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# Commuting contractive families

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**Abstract.** A family $f_1, \dots, f_n$ of operators on a complete metric space $X$ is called contractive if there exists a positive $\lambda < 1$ such that for any $x, y$ in $X$ we have $d(f_i(x), f_i(y)) \le \lambda d(x, y)$ for some $i$. Austin conjectured that any commuting contractive family of operators has a common fixed point, and he proved this for the case of two operators. We show that Austin's conjecture is true for three operators, provided that $\lambda$ is sufficiently small.

**1. Introduction.** Let $(X, d)$ be a (non-empty) complete metric space. Given $n$ functions $f_1, \dots, f_n : X \to X$ and $\lambda \in (0, 1)$, we call $\{f_1, \dots, f_n\}$ a $\lambda$-*contractive family* if for any points $x, y$ in $X$ there is $i$ such that $d(f_i(x), f_i(y)) \le \lambda d(x, y)$. We say that $\{f_1, \dots, f_n\}$ is a *contractive family* if it is $\lambda$-contractive for some $\lambda \in (0, 1)$. Furthermore, when $f$ is a function on $X$ and $\{f\}$ is a contractive family, we say that $f$ is a *contraction*. Recall the well-known theorem of Banach [3] stating that any contraction on a complete metric space has a fixed point. Finally, by an *operator* $f$ on $X$ we mean a continuous function $f : X \to X$.

Stein [7] considered various possible generalizations of this result. In particular, he conjectured that for any contractive family $\{f_1, \dots, f_n\}$ of operators on a complete metric space there is a composition of the functions $f_i$ (i.e. some word in $f_1, \dots, f_n$) with a fixed point. We refer to this statement as *Stein's conjecture*. However, in [2], Austin showed that this is in fact false. Recently, the present author showed [6] that Stein's conjecture fails even for compact metric spaces. But Austin also showed that if $n = 2$ and $f_1$ and $f_2$ commute, the conjecture of Stein does hold.

With this in mind, we say that $\{f_1, \dots, f_n\}$ is *commuting* if any $f_i$ and $f_j$ commute.

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Theorem 1.1 ([2]). *Suppose that $\{f, g\}$ is a commuting contractive family of operators on a complete metric space. Then $f$ and $g$ have a common fixed point.*

Let us mention another result in this direction, which was proved by Arvanitakis [1] and by Merryfield and Stein [5].

Theorem 1.2 ([1], [5], Generalized Banach Contraction Theorem). *Let $f$ be a function from a complete metric space to itself such that $\{f, f^2, \dots, f^n\}$ is a contractive family. Then $f$ has a fixed point.*

Note that there is no assumption of continuity in the statement of Theorem 1.2. We also remark that Merryfield, Rothschild and Stein [4] proved this theorem for the case of operators. Furthermore, Austin raised a question which is a version of Stein's conjecture, and generalizes these two theorems in the context of operators.

Conjecture 1.3 ([2]). *Suppose that $\{f_1, \dots, f_n\}$ is a commuting contractive family of operators on a complete metric space. Then $f_1, \dots, f_n$ have a common fixed point.*

Let us now state the result that we establish here, which proves the case $n = 3$ and $\lambda$ sufficiently small:

Theorem 1.4. *Let $(X, d)$ be a complete metric space and let $\{f_1, f_2, f_3\}$ be a commuting $\lambda$-contractive family of operators on $X$, for a given $\lambda$ in $(0, 10^{-23})$. Then $f_1, f_2, f_3$ have a common fixed point.*

We remark that such a fixed point is necessarily unique.

**2. Main goal, notation and definitions.** In this section we will give a statement that implies the theorem we want to prove and we will establish the notation and definitions that will be used throughout the proof. We write $\mathbb{N}_0$ for the set of non-negative integers, and for a positive integer $N$, $[N]$ stands for the set $\{1, \dots, N\}$.

When $a$ is an ordered triple of non-negative integers and $x \in X$, we define $a(x) = f_1^{a_1} \circ f_2^{a_2} \circ f_3^{a_3}(x)$. Since our functions commute, we have $a(b(x)) = (a + b)(x)$ for all $a, b \in \mathbb{N}_0^3$.

Pick an arbitrary point $p_0 \in X$, and define a new pseudometric space (abusing the notation slightly) $G(p_0) = (\mathbb{N}_0^3, d)$, where $d(a, b) = d(a(p_0), b(p_0))$ for $a, b \in \mathbb{N}_0^3$. Therefore, we will actually work on an integer grid instead. Define $e_i$ to be the triple with 1 at position $i$, and zeros elsewhere.

Now, we will prove a few basic claims which will tell us what in fact our main goal is.

Proposition 2.1. *Let $(X,d)$ be a complete metric space and let $\lambda \in (0, 10^{-23})$, and suppose that $f_1, f_2, f_3 : X \to X$ form a commuting $\lambda$-contractive family. Then for some $i$, $f_i$ has a fixed point.*

Proposition 2.2. *If Proposition* 2.1 *holds, so does Theorem* 1.4.

*Proof.* Without loss of generality, $f_1$ has a fixed point $x$. Define $X_1$ to be the set of all fixed points of $f_1$. It is a closed subspace of $X$, hence complete. Further, $s \in S_1$ implies $f_1(f_i(s)) = f_i(f_1(s)) = f_i(s)$, so $f_i(s) \in S_1$, hence the other two functions preserve $S_1$ and form a $\lambda$-contractive family themselves, so $f_2$ has a fixed point in $S_1$; repeat the same argument once more to obtain a common fixed point. ■

Proposition 2.3. *Let $(\mathbb{N}_0^3, d)$ be a pseudometric space and suppose that $\lambda \in (0, 10^{-23})$ is such that given any $a, b \in \mathbb{N}_0^3$, there is $i \in [3]$ for which $\lambda d(a,b) \geq d(a+e_i, b+e_i)$. Then there is a Cauchy sequence $(x_n)_{n\geq 1}$ in this space such that $x_{n+1} - x_n$ is always an element of $\{e_1, e_2, e_3\}$.*

Proposition 2.4. *If Proposition* 2.3 *holds, so does Theorem* 1.4.

*Proof.* It suffices to show that Proposition 2.3 implies Proposition 2.1. Let $(X,d)$ and $f_1, f_2, f_3$ be as in Proposition 2.1. Pick an $p_0 \in X$, and consider the pseudometric space $G(p_0)$ defined before. By Proposition 2.3, there is a Cauchy sequence $(x_n)_{n\geq 1}$ with the property there. So, $(x_n(p_0))$ is Cauchy in $X$. Without loss of generality, we have change by $e_1$ infinitely often, say $x_{n_i+1} = x_{n_i} + e_1$ for $i \geq 1$. As $X$ is complete, $x_n(p_0)$ converges to some $x$. Hence $x_{n_i}(p_0)$ converges to $x$, and so does $f_1(x_{n_i}(p_0))$; but $f_1$ is continuous, thus $f_1(x) = x$. ■

Therefore, Proposition 2.3 is what is sufficient to prove. The integer grid has a lot of structure itself, and the following definitions aim to capture some of it and to help us establish the claim.

Let $x$ be a point in the grid. Define $\rho(x)$ to be the maximum of $d(x, x+e_1)$, $d(x, x+e_2)$, $d(x, x+e_3)$. As we shall see in the following section, $\rho$ will be of fundamental importance. Given $x$ in the grid, we define $N(x) = \{x+e_1, x+e_2, x+e_3\}$ and refer to this set as the *neighbourhood* of $x$.

Let $S$ be a subset of the grid. Given $k \in [3]$, we say that $S$ is a *$k$-way set* if for all $s \in S$, precisely $k$ elements of $N(s)$ are in $S$. We denote the unique 3-way set starting from $x$ by $\langle x \rangle_3 = \{x + k : k \in \mathbb{N}_0^3\}$.

**3. Overview of the proof of Proposition 2.3.** The proof of Proposition 2.3 will occupy most of the remainder of the paper. To elucidate the proof, we will structure it in a few parts. The short first section will show our strategy along with some basic ideas. The second part will be about $k$-way sets and how they interact with each other. Afterwards, we shall deal with the local structure, namely we shall prove existence or non-existence of certain finite sets of points, and our main means will be $k$-way sets. Finally, after we have sufficiently clarified the local structure, we will be able to obtain the final contradiction. Let us now be more precise and elaborate on these parts of the proof.

First of all, we shall establish a few basic facts about $\rho$, most importantly $\mu = \inf \rho(x) > 0$, where $x$ ranges over all points. Important for the proof of this statement is a lemma that says $d(x, y) \leq (\rho(x)+\rho(y))/(1-\lambda)$. This fact will be the pillar of the proof, and it will be used several times. The basic idea introduced in this section is to create sets of points by contracting with some previously chosen ones (by *contracting* a pair $x, y$ we mean choosing a suitable function $f$ in our family such that $d(f(x), f(y)) \leq \lambda d(x, y)$). By doing so, we will be able to construct $k$-way sets of bounded diameter.

After that, we shall prove a few propositions about $k$-way sets. For example, every 3-way set of bounded diameter contains a 2-way subset of much smaller diameter, in a precise sense. At first glance, it seems that we have lost a dimension by doing this; however, we shall also show that if we have a 2-way set of sufficiently small diameter, we can obtain a 3-way set of small diameter as well. So, for example, given $K$ and provided $\lambda$ is small enough, we cannot have 3-way sets of diameter $K\mu$, and we cannot have 2-way sets of diameter $\lambda K\mu$ inside every 3-set. From this point on we shall combine the results and approaches of these two parts in the proof. Most of the claims we establish afterwards will either show that a certain finite configuration (by which we mean a finite set of points with suitable mutual distances) exists or does not exist, and we do so by supposing the contrary, contracting new points with the given ones and finding suitable $k$-way sets, which give us a contradiction.

As the basic example of this method, we note that each point $x$ induces a 1-way set of diameter at most $2\rho(x)/(1-\lambda)$, more importantly, such a set exists in every 3-way set. With a greater number of suitable points we are able to induce bounded $k$-way sets for larger $k$. Using the facts established, we prove the existence or non-existence of specific finite sets.

Gradually, we learn more about the local structure of the grid considered. For example, for some constant $C$ (independent of $\lambda$) we have $y$ with $\rho(y) \leq C\mu$ and $d(y + e_1, y + e_2) \leq \lambda C\mu$, provided $\lambda$ is small enough. Similarly, we shall establish that there is no point $y$ with $\rho(y) \leq C\mu$ and $\operatorname{diam} N(y) \leq \lambda C\mu$, for suitable $\lambda, C$. Such points will be used several times in the later part of the proof and in the final argument to reach a contradiction.

At this point we introduce the notion of a *diagram* of a point $x$, giving the information about the contractions in $\{x\} \cup N(x)$. The diagrams will be shown in figures, and usually dashed lines will imply that the corresponding edge is the result of a contraction. In Figure 1 we give an example of two

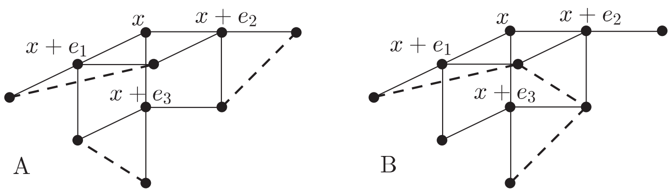


Fig. 1. Examples of diagrams

diagrams [1]; the left one, denoted by A, tells us among other things that $x+e_1, x+e_2$ are *contracted by* 1 (i.e. $d(x+2e_1, x+e_1+e_2) \leq \lambda d(x+e_1, x+e_2)$). The claims established so far allow us to have a very restricted number of possibilities for diagrams, and one of the possible strategies will then be to classify the diagrams, see how they fit together and establish the existence of a 1-way Cauchy sequence. The most important claim that we use to reject diagrams is the following proposition.

Proposition 3.1. *Given $K \geq 1$, suppose we have $x_0, x_1, x_2, x_3$ such that* $\operatorname{diam}\{x_i + e_j : i, j \in [3], i \neq j\} \leq \lambda K\mu$. *Furthermore, suppose $\rho(x_0) \leq K\mu$ and that $d(x_0, x_i) \leq K\mu$ for $i \in [3]$. Let $\{a, b, c\} = [3]$. Provided $\lambda < 1/(820 C_1 K)$, whenever there is a point $x$ which satisfies $d(x+e_a, x+e_b) \leq \lambda K\mu$ and $d(x, x_0) \leq K\mu$, then $d(x+e_c, x_c+e_c) \leq 16\lambda K\mu$.*

The final part of the proof is based on the following claim:

Proposition 3.2. *Fix $x_0$ with $\rho(x_0) < 2\mu$. For $K \geq 1$ and $i \in [3]$, define $S_i(K, x_0) = \{y : d(x_0, y) \leq K\mu,\ d(y, y+e_i) \leq K\mu\}$. Provided $1 > 10\lambda K C_1$, in every $\langle z\rangle_3$ there is $t$ such that $d(t, x_0) \leq 3K\mu$, but for some $i$ we have $s \overset{i}{\not\to} t$ whenever $s \in S_i(K, x_0)$.*

Using the point $t$ whose existence is provided, we shall discuss the cases of $d(t+e_1, t+e_2)$ being large or small. Both help us to reject many diagrams and then to establish a contradiction in a straightforward manner.

To sum up, the basic principle here is that contractions ensure that we get specific finite sets. On the other hand, certain finite sets empower contractions further, allowing us to construct $k$-way sets of small diameter. Therefore, if we are to establish a contradiction, we can expect a dichotomy: either we get finite sets that imply global structure that is easy to work with, or we do not have such sets, and we impose strong restrictions on the local structure of the grid.

We are now ready to start the proof of Proposition 2.3. The proof will run for most of the paper, ending in Section 8.

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[1] In other figures we shall not denote the points on the diagram itself; however, the coordinate axes will always be the same.

*Proof of Proposition 2.3.* Suppose the contrary: there is no 1-way Cauchy sequence in the given pseudometric space on $\mathbb{N}_0^3$. This condition will, as we shall see, imply a lot about the structure of the space, and we will start by getting more familiar with the function $\rho$, which will, as already remarked, play a fundamental role.

**4. Basic finite contractive configurations arguments and properties of $\rho$.** In this section we establish a few claims about $\rho$, together with some claims which will come in handy at several places of the proof.

LEMMA 4.1 (Farthest neighbour inequality, FNI). *Given $x, y$ in the grid we have $d(x,y) \le (\rho(x)+\rho(y))/(1-\lambda)$.*

*Proof.* Let $i$ be such that $\lambda d(x,y) \ge d(x+e_i, y+e_i)$, which we denote from now on by $x \overset{i}{\frown} y$, and say that $i$ *contracts* $x, y$ (2). Using the triangle inequality a few times yields $d(x,y) \le d(x, x+e_i) + d(x+e_i, y+e_i) + d(y+e_i, y) \le \lambda d(x,y) + \rho(x) + \rho(y)$, which implies the result. ∎

Similarly to $x \overset{i}{\frown} y$, we write $x \overset{i}{\not\frown} y$ to mean that $d(x+e_i, y+e_i) > \lambda d(x,y)$.

LEMMA 4.2. *Let $x, y$ be any two points in the grid. Then we can find a 1-way subset $S$ such that $y \in S$ and given $\epsilon > 0$ we have $d(s,x) \le \frac{1}{1-\lambda}\rho(x) + \epsilon$ for all but finitely many $s \in S$.*

*Proof.* Consider the sequence $(x_n)_{n\ge 0}$ defined inductively by $x_0 = y$ and for any $k \ge 0$, $x_{k+1} = x_k + e_i$ when $i$ contracts $x$ and $x_k$. By induction on $k$ we shall prove that $d(x, x_k) \le \lambda^k d(x,y) + \rho(x)/(1-\lambda)$.

The case $k = 0$ is clear as $\rho(x_0) \ge 0$. If the claim holds for some $k$ and $x_k \overset{i}{\frown} x$, then by the triangle inequality we have

$$\begin{aligned} d(x_{k+1}, x) &\le d(x_{k+1}, x+e_i) + d(x+e_i, x) \le \lambda d(x_k, x) + \rho(x_0) \\ &\le \lambda^{k+1} d(x,y) + \lambda\rho(x_0)/(1-\lambda) + \rho(x_0) \\ &\le \lambda^{k+1} d(x,y) + \rho(x_0)/(1-\lambda), \end{aligned}$$

as desired.

Now, take $n$ sufficiently large so that $\rho(x_0)/(1-\lambda) + \lambda^n d(x,y) \le \frac{1}{1-\lambda}\rho(x_0) + \epsilon$. Hence $d(x_k, x) \le \frac{1}{1-\lambda}\rho(x_0) + \epsilon$ for all $k \ge n$, so choose $S = (x_k)_{k\ge 0}$. ∎

PROPOSITION 4.3. *Given any $x$ in the grid, we have $\rho(x) > 0$.*

*Proof.* Suppose $\rho(x) = 0$ for some $x$. Then Lemma 4.2 immediately gives a 1-way Cauchy sequence, which is a contradiction. ∎

PROPOSITION 4.4. *The infimum* $\inf\{\rho(x) : x \in \mathbb{N}_0^3\}$ *is positive.*

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(2) We also say $x, y$ is *contracted by $i$*, or $x, y$ is *contracted in the direction $e_i$*.

This result is one of the crucial structural properties for the rest of the proof, and having it in mind, we will try either to find small $\rho$, or to use the structure implied to get a Cauchy sequence, which will yield a contradiction. To prove this statement, we use Lemma 4.2, the difference being that we now contract with many different points of small $\rho$ instead of just one.

*Proof of Proposition 4.4.* Suppose, contrary to our claim, that there is a sequence $(y_n)_{n\geq 1}$ such that $\rho(y_n) < 1/n$. As $\rho$ is always positive, we can assume that all elements of the sequence are distinct.

We define a 1-way sequence $(x_k)_{k\geq 0}$ as follows: start from an arbitrary $x_0$ and contract with $y_1$ as in the proof of Lemma 4.2 until we get a point $x_{k_1}$ with $d(x_{k_1}, y_1) \leq 2\rho(y_1)/(1-\lambda)$ (such a point exists by Lemma 4.2). Now, start from $x_{k_1}$ and contract with $y_2$ until we reach $x_{k_2}$ with $d(x_{k_2}, y_2) \leq 2\rho(y_2)/(1-\lambda)$. We require that $k_{i+1} > k_i$ for all possible $i$, so that, proceeding in this way, one defines the whole sequence. Recalling the estimates in the proof of Lemma 4.2, we see that for $k_i \leq j \leq k_{i+1}$ we have

$$\begin{aligned} d(x_j, y_{i+1}) &\leq d(x_{k_i}, y_{i+1}) + \rho(y_{i+1})/(1-\lambda) \\ &\leq d(x_{k_i}, y_i) + d(y_i, y_{i+1}) + \rho(y_{i+1})/(1-\lambda). \end{aligned}$$

So by FNI, we see that $d(x_j, y_{i+1}) \leq (3\rho(y_i) + 2\rho(y_{i+1}))/(1-\lambda) \leq \frac{5}{i(1-\lambda)}$. Hence, if we are given any other $x_{j'}$ with $k_{i'} \leq j' \leq k_{i'+1}$, by the triangle inequality and FNI we see that $d(x_j, x_{j'}) \leq \frac{6}{1-\lambda}(1/i + 1/i')$, which is enough to show that the constructed sequence is 1-way Cauchy. ∎

We will denote $\mu = \inf \rho$, where the infimum is taken over the whole grid. We have just proved $\mu > 0$.

**5. Properties of and relationships between $k$-way sets.** The following propositions reflect the nature of $k$-way sets. These both confirm their importance for the problem and will prove useful at various places of the proof.

Proposition 5.1. *If $\langle\alpha\rangle_3$ is a 3-way set of diameter $D$, then it contains a 2-way subset of diameter not greater than $\lambda C_1 D$, where $C_1 = 49158$.*

*Proof.* This will be a consequence of Proposition 5.2 and Lemma 5.4, each needing an auxiliary lemma. Let us start by establishing:

Proposition 5.2. *If the conclusion of Proposition 5.1 does not hold, then given $x, y \in \langle\alpha\rangle_3$ and distinct $i, j \in [3]$, there is $z \in \langle\alpha\rangle_3$ with $d(x, z+e_i) > 2\lambda D$ and $d(y, z+e_j) > 2\lambda D$.*

The purpose of this proposition is to provide us with a finite set of points which will then be used to induce a 2-way set of the desired diameter, by contractions. To prove this claim, we examine two cases: $d(x,y) > 5\lambda D$ and when $d(x,y) \leq 5\lambda D$.

*Proof of Proposition 5.2.*

CASE 1: $d(x,y) > 5\lambda D$. We shall actually prove something more general: if $d(x,y) > 5\lambda D$ and we cannot find a desired point $z$, then we get a 3-way subset $T$ of $\langle\alpha\rangle_3$ of diameter not greater than $4\lambda D$.

Suppose there is no such $z$, hence for all $z \in \langle\alpha\rangle_3$ either $d(x, z+e_1) \leq 2\lambda D$ or $d(y, z+e_2) \leq 2\lambda D$. We can colour all points $t$ in this 3-way set by $c(t) = 1$ if $d(t,x) \leq 2\lambda D$, $c(t) = 2$ if $d(t,y) \leq 2\lambda D$, and $c(t) = 3$ otherwise. This is well-defined, as the triangle inequality prevents the first two conditions from holding simultaneously. Thus, for any $z$, either $c(z+e_1) = 1$ or $c(z+e_2) = 2$. Also given any two points $z, t$ in the grid such that $t \overset{j}{\frown} z$, and whose neighbours take only colours 1 and 2, it cannot be that $c(z+e_j) \neq c(t+e_j)$, as otherwise we would have e.g. $c(t+e_j) = 1$, $c(z+e_j) = 2$. Then $d(x,y) \leq d(x, t+e_j) + d(t+e_j, z+e_j) + d(z+e_j, y) \leq 5\lambda D$, a contradiction. Thus for any such $z$ and $t$, there is an $i$ such that $c(t+e_i) = c(z+e_i)$.

The following auxiliary lemma tells us that all such colourings are essentially trivial. (Note that we are still in Case 1 of the proof of Proposition 5.2.)

LEMMA 5.3. *Let $c : \langle\beta\rangle_3 \to [3]$ be a colouring such that:*

1. *Given $z \in \langle\beta\rangle_3$, either $c(z+e_1) = 1$ or $c(z+e_2) = 2$.*
2. *Given $z, t \in \langle\beta\rangle_3$ such that the neighbours of $z, t$ take only colours 1 and 2, we have $c(z+e_i) = c(t+e_i)$ for some $i$.*

*Then there is a 3-way subset of $\langle\beta\rangle_3$ which is either entirely coloured by 1, or entirely coloured by 2.*

*Proof.* We denote coordinates by superscripts. Given non-negative integers $a \geq \beta^{(3)}$ and $b \geq \beta^{(1)} + \beta^{(2)}$ denote

$$\mathcal{L}(a,b) = \{z \in \mathbb{N}_0^3 : z^{(3)} = a,\ z^{(1)} + z^{(2)} = b\}.$$

Such a line must be coloured as $c(b-\beta_2, \beta_2, a) = 1, c(b-\beta_2-1, \beta_2+1, a) = 1, \ldots, c(t+e_1-e_2) = 1, c(t)$ arbitrary, $c(t+e_2-e_1) = 2, \ldots, c(\beta_1, b-\beta_1, a) = 2$, for some point $t$. If all $z$ in $\mathcal{L}(a,b)$ with $z^{(1)} \geq \beta^{(1)} + 3$, $z^{(2)} \geq \beta^{(2)} + 3$ are coloured by 1, say that $\mathcal{L}(a,b)$ is a 1-*line*. Similarly, if they are coloured by 2, call it a 2-*line*, and otherwise a 1,2-*line*.

Observe that if $\mathcal{L}(a,b)$ is a 1,2-line for $a \geq \beta^{(3)}$ and $b > \beta^{(1)} + \beta^{(2)} + 10$, then $\mathcal{L}(a+1, b-1)$ is not a 1,2-line, for otherwise we have:

1. $(\beta^{(1)}, b-\beta^{(1)}, a)$, $(\beta^{(1)}+1, b-\beta^{(1)}-1, a)$, $(\beta^{(1)}, b-\beta^{(1)}-1, a+1)$ are coloured by 1,
2. $(b-\beta^{(2)}, \beta^{(2)}, a)$, $(b-\beta^{(2)}-1, \beta^{(2)}+1, a)$, $(b-\beta^{(2)}-1, \beta^{(2)}, a+1)$ are coloured by 2,

which is impossible by the second property of the colouring.

Suppose we have a 1,2-line $\mathcal{L}(a,b)$ for $a > \beta^{(3)}$, $b > \beta^{(1)} + \beta^{(2)} + 20$. Then $\mathcal{L}(a+1, b-1)$ and $\mathcal{L}(a-1, b+1)$ are either 1- or 2-lines. But as above, we can exhibit $x', y'$ such that $x'+e_1, x'+e_2, y'+e_3$ are of colour 1, while $y'+e_1, y'+e_2, x'+e_3$ are of colour 2, or we can find $x', y'$ for which $x'+e_1, x'+e_2, x'+e_3$ have $c = 1$, and $y'+e_1, y'+e_2, y'+e_3$ are coloured by 2. So, there can be no such 1,2-lines. Further, by the same arguments we see that $\mathcal{L}(a, s-a)$ for fixed $s$ must all be 1-lines or all 2-lines, for $a > \beta_3+1$, and that in fact only one of these possibilities can occur, hence we are done. ∎

Applying Lemma 5.3 immediately solves Case 1 of the proof.

CASE 2: $d(x,y) \le 5\lambda D$. Suppose the contrary. Then in particular, for any $z$, we have $d(x, z+e_1) \le 7\lambda D$ or $d(x, z+e_2) \le 7\lambda D$. Further, we must have $z$ such that $d(x, z+e_{i_1}), d(x, z+e_{i_2}) > 10\lambda D$ for some distinct $i_1, i_2 \in [3]$. Take such a $z$, and without loss of generality $i_1 = 2$, $i_2 = 3$. So $d(z+e_1, x) \le 7\lambda D$. Hence $d(z+(-1,1,1), x) \le 7\lambda D$ and contracting $z, z+(-1,0,1)$ gives $d(z+(-1,0,2), x) > 9\lambda D$. Now contract $z, z+(-1,1,0)$ to get $d(z, z+(-1,2,0)) > 9\lambda D$. However, this is a contradiction, as both $z+(-1,1,0)+e_1$ and $z+(-1,1,0)+e_2$ are too far from $x$.

Having settled both cases, we have completed the proof of Proposition 3.2. ∎

If there is $x \in \langle\alpha\rangle_3$ such that for some $x' \in \langle\alpha\rangle_3$ and for all $y \in \langle x'\rangle_3$ we have $d(x,y) \le 5\lambda D$, we are done. Hence, we can assume that for all $x, x' \in \langle\alpha\rangle_3$ there is $y \in \langle x'\rangle_3$ which violates the above distance condition.

Take now an arbitrary $x_0 \in \langle\alpha\rangle_3$. Due to the observation we have just made, for any $i \in [3]$ there is an $x_i \ne x_0$ such that $d(x_i+e_i, x_0+e_i) > 5\lambda D$; to be on the safe side, assume that the neighbourhoods of $x_0, x_1, x_2, x_3$ are pairwise disjoint. Now, by Proposition 5.2, given $i \ne j$ in [3], we can find $x_{i,j} \in \langle\alpha\rangle_3$ such that $d(x_{i,j}+e_i, x_0+e_i), d(x_{i,j}+e_j, x_i+e_j) > 2\lambda D$. Now, let $y$ be any element of the 3-way set generated by $\alpha$. Take $i$ which contracts $x_0, y$, implying $d(x_0+e_i, y+e_i) \le \lambda D$. Hence by the triangle inequality $d(x_i+e_i, y+e_i) > \lambda D$, so $x_i, y$ must be contracted by some $j \ne i$. Using the triangle inequality once more, we get $d(x_{i,j}+e_j, y+e_j) > \lambda D$ and by construction $d(x_{i,j}+e_i, y+e_i) \ge d(x_{i,j}+e_i, x_0+e_i) - d(x_0+e_i, y+e_i) > \lambda D$, therefore for $k \ne i, j$ we have $d(y+e_k, x_{i,j}+e_k) \le \lambda D$. We are now ready to conclude that there is a finite set of points $P$ such that whenever $y \in \langle\alpha\rangle_3$ is given, then for each $i \in [3]$ there is a point $p \in P$ with $d(p, y+e_i) \le \lambda D$. Here $P$ consists of $N(x_0)$, $x_i + e_j$ and $x_{i,j} + e_k$ for suitable indices $i \ne j \ne k \ne i$, in particular $|P| = 15$.

LEMMA 5.4. *Suppose we are given a 3-way set $\langle\beta\rangle_3 = \bigcup_{i=1}^k A_i$ of diameter $C$, where $\operatorname{diam} A_i \le \lambda r C$ for each $i$. Then there is a constant $K_{k,r}$ (i.e. does not depend on $\lambda$ or $C$) such that $\langle\beta\rangle_3$ has a 2-way subset of di-*

*ameter at most $K_{k,r}\lambda C$. Further, we can take $K_{1,r} = r$, $K_{2,r} = 2r + 8$, $K_{k+1,r} = K_{k,2r+1}$ for all $r$ and $k \geq 2$.*

*Proof.* We use induction on $k$. When $k = 1$, there is nothing to prove, and $K_{1,r} = r$. Suppose $k = 2$.

Before we proceed, we need to establish:

LEMMA 5.5. *Consider a 3-colouring of the edges of a complete graph $G$ whose vertex set consists of all positive integers, namely $c : \{\{a, b\} : a \neq b, a, b \in \mathbb{N}\} \to [3]$. Then we can find sets $A, B$ with union $\mathbb{N}$ such that for some colours $c_A, c_B$ we have $\operatorname{diam}_{c_A} G[A], \operatorname{diam}_{c_B} G[B] \leq 8$. (Here $\operatorname{diam}_{c_0}$ means the diameter of the graph induced by colour $c_0$.) Furthermore, we can assume $A$ and $B$ are non-disjoint when $c_a \neq c_b$.*

*Proof.* Let $x$ be any vertex. Define $A_i = \{a : c(a, x) = i\}$ for $i \in [3]$, the monochromatic neighbourhood of colour $i$ of $x$. We shall start by looking at the sets $A_i$ if these are not appropriate, we shall look at similar candidates for $A, B$ until we find the right pair of sets. The following simple fact will play a key role: if $X, Y$ intersect and $\operatorname{diam}_c G[X], \operatorname{diam}_c G[Y]$ are both finite, then $\operatorname{diam}_c G[X \cup Y] \leq \operatorname{diam}_c G[X] + \operatorname{diam}_c G[Y]$.

Firstly, if any of the sets $A_i$ is empty, then taking $A_j \cup \{x\}$ and $A_k \cup \{x\}$ for the other two indices $j, k$ proves the lemma. So assume that all $A_i$ are non-empty. The next idea is to try to 'absorb' all the vertices into two of the sets $A_i$. To be more precise, let $B_{i,j} = \{a_i \in A_i : \forall a_j \in A_j,\ c(a_i, a_j) \neq j\}$ for distinct $i, j \in [3]$. Then

$$\operatorname{diam}_i(\{x\} \cup A_i \cup (A_j \setminus B_{j,i})) \leq 4$$

for all distinct $i, j$ (which is what we meant by 'absorbing vertices' above). Observe that if $\{i, j, k\} = [3]$ and $B_{j,i}$ and $B_{j,k}$ are disjoint, then $A_j \setminus B_{j,i}$ and $A_j \setminus B_{j,k}$ cover the whole $A_j$ so we can take $c_A = i$, $c_B = k$ and $A = \{x\} \cup A_i \cup (A_j \setminus B_{j,i})$, $B = \{x\} \cup A_k \cup (A_j \setminus B_{j,k})$. Hence, we may assume that $B_{j,i}$ and $B_{j,k}$ intersect, and in particular are non-empty.

Observe also that for $\{i, j, k\} = [3]$, if we are given $a_i \in B_{i,j}$, $a_j \in B_{j,i}$ then $c(a_i, a_j) \neq i, j$, so $c(a_i, a_j) = k$. This implies $\operatorname{diam}_k G[B_{i,j} \cup B_{j,i}] \leq 2$. We shall exploit this fact to finish the proof.

Now pick an $a_3 \in B_{3,1} \cap B_{3,2}$. If $c(a_1, a_3) = 3$ for some $a_1 \in B_{1,2}$, then $\operatorname{diam}_3(B_{1,2} \cup B_{2,1} \cup A_3 \cup \{x\}) \leq 5$ and $\operatorname{diam}_1(A_1 \cup (A_2 \setminus B_{2,1}) \cup \{x\}) \leq 4$, so we are done. The same argument works for $a_3$ and $B_{2,1}$, allowing us to assume that no edge between $B_{1,2} \cup B_{2,1}$ and $a_3$ is coloured by 3. Therefore, since $a_3 \in B_{3,1} \cap B_{3,2}$, we have $c(B_{1,2}, a_3) = 2$ and $c(B_{2,1}, a_3) = 1$.

Recall that previously we tried to absorb the vertices of $A_1$ to $A_2$ to have a set of bounded diameter in colour 2, but this failed for the set $B_{1,2}$. Now, we have $c(B_{1,2}, a_3) = 2$, so we can once again try the same idea, by looking for an edge of colour 2 between $a_3$ and $A_1 \setminus B_{1,2}$ (vertices of which are joined by an edge of colour 2 to something in $A_2$).

Suppose that $c(a_1, a_3) = 2$ for some $a_1 \in A_1 \setminus B_{1,2}$. Then $\mathrm{diam}_2(A_1 \cup A_2 \cup \{x\} \cup \{a_3\}) \le 8$, and taking $A_3 \cup \{x\}$ for the other set proves the lemma. Analogously, the lemma is proved if $c(a_2, a_3) = 1$ for some $a_2 \in A_2 \setminus B_{2,1}$.

Finally, since $a_3 \in B_{3,1} \cap B_{3,2}$, we may assume that $c(A_1 \setminus B_{1,2}, a_3) = 3$ and $c(A_2 \setminus B_{2,1}, a_3) = 3$. Observing that $\mathrm{diam}_3(B_{1,2} \cup B_{2,1}) \le 2$ and $\mathrm{diam}_3(\mathbb{N} \setminus B_{1,2} \setminus B_{2,1}) \le 4$ completes the proof. ∎

We refer to $\mathrm{diam}_c$ as the *monochromatic diameter* for $c$.

Consider the complete graph on $\langle \beta \rangle_3$ along with an edge 3-colouring $c$ such that $x \overset{c(xy)}{\frown} y$. From Lemma 5.5, we have sets $B_1, B_2$ whose union is $\langle \beta \rangle_3$, and whose monochromatic diameters for some colours are at most 8, that is, by the triangle inequality $\mathrm{diam}(B_1 + e_{i_1}), \mathrm{diam}(B_2 + e_{i_2}) \le 8\lambda C$ for some $i_1, i_2$. If $i_1 = i_2$ we are done, hence we can assume these are different, and in fact without loss of generality $i_1 = 1$, $i_2 = 2$. If $A_1, A_2$ intersect, then the diameter of their union is not greater than $2r\lambda C$, proving the claim. Therefore, we shall consider only the situation when these are disjoint. Similarly, if $B_1 + e_1$ intersects both $A_1, A_2$, then by the triangle inequality, $\mathrm{diam}\,\langle \beta \rangle_3 \le (2r+8)\lambda C$, so without loss of generality $B_1 + e_1 \subset A_1$. Depending on which of the two sets contains $B_2 + e_2$, we distinguish the following cases:

Case 1: $A_1 \supset B_2 + e_2$. We now show that $A_1$ has a 2-way subset, whose diameter is then bounded by the diameter of $A_1$, which suffices to prove the claim. Suppose $a \in A_1$. Then $a \in B_i$ for some $i$, hence $a + e_1$ or $a + e_2$ is in $A_1$. If both are, there is nothing to prove. Otherwise, the other point must be in $A_2$, say $a + e_1 \in A_1$, $a + e_2 \in A_2$. Suppose $a + e_3 \in A_2$ as well. Then $a + e_2 - e_1, a + e_3 - e_1 \in B_2$, thus $a + (-1, 2, 0), a + (-1, 1, 1) \in A_1$, hence contracting $a, a - e_1 + e_2$ gives $d(A_1, A_2) \le \lambda C$. Otherwise $a + e_3 \in A_1$, hence we are done.

Case 2: $A_2 \supset B_2 + e_2$. Colour a point by $i$ if it belongs to $A_i$. Such a colouring satisfies the hypothesis of Lemma 5.3 since given a point $y$, either $y + e_1$ is coloured by 1, or $y + e_2$ is coloured by 2, and the second condition is also satisfied (or after contraction we get $d(A_1, A_2) \le \lambda C$ so we are done). Hence, we have a colouring that is essentially trivial, proving the claim.

Suppose the claim holds for some $k \ge 2$, and we have $k + 1$ sets. As before, we can assume that these are disjoint and thus define a colouring $c$ such that $y \in A_{c(y)}$. Further, we can assume that $d(A_i, A_j) > \lambda C$ for distinct $i, j$. Moreover, we have $A_i \cap \langle \beta + (1, 1, 1) \rangle_3 \ne \emptyset$, as otherwise we are done by considering $\beta + (1, 1, 1)$ instead of $\beta$.

Let $z \in \langle \beta \rangle_3$. Define the *signature* of $z$ as

$$\sigma(z) = (c(z+e_1), c(z+e_2), c(z+e_3)).$$

By the discussion above, given $i \in [k+1]$, $l \in [3]$ we have a point $z$ such that $\sigma(z)^{(l)} = i$. Also, whenever $z, z'$ are two points in our 3-way set, we must have $\sigma(z)^{(i)} = \sigma(z')^{(i)}$ for some $i$, for otherwise we violate the condition on the distance between the sets $A_j$.

Let $(a,b,c)$ be a signature. Suppose there were another signature $(p,d,e)$ where $b \neq d$, $c \neq e$, which implies that $p = a$. Since $k+1 \geq 3$, there are signatures $(g_1,h_1,j_1), (g_2,h_2,j_2)$ where $g_1, g_2, a$ are distinct. Then $(h_1,j_1) = (h_2,j_2) \in \{(b,e),(d,c)\}$; without loss of generality these are $(b,e)$. Hence, for any $z$ we have $\sigma(z)^{(2)} = b$ or $\sigma(z)^{(3)} = e$. Now, define a new colouring $c'$ of $\langle \beta \rangle_3$: if a point $p$ was coloured by $b$ set $c'(p) = 1$, if it was coloured by $e$ set $c'(p) = 2$, otherwise $c'(p) = 3$. Recalling the previous observations we see that $c'$ satisfies the necessary assumptions in Lemma 5.3, and we apply it (formally changing the coordinates first) to finish the proof.

Otherwise, any two signatures must coincide on at least two coordinates. In particular, the only possible ones are $(\cdot,b,c),(a,\cdot,c),(a,b,\cdot)$ where the dot can be any member of $[k+1]$. If $a \neq b,c$, we have $\sigma(z+(1,0,-1)) = (a,b,a)$ and $\sigma(z+(1,-1,0)) = (a,a,c)$. Thus $\sigma(z+(2,-1,-1))^{(2)} = \sigma(z+(2,-1,-1))^{(3)} = a$, which is impossible. Similarly $b \in \{a,c\}$, $c \in \{a,b\}$, hence $a = b = c$, and so $A_a$ is a 2-way set with the desired diameter. ∎

By Lemma 5.4, there is a 2-way set $T$ with $\operatorname{diam} T \leq K_{15,2}\lambda D$. Setting $s = 3$, we have $K_{15,s-1} = K_{14,2s-1} = K_{13,2^2 s-1} = \cdots = K_{2,2^{13}s-1} = 2^{14}\cdot 3+6 = 49158$, as desired. ∎

We say that a set of points of the grid $Q$ is a *quarter-plane* if there are distinct $i_1, i_2 \in [3]$ such that $Q = \{t + ae_{i_1} + be_{i_2} : a,b \in \mathbb{N}_0\}$ for some point $t$.

Proposition 5.6. *Suppose $\lambda < 1/4$ and there is a 2-way set $S$ of diameter $D$. Provided $m_1 = \inf_{s\in S} \rho(s) > D(2+\lambda)$, $S$ contains a quarter-plane subset $Q$.*

*Proof.* Without loss of generality, we can assume that $S$ has a point $p$ such that $S \subset \langle p \rangle_3$, and all points $s$ of $S$ except $p$ have a unique point $s'$ such that $s \in N(s')$. This is because we can always pick such a subset of $S$, and it suffices to prove the statement in that situation. We say that such a $k$-way set is *spreading* (from $p$).

Case 1: *For all $i \in [3]$, there is $x$ with $x + e_i$ not in $S$.* Let $x,y \in S$ be points such that $x+e_i, y+e_j \notin S$ with $i \neq j$. Take $k$ so that $\{i,j,k\} = [3]$. Then if $x \overset{i}{\frown} y$, by the triangle inequality we have $m_1 \leq d(x,x+e_i) \leq d(x,y) + d(y,y+e_i) + d(y+e_i,x+e_i) \leq (2+\lambda)D$, a contradiction. Similarly we reject $x \overset{j}{\frown} y$, hence $x \overset{k}{\frown} y$. Thus, if we define $A_l = \{s \in S : s = t + e_l$ for some $t \in S\}$, these are all of diameter $\leq 2\lambda D$.

Suppose $A_1$ and $A_2$ are disjoint. Consider $x$ such that $x + e_3 \notin S$. If $x + e_1 + e_2 \in S$, it is in both $A_1$ and $A_2$, which is impossible. Hence, $x + e_1 + e_3, x + e_2 + e_3 \in S$, thus $x + e_3 + e_1 + e_2$ is not in $S$, so we can repeat the argument to get all the $x + (1, 0, n)$ and $x + (0, 1, n)$ in $S$. Now, by the triangle inequality we must have $x + (1, 0, n) \overset{3}{\frown} x + (0, 1, n)$, $x + (1, 0, n) \overset{3}{\frown} x + (0, 1, n + 1)$, for all non-negative $n$, so $(x + (1, 0, n))_{n \geq 1}$ is Cauchy, a contradiction. Thus $A_1, A_2$ intersect, and similarly $A_1$ and $A_2$ intersect $A_3$; therefore, the union $T$ of these, which is 2-way (as every point of $S$ belongs to some $A_i$, except the starting one), has $\operatorname{diam} T \leq 4\lambda D$.

CASE 2: *there is $i$ such that for any $x \in S$, $x + e_i$ is in $S$.* Without loss of generality, we assume $i = 3$. Pick any $x_0$ in $S$ and set $a = (x_0)^{(3)}$. Starting from $x_0$ we can form a sequence $(x_n)_{n \geq 0}$ such that $\{x_{n+1}\} = S \cap \{x_n + e_1, x_n + e_2\}$. Suppose we have $x, y$ among these such that $x + e_1, y + e_2 \in S$. Hence, $x + (1, 0, n), x + (0, 0, n), y + (0, 1, n), y + (0, 0, n)$ belong to $S$ for all non-negative $n$, thus $x + (0, 1, n), y + (1, 0, n)$ are never elements of $S$. Now, contracting the pairs $x + (0, 0, n), y + (0, 0, n)$ and $x + (0, 0, n + 1), y + (0, 0, n)$ gives a 1-way Cauchy sequence as in Case 1. If there are no such $x, y$ then $S$ contains a quarter-plane.

Therefore, if we ever get into Case 2, we are done. Hence, let $S_1 = S$. Then by Case 1, we have a 2-way $S_2$ subset of $S_1$, which we can assume to be spreading, by the same arguments as those for $S$. It also satisfies the necessary hypothesis of this claim, so we can apply Case 1 once more to obtain a 2-way set $S_3 \subset S_2$. Proceeding in the same manner, we obtain a sequence of spreading 2-way sets $S_1 \supset S_2 \supset \cdots$ whose diameters tend to zero, so just pick a point in each of them, and then find a 1-way Cauchy sequence containing these to reach a contradiction. ■

PROPOSITION 5.7. *Let $\{i_1, i_2, i_3\} = [3]$. Suppose we have a quarter-plane $S = \{\alpha + me_{i_1} + ne_{i_2} : m, n \in \mathbb{N}_0\}$ of diameter $D$, and let $R = \inf_S \rho$. Provided $\lambda < 1/3$ and $D(1 - \lambda^2) < (1 - 4\lambda)R$, there is a 3-way set of diameter at most $2\lambda\big(\frac{2}{1-\lambda}D + \frac{1+2\lambda}{1-\lambda}R\big)$.*

*Proof.* Without loss of generality, $i_3 = 1$. Observe that for any $s \in S$ we must have $\rho(s) = d(s, s + e_1)$. The reason is that both $s + e_2, s + e_3$ are in $S$ and so $d(s, s + e_2), d(s, s + e_3) \leq \operatorname{diam} S = D$, but $\max\{d(s, s + e_1), d(s, s + e_2), d(s, s + e_3)\} = \rho(s) \geq R > D$.

Let $x_n \in S$ with $\rho(x_n) < (1 + 1/n)R \leq 2R$. As $\lambda < 1/2$, we must have $x_n \overset{1}{\frown} x_n + e_1$. Furthermore, suppose $i \neq 1$ contracts $y, x_n + e_1$ for some $y$ in $S$. Thus $x_n + e_i \in S$ and so $\rho(x_n + e_i) = d(x_n + e_i, x_n + e_1 + e_i)$. Then, by the triangle inequality, $\rho(x_n + e_i) = d(x_n + e_i, x_n + e_1 + e_i) \leq d(x_n + e_i,$

$y+e_i)+d(y+e_i,x_n+e_1+e_i) \leq \lambda d(y,x_n+e_1)+D \leq \lambda 2R+(1+\lambda)D < R$, therefore $y \overset{1}{\frown} x_n+e_1$. Hence $\rho(y) \leq d(y,x_n)+d(x_n,x_n+2e_1)+d(x_n+2e_1, y+e_1) \leq D+R(1+1/n)(1+\lambda)+\lambda(D+R(1+1/n))$ for all $n$, hence $\rho(y) \leq D(1+\lambda)+R(1+2\lambda) < 2R$.

Now we claim that for all $y \in S$ and all $k \geq 1$, we have $y \overset{1}{\frown} y+ke_1$, which we prove by induction on $k$. For $k=1$, this is immediate as otherwise there is $y$ with $\rho(y) < 2\lambda R < R$.

Suppose the claim holds for some $k \geq 1$. Then for any $y$ and $l \leq k+1$ we have

$$\begin{aligned} d(y,y+le_1) &\leq d(y,y+e_1)+d(y+e_1,y+le_1) \leq \rho(y)+\lambda d(y,y+(l-1)e_1) \\ &\leq \cdots \leq \rho(y)(1+\lambda+\cdots+\lambda^{l-1}) < \rho(y)/(1-\lambda). \end{aligned}$$

Also, $d(y,y+le_1) \geq d(y,y+e_1)-d(y+e_1,y+le_1) \geq \rho(y)-\lambda d(y,y+(l-1)e_1) > \rho(y)\frac{1-2\lambda}{1-\lambda}$. As $\lambda < 1-2\lambda$, we see that 1 always contracts $y, y+(k+1)e_1$. In particular, $\rho(y)\frac{1-2\lambda}{1-\lambda} < d(y,y+ke_1) < \rho(y)/(1-\lambda)$.

Fix any $x \in S$. Now, suppose $x \overset{i}{\frown} y+ke_1$ for some $i \neq 1$. Then

$$\begin{aligned} R\frac{1-2\lambda}{1-\lambda} &\leq \rho(y+e_i)\frac{1-2\lambda}{1-\lambda} < d(y+e_i,y+e_i+ke_1) \\ &\leq d(y+e_i,x+e_i)+\lambda(d(x,y)+d(y,y+ke_1)) \\ &\leq D(1+\lambda)+\lambda\rho(y)/(1-\lambda) < (1+\lambda)D+\frac{2\lambda}{1-\lambda}R, \end{aligned}$$

a contradiction. Hence, by looking at distance from $x+e_1$, we see that $\operatorname{diam}\{\alpha+(a,b,c) : a \geq 2, b,c \geq 0\} \leq 2\lambda(D+D(1+\lambda)/(1-\lambda)+R(1+2\lambda)/(1-\lambda))$, as required. ∎

In order to make the calculations easier, we use the following corollary.

Corollary 5.8. *Suppose we have a 2-way set $S$ of diameter $D$, and $R=\inf_{s\in S}\rho(s)$. Provided $\lambda < 1/9$ and $R > (2+\lambda)D$, there is a 3-way set of diameter at most $6\lambda R$.*

*Proof.* Firstly, apply Proposition 5.6 to find a quarter-plane inside the given 2-way set. Since $R(1-4\lambda) > R/(2+\lambda) > D > (1-\lambda^2)D$ and $\lambda < 1/3$, we can apply Proposition 5.7 to obtain a 3-way set of diameter at most $2\lambda\big(\frac{2}{1-\lambda}D+\frac{1+2\lambda}{1-\lambda}R\big)$. An easy calculation shows that this expression is smaller than $\lambda(5D+3R) < 6\lambda R$. ∎

Recall that we defined $\mu = \inf_x \rho(x)$, where $x$ ranges over the whole grid. Recall also that $\mu > 0$ by Proposition 4.4.

Proposition 5.9. *Given $K$, provided $1 > (2+\lambda)\lambda K C_1$, all 3-way sets have diameter greater than $K\mu$.*

*Proof.* This is clear for $K < 1$, so assume $K \geq 1$ and in particular $\lambda < 1/9$. Suppose $T$ is a 3-way set of diameter $D \leq K\mu$. By Proposition 5.1, there is a 2-way set $S \subset T$ with $\operatorname{diam} S \leq \lambda C_1 K\mu$. Therefore by Corollary 5.8, as $\lambda C_1 K\mu < \mu/(2+\lambda)$, we have a 3-way set of diameter not greater than $6\lambda K\mu < \mu$, a contradiction. ∎

PROPOSITION 5.10. *Given $K$, provided $\lambda < 1/9, 1/(3K)$, all 2-way sets have diameter greater than $\lambda K\mu$.*

*Proof.* Pick a set $S_0$ contradicting the conclusion. Since $K\lambda\mu(2+\lambda) < \mu$, we have a 3-way set $T_1$ with $r_1 = \operatorname{diam} T_1$ by Corollary 5.8. Now take a 2-way subset $S_1 \subset T_1$ with $\operatorname{diam} S_1 \leq K\lambda\mu$, so we have a 3-way set $T_2$ of diameter not greater than $r_2 = 6\lambda r_1$. Repeating this argument, for each $k \geq 1$ we can find a 3-way set $T_k$ with diameter bounded by $r_k$, where $r_{k+1} = 6\lambda r_k$. But then we must have $r_k < \mu$ for some $k$, a contradiction. ∎

Note that the only way for a 2-way subset not to have elements in every $\langle (n,n,n)\rangle_3$ is to be contained in a union of finitely many quarter-planes.

**6. Finite contractive structures.** Recall the proofs of Proposition 5.1 and Lemma 4.2. There we fixed a finite set $S$ of points, and then contracted various points with points in $S$ to obtain $k$-way sets. The following claims pursue this approach further. In this subsection, we also show that we cannot have some configurations of points.

PROPOSITION 6.1. *Suppose $K \geq 1$ and $\lambda < 1/(24K)$. Then we cannot have a point $x_0$ in the grid with $\rho(x_0) \leq K\mu$ such that $N(x_0) = \{x_1, x_2, x_3\}$ with* $\operatorname{diam}(N(x_0) \cup \{x_i + e_j : i, j \in [3],\ i \neq j\}) \leq \lambda K\mu$.

When using this proposition (and to obtain a contradiction in the proofs to follow), we say that we are applying Proposition 6.1 to $(x_0; x_1, x_2, x_3)$ with constant $K$.

*Proof of Proposition 6.1.* Suppose we do have points as described in the statement. By Lemma 4.2, in each 3-way set we have a point $t$ such that $d(t, x_0) \leq 2K\mu$. Consider the contractions of $t$ with $x_0, x_1, x_2, x_3$; our main aim is to obtain a 2-way set of a small diameter and then to use Proposition 5.10 to reach a contradiction.

Observe that from the assumptions of the proposition, for any $\{i,j,k\} = [3]$, we have

$$\begin{aligned}\max\{d(x_i, x_i + e_i), d(x_i, x_i + e_j), d(x_i, x_i + e_k))\}\\ = \rho(x_i) \geq \mu > \lambda K\mu \geq \max\{d(x_i, x_i + e_j), d(x_i, x_i + e_k)\}.\end{aligned}$$

Thus $\rho(x_i) = d(x_i, x_i + e_i)$ for all $i \in [3]$.

Suppose first that $t \overset{i}{\frown} x_i$ for all $i \in [3]$. Take $i$ so that $t \overset{i}{\frown} x_0$. Then

$$\begin{aligned}\rho(x_i) = d(x_i, x_i + e_i) &\le d(x_i, x_0 + e_i) + d(x_0 + e_i, t + e_i) + d(t + e_i, x_i + e_i)\\ &\le \lambda K\mu + \lambda d(x_0, t) + \lambda d(x_i, t) \le 6\lambda K\mu < \mu,\end{aligned}$$

which is impossible.

Thus, there are distinct $i, j \in [3]$ with $t \overset{j}{\frown} x_i$. If $j$ were to contract $t, x_j$, we would get

$$\begin{aligned}\rho(x_j) &= d(x_j, x_j + e_j)\\ &\le d(x_j, x_i + e_j) + d(x_i + e_j, t + e_j) + d(t + e_j, x_j + e_j)\\ &\le \lambda K\mu + \lambda d(x_i, t) + \lambda d(t, x_j) \le 7\lambda K\mu < \mu,\end{aligned}$$

which is impossible. Therefore, for some $k \neq j$, we have $t \overset{k}{\frown} x_j$. In particular, $d(t+e_j, x_1+e_2) \le d(t+e_j, x_i+e_j)+d(x_i+e_j, x_1+e_2) \le \lambda d(t, x_i)+\lambda K\mu \le 4\lambda K\mu$, and in a similar fashion $d(t + e_k, x_1 + e_2) \le 4\lambda K\mu$. Furthermore by the triangle inequality, both $t+e_j$ and $t+e_k$ are at most $K\mu+4\lambda K\mu \le 2K\mu$ away from $x_0$, so we can apply to these points the same arguments as we did for $t$. Hence, we obtain a bounded 2-way set of diameter at most $4K\mu$. But, considering all the points of the 2-way set except $t$ and their distance from $x_1 + e_2$, this is actually a 2-way set of diameter at most $8\lambda K\mu$, and we have such a set in every 3-way subset of the grid. Now, we apply Proposition 5.10 to obtain a contradiction, since $\lambda < 1/(24K)$ and $K \ge 1$. ∎

Proposition 6.2. *Given $K \ge 1$, provided $\lambda < 1/(78K), 1/(13C_1)$, there is no $x$ such that $\rho(x) \le K\mu$, but $\rho(x + e_i) > 7K\mu$ for all $i \in [3]$.*

Sometimes we refer to a pair of points $a, b$ in the grid as the *edge* $a, b$, and by the *length* of the edge $a, b$ we mean $d(a, b)$. The points $a$ and $b$ are the *endpoints* of the edge $a, b$.

*Proof of Proposition 6.2.* Suppose there is such an $x$. Consider the contractions of $x + e_i, x + e_j$ for $i \neq j$ and suppose that two such pairs are contracted by the same $k$. Thus $\operatorname{diam}\{x+e_k+e_1, x+e_k+e_2, x+e_k+e_3\} \le 4\lambda K\mu$. Now, contract $x, x+e_k$ to get $\rho(x+e_k) \le (2+5\lambda)K\mu < 3K\mu$, a contradiction. So, the pairs described above must be contracted in different directions. Further, we can make a distinction between the *short* edges of the form $a, a+e_i$ and the *long* edges $a + e_i, a + e_j$, where $a$ is any point of the grid and $i, j$ are distinct integers in [3] ([3]). For every such long edge we have a unique short *orthogonal* edge $a, a + e_k$ where $\{i, j, k\} = [3]$. Observe that if we have a short edge and a long edge in $\{x\} \cup N(x)$ which are not orthogonal, but both contracted by some $i$, then we must have another such pair, contracted

([3]) Note that ‘short’ and ‘long’ have nothing to do with the length of an edge previously defined, but actually just describe how these edges appear in the figures in the proofs.

by some $j \neq i$. One can show this by looking at the short edge $e$ which is orthogonal to the long one in a given pair of edges contracted by $i$.

If we write $[3] = \{i, j, k\}$, then $j$ contracts one long edge, and so does $k$. But now consider the orthogonal short edge $e$ described above. It cannot be contracted by $i$, for otherwise $\rho(x + e_i)$ is too small. Thus, it gives another pair as desired. Having shown this, we have two cases, with at least two such pairs (i.e. non-orthogonal short and long edges contracted in the same direction), or no such pairs.

CASE 1: *There are at least two such pairs.* In Figure 2, we show the possibilities for contractions; the edges shown as dashed lines have length at most $3K\lambda\mu$. Here we actually consider possible contractions and then

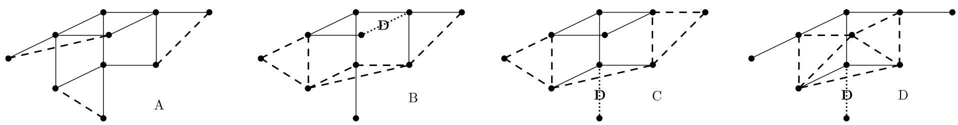


Fig. 2. Case 1

apply triangle inequalities. This way, we obtain very few possible diagrams. We only list the possible configurations up to rotation or reflection, as the same arguments go through. In diagram A, by short edge contractions we get $\rho(x + e_i) \leq 3K\mu$ for some $i$, which gives the claim. Dotted lines marked with $\mathbf{D}$ will be called *D-lines*. On the other hand, in diagrams B, C and D we claim that either we are done, or the dotted lines marked with $\mathbf{D}$ are of length at most $9\lambda K\mu$. Once this is established, we have $\rho(x+e_i) \leq 3K\mu$ for some $i$, resulting in a contradiction.

For each $i \in [3]$, let $x_i \in N(x)$ be such that $x_i + e_i$ is not an endpoint of a long edge shown as a dashed line. By Lemma 4.2, in each 3-way set we have a point $t$ with $d(x, t) \leq 2\rho(x)/(1 - \lambda) \leq 3K\mu$. Observe that from the diagrams we have $d(x_i, x_i + e_j) \leq (2 + 6\lambda)K\mu$ whenever $i \neq j$. Further, we cannot have $x_i \stackrel{i}{\frown} t$ for all $i$, since otherwise we get a contradiction by considering the contraction $x \stackrel{j}{\frown} t$. If $x + e_j$ is an endpoint of an edge shown as a D-line, and $x + e_j + e_l$ is the other endpoint, we have $x + e_l = x_k$, hence $d(x + e_l + e_j, x + e_j) \leq \lambda(d(x + e_j, t) + d(t, x)) \leq 7\lambda K\mu$, which is impossible. Thus, $x + e_j$ is not on a D-line edge, which gives $\rho(x_j) = d(x_j, x_j + e_j) \leq d(x_j, x) + d(x, x + e_j) + d(x + e_j, t + e_j) + d(t + e_j, x_j + e_j) \leq (2 + 7\lambda)K\mu$. Previous arguments imply that we must have $i \neq j$ with $x_i \stackrel{j}{\frown} t$, and hence $x_j \stackrel{j}{\not\frown} t$ (otherwise $\rho(x_j) \leq (2+14\lambda)K\mu$), so given such a $t$ we get $t+e_a, t+e_b$, $a \neq b$, at most $13\lambda K\mu$ away from $x_1 + e_2$ and at most $3K\mu$ away from $x$, by the triangle inequality. Hence, in every $\langle z\rangle_3$ we get a 2-way subset of diameter not greater than $\lambda 26K\mu$, yielding a contradiction, due to $\lambda < 1/(78K)$ and

Proposition 5.10. Hence, edges shown as D-lines satisfy the desired length condition.

Case 2: *There are no such pairs.* The possible cases up to rotation or reflection are shown in Figure 3, where the short edges shown as dashed

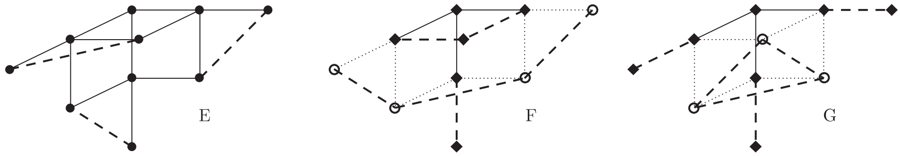


Fig. 3. Case 2

lines are of length at most $\lambda K\mu$, while the long ones are of length $2\lambda K\mu$. As above, diagram A gives $\rho(x+e_i) \leq 3K\mu$ immediately. On the other hand, we can consider points shown as black squares and empty circles in the other two possibilities. We call a point *black* if it is a black square, and *white* if it is an empty circle. In the course of the proof, we shall colour more points black and white. Let $r$ be the minimal length of dotted edges in Figure 3, and $r'$ the maximal one. Then $r' \leq r + 2K\mu + 6\lambda K\mu$. Furthermore, given $i \in [3]$ we have $7K\mu < \rho(x+e_i) \leq r' < r + 3K\mu$, so $r > 4K\mu$.

Consider $t$ such that $d(x,t) \leq 2r$. Let $j$ contract $x+e_i, t$, so we have

$$\begin{aligned} d(t+e_j, x) &\leq d(t+e_j, x+e_i+e_j) + d(x+e_i+e_j, x+e_i) + d(x+e_i, x) \\ &\leq \lambda d(t, x+e_i) + \rho(x+e_i) + \rho(x) \\ &\leq \lambda(d(t,x) + d(x, x+e_i)) + r' + K\mu \\ &\leq 2\lambda r + \lambda K\mu + r + 2K\mu + 6\lambda K\mu + K\mu \\ &\leq (1+2\lambda)r + (3+8\lambda)K\mu < (1+2\lambda+(3+8\lambda)/4)r \leq 2r, \end{aligned}$$

since $\lambda < 1/16$. Similarly if $j$ contracts $x, t$ we have $d(t+e_j, x) \leq d(t+e_j, x+e_j) + d(x+e_j, x) \leq 2\lambda r + K\mu \leq 2r$ as well. Further, observe that if $t+e_j$ is the result of contraction as before, then we have a point $a \in N(x) \cup \{x + e_i + e_j : i, j \in [3]\}$ with $d(t+e_j, a) \leq \lambda(2r + K\mu)$. Restrict our attention to the black points (shown as black squares) and white points (empty circles) in Figure 3. We have $\operatorname{diam}\{\text{white points}\} \leq 6\lambda K\mu$, $\operatorname{diam}\{\text{black points}\} \leq (2+2\lambda)K\mu$ and the distance from any white point to any black point is at least $r - K\mu - 4\lambda K\mu$. Take a point $t$ at most $2r$ away from $x$ (note that by Lemma 4.2 such a point exists in every 3-way set). Consider contractions with $\{x\} \cup N(x)$ and suppose that $t+e_i, w$ and $t+e_i, b$ are results of these operations, where $w$ is white and $g$ is black. Then, by the triangle inequality, $r - K\mu - 4\lambda K\mu \leq d(w,b) \leq d(w, t+e_i) + d(t+e_i, b) \leq 2\lambda(2r + K\mu)$, a contradiction. For any given $i \in [3]$ let $x_i$ stand for the point of $N(x)$ such that $d(x_i, x_i + e_i) \leq \lambda K\mu$, thus $N(x) = \{x_1, x_2, x_3\}$. Let $t \overset{i}{\frown} x$. Then take

$j \in [3]$ distinct from $i$. We see that $x_j + e_i$ is white, while $x + e_i$ is black, hence $i$ does not contract $t, x_j$. Let $k \neq i$ contract $t, x_j$ and let $l$ be such that $\{i, j, l\} = [3]$. If $k = j$ then similarly we see that $x_l \overset{l}{\frown} t$, while in the other case $k = l$ and $x_l \overset{j}{\frown} t$. Hence, in conjunction with the previous arguments, we obtain a 3-way set of diameter at most $4r$.

Furthermore, recall that given pairs $t + e_i, p$ and $t + e_i, q$, which are results of contracting $t$ with $x$ or a point in $N(x)$, we must have $p$ and $q$ of the same colour. As each of $t + e_1$, $t + e_2$ and $t + e_3$ is a result of such a contraction, we can extend the 2-colouring of the points in diagrams F and G to all points of $\langle x \rangle_3$, namely $c : \langle x \rangle_3 \to \{\text{black}, \text{white}\}$, with $t + e_i$ being coloured black if $p$ described above is black in the original colouring, and white otherwise.

Now, the distance between any black point and any white point in the extended colouring is at least $r - K\mu - 4\lambda K\mu - 2\lambda(2r + K\mu) = (1 - 4\lambda)r - (1 + 6\lambda)K\mu$. Recall Proposition 5.1, which guarantees the existence of a 2-way set $S \subset \langle x \rangle_3$ of diameter at most $4\lambda C_1 r$, from which we infer that $S$ is monochromatic, since $4\lambda C_1 r < (1 - 4\lambda)r - (1 + 6\lambda)K\mu$.

CASE 2.1: *$S$ is black.* Consider any $t \in \langle x \rangle_3$ which has two black neighbours $t+e_{i_1}, t+e_{i_2}$, where $i_1 \neq i_2$. Then, letting $i_3$ be the third direction, that is, $[3] = \{i_1, i_2, i_3\}$, we have $t \overset{i_3}{\frown} x_{i_3}$, since the points of $N(x_{i_3}) \setminus \{x_{i_3} + e_{i_3}\}$ are white. Hence, $N(t)$ is black for any $t \in S$. Furthermore, by the same arguments, $t \overset{i}{\frown} x_i$ for all $i \in [3]$. Now, if $t$ is in $S$, and without loss of generality so are $t + e_1, t + e_2$, then $N(t + e_1), N(t + e_2)$ are black, so at least two elements of $N(t + e_3)$ are black too, implying that $N(t + e_3)$ is black. But now looking at $t$ gives $t \overset{3}{\frown} x_3$, and similarly looking at $t + e_1, t + e_2, t + e_3$ tells us that 3 contracts $t + e_1, t + e_2, t + e_3$ with $x_3$.

Let $s$ be the distance from such a $t$ to $x$. Then, for all $i \in [3]$, we have a black point $p$ in $\{x\} \cup N(x)$, which is contracted with $t$ by $i$, so that $p + e_i$ is black as well. Now, by the triangle inequality,

$$\begin{aligned} d(x, t + e_i) &\leq d(x, p) + d(p, p + e_i) + d(p + e_i, t + e_i) \\ &\leq d(x, p) + d(p, p + e_i) + \lambda d(p, t) \\ &\leq \lambda d(t, x) + (1 + \lambda) d(x, p) + d(p, p + e_i) \leq \lambda s + (2 + \lambda)(K\mu). \end{aligned}$$

As in the proof of Lemma 4.2, there is $t \in S$ such that $d(t, x) < 3K\mu$. From the estimates just made, $d(t + e_i, x) < 3K\mu$ for all $i \in [3]$. Without loss of generality $t, t + e_1, t + e_2 \in S$. Recalling that this implies $d(t + e_3, x_3)$, $d(t + e_3 + e_i, x_3) < 3\lambda K\mu$ where $i$ takes all the values in [3] shows that $\rho(t + e_3) < 6\lambda K\mu$, a contradiction.

CASE 2.2: *$S$ is white.* If $t \in S$, then the point in $N(t) \setminus S$ is black, by contracting $t, x$. Hence, by Proposition 5.8, we must have a 3-way set inside

$\langle x\rangle_3$ of diameter at most $6\lambda r$, since $(1-4\lambda)r-(1+6\lambda)K\mu > (1-4\lambda)r-(1+6\lambda)r/4 > 2r/3 > (2+\lambda)\lambda 4C_1 r$, since $\lambda < 1/(13C_1)$. But such a set has at least one black point, so it must have only black points, and we have a contradiction as in Case 2.1. ∎

PROPOSITION 6.3. *Given $K \geq 1$, provided $\lambda < 1/(41KC_1)$, there is no $x$ with $\rho(x) \leq K\mu$ and* $\operatorname{diam} N(x) \leq \lambda K\mu$.

*Proof.* Suppose we have such an $x$. We start by observing that two pairs of the form $x+e_i, x+e_j$ cannot be contracted by the same $k$. Otherwise, since $\operatorname{diam} N(x) \leq \lambda K\mu$, after an application of the triangle inequality, we also have $N(x+e_k) \leq 2\lambda^2 K\mu$. Let $t$ be such that $x \overset{t}{\frown} x+e_k$. Then

$$\begin{aligned}
&d(x+e_k, x+2e_k)\\
&\qquad \leq d(x+e_k, x+e_t) + d(x+e_t, x+e_k+e_t) + d(x+e_k+e_t, x+2e_k)\\
&\qquad \leq \operatorname{diam} N(x) + \lambda d(x, x+e_k) + \operatorname{diam} N(x+e_k)\\
&\qquad \leq \lambda K\mu + \lambda K\mu + 2\lambda^2 K\mu < 4\lambda K\mu.
\end{aligned}$$

But then, for any $s \in [3]$, we have $d(x+e_k, x+e_k+e_s) \leq d(x+e_k, x+e_k+e_k) + \operatorname{diam} N(x+e_k) < 6\lambda K\mu < \mu$, implying that $\rho(x+e_k) < \mu$, which is impossible.

Thus, all three pairs of the form $x+e_i, x+e_j$ are contracted in different directions, hence we can distinguish the following cases (up to symmetry):

CASE 1. The results of contractions are shown as dashed lines in Figure 4, diagram A. It is not hard to see that after contracting all pairs $x, x+e_i$, we get $\rho(x+e_j) < \mu$ for some $j$, a contradiction.

CASE 2. The results of contractions are shown as dashed lines in Figure 4, diagram B. By considering contractions of pairs $x, x+e_i$, we get either $\rho(x+e_j) < \mu$ for some $j$, or diagrams B.1, B.2 in Figure 4, where dashed lines now indicate lengths at most $3\lambda K\mu$.

CASE 3. The results of contractions are shown as dashed lines in Figure 4, diagram C. By considering contractions of pairs $x, x+e_i$, we get either $\rho(x+e_j) < \mu$ for some $j$, or diagrams C.1, C.2 in Figure 4, where now a dashed line implies length at most $3\lambda K\mu$.

Firstly, we will use Proposition 6.1 to reject B.1 and C.1. In these two diagrams, for each $i \in [3]$ we can find a unique $x_i \in N(x)$ such that $\rho(x_i) = d(x_i, x_i+e_i)$. Then $\operatorname{diam}(N(x) \cup \{x_i+e_j : i,j \in [3], i \neq j\}) \leq 15\lambda K\mu$. Also $\rho(x) \leq K\mu$, hence we can apply Proposition 6.1 to $(x; x_1, x_2, x_3)$ with constant $15K$ to obtain a contradiction, since $\lambda < 1/(360K)$.

Observe that in diagrams B.2 and C.2 we can denote $N(x) = \{x_1, x_2, x_3\}$ so that $d(x_i, x_i+e_i) \leq 3\lambda K\mu$. By Proposition 6.2, $\rho(x_i) \leq (7+7\lambda)K\mu$

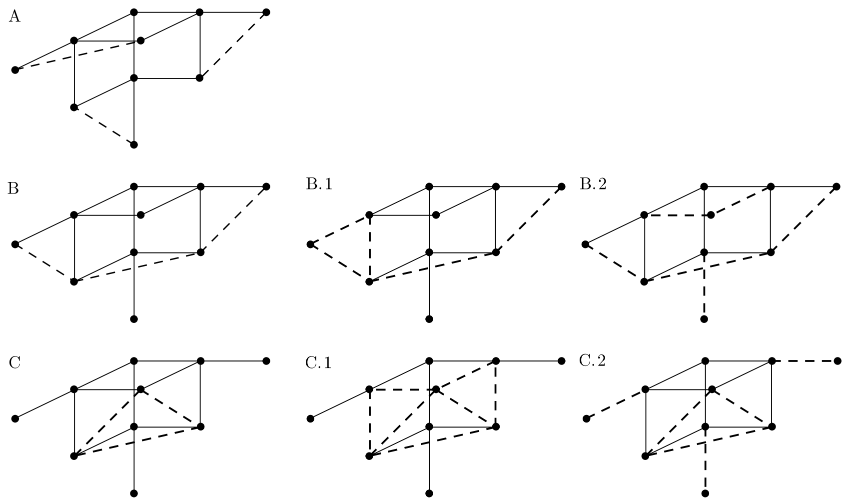


Fig. 4. Possible distances in the proof of Proposition 6.3

for all $i \in [3]$, as $\lambda < 1/(78K), 1/(13C_1)$. Now, start from a point $t$ with $d(t,x) \le 2\rho(x)/(1-\lambda) \le 2K\mu/(1-\lambda) \le 10K\mu$, which exists by Lemma 4.2. Take any $p \in \{x\} \cup N(x)$ and contract with $t$. If $t \overset{i}{\frown} p$, then

$$\begin{aligned} d(t+e_i,x) &\le d(t+e_i,p+e_i) + d(p+e_i,p) + d(p,x) \\ &\le \lambda d(t,p) + d(p+e_i,p) + d(p,x) \\ &\le \lambda(d(t,x)+d(x,p)) + d(p+e_i,p) + d(p,x) \\ &\le \lambda 10K\mu + (7+7\lambda)K\mu + (1+\lambda)K\mu \le 10K\mu. \end{aligned}$$

Contract such a point $t$ with $x$ by some $i$. Write $[3] = \{i,j,k\}$ and consider the contraction of $t, x_j$. It is not $i$ that contracts this couple of points, as otherwise $\rho(x_j) < \mu$. If it is $j$, then we can see that $x_k \overset{k}{\frown} t$, and if it is $k$, then $x_k \overset{j}{\frown} t$. Hence, all the points of $N(t)$ are at most $10K\mu$ away from $x$, so we can repeat the argument to obtain a bounded 3-way set of diameter at most $20K\mu$. However, we get a contradiction by Proposition 5.9, since $1 > 41KC_1\lambda$. ∎

Proposition 6.4. *Given $K \ge 1$, suppose we have $x_0, x_1, x_2, x_3$ such that* $\operatorname{diam}\{x_i + e_j : i,j \in [3],\ i \ne j\} \le \lambda K\mu$. *Furthermore, suppose $\rho(x_0) \le K\mu$ and $d(x_0,x_i) \le K\mu$ for $i \in [3]$. Let $\{a,b,c\} = [3]$. Provided $\lambda < 1/(820C_1K)$, whenever there is a point $x$ which satisfies $d(x+e_a, x+e_b) \le \lambda K\mu$ and $d(x,x_0) \le K\mu$, then $d(x+e_c, x_c+e_c) \le 16\lambda K\mu$.*

Note that this is Proposition 3.1 in the overview of the proof. When using this proposition, we say that we are applying Proposition 6.4 to $(x_0; x_1, x_2, x_3; x)$ with constant $K$.

*Proof of Proposition 6.4.* Suppose the contrary. Without loss of generality, we may assume $a = 1$, $b = 2$, $c = 3$. Let us first establish $d(x, x + e_1)$, $d(x, x + e_2) \leq 3K\mu$. As $d(x + e_3, x_3 + e_3) > 16\lambda K\mu$, either 1 or 2 contracts $x, x_3$. Similarly, we cannot have $x \overset{3}{\frown} x_0$ and $x_0 \overset{3}{\frown} x_3$ simultaneously. If $x \overset{3}{\frown} x_0$ then $x_0 \overset{i}{\frown} x_3$ for some $i \in [2]$, and recall $x \overset{j}{\frown} x_3$ for some $j \in [2]$, so

$$\begin{aligned} d(x, x + e_1) &\leq d(x, x_0) + d(x_0, x_0 + e_i) + d(x_0 + e_i, x_3 + e_i) \\ &\quad + d(x_3 + e_i, x_3 + e_j) + d(x_3 + e_j, x + e_j) + d(x + e_j, x + e_1) \\ &\leq K\mu + K\mu + \lambda K\mu + \lambda K\mu + 2\lambda K\mu + \lambda K\mu < 3K\mu, \end{aligned}$$

and in the same way $d(x, x + e_2) < 3K\mu$. On the other hand, if $x \overset{i}{\frown} x_0$ for $i \in [2]$ we get $d(x, x + e_j) \leq d(x, x_0) + d(x_0, x_0 + e_i) + d(x_0 + e_i, x + e_i) + d(x + e_i, x + e_j) \leq K\mu + K\mu + \lambda K\mu + \lambda K\mu < 3K\mu$ for any $j \in [2]$.

Similarly, observe that $\operatorname{diam}\{x_1, x_2, x_3\} \cup \{x_i + e_j : i, j \in [3], i \neq j\} \leq 5K\mu$. This certainly holds if there are distinct $i, j \in [3]$ with $x_0 \overset{j}{\frown} x_i$, as then $d(x_i, x_i + e_j) \leq d(x_i, x_0) + d(x_0, x_0 + e_j) + d(x_0 + e_j, x_i + e_j) \leq (2 + \lambda)K\mu$, and the claim about the diameter follows. Hence, suppose that for all $i \in [3]$ contractions are $x_0 \overset{i}{\frown} x_i$. Then we cannot have $x_0 \overset{3}{\frown} x$, so suppose $x_0 \overset{j}{\frown} x$ and $x \overset{k}{\frown} x_3$, where $j, k \in [2]$. Now, by the triangle inequality,

$$\begin{aligned} d(x_3 + e_k, x_3) &\leq d(x_3 + e_k, x + e_k) + d(x + e_k, x + e_j) + d(x + e_j, x_0 + e_j) \\ &\quad + d(x_0 + e_j, x_0) + d(x_0, x_3) \\ &\leq 2\lambda K\mu + \lambda K\mu + \lambda K\mu + K\mu + K\mu = (2 + 4\lambda)K\mu, \end{aligned}$$

so once again we have the desired bound on the diameter.

Now, by Lemma 4.2, in every 3-way set we have a point $t$ with $d(t, x_0) \leq 7K\mu$. Suppose that for some distinct $i, j \in [3]$ we have $t \overset{i}{\frown} x_i$ and $t \overset{i}{\frown} x_j$. Then $d(x_i+e_i, x_i+e_k) \leq d(x_i+e_i, t+e_i)+d(t+e_i, x_j+e_i)+d(x_j+e_i, x_i+e_k) \leq 17\lambda K\mu$ for any $k \neq i$. Hence $\operatorname{diam} N(x_i) \leq 17\lambda K\mu$. However, contract $x_0, x_i$ to see that $\rho(x_i) \leq (2+18\lambda)K\mu < 17K\mu$. But we can apply Proposition 6.3, as $\lambda < (17.41KC_1)$, to obtain a contradiction. Hence, we cannot have $x_i \overset{i}{\frown} t$ and $x_j \overset{i}{\frown} t$.

Suppose that for every such $t$ we have distinct $i, j \in [3]$ with $t \overset{i}{\frown} x_j$. Then, by the previous observation, we see that $t \overset{k}{\frown} x_i$ for some $k \neq i$. Hence $d(t+e_i, x_0) \leq d(t+e_i, x_j+e_i)+d(x_j+e_i, x_j)+d(x_j, x_0) \leq 8\lambda K\mu+6K\mu \leq 7K\mu$ and similarly for $t + e_k$. So, we can apply the same arguments to newly obtained points, and proceeding in this manner we construct a bounded 2-way set. However, the points that we construct after $t$ are at most $9\lambda K\mu$ away from $x_1 + e_2$, hence we get a 2-way set of diameter at most $18\lambda K\mu$. This contradicts Proposition 5.10, as we have such a point $t$ in every 3-way set and $\lambda < 1/(54K)$.

With this in mind, we see that in every 3-way set there is a point $t$ with $d(x_0, t) \le 7K\mu$ but $t \overset{i}{\frown} x_i$ for all $i \in [3]$. Contract such a $t$ with $x$. It cannot be by 3, as then $d(x + e_3, x_3 + e_3) \le 16\lambda K\mu$, so without loss of generality we have $x \overset{1}{\frown} t$. But then for any $j \in \{2, 3\}$ and $k \in [2]$ that contracts $x$ and $x_3$ we obtain

$$\begin{aligned} d(x_1 + e_1, x_1 + e_j) &\le d(x_1 + e_1, t + e_1) + d(t + e_1, x + e_1) + d(x + e_1, x + e_k) \\ &\quad + d(x + e_k, x_3 + e_k) + d(x_3 + e_k, x_1 + e_j) \\ &\le 8\lambda K\mu + 8\lambda K\mu + \lambda K\mu + 2\lambda K\mu + \lambda K\mu = 20\lambda K\mu, \end{aligned}$$

giving $\operatorname{diam} N(x_1) \le 20\lambda K\mu$ and as before $\rho(x_1) \le 20K\mu$. Applying Proposition 6.3 establishes the final contradiction, as $\lambda < 1/(820C_1K)$. ■

**7. Existence of certain finite configurations.** Our next aim is to show that, provided $\lambda$ is sufficiently small, certain finite configurations must exist. Recalling Proposition 6.3, we see that we are approaching the final contradiction in the proof of Proposition 2.3.

PROPOSITION 7.1. *Provided* $\lambda < 1/(5\times 10^{12})$, *there is a point* $x$ *such that* $\rho(x) \le C_2\mu$ *and* $\operatorname{diam}\{x, x + e_i, x + e_j\} \le \lambda C_2\mu$ *for some distinct* $i, j \in [3]$. *Here* $C_2 = 100000$.

*Proof.* Suppose the contrary. First we will establish the existence of an auxiliary point $y$ with $\rho(y) \le 15\mu$ and $d(y, y + e_i) \le 192\lambda\mu$, $d(y + e_j, y + e_k) \le 4\lambda\mu$ for some $\{i, j, k\} = [3]$. Pick any $t$ with $\rho(t) \le 2\mu$ and consider contractions $\{t\} \cup N(t)$. As before, up to symmetry, we have diagrams A, B and C in Figure 5 as possibilities for contractions of pairs of the form

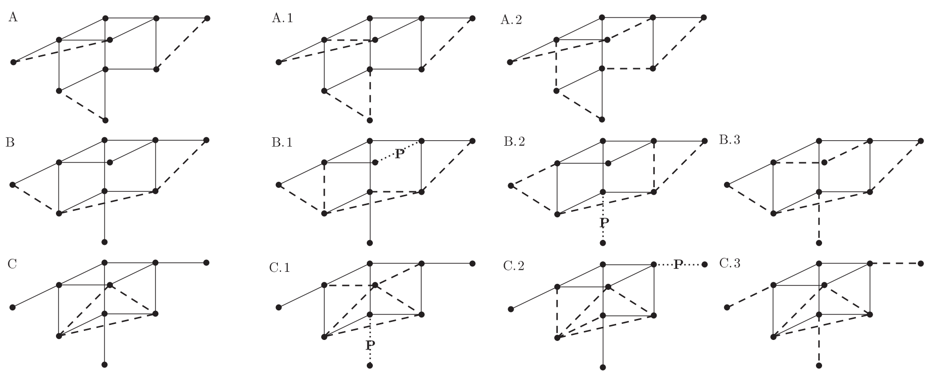


Fig. 5. Possible contractions in the proof of existence of an auxiliary point

$t + e_a, t + e_b$, since no two such long edges can be contracted by the same $i$. If an edge is a dashed line in Figure 5, then it is the result of a contraction of some pair of points in $\{t\} \cup N(x)$. The dotted lines marked with **P** represent the edges that will be results of applying Proposition 6.4.

Case 1. Suppose that we have diagram A. We see that we have diagrams A.1 and A.2 up to symmetry, or otherwise some $\rho(z)$ is too small. However, diagram A.1 is impossible since $\rho(t+e_1) \leq C_2\mu$ and $\operatorname{diam}\{t+e_1, t+e_1+e_1, t+e_1+e_2\} \leq \lambda C_2\mu$, which does not occur by assumption. Hence, diagram A.2 must occur, so we have $y$ with $\rho(y) \leq (4+6\lambda)\mu$, $d(y, y+e_3) \leq 2\lambda\mu$ and $d(y+e_1, y+e_2) \leq 4\lambda\mu$.

Case 2. Suppose that we have diagram B. As above, we can distinguish diagrams B.1, B.2, B.3, up to symmetry. First of all, if we have diagram B.3, we can apply Proposition 6.2 to $t$, as $\lambda < 1/(13C_1), 1/(78\cdot 2)$, to obtain $\rho(t+e_i) \leq 14\mu$ for some $i$. This yields $\rho(y+e_3) \leq 15\mu$, $d(y+e_3+e_1, y+e_3+e_2) \leq 4\lambda\mu$, $d(y+e_3, y+2e_3) \leq 2\lambda\mu$, as desired.

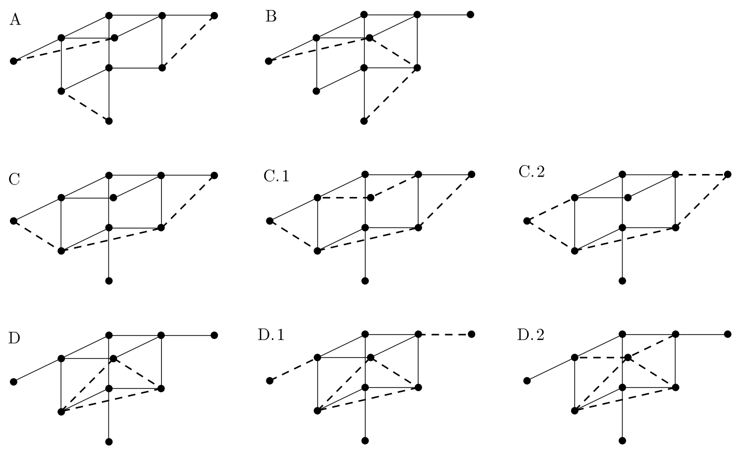


Fig. 6. Possible contractions in the neighbourhood of an auxiliary point

Consider now diagrams B.1 and B.2. We can denote $N(t) = \{t_1, t_2, t_3\}$ so that $t_1+e_2, t_1+e_3$ is a result of a contraction in $N(t)$, and so on. Observe that $\operatorname{diam}\{t_i + e_j : i, j \in [3],\ i \neq j\} \leq 12\lambda\mu$ and $\rho(t) \leq 2\mu$, and in diagram B.1, $d(t+e_1, t+e_3) \leq 8\lambda\mu$, while in diagram B.2, $d(t+e_1, t+e_2) \leq 10\lambda\mu$; we can apply Proposition 6.4, as $\lambda < (9840C_1)$, to $(t; t_1, t_2, t_3; t)$ with constant 12 to see that $d(t+e_2, t_2+e_2) \leq 12\cdot 16\lambda\mu = 192\lambda\mu$ in diagram B.1 and $d(t+e_3, t_3+e_3) \leq 192\lambda\mu$. Hence, $t+e_2$ in diagram B.1 and $t+e_3$ in diagram B.2 are the desired points.

Case 3. As in the previous case, we are able to reach the same conclusion using similar arguments.

To sum up, without loss of generality, we can assume that there is $y_0$ with $\rho(y_0) \le 15\mu$, $d(y_0+e_1, y_0+e_2) \le 4\lambda\mu$ and $d(y_0, y_0+e_3) \le 192\lambda\mu$. We shall now use this point to obtain a contradiction.

Let $K = 20000$, and consider those points $y$ which satisfy $\rho(y) \le K\mu$, $d(y+e_i, y+e_j) \le \lambda K\mu$ and $d(y, y+e_k) \le \lambda K\mu$ for some $\{i,j,k\} = [3]$. We know that $y_0$ is one such point. Contract first the pairs inside $N(y)$, that is, the long edges. As a few times before, it is not hard to see that for $i = 1$, $j = 2$, $k = 3$ we can only have diagrams A, B, C and D of Figure 6 (if an edge is shown as a dashed line, that implies that it is a result of a contraction) and diagrams symmetric to these for different values of $i, j, k$. However, we can immediately reject diagram A, for if a point $y$ had diagram A, by contracting the short edges we obtain either a point $t \in N(y)$ with $\rho(t) \le 3K\mu$ and $\operatorname{diam}\{t, t+e_i, t+e_j\} \le 3\lambda K\mu$, or a point $t \in N(y)$ with $\rho(y) \le 4\lambda K\mu < \mu$, both resulting in a contradiction. Furthermore, if we were given a diagram B, then we can immediately apply Proposition 6.4 to $(y; y+e_3, y+e_2, y+e_1; y)$ with constant $6K$, as $\lambda < 1/(4920C_1K)$, which gives $d(y+e_3, y+e_1+e_3) \le 96\lambda K\mu$. Then we must have $y \overset{2}{\frown} y+e_3$, hence $\rho(y+e_1) \le (1+97\lambda)K\mu < (K+1)\mu, \operatorname{diam}\{y+e_1, y+e_1+e_1, y+e_1+e_2\} \le 5\lambda K\mu$, giving a contradiction once more.

Therefore, we must end up with either diagram C or D. Also observe that $y+e_i \overset{k}{\frown} y+e_j$ must then hold for any $y$ with the properties stated above. Furthermore we must have $d(y+e_k, y+2e_k) \le 96\lambda K\mu$, as we can apply Proposition 6.4 to $(y; y_1, y_2, y_3; y)$, where $\{y_1, y_2, y_3\} = N(y)$ with constant $6K$. From this, we conclude that neither $y \overset{k}{\frown} y+e_i$ nor $y \overset{k}{\frown} y+e_j$. Also we cannot have $y \overset{i}{\frown} y+e_i$ and $y \overset{i}{\frown} y+e_j$ simultaneously, as then $\rho(y+e_i) < \mu$, and similarly we cannot have both $y \overset{j}{\frown} y+e_i$ and $y \overset{j}{\frown} y+e_j$. Hence, contracting the short edges implies that in fact we can only have diagrams C.1, C.2, D.1 or D.2.

Observe that actually we can only have either C.1 and D.1, or C.2 and D.2. This is because if we had $y_1$ with diagram C.1 or D.1, and a point $y_2$ with diagram C.2 or D.2, we could first find the unique $e_i, e_j$ such that $d(y_1+e_i, y_1+e_i+e_1) \le \lambda K\mu$ and $d(y_2+e_j, y_2+e_j+e_1) = \rho(y_2+e_j)$. Now, apply Proposition 6.4 to $(y_1; y_1+e_i, y_1+e_k, y_1+e_3; y_2+e_j)$ with constant $6K$, where $k \in [2]$, $k \ne i$, to obtain

$$\begin{aligned}\rho(y_2+e_j) &= d(y_2+e_j, y_2+e_j+e_1)\\ &\le d(y_2+e_j, y_2) + d(y_2, y_1) + d(y_1, y_1+e_i)\\ &\quad + d(y_1+e_i, y_1+e_i+e_1) + d(y_1+e_i+e_1, y_2+e_j+e_1)\\ &\le K\mu + 2K\mu/(1-\lambda) + K\mu + \lambda K\mu + 96\lambda K\mu \le 5K\mu,\end{aligned}$$

while $\operatorname{diam}\{y_2+e_j, y_2+e_j+e_2, y_2+e_j+e_3\} \le 3\lambda K\mu$, a contradiction. Thus, we shall consider the following cases.

Case 1: *We only have diagrams C.1 and D.1.* Suppose we had $y$ with $\rho(y) \leq K\mu/10$, $d(y+e_i, y+e_j) \leq \lambda K\mu/10$, $d(y, y+e_k) \leq \lambda K\mu/10$, for some $\{i,j,k\} = [3]$ that gave us diagram C.1 after contractions in $\{y\} \cup N(y)$. Without loss of generality, we take $i = 1$, $j = 2$ and $k = 3$. Then, by Proposition 6.2 and the triangle inequality, we get $\rho(y+e_1), \rho(y+e_2) \leq K\mu$. In conjunction with $d(y+e_1+e_1, y+e_1+e_3), d(y+e_2+e_2, y+e_2+e_3) \leq \lambda K\mu/5$ and $d(y+e_1, y+e_1+e_2), d(y+e_2, y+e_2+e_1) \leq \lambda K\mu/10$, we see that $y+e_1, y+e_2$ are points whose neighbourhoods contracting gives one of the diagrams considered, in particular $y+e_1+e_1 \overset{2}{\frown} y+e_1+e_3$ and $y+e_2+e_2 \overset{1}{\frown} y+e_2+e_3$. But contracting $y+e_1+e_2$ with $y$ gives $\rho(y+e_1+e_2) \leq K\mu/5 < K\mu$ and $\operatorname{diam} N(y+e_1+e_2) \leq \lambda^2 K\mu < \lambda K\mu$, which contradicts Proposition 6.3, since $\lambda < 1/(41C_1K)$.

Hence, as long as $y$ satisfies $\rho(y) \leq K\mu/10$, $d(y+e_i, y+e_j) \leq \lambda K\mu/10$, $d(y, y+e_k) \leq \lambda K\mu/10$ for some $\{i,j,k\} = [3]$, it must have diagram D.1. Start from $y_0$. Then $d(y_0+e_3+e_1, y_0+e_3+e_2) \leq \lambda^2 K\mu$, $d(y_0+e_3, y_0+2e_3) \leq \lambda^2 K\mu$. Now, apply Proposition 6.2 to $y_0$ to see that $\rho(y_0+e_3) \leq 8\rho(y_0)$. Therefore, contractions around $y_0+e_3$ give us diagram D.1. But contract $y_0+e_1, y_0+e_1+e_3$ to obtain $\rho(y_0+e_1+e_3) < \mu$ or $\rho(y_0+e_1) < \mu$.

Case 2: *We can only have diagrams C.2 and D.2.* Start from $y_0$ and define $y_n = y_0 + ne_3$ for all $n \geq 1$. By induction on $n$ we shall prove that $\rho(y_n) \leq 16\mu$, $d(y_n+e_1, y_n+e_2) \leq 4\lambda^{n+1}\mu$, $d(y_n+e_1, y_{n+1}+e_1) \leq (45+8n)\lambda^{n+1}\mu$, $d(y_n+e_2, y_{n+1}+e_2) \leq (45+8n)\lambda^{n+1}\mu$, $d(y_n, y_n+e_3) \leq 2000\lambda\mu$.

For $n = 0$ the claim holds, since $y_0$ has diagram C.2 or D.2. Suppose the claim holds for some $n \geq 0$. Then $y_0$ must have diagram C.2 or D.2, so $y_n+e_1 \overset{3}{\frown} y_n+e_2$, giving $d(y_{n+1}+e_1, y_{n+1}+e_2) \leq \lambda d(y_n+e_1, y_n+e_2) \leq 4\lambda^{n+1}\mu$. We can apply Proposition 6.4 to $(y_0; y_0+e_2, y_0+e_1, y_0+e_3; y_n)$ or $(y_0; y_0+e_1, y_0+e_2, y_0+e_3; y_n)$ (depending on the diagram of $y_0$), and to $(y_0; y_0+e_2, y_0+e_1, y_0+e_3; y_{n+1})$ or $(y_0; y_0+e_1, y_0+e_2, y_0+e_3; y_{n+1})$ with constant 60, so we get $d(y_0+e_3, y_n+e_3), d(y_0+e_3, y_{n+1}+e_3) \leq 960\lambda\mu$, thus $d(y_{n+1}, y_{n+1}+e_3) \leq 2000\lambda\mu$. So $\rho(y_{n+1}) \leq (1+3\lambda)\rho(y_n) \leq 17\mu$, so $y_{n+1}$ has diagram C.2 or D.2. If the diagrams of $y_n$ and $y_{n+1}$ are distinct, then $y_n+e_1 \overset{3}{\frown} y_{n+1}+e_1$ and $y_n+e_2 \overset{3}{\frown} y_{n+1}+e_2$, so the inequalities for $d(y_{n+1}+e_1, y_{n+2}+e_1)$ and $d(y_{n+1}+e_2, y_{n+2}+e_2)$ follow. Otherwise, $y_n+e_1 \overset{3}{\frown} y_{n+1}+e_2$ and $y_n+e_2 \overset{3}{\frown} y_{n+1}+e_1$, so

$$\begin{aligned}
d(y_{n+1}+e_1, y_{n+2}+e_1) &\leq d(y_{n+1}+e_1, y_{n+1}+e_2) + d(y_{n+1}+e_2, y_{n+2}+e_1)\\
&\leq 4\lambda^{n+2}\mu + \lambda(d(y_n+e_2, y_n+e_1) + d(y_n+e_1, y_{n+1}+e_1))\\
&\leq 8\lambda^{n+2}\mu + \lambda(45+8n)\lambda^{n+1} = (45+8(n+1))\lambda^{n+2}\mu.
\end{aligned}$$

The inequality for $d(y_{n+1}+e_2, y_{n+2}+e_2)$ is proved in the same spirit.

Finally, by the triangle inequality we get $d(y_0+e_1, y_{n+1}+e_1) \le d(y_0+e_1, y_1+e_2)+d(y_1+e_2, y_1+e_1)+d(y_1+e_1, y_2+e_2)+\cdots+d(y_n+e_1, y_{n+1}+e_1) < 50\lambda\mu$. Also $d(y_0, y_{n+1}) \le d(y_0, y_0+e_3)+d(y_0+e_3, y_n+e_3) \le 192\lambda\mu+960\lambda\mu = 1152\lambda\mu$. Combining these conclusions further implies $\rho(y_{n+1}) \le 16\mu$, as desired. Having established this claim, we can see that $(y_n + e_1)_{n\ge 0}$ is a 1-way Cauchy sequence, which is the final contradiction in this proof. ∎

Proposition 7.2. *Set* $C_3 = 240000000000$ *and* $C_{3,1} = 19000000000$*, and let* $i, j \in [3]$ *be distinct. If* $\lambda < 1/(7380 C_1 C_{3,1})$*, then there is* $x$ *such that* $\rho(x) \le C_3\mu$ *and* $d(x+e_i, x+e_j) \le \lambda C_3\mu$*.*

*Proof.* The statement will be a consequence of a few lemmata, the last one being Lemma 7.8. It suffices to prove the claim for $i = 1$, $j = 2$. Suppose there is no such $x$. Consider those $y$ which satisfy $\rho(y) \le C_{3,1}\mu$ and $d(y+e_3, y+e_i) \le \lambda C_{3,1}\mu$; call them $C_{3,1}$-*good*, and more generally use this definition for an arbitrary constant instead of $C_{3,1}$. We already know that such a $y$ exists by Proposition 7.1. We show the possible diagrams of contractions in $\{y\} \cup N(y)$ for such a point in Figure 7 for $i = 1$. If an edge is shown as a

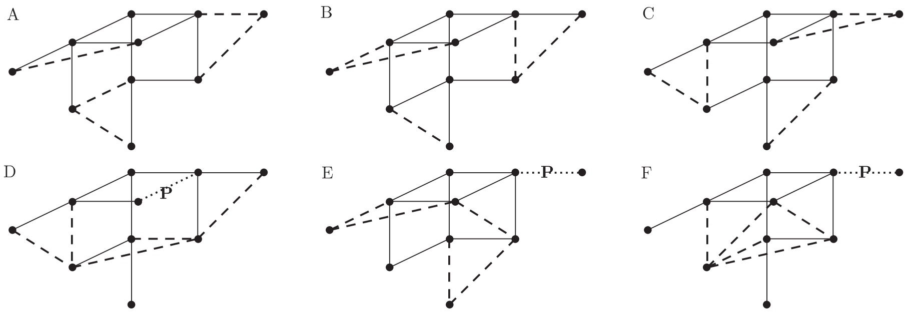


Fig. 7. Possible diagrams in the proof of Proposition 7.2

dashed line, then it is a result of a contraction. Furthermore, dotted lines marked with **P** indicate edges whose length will be the result of applying Proposition 6.4. It is not hard to show that these are the only possible diagrams, but for the sake of completeness we include an Appendix on the contraction diagrams, which in particular provides an explanation for Figure 7. Diagrams symmetric to these for the case $i = 2$ are denoted by A′, B′, etc.

Our aim is to reject diagrams one by one. We shall start by discarding diagram A, and the same method will then be used for the others. As we shall see, we can first apply the propositions proved so far to discard many diagrams in the presence of one given, and then the remaining ones can be fitted together so that we obtain a 1-way Cauchy sequence.

LEMMA 7.3. *Set $C_{3,2} = 3100000000$. There is no $C_{3,2}$-good $y$ such that contractions give diagram* A *or* A′ *for $y$.*

*Proof.* Suppose we do have such a point $y$, and without loss of generality $d(y+e_1, y+e_3) \le \lambda C_{3,2}\mu$. Firstly, suppose that there is another point $z$ that is $C_{3,1}$-good, but whose diagram is D, D′, E, E′, F or F′. By FNI we have $d(y,z) \le (C_{3,2}+C_{3,1})\mu/(1-\lambda) < 2C_{3,1}\mu$. Then, for a suitable choice $\{z_1, z_2, z_3\} = N(z)$, we can apply Proposition 6.4 to $(z; z_1, z_2, z_3; y+e_1)$ with constant $6C_{3,1}$ to get

$$\begin{aligned} d(z_3, z_3+e_3) &\le d(z_3, z) + d(z,y) + d(y, y+e_1+e_3) \\ &\quad + d(y+e_1+e_3, z_3+e_3) \\ &\le C_{3,1}\mu + 2C_{3,1}\mu + (1+\lambda)C_{3,2}\mu + 96\lambda C_{3,1}\mu < 4C_{3,1}\mu. \end{aligned}$$

Hence, $\rho(z_3) \le 4C_{3,1}\mu$, except when the diagram is D or D′, so we must apply Proposition 6.2 to $z$ first, to obtain $\rho(z_3) \le 10C_{3,1}\mu$. Also, $d(z_3+e_1, z_3+e_2) \le 2\lambda C_{3,1}\mu$, but such a point $z$ cannot exist by the assumptions.

Now, take an arbitrary $(C_{3,1}/3)$-good point $z$ with diagram A. Consider the point $z+e_3$. We have $\rho(z+e_3) \le (2+3\lambda)\rho(z)$, $d(z+e_3+e_1, z+e_3+e_3) \le 2\lambda\rho(z)$, so $z+e_3$ is $C_{3,1}$-good, so its diagram is A, B or C (it cannot be symmetric to these, as then $\rho(z+e_3) < \mu$). If it were B, then contracting the pair $z+e_1, z+e_3+e_2$ would give an immediate contradiction, for we would obtain $\rho(z+e_1) < C_{3,1}\mu$, $\rho(z+e_2) < \mu$ or $\rho(z+e_2+e_3) < \mu$. Similarly, it cannot be C, since contracting the same pair of points would give the contradiction once again, as it would yield $\rho(z+e_2) < \mu$ or $\rho(z+2e_3) < \mu$. Therefore, whenever we have a $(C_{3,1}/3)$-good point $z$ with diagram A, then $z+e_3$ is $C_{3,1}$-good and has the same diagram.

Now, start from the $y$ given, and define $y_n = y + ne_3$ for $n \ge 0$. We shall show that $(y_n)_{n\ge 0}$ is a Cauchy sequence and hence obtain a contradiction. We claim $d(y_n, y_n+e_1) \le \lambda^n C_{3,2}\mu$, $d(y_n+e_1, y_{n+1}) \le \lambda^{n+1}C_{3,2}\mu$, $\rho(y_n) < (2+10\lambda)C_{3,2}\mu$ and the diagram of $y_n$ is A. This is clearly true for $n = 0$.

Suppose that the claim holds for $n \ge 0$. Note $d(y_0, y_{n+1}) \le d(y_0, y_1) + d(y_1, y_2) + \cdots + d(y_n, y_{n+1}) \le C_{3,2}\mu + 2\lambda C_{3,2}\mu + 2\lambda^2 C_{3,2}\mu + \cdots < (1+3\lambda)C_{3,2}\mu$. The fact that $y_n$ has diagram A and is in fact $C_{3,1}/3$-good implies that $y_{n+1}$ is $C_{3,1}$-good and itself has diagram A. Further, $y_n \overset{3}{\frown} y_n + e_1$ and $y_n + e_1 \overset{3}{\frown} y_n + e_3$. This is then sufficient to obtain the next two inequalities: $d(y_0+e_2, y_{n+1}) < C_{3,2}\mu + 3C_{3,2}\mu/(1-\lambda) < 5C_{3,2}\mu$. Hence, we must have $y_0 + e_2 \overset{2}{\frown} y_{n+1}$, for otherwise $\rho(y_0+e_1) < \mu$ or $\rho(y_1) < \mu$. So $d(y_0, y_{n+1}+e_2) \le d(y_0, y_0+2e_2) + \lambda d(y_0+e_2, y_{n+1}) \le C_{3,2}\mu + \lambda C_{3,2}\mu + 5\lambda C_{3,2}\mu$, from which we can infer $\rho(y_{n+1}) = d(y_{n+1}, y_{n+1}+e_2) \le d(y_{n+1}, y_0) + d(y_0, y_0+e_2) + d(y_0+e_2, y_0+2e_3) + d(y_0+2e_2, y_{n+1}+e_2) < (2+10\lambda)C_{3,2}\mu$, as claimed.

From this we immediately see that $(y_n)_{n\ge 0}$ is a Cauchy sequence. ∎

LEMMA 7.4. *Set $C_{3,3} = 1029000000$. There is no $C_{3,3}$-good point $y$ with diagram* E *or* E′.

*Proof.* Suppose the contrary, and without loss of generality $d(y + e_1, y + e_3) \le \lambda C_{3,3}\mu$. Firstly, suppose there is a $C_{3,2}$-good point $z$ with $d(z + e_1, z + e_3) \le \lambda C_{3,2}\mu$ and a diagram among B, C, D. Take $p = z + e_2$ for diagrams B, D, $p = z + e_3$ for C, and apply Proposition 6.4 with constant $3C_{3,2}$ (for $d(p, y) \le 3C_{3,2}\mu$ and for such a constant the other necessary assumptions also hold) to $(y; y + e_3, y + e_2, y + e_1; p)$ to obtain a contradiction at $y + e_1$, as it has $d(y + e_1 + e_1, y + e_1 + e_2) \le 2C_{3,3}\lambda\mu$ and

$$\begin{aligned}\rho(y + e_1) &= d(y + e_1, y + e_1 + e_3)\\ &\le d(y + e_1, y) + d(y, z) + d(z, p + e_1) + d(p + e_1, y + e_3 + e_1)\\ &\le C_{3,3}\mu + (C_{3,3} + C_{3,2})\mu/(1 - \lambda) + 2C_{3,2}\mu + 48\lambda C_{3,2} < C_{3,1}\mu.\end{aligned}$$

Hence, any such $C_{3,2}$-good point $z$ can only have diagram E or F.

Now, return to the point $y$ and define $y_n = y+ne_2$ for all $n \ge 0$. We shall show that $(y_n)_{n\ge 0}$ is a Cauchy sequence. By induction on $n$ we shall show $d(y_n + e_1, y_n + e_3) \le \lambda^{n+1}C_{3,3}\mu$, $d(y_n + e_3, y_{n+1} + e_3) \le (3 + 2n)\lambda^{n+1}C_{3,3}\mu$ and $\rho(y_n) < 3C_{3,3}\mu$. The case $n = 0$ is clear.

Suppose the claim holds for some $n \ge 0$. Firstly, $y_n$ is $C_{3,2}$-good, so it has diagram E or F, so in particular $d(y_{n+1}+e_1, y_{n+1}+e_3) \le \lambda d(y_n+e_1, y_n+e_3) \le \lambda^{n+1}C_{3,3}\mu$. Applying the triangle inequality gives

$$\begin{aligned}d(y_{n+1} + e_3, y_0 + e_3) &\le d(y_{n+1} + e_3, y_n + e_3) + \cdots + d(y_1 + e_3, y_0 + e_3)\\ &\le (5 + 2n)\lambda^{n+1}C_{3,3}\mu + \cdots + 3\lambda C_{3,3}\mu\\ &< 3\lambda C_{3,3}\mu/(1 - 2\lambda).\end{aligned}$$

Further, since $d(y_0, y_{n+1}) \le d(y_0, y_n)+d(y_n, y_{n+1}) \le (\rho(y_0)+\rho(y_n))/(1-\lambda)+\rho(y_n) \le 8C_{3,3}\mu$, apply Proposition 6.4 to $(y; y + e_3, y + e_2, y + e_1; y_{n+1})$ with constant $8C_{3,3}$, which gives $d(y_1, y_{n+2}) \le 128\lambda C_{3,3}\mu$. Therefore, $\rho(y_{n+1}) < 3C_{3,3}\mu$, in particular $y_{n+1}$ is $C_{3,2}$-good, hence its diagram can also only be E or F. If $y_n$ and $y_{n+1}$ have the same diagram, then contract $y_n + e_1, y_{n+1} + e_3$, otherwise $y_n + e_3, y_{n+1} + e_3$. These must be contracted by 2, so using the triangle inequality gives in the former case

$$\begin{aligned}d(y_{n+1} + e_3, y_{n+2} + e_3) &\le d(y_{n+1} + e_3, y_{n+1} + e_1) + d(y_{n+1} + e_1, y_{n+2} + e_3)\\ &\le \lambda^{n+2}C_{3,3}\mu + \lambda d(y_n + e_1, y_{n+1} + e_3)\\ &\le \lambda^{n+2}C_{3,3}\mu + \lambda(d(y_n + e_1, y_n + e_3) + d(y_n + e_3, y_{n+1} + e_3))\\ &\le 2\lambda^{n+2}C_{3,3}\mu + \lambda d(y_n + e_3, y_{n+1} + e_3)\\ &\le (5 + 2n)\lambda^{n+2}C_{3,3}\mu,\end{aligned}$$

as desired. In the latter case we are immediately done.

Furthermore this claim implies that $(y_n + e_1)_{n\geq 0}$ is a Cauchy sequence, so we obtain a contradiction. ∎

Lemma 7.5. *Set* $C_{3,4} = 147000000$. *There is no* $C_{3,4}$*-good point* $y$ *with diagram* F *or* F′.

*Proof.* Suppose there is such a point $y$ and without loss of generality $d(y + e_1, y + e_3) \leq \lambda C_{3,4}\mu$.

Suppose that we have a point $z$ that is $3C_{3,4}$-good with diagram F and satisfies $d(z+e_1, z+e_3) \leq 3\lambda C_{3,4}\mu$, while $z+e_2$ being $C_{3,3}$-good has diagram B, C or D. If it were B, we would immediately obtain a contradiction by contracting $z+e_1, z+2e_2$, and if it were C, contracting $z+e_1, z+e_2+e_3$ would once again end the proof, both giving a point $p$ with $\rho(p) < \mu$, so suppose that it were D. Apply Proposition 6.4 to $(z; z + e_1, z + e_2, z + e_3; z + e_2 + e_3)$ and to $(z; z + e_1, z + e_2, z + e_3; z + 2e_2)$ with constant $12C_{3,4}$. Now $z + e_1$ is $7C_{3,4}$-good, so it has diagram B′, C′, D′ or F′. However, $\operatorname{diam}\{z + e_1 + e_2, z + 2e_1 + e_2, z + e_1 + e_2 + e_3\} \leq 4\lambda C_{3,4}\mu$, so it must in fact be F′. Apply Proposition 6.4 to $(z; z + e_1, z + e_2, z + e_3; z + e_1 + e_3)$ with constant $12C_{3,4}$. Thus $z + e_3 \overset{3}{\frown} z + 2e_3$. Write $r = d(z + e_3, z + 2e_3)$, so we see that FNI implies $r - \rho(z) \leq d(z, z + 2e_3) \leq \lambda(r + \rho(z))/(1 - \lambda)$, but $r \geq C_3\mu$ and $\rho(z) \leq 3C_{3,4}\mu$ give a contradiction.

Hence, whenever $z$ is a $3C_{3,4}$-good point with diagram F, then $z + e_2$ is $C_{3,3}$-good and has the same diagram. Now $(y + ne_2)_{n\geq 0}$ is Cauchy by the arguments from the proof of Lemma 7.4, since there we allow both E and F. ∎

Lemma 7.6. *Set* $C_{3,5} = 21000000$. *There is no* $C_{3,5}$*-good point* $y$ *with diagram* D *or* D′.

*Proof.* Suppose there is such a point $y$ and without loss of generality $d(y + e_1, y + e_3) \leq \lambda C_{3,5}\mu$.

Now consider a $3C_{3,5}$-good point $z$ with diagram D and $d(z+e_1, z+e_3) \leq \lambda 3C_{3,5}\mu$. Since $z + e_1$ is $C_{3,4}$-good, it can only have diagram B, C or D. If it were not D, contract $z + e_2, z + e_1 + e_3$ for the sake of contradiction: if it were B we would get $\rho(z + e_2) < \mu$ or $\rho(z + 2e_1) \leq C_3\mu$, but $d(z + 2e_1 + e_1, z + 2e_1 + e_2) \leq 2\lambda C_{3,5}\mu$, and if it were C, we would obtain $\rho(z + e_1 + e_3) < \mu$ or $\rho(z + 2e_1) < \mu$. Hence, whenever $z$ has the given properties, $z + e_1$ has diagram D.

Now, return to $y$, and consider the sequence $y_n = y + ne_1$ for $n \geq 0$. By induction on $n$, we shall show that $\rho(y_n) \leq 3C_{3,5}\mu, d(y_n, y_n + e_3) \leq \lambda^n C_{3,5}\mu$ and $d(y_n + e_3, y_{n+1}) \leq \lambda^{n+1}C_{3,5}\mu$. This is clearly true for $n = 0$.

Suppose that the claim holds for some $n \geq 0$. Then $y_n$ is $3C_{3,5}$-good, so it has diagram D. Hence, $y_n \overset{1}{\frown} y_n + e_3$ and $y_{n+1} \overset{1}{\frown} y_n + e_3$, which establishes two of the necessary inequalities. Also, by the triangle inequality

$d(y_{n+1}, y_0) \leq C_{3,5}\mu + 2\lambda C_{3,5}\mu/(1-\lambda)$, so we can apply Proposition 6.4 to $(y_0; y_0 + e_2, y_0 + e_1, y_0 + e_3; y_{n+1})$ with constant $6C_{3,5}$ to get $d(y_{n+1} + e_2, y_0 + e_1 + e_2) \leq 96C_{3,5}\mu$, in particular $\rho(y_{n+1}) \leq 3C_{3,5}\mu$, as desired.

Now it follows that $(y_n)_{n\geq 0}$ is a Cauchy sequence. ∎

Lemma 7.7. *Set $C_{3,6} = 3000000$. There is no $C_{3,6}$-good point $y$ with diagram C or C′.*

*Proof.* Suppose there is such a point $y$ and without loss of generality $d(y + e_1, y + e_3) \leq \lambda C_{3,6}\mu$.

Firstly, suppose that we have a $3C_{3,6}$-good point $z$ such that $d(z + e_1, z+e_3) \leq 3\lambda C_{3,6}\mu$, and $z+e_1$ has diagram B. We shall obtain a contradiction by considering contractions. First, observe that $z + e_3 \overset{3}{\frown} z + e_1 + e_3$. Note that $d(z + 2e_1, z + 2e_1 + e_3) > C_3\mu$, so $d(z + e_3, z + 2e_3) \geq d(z + 2e_1, z+2e_1+e_3) - d(z+e_3, z+2e_1) - d(z+2e_3, z+2e_1+e_3) > C_3\mu - 24\lambda C_{3,6}\mu$.

Case 1. Suppose that $z+e_3 \overset{2}{\frown} z+2e_3$. We see that $z+e_2+e_3, z+2e_3$ is not contracted by 1, and from FNI, we must have $\rho(z + 2e_3) \geq (1-\lambda) d(z, z+2e_3) - \rho(z) \geq (1-\lambda)d(z+e_3, z+2e_3) - (2-\lambda)\rho(z)$, thus $z+e_2+e_3, z+2e_3$ is not contracted by 3 either, hence $z+e_2+e_3 \overset{2}{\frown} z+2e_3$. Now suppose that $z+2e_2 \overset{3}{\frown} z+e_1+e_2$. Then $d(z+e_3, z+2e_3) \leq d(z+e_3, z+e_2+2e_3) + d(z+e_2+2e_3, z+e_2+e_3) + d(z+e_2+e_3, z+2e_3)$, so $d(z+e_3, z+2e_3)(1-\lambda) \leq 3\rho(z)$, which is impossible.

Therefore we must have $z + 2e_2 \overset{2}{\frown} z + e_1 + e_2$ and $z + 2e_1 \overset{3}{\frown} z + 2e_2$, otherwise $\rho(z + e_1 + e_2) < \mu$. Finally, contract $z + 2e_1$with $z + 2e_3$ to get $\rho(z + 2e_1) < \mu$ or $\rho(z + 2e_3) < \mu$.

Case 2. Suppose that $z + e_3 \overset{3}{\frown} z + 2e_3$. By FNI applied to $z, z + 2e_3$ we see that $\rho(z + 2e_3) \geq (1-\lambda)d(z + e_3, z + 2e_3) - (2-\lambda)\rho(z)$, hence $\rho(z+2e_3) = d(z+2e_3, z+e_2+2e_3) \geq (1-\lambda)d(z+e_3, z+2e_3) - (2-\lambda)\rho(z)$. So we have $z + 2e_3 \overset{2}{\frown} z + e_2 + e_3$. Also $z + 2e_2 \overset{2}{\frown} z + e_3$, from which we see that $z + 2e_2 \overset{2}{\frown} z + 2e_1$, a contradiction.

Thus, whenever we have a point $z$ as described, we must have $z + e_1$ with diagram C as well. Now, set $y_n = y + ne_1$ for $n \geq 0$. We claim that $d(y_n, y_n + e_3) \leq \lambda^n C_{3,6}\mu, d(y_n + e_3, y_{n+1}) \leq \lambda^{n+1} C_{3,6}\mu$, $\rho(y_n) \leq 3C_{3,6}\mu$ and $y_n$ has diagram C. This is clear for $n = 0$.

Suppose the claim holds for some $n \geq 0$, so $y_n$ must have diagram C, from which the first two inequalities follow. Observe that $d(y_{n+1}, y_0) < \rho(y_0) + 2\lambda\rho(y_0)/(1-\lambda)$ and $d(y_0+e_2, y_0+e_2+e_3) > C_3\mu$, so $y_{n+1} \overset{2}{\frown} y_0+e_2$. Therefore $\rho(y_{n+1}) < 3\rho(y_0) \leq 3C_{3,6}\mu$, which gives the rest of the claim, as $y_{n+1} = y_n + e_1$ must have diagram C, by the previous conclusions.

Hence $(y_n)_{n\geq 0}$ is a 1-way Cauchy sequence, which is a contradiction. ∎

Lemma 7.8. *Set $C_{3,7} = 100000$. There is no $C_{3,7}$-good point $y$ with diagram* B *or* B′.

*Proof.* Suppose there is such a point $y$ and without loss of generality $d(y + e_1, y + e_3) \leq \lambda C_{3,7}\mu$.

Consider a $6C_{3,7}$-good point $z$ which has $d(z+e_1, z+e_3) \leq 6\lambda C_{3,7}\mu$, and which therefore must have diagram B. We have $\rho(z + e_2) \leq (2 + 3\lambda)\rho(z)$, $d(z+e_2+e_2, z+e_2+e_3) \leq 2\lambda\rho(z)$, so $z+e_2$ is $C_{3,6}$-good, so has diagram B′. Observe that $z + e_3 \overset{3}{\frown} z + e_2 + e_3$ as $d(z + (1, 0, 1), z + (1, 1, 1)) \geq R - 4\lambda\rho(z)$ and $d(z+(0, 1, 1), z+(0, 2, 1)) > C_3\mu - 2\lambda\rho(z)$, where $R = d(z+e_1, z+e_1+e_3) > C_3\mu$. Also $z+e_1 \overset{3}{\frown} z+e_3$ since $z$ has diagram B. Similarly, since $z + e_2$ has diagram B′, we must have $z + 2e_2 \overset{3}{\frown} z + e_2 + e_3$. Furthermore $\rho(z + e_1 + e_2) \leq (2 + 3\lambda)\rho(z + e_2) \leq (2 + 3\lambda)^2\rho(z)$, $d(z + e_1 + e_2 + e_1, z + e_1 + e_2 + e_3) \leq 2\lambda\rho(z + e_2) \leq 5\lambda\rho(z)$, so $z + e_1 + e_2$ is $C_{3,6}$-good, hence has diagram B, from which we infer $z + (0, 1, 1) \overset{3}{\frown} z + (1, 1, 1)$.

Suppose that $z + e_1 + e_3 \overset{3}{\frown} z + e_2 + e_3$, so $d(z + (1, 0, 2), z + (0, 1, 2)) \leq \lambda(R+3\rho(z))$ and $d(z+(1, 0, 2), z+(0, 0, 2)) \leq \lambda(R+6\rho(z))$. Thus $d(z+(1, 0, 1), z + (1, 0, 2)) \leq \lambda(R + 8\rho(z))$, hence $z \overset{3}{\frown} z + 2e_3$, which implies $d(z + 2e_3, z+3e_3) \geq R(1-\lambda) - 3\rho(z)$, so $z+e_1+e_3 \overset{1}{\frown} z+2e_3$ (if $z+e_1+e_3 \overset{2}{\frown} z+2e_3$, then $\rho(z+2e_1+e_2) < \mu$), giving $d(z+(2, 0, 1), z+(1, 0, 1)) \leq \lambda(R+10\rho(z))$. Also $z + (1, 1, 0) \overset{1}{\frown} z + (2, 0, 0)$ and $z + 2e_1 \overset{3}{\frown} z + 2e_2$, but then contracting $z + 2e_1, z + 2e_3$ results in a contradiction.

Thus $z+(1, 0, 1) \overset{1}{\frown} z+(0, 1, 1)$, as otherwise $R(1-\lambda) \leq 2C_{3,7}\mu$, which is not possible. From the fact that $z$ has diagram B, we have $z \overset{1}{\frown} z+e_1$. Also, we must have $z+e_1 \overset{1}{\frown} z+e_1+e_2$. As $d(z+e_1+e_3, z+2e_1+e_3) \geq (1-\lambda)R - 7\lambda\rho(z)$, we cannot have $z + e_1 \overset{3}{\frown} z + 2e_1$. Suppose that $z + e_1 \overset{1}{\frown} z + 2e_1$; then contracting $z + 2e_2, z + e_1$ and $z + 2e_2, z + 2e_1$ (both must be in the direction $e_3$) gives $d(z + e_1 + e_3, z + 2e_1 + e_3) \leq 6\lambda\rho(z)$, a contradiction.

We conclude that $z \overset{1}{\frown} z + e_1$, $z + e_1 \overset{1}{\frown} z + e_1 + e_2$ and $z + e_1 \overset{2}{\frown} z + 2e_1$, for such a $z$. By symmetry, when $d(z + e_2, z + e_3) \leq 6\lambda C_{3,7}\mu$ holds instead of $d(z + e_1, z + e_3) \leq 6\lambda C_{3,7}$, we must have $z \overset{2}{\frown} z + e_2$, $z + e_2 \overset{2}{\frown} z + e_1 + e_2$ and $z + e_2 \overset{1}{\frown} z + 2e_2$.

Return now to the point $y$ and consider the sequence given as $y_0 = y$, $y_{k+1} = y + e_2$ when $k$ is even, otherwise $y_{k+1} = y + e_1$. By induction on $k$ we shall prove $\rho(y_k) \leq 3C_{3,7}\mu$, $d(y_k, y_{k+2}) \leq 3\lambda^k \frac{1+\lambda^2}{1-\lambda} C_{3,7}\mu$, $d(y_k, y_k + e_3) \leq \lambda^k C_{3,7}\mu$ and $d(y_k, y_k + e_1) \leq 3\lambda^k C_{3,7}\mu$ for even $k$, and $d(y_k, y_k + e_2) \leq 3\lambda^k C_{3,7}\mu$ for odd $k$.

When $k = 0$, the claim clearly holds. Suppose that it is true for all values less than or equal to some even $k \geq 0$. We shall argue in the case when $k$ is even; the same argument works in the opposite situation. By the triangle inequality, we have $d(y_0, y_i) \le 3\frac{1+\lambda^2}{(1-\lambda)(1-\lambda^2)}C_{3,7}\mu$ for even $i \le k+2$, and $d(y_1, y_i) \le 3\lambda\frac{1+\lambda^2}{(1-\lambda)(1-\lambda^2)}C_{3,7}\mu$ for odd $i \le k+2$. In particular, as $y_k$ is $C_{3,6}$-good, it has diagram B, so

$$\begin{aligned}\rho(y_{k+1}) = d(y_{k+1}, y_{k+2}) &\le d(y_{k+1}, y_1) + \rho(y_0) + d(y_0, y_{k+2}) \\ &\le 3(1+\lambda)\frac{1+\lambda^2}{(1-\lambda)(1-\lambda^2)}C_{3,7}\mu + C_{3,7}\mu \le 5C_{3,7}\mu\end{aligned}$$

and $d(y_{k+1}+e_2, y_{k+1}+e_3) \le 2\lambda\rho(y_k) \le 10\lambda C_{3,7}\mu$. Then $y_{k+1}$ is $10C_{3,7}$-good, so it must have diagram B$'$. From the contractions implied by this diagram described previously, we get $d(y_{k+1}, y_{k+1}+e_3) \le \lambda^{k+1}C_{3,7}\mu$. Moreover, $y_{k+1} \overset{2}{\frown} y_{k+1}+e_2$, $y_{k+1}+e_2 \overset{2}{\frown} y_{k+1}+e_1+e_2$ and $y_{k+1}+e_2 \overset{1}{\frown} y_{k+1}+2e_2$. Therefore

$$\begin{aligned}d(y_{k+1}+e_2, y_{k+3}) &\le d(y_{k+1}+e_2, y_{k+1}+2e_2) \\ &\qquad + d(y_{k+1}+2e_2, y_{k+1}+2e_2+e_1) + d(y_{k+1}+2e_2+e_1, y_{k+3}) \\ &\le \lambda d(y_{k+1}+e_2, y_{k+3}) + (1+\lambda)d(y_{k+1}+e_2, y_{k+1}+2e_2) \\ &\le \lambda d(y_{k+1}+e_2, y_{k+3}) + \lambda(1+\lambda)d(y_{k+1}, y_{k+1}+e_2).\end{aligned}$$

Hence $d(y_{k+1}, y_{k+3}) \le \frac{1+\lambda^2}{1-\lambda}d(y_{k+1}, y_{k+1}+e_2)$, proving the claim.

Now, we infer that $y_0, y_0+e_1, y_1, y_1+e_1, y_2, \ldots$ is a 1-way Cauchy sequence, which is a contradiction. ■

But now, Proposition 7.1 provides us with a $C_{3,7}$-good point, which however cannot exist because of the lemmata we have shown during this proof. ■

**8. Final contradiction.** In the remainder of the proof of Proposition 2.3, an important role will be played by the sets

$$S_i(K, x_0) = \{y : d(x_0, y) \le K\mu,\ d(y, y+e_i) \le K\mu\},$$

defined for any point $x_0$, constant $K$ and $i \in [3]$. Given any point $t$, the set $S_i(K, x_0)$ serves to give approximate versions of contractions of $x_0$ and $t$ in the direction $i$, in the following sense. If $t \overset{i}{\frown} y$ for some $y \in S_i(K, x_0)$, then

$$\begin{aligned}d(x_0, t+e_i) &\le d(x_0, y) + d(y, y+e_i) + d(y+e_i, t+e_i) \\ &\le K\mu + K\mu + \lambda d(y, t) \le 2K\mu + \lambda(d(y, x_0) + d(x_0, t)) \\ &\le (2+\lambda)K\mu + \lambda d(x_0, t).\end{aligned}$$

Using this idea, unless $t$ never contracts with $S_i(K, x_0)$ in the direction $i$ for some $i$, we can get 3-way sets of small diameter, as we shall see in the proof of the next proposition.

An additional benefit of using these sets is that they usually do not only consist of $x_0$ (note $x_0 \in S_i(K, x_0)$ if $\rho(x_0) \le K\mu$), and for example, under certain circumstances, we can find a point $y$ with $y, y+e_3 \in S_3(K, x_0)$. Such points will then be used in proving Propositions 8.3 and 8.4, which combined with the following proposition finish the main proof of this paper.

Recall that $x \overset{i}{\not\frown} y$ means that $d(x+e_i, y+e_i) > \lambda d(x,y)$.

Proposition 8.1. *Fix $x_0$ with $\rho(x_0) < 2\mu$. Given $K \geq 2$, when $i \in [3]$, define $S_i(K, x_0) = \{y : d(x_0, y) \leq K\mu,\ d(y, y+e_i) \leq K\mu\}$. Provided $1 > 2\lambda K C_1 (2+\lambda)^2/(1-\lambda)$, in every $\langle z\rangle_3$ there is $t$ such that $d(t, x_0) \leq \frac{2+\lambda}{1-\lambda} K\mu$, but for some $i$ we have $s \overset{i}{\not\frown} t$ whenever $s \in S_i(K, x_0)$.*

*Proof.* First of all, we have $x_0 \in S_1(K, x_0), S_2(K, x_0), S_3(K, x_0)$, making these non-empty, as $K\mu \geq \rho(x_0) \geq d(x_0, x_0+e_i)$ for all $i \in [3]$. Suppose that, contrary to our statement, there is $z$ without any $t$ as described above. Since $\frac{2+\lambda}{1-\lambda} K\mu > \rho(x_0)/(1-\lambda)$, we know that there is $y \in \langle z\rangle_3$ such that $d(x_0, y) \leq \frac{2+\lambda}{1-\lambda} K\mu$, by Lemma 4.2. Then we have $s_1 \in S_1$ such that $s_1 \overset{1}{\frown} y$. Hence $d(y+e_1, x_0) \leq d(y+e_1, s_1+e_1) + d(s_1+e_1, s_1) + d(s_1, x_0) \leq \lambda(d(y, x_0) + d(x_0, s_1)) + 2K\mu \leq \lambda\big(\frac{2+\lambda}{1-\lambda} K\mu + K\mu\big) + 2K\mu = \frac{2+\lambda}{1-\lambda} K\mu$. Similarly, we get the same result for $y+e_2, y+e_3$, and so we have constructed a 3-way set of diameter not greater $2\frac{2+\lambda}{1-\lambda} K\mu$, but there are no such sets since $1 > 2\lambda K C_1(2+\lambda)^2/(1-\lambda)$ by Proposition 5.9, giving a contradiction. ∎

As before, we use tighter constraints on $\lambda$. Here we use the fact that $\lambda < 1/10$ implies $(2+\lambda)/(1-\lambda) < 3$ and $(2+\lambda)^2/(1-\lambda) < 5$. The corollary below is Proposition 3.2 described in the overview of the proof.

Corollary 8.2. *Fix $x_0$ with $\rho(x_0) < 2\mu$. Given $K \geq 2$, provided $1 > 10\lambda K C_1$, in every $\langle z\rangle_3$ there is $t$ such that $d(t, x_0) \leq 3K\mu$, but for some $i \in [3]$ we have $s \overset{i}{\not\frown} t$ whenever $s \in S_i(K, x_0)$.*

Based on this, we shall reach the final contradiction in the proof of Proposition 2.3. To do so, we shall consider the possible cases for $d(t+e_j, t+e_k)$ where $\{i,j,k\} = [3]$ and $t$ is given by Corollary 8.2. Namely, suppose that $d(t+e_j, t+e_k)$ is small enough, and in fact $j = 1$, $k = 2$, $i = 3$. Then whenever we have $y \in S_3(K, x_0)$ with $d(y+e_1, y+e_2)$ small, we shall have $\operatorname{diam}\{y+e_1, y+e_2, t+e_1, t+e_2\}$ small as well. On the other hand, if $d(t+e_1, t+e_2)$ is large, and $y_1, y_2 \in S_3(K, x_0)$ with $d(y_1+e_1, y_1+e_2), d(y_2+e_1, y_2+e_2)$ small but $d(y_1+e_1, y_2+e_1)$ large, we shall have pairs $t, y_1$ and $t, y_2$ contracted by different values in $\{1,2\}$. Of course, we need to specify what we mean by small and large in this context, and this is done in the following two propositions.

Proposition 8.3. *Let $C_4 = 16C_3$. Fix $x_0$ with $\rho(x_0) < 2\mu$. Let $\{i,j,k\} = [3]$. Given $K$, provided $\lambda < 1/(44C_3 + 6C_4 + K), 1/(34440 C_1 C_3)$, we have*

*$d(t+e_j, t+e_k) > K\lambda\mu$ when $t$ is such that $d(t, x_0) \leq 3C_4\mu$ and $s \stackrel{i}{\not\nearrow} t$ whenever $s \in S_i(C_4, x_0)$.*

PROPOSITION 8.4. *Let $C_5 = 1000C_3$. Fix $x_0$ with $\rho(x_0) < 2\mu$. Let $\{i, j, k\} = [3]$. Provided $\lambda < 1/(8200000C_1C_3)$, we have $d(t+e_j, t+e_k) \leq 10C_5\lambda\mu$ when $t$ is such that $d(t, x_0) \leq 3C_5\mu$ and $s \stackrel{i}{\not\nearrow} t$ whenever $s \in S_i(C_5, x_0)$.*

Once we have shown these propositions, we just need to take $\lambda$ small enough so that they both hold. Now, let us prove a lemma that classifies the possible relevant diagrams, which will be used in further arguments. Once that is done, we proceed to establish the propositions.

LEMMA 8.5. *Let $K \geq 1$ and $\lambda < 1/(4920KC_1)$. Suppose that we have a point $y$ with $\rho(y) \leq K\mu$ and $d(y+e_1, y+e_2) \leq \lambda K\mu$. Then $y$ must have one of the diagrams shown in Figure 8 (up to symmetry).*

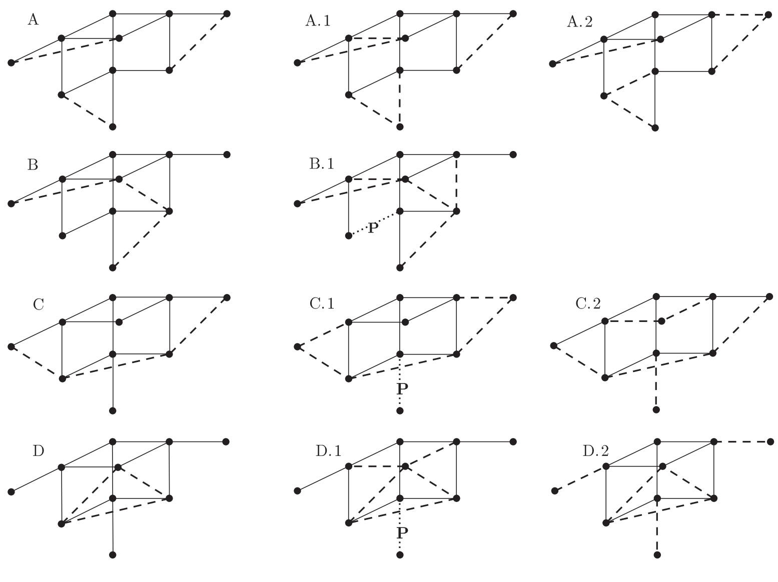


Fig. 8. Possible diagrams for $\rho(y) \leq K\mu$ and $d(y+e_1, y+e_2) \leq \lambda K\mu$

*Proof.* Contracting the long edges in $N(y) \cup \{y\}$ can only, up to symmetry, give us diagrams A, B, C and D, as described in the first part of the Appendix, with the requirement $1/(164KC_1) > \lambda$. Observe that in B, C and D, we can apply Proposition 6.4 to $(y; y+e_3, y+e_2, y+e_1; y+e_1)$, $(y; y+e_2, y+e_1, y+e_3; y+e_3)$ and $(y; y+e_1, y+e_2, y+e_3; y+e_3)$ respectively with constant $6K$, as long as $\lambda < 1/(4920KC_1)$. Further, by contracting the short edges, we can only obtain diagrams A.1, A.2, B.1, etc. in Figure 8, up to symmetry, as otherwise we obtain a point $p \in \{y\} \cup N(y)$ with $\rho(p) < \mu$. ∎

*Proof of Proposition 8.3.* We prove the claim for $i = 3$, $j = 2$, $k = 1$; the other cases follow by symmetry. Suppose that for some $K$ and $\lambda < 1/(44C_3+6C_4+K), 1/(34440C_1C_3)$, we have $t_0$ such that $d(t_0+e_1, t_0+e_2) \leq K\lambda\mu$, $d(t_0, x_0) \leq 3C_4\mu$ and $s \overset{3}{\not\frown} t_0$ whenever $s \in S_3(C_4, x_0)$, where $x_0$ is a point with $\rho(x_0) < 2\mu$.

Consider now a point $y$ with $\rho(y) \leq 7C_3\mu$, $d(y + e_1, y + e_2) \leq 7\lambda C_3\mu$; its existence is granted by Proposition 7.2. Apply Lemma 8.5 to $y$. Now we shall discard some of the diagrams by contractions with $t_0$. Suppose that $y$ had diagram A.1. By FNI, $d(y, x_0) \leq (7C_3 + 2)\mu/(1 - \lambda) \leq 8C_3\mu$, and so $y, y + e_3 \in S_3(C_4, x_0)$. Hence $y, t_0$ and $y + e_3, t_0$ would be contracted by 1 or 2. However, from this we see that if $y + e_3 \overset{1}{\frown} t_0$ we get

$$\begin{aligned}
\rho(y + e_1) &= d(y + e_1, y + e_3 + e_1) \\
&\leq d(y + e_1, t_0 + e_1) + d(t_0 + e_1, y + e_3 + e_1) \\
&\leq d(y + e_1, y + e_2) + \lambda d(t_0, y) \\
&\quad + d(t_0 + e_1, t_0 + e_2) + \lambda(d(t_0, y) + d(y, y + e_3)) \\
&\leq 7\lambda C_3\mu + 3\lambda C_4\mu + \lambda d(x_0, y) + \lambda K\mu \\
&\quad + 3\lambda C_4\mu + \lambda d(x_0, y) + 7\lambda C_3\mu \\
&\leq \lambda(30C_3 + 6C_4 + K)\mu < \mu.
\end{aligned}$$

On the other hand, if $y + e_3 \overset{2}{\frown} t_0$, we get

$$\begin{aligned}
\rho(y + e_2) &\leq d(y + e_3 + e_2, y + e_2) + d(y + e_2 + e_3, y + e_2 + e_2) \\
&\leq d(y + e_3 + e_2, t_0 + e_2) + d(t_0 + e_2, y + e_2) + 14\lambda C_3\mu \\
&\leq \lambda(d(t_0, x_0) + d(x_0, y) + d(y, y + e_3)) + d(y + e_1, y + e_2) \\
&\quad + d(t_0 + e_1, t_0 + e_2) + \lambda d(y, t_0) + 14\lambda C_3 \\
&\leq 3\lambda C_4\mu + 8\lambda C_3\mu + 7\lambda C_3\mu + 7\lambda C_3\mu + \lambda K\mu + 3\lambda C_4\mu \\
&\quad + 8\lambda C_3\mu + 14\lambda C_3\mu \\
&\leq \lambda(44C_3 + 6C_4 + K)\mu < \mu.
\end{aligned}$$

Similarly, if it were A.2 instead of A.1, we would have $y, y + e_1 \in S_3(C_4, x_0)$ and so contracting these two points with $t_0$ would give

$$\begin{aligned}
\rho(y + e_2) &= d(y + e_2 + e_1, y + e_2) \\
&\leq d(y + e_1 + e_2, y + 2e_1) + \lambda d(y + e_1, t_0) \\
&\quad + d(y + e_1, y + e_2) + \lambda d(y, t_0) + \lambda K\mu \\
&\leq \lambda(14C_3 + 15C_3 + 3C_4 + 8C_3 + 3C_4 + K)\mu < \mu.
\end{aligned}$$

Now consider diagrