaccessibles, such that for every countable $A \in V[G]$, we have $|A \cap S| < \aleph_0$.

Let X be a discrete set of measurable cardinals below κ of size κ, and for each $\xi \in X$ fix a normal measure U_ξ on ξ. For each $\xi \in X$ we define two forcing notions $\mathbb{P}_\xi$ and $\mathbb{Q}_\xi$ as follows.

Remark 5.3. In the following definitions we let $\sup(X \cap \xi) = \omega$ for $\xi = \min X$.

A condition in $\mathbb{P}_\xi$ is of the form $p = \langle s_\xi, A_\xi, f_\xi \rangle$ where

(1) $s_\xi \in [\xi \setminus \sup(X \cap \xi)^+]^{<2}$,
(2) if $s_\xi \neq \emptyset$ then $s_\xi(0)$ is an inaccessible cardinal,
(3) $A_\xi \in U_\xi$,
(4) $\max s_\xi < \min A_\xi$,
(5) $s_\xi = \emptyset \Rightarrow f_\xi \in \mathrm{Col}(\sup(X \cap \xi)^+, {<}\xi)$, where $\mathrm{Col}(\sup(X \cap \xi)^+, {<}\xi)$ is the Lévy collapse for collapsing all cardinals less than ξ to $\sup(X \cap \xi)^+$, and making ξ become the successor of $\sup(X \cap \xi)^+$,
(6) $s_\xi \neq \emptyset \Rightarrow f_\xi = \langle f^1_\xi, f^2_\xi \rangle$, where $f^1_\xi \in \mathrm{Col}(\sup(X \cap \xi)^+, {<}\, s_\xi(0))$ and $f^2_\xi \in \mathrm{Col}((s_\xi(0))^+, {<}\xi)$.

For $p, q \in \mathbb{P}_\xi$, $p = \langle s_\xi, A_\xi, f_\xi \rangle$ and $q = \langle t_\xi, B_\xi, g_\xi \rangle$ we define $p \le q$ iff

(1) s_ξ end extends t_ξ,
(2) $A_\xi \cup (s_\xi \setminus t_\xi) \subseteq B_\xi$,
(3) $t_\xi = s_\xi = \emptyset \Rightarrow f_\xi \le g_\xi$,
(4) $t_\xi = \emptyset$ and $s_\xi \neq \emptyset \Rightarrow \sup(\mathrm{ran}(g_\xi)) < s_\xi(0)$ and $f^1_\xi \le g_\xi$,
(5) $t_\xi \neq \emptyset \Rightarrow f^1_\xi \le g^1_\xi$ and $f^2_\xi \le g^2_\xi$ (note that in this case $s_\xi = t_\xi$).

We also define $p \le^* q$ (p is a Prikry or a direct extension of q) iff

(1) $p \le q$,
(2) $s_\xi = t_\xi$.

The proof of the following lemma is essentially the same as in [G2], [Ma].

Lemma 5.4. (*GCH*)

(a) $\mathbb{P}_\xi$ *satisfies the* ξ^+*-c.c.*
(b) *Suppose* $p = \langle s_\xi, A_\xi, f_\xi\rangle \in \mathbb{P}_\xi$, *and* $l(s_\xi) = 1$ (*where* $l(s_\xi)$ *is the length of* s_ξ). *Then* $\mathbb{P}_\xi/p = \{q \in \mathbb{P}_\xi : q \le p\}$ *satisfies the* ξ*-c.c.*
(c) $(\mathbb{P}_\xi, \le, \le^*)$ *satisfies the Prikry property, i.e., given* $p \in \mathbb{P}$ *and a sentence* σ *of the forcing language for* $(\mathbb{P}, \le)$, *there exists* $q \le^* p$ *which decides* σ.
(d) *Let* G_ξ *be* $\mathbb{P}_\xi$*-generic over* V *and* $\langle s_\xi(0)\rangle$ *be the one-element sequence added by* G_ξ. *Then in* $V[G_\xi]$, *GCH holds, and the only cardinals which are collapsed are the cardinals in the intervals* $(\sup(X\cap\xi)^{++}, s_\xi(0))$ *and* $(s_\xi(0)^{++}, \xi)$, *which are collapsed to* $\sup(X \cap \xi)^+$ *and* $s_\xi(0)^+$ *respectively.*

We now define the forcing notion $\mathbb{Q}_\xi$. A condition in $\mathbb{Q}_\xi$ is of the form $p = \langle s_\xi, A_\xi, f_\xi\rangle$ where

(1) $s_\xi \in [\xi \setminus \sup(X \cap \xi)^+]^{<3}$,
(2) if $s_\xi \neq \emptyset$ then for all $i < l(s_\xi)$, $s_\xi(i)$ is an inaccessible cardinal,
(3) $A_\xi \in U_\xi$,
(4) $\max s_\xi < \min A_\xi$,
(5) $s_\xi = \emptyset \Rightarrow f_\xi \in \mathrm{Col}(\sup(X \cap \xi)^+, {<}\xi)$,
(6) $s_\xi \neq \emptyset \Rightarrow f_\xi = \langle f^1_\xi, f^2_\xi\rangle$, where $f^1_\xi \in \mathrm{Col}(\sup(X \cap \xi)^+, < s_\xi(0))$ and $f^2_\xi \in \mathrm{Col}((s_\xi(0))^+, {<}\xi)$.

For $p, q \in \mathbb{Q}_\xi$, $p = \langle s_\xi, A_\xi, f_\xi\rangle$ and $q = \langle t_\xi, B_\xi, g_\xi\rangle$, we define $p \le q$ iff

(1) s_ξ end extends t_ξ,
(2) $A_\xi \cup (s_\xi \setminus t_\xi) \subseteq B_\xi$,
(3) $t_\xi = s_\xi = \emptyset \Rightarrow f_\xi \le g_\xi$,
(4) $t_\xi = \emptyset$ and $s_\xi \neq \emptyset \Rightarrow \sup(\mathrm{ran}(g_\xi)) < s_\xi(0)$ and $f^1_\xi \le g_\xi$,
(5) $t_\xi \neq \emptyset$ and $s_\xi = t_\xi \Rightarrow f^1_\xi \le g^1_\xi$ and $f^2_\xi \le g^2_\xi$,
(6) $t_\xi \neq \emptyset$ and $s_\xi \neq t_\xi \Rightarrow \sup(\mathrm{ran}(g^2_\xi)) < s_\xi(1)$, $f^1_\xi \le g^1_\xi$ and $f^2_\xi \le g^2_\xi$.

We also define $p \le^* q$ iff

(1) $p \le q$,
(2) $s_\xi = t_\xi$.

As above we have the following.

Lemma 5.5. (*GCH*)

(a) $\mathbb{Q}_\xi$ *satisfies the* ξ^+*-c.c.*
(b) *Suppose* $p = \langle s_\xi, A_\xi, f_\xi\rangle \in \mathbb{Q}_\xi$, $l(s_\xi) = 2$. *Then* $\mathbb{Q}_\xi/p = \{q \in \mathbb{Q}_\xi : q \le p\}$ *satisfies the* ξ*-c.c.*
(c) $(\mathbb{Q}_\xi, \le, \le^*)$ *satisfies the Prikry property.*

(d) *Let H_ξ be $\mathbb{Q}_\xi$-generic over V and $\langle s_\xi(0), s_\xi(1)\rangle$ be the two-element sequence added by H_ξ. Then in $V[H_\xi]$, GCH holds, and the only cardinals which are collapsed are the cardinals in the intervals* $(\sup(X\cap\xi)^{++}, s_\xi(0))$ *and* $(s_\xi(0)^{++}, \xi)$, *which are collapsed to* $\sup(X\cap\xi)^+$ *and* $s_\xi(0)^+$ *respectively.*

Now let $\mathbb{P}$ be the Magidor iteration of the forcings $\mathbb{P}_\xi$, $\xi\in X$, and $\mathbb{Q}$ be the Magidor iteration of the forcings $\mathbb{Q}_\xi$, $\xi\in X$. Since the set X is discrete, we can view each condition in $\mathbb{P}$ as a sequence $p=\langle\langle s_\xi, A_\xi, f_\xi\rangle : \xi\in X\rangle$ where for each $\xi\in X$, $\langle s_\xi, A_\xi, f_\xi\rangle\in\mathbb{P}_\xi$ and $\mathrm{supp}(p)=\{\xi : s_\xi\neq\emptyset\}$ is finite. Similarly, each condition in $\mathbb{Q}$ can be viewed as a sequence $p=\langle\langle s_\xi, A_\xi, f_\xi\rangle : \xi\in X\rangle$ where for each $\xi\in X$, $\langle s_\xi, A_\xi, f_\xi\rangle\in\mathbb{Q}_\xi$ and $\mathrm{supp}(p)$ is finite (for more information see [G2], [Kr], [Ma]).

Notation 5.6. If p is as above, then we write $p(\xi)$ for $\langle s_\xi, A_\xi, f_\xi\rangle$.

We also define $\pi:\mathbb{Q}\to\mathbb{P}$ by

$$\pi(\langle\langle s_\xi, A_\xi, f_\xi\rangle : \xi\in X\rangle)=\langle\langle s_\xi\restriction 1, A_\xi, f_\xi\rangle : \xi\in X\rangle.$$

It is clear that π is well-defined.

Lemma 5.7. *π is a projection, i.e.,*

(a) $\pi(1_{\mathbb{Q}})=1_{\mathbb{P}}$,
(b) *π is order preserving,*
(c) *if $p\in\mathbb{Q}$, $q\in\mathbb{P}$ and $q\le\pi(p)$, then there is $r\le p$ in $\mathbb{Q}$ such that $\pi(r)\le q$.*

Now let H be $\mathbb{Q}$-generic over V and let $G=\pi''H$ be the filter generated by $\pi''H$. Then G is $\mathbb{P}$-generic over V.

Lemma 5.8.

(a) *If $\langle\tau_\xi : \xi\in X\rangle$ and $\langle\langle\eta^0_\xi, \eta^1_\xi\rangle : \xi\in X\rangle$ are the Prikry sequences added by G and H respectively, then $\tau_\xi=\eta^0_\xi$ for all $\xi\in X$.*
(b) *The models $V[G]$ and $V[H]$ satisfy GCH, have the same cardinals and reals, and furthermore the only cardinals of V below κ which are preserved are* $\{\omega,\omega_1\}\cup\lim(X)\cup\{\tau_\xi,\tau_\xi^+,\xi,\xi^+ : \xi\in X\}$.
(c) *κ is the first fixed point of the $\aleph$-function in $V[G]$ (and hence in $V[H]$).*

Proof. (a) and (b) follow easily from Lemmas 5.4 and 5.5 and the definition of the projection π. Let us prove (c). It is clear that κ is a fixed point of the $\aleph$-function in $V[G]$. On the other hand, by (b) the only cardinals of V below κ which are preserved in $V[G]$ are $\{\omega,\omega_1\}\cup\lim(X)\cup\{\tau_\xi,\tau_\xi^+,\xi,\xi^+ : \xi\in X\}$, and so if $\lambda<\kappa$ is a limit cardinal in $V[G]$ then $\lambda\in\lim(X)$. But by our assumption on κ, if $\lambda\in\lim(X)$ then $X\cap\lambda$ has order type less than λ, and hence $(\{\omega,\omega_1\}\cup\lim(X)\cup\{\tau_\xi,\tau_\xi^+,\xi,\xi^+ : \xi\in X\})\cap\lambda$ has order type less than $\aleph_\lambda$. Thus $\lambda<\aleph_\lambda$. ■

Let $\mathbb{Q}/G=\{p\in\mathbb{Q} : \pi(p)\in G\}$. Then $V[H]$ can be viewed as a generic extension of $V[G]$ by $\mathbb{Q}/G$.

Lemma 5.9. *$\mathbb{Q}/G$ is cone homogeneous: for p and q in $\mathbb{Q}/G$ there exist $p^*\le p$, $q^*\le q$ and an isomorphism $\rho : (\mathbb{Q}/G)/p^*\to(\mathbb{Q}/G)/q^*$.*

Proof. Suppose $p,q\in\mathbb{Q}/G$. Extend p and q to $p^*=\langle\langle s_\xi,A_\xi,f_\xi\rangle : \xi\in X\rangle$ and $q^*=\langle\langle t_\xi,B_\xi,g_\xi\rangle : \xi\in X\rangle$ respectively, so that the following conditions are satisfied:

(1) $\mathrm{supp}(p^*)=\mathrm{supp}(q^*)$. Call this common support K.
(2) For every $\xi\in K$, $l(s_\xi)=l(t_\xi)=2$. Note that then for every $\xi\in K$, we have $s_\xi(0)=t_\xi(0)=\tau_\xi$, $f_\xi=\langle f_\xi^1,f_\xi^2\rangle$ and $g_\xi=\langle g_\xi^1,g_\xi^2\rangle$, where $f_\xi^1,g_\xi^1\in\mathrm{Col}(\sup(X\cap\xi)^+,{<}\tau_\xi)$ and $f_\xi^2,g_\xi^2\in\mathrm{Col}(\tau_\xi^+,{<}\xi)$.
(3) For every $\xi\in K$, $A_\xi=B_\xi$.
(4) For every $\xi\in K$, $\mathrm{dom}(f_\xi^1)=\mathrm{dom}(g_\xi^1)$ and $\mathrm{dom}(f_\xi^2)=\mathrm{dom}(g_\xi^2)$.
(5) For every $\xi\in K$, there exists an automorphism ρ_ξ^1 of $\mathrm{Col}(\sup(X\cap\xi)^+,{<}\tau_\xi)$ such that $\rho_\xi^1(f_\xi^1)=g_\xi^1$.
(6) For every $\xi\in K$, there exists an automorphism ρ_ξ^2 of $\mathrm{Col}(\tau_\xi^+,{<}\xi)$ such that $\rho_\xi^2(f_\xi^2)=g_\xi^2$.

Note that clauses (5) and (6) are possible, as the corresponding forcing notions are homogeneous.

We now define $\rho : (\mathbb{Q}/G)/p^*\to(\mathbb{Q}/G)/q^*$ as follows. Suppose $r\in\mathbb{Q}/G$, $r\le p^*$. Let $r=\langle\langle r_\xi,C_\xi,h_\xi\rangle : \xi\in X\rangle$. Then for every $\xi\in K$, $r_\xi=s_\xi$ and $h_\xi=\langle h_\xi^1,h_\xi^2\rangle$, where $h_\xi^1\in\mathrm{Col}(\sup(X\cap\xi)^+,{<}\tau_\xi)$ and $h_\xi^2\in\mathrm{Col}(\tau_\xi^+,{<}\xi)$. Let

$$\rho(r)=\big\langle\langle t_\xi,C_\xi,\langle\rho_\xi^1(h_\xi^1),\rho_\xi^2(h_\xi^2)\rangle\rangle : \xi\in K\big\rangle^\frown\langle\langle r_\xi,C_\xi,h_\xi\rangle : \xi\in X\setminus K\rangle.$$

It is easily seen that ρ is an isomorphism from $(\mathbb{Q}/G)/p^*$ to $(\mathbb{Q}/G)/q^*$. ■

Lemma 5.10. *Let $S=\{\eta_\xi^1 : \xi\in X\}$. Then S is a subset of κ of size κ, and $|A\cap S|<\aleph_0$ for every countable set $A\in V[G]$.*

Remark 5.11. (a) Since $V[G]$ and $V[H]$ have the same reals, it suffices to prove the lemma for $A\subseteq S$, $A\in V[G]$. In fact suppose that the lemma is true for all countable $A\subseteq S$, $A\in V[G]$. If the lemma fails, then for some countable set $B\in V[G]$, $|B\cap S|=\aleph_0$. Let $g:\omega\to B$ be a bijection in $V[G]$. Then $g^{-1}[B\cap S]$ is a subset of ω which is in $V[H]$, and hence in $V[G]$. Thus $B\cap S\in V[G]$. Hence we find a countable subset $A\subseteq S$ in $V[G]$, namely $B\cap S$, for which the lemma fails, which contradicts our initial assumption.

(b) In what follows we say A *codes* ξ (for $\xi\in X$) if $\eta_\xi^1\in A$.

Proof of Lemma 5.10. Let $\underset{\sim}{S}$ be a $\mathbb{Q}/G$-name for S. Also let $p_0 \in H\cap\mathbb{Q}/G$ be such that $p_0 \Vdash^{V[G]}_{\mathbb{Q}/G} \ulcorner \check{A} \subseteq \underset{\sim}{S}$ is countable$\urcorner$.

CLAIM 5.12. *For every $p \in \mathbb{Q}/G$ and every $\xi \in X\setminus\mathrm{supp}(p)$ there is $q \le p$ in $\mathbb{Q}/G$ such that $\xi \in \mathrm{supp}(q)$ and, if $q(\xi) = \langle s_\xi, A_\xi, f_\xi\rangle$, then $l(s_\xi) = 2$ and $q \Vdash^{V[G]}_{\mathbb{Q}/G} \ulcorner s_\xi(1) \notin \check{A}\urcorner$.*

Proof. Let p and ξ be as in the claim. First pick $\langle\langle\langle t_\xi(0)\rangle, A_\xi, f_\xi\rangle\rangle \in G$, and then let $q = p^\frown\langle\langle s_\xi, A_\xi, f_\xi\rangle\rangle$, where $s_\xi(0) = t_\xi(0) = \tau_\xi$, $s_\xi(1) < \xi$ is large enough so that $s_\xi(1) \notin A$, $\sup(\mathrm{ran}(f^2_\xi)) < s_\xi(1)$ and $s_\xi(1)$ is inaccessible. Then $\pi(\langle\langle s_\xi, A_\xi, f_\xi\rangle\rangle) = \langle\langle\langle t_\xi(0)\rangle, A_\xi, f_\xi\rangle\rangle \in G$. On the other hand $\pi(p) \in G$. Let $r \in G$, $r \le \pi(p)$, $\langle\langle\langle t_\xi(0)\rangle, A_\xi, f_\xi\rangle\rangle$. Then $r \le \pi(q)$, hence $\pi(q) \in G$. This implies that $q \in \mathbb{Q}/G$. Clearly q satisfies the requirements of the claim. ∎

It follows that the set

$$D = \{p \in \mathbb{Q}/G : \forall\xi \in X \setminus \mathrm{supp}(p) \text{ there exists } q \le p \text{ as in the above claim}\}$$

is dense open in $\mathbb{Q}/G$. Let $p \in H \cap D$. We can assume that $p \le p_0$. We show that $p \Vdash^{V[G]}_{\mathbb{Q}/G} \ulcorner$if $\check{A}$ codes ξ then $\xi \in \mathrm{supp}(p)\urcorner$. To see this suppose that $\xi \in X\setminus\mathrm{supp}(p)$. Thus by Claim 5.12 we can find $q \le p$ in $\mathbb{Q}/G$ such that $\xi \in \mathrm{supp}(q)$ and, if $q(\xi) = \langle s_\xi, A_\xi, f_\xi\rangle$ then $l(s_\xi) = 2$ and $q \Vdash^{V[G]}_{\mathbb{Q}/G} \ulcorner s_\xi(1) \notin \check{A}\urcorner$. It then follows that $\sim p \Vdash^{V[G]}_{\mathbb{Q}/G} \ulcorner s_\xi(1) \in \check{A}\urcorner$. But then by the cone homogeneity of $\mathbb{Q}/G$ we have $p \Vdash^{V[G]}_{\mathbb{Q}/G} \ulcorner s_\xi(1) \notin \check{A}\urcorner$ ([20]). Hence $p \Vdash^{V[G]}_{\mathbb{Q}/G} \ulcorner\check{A}$ does not code $\xi\urcorner$. This means that $p \Vdash^{V[G]}_{\mathbb{Q}/G} \ulcorner \check{A} \subseteq \{s_\xi(1) : \xi \in \mathrm{supp}(p)\} = \{\eta^1_\xi : \xi \in \mathrm{supp}(p)\}\urcorner$. Lemma 5.10 follows by noting that $p \in H$, and since the Magidor iteration is used, the support of any condition is finite. ∎

Theorem 5.1 follows. ∎

The following theorem can be proved by combining the methods of the proofs of Theorems 3.1 and 5.1.

THEOREM 5.13. *Suppose GCH holds, and κ is the least singular cardinal of cofinality ω which is a limit of κ-many measurable cardinals. Also let $V[G]$ and $V[H]$ be the models constructed in the proof of Theorem* 5.1. *Then there is a generic extension $V[H][K]$ of $V[H]$ that is cardinal preserving, not adding a real, and such that in $V[H][K]$ there exists a splitting $\langle S_\sigma : \sigma < \kappa\rangle$ of κ^+ into sets of size κ^+ such that for every countable set $I \in V[G]$ and $\sigma < \kappa$, we have $|I \cap S_\sigma| < \aleph_0$.*

([20]) If not, then for some $p' \le p$, $p' \Vdash^{V[G]}_{\mathbb{Q}/G} \ulcorner s_\xi(1) \in \check{A}\urcorner$. By cone homogeneity of $\mathbb{Q}/G$ we can find $q^* \le q$, $p^* \le p'$ and an isomorphism $\rho : (\mathbb{Q}/G)/p^* \to (\mathbb{Q}/G)/q^*$. But then by standard forcing arguments and the fact that $q^* \Vdash^{V[G]}_{\mathbb{Q}/G} \ulcorner s_\xi(1) \notin \check{A}\urcorner$, we can conclude that $p^* \Vdash^{V[G]}_{\mathbb{Q}/G} \ulcorner s_\xi(1) \notin \check{A}\urcorner$, which is impossible as $p^* \le p'$ and $p' \Vdash^{V[G]}_{\mathbb{Q}/G} \ulcorner s_\xi(1) \in \check{A}\urcorner$.

Proof. Work over $V[H]$ and force the splitting $\langle S_\sigma : \sigma < \kappa \rangle$ as in the proof of Theorem 3.1, with $V, V[G]$ used there replaced by $V[G], V[H]$ here respectively. The role of the sequence $\bigcup_{\xi \in X} x_\xi$ in the proof of Theorem 3.1 is now played by the sequence $S = \{\eta^1_\xi : \xi \in X\}$. ∎

COROLLARY 5.14. *Suppose GCH holds and there exists a cardinal κ which has cofinality ω and is a limit of κ-many measurable cardinals. Then there is a pair (V_1, V_2) of models of ZFC, $V_1 \subseteq V_2$, such that:*

(a) *V_1 and V_2 have the same cardinals and reals.*
(b) *κ is the first fixed point of the $\aleph$-function in V_1 (and hence in V_2).*
(c) *Adding κ-many Cohen reals over V_2 adds κ^+-many Cohen reals over V_1.*

Proof. Let $V_1 = V[G]$ and $V_2 = V[H][K]$, where $V[G], V[H][K]$ are as in Theorem 5.13. The result follows from Remark 2.2 and Theorem 5.13. ∎

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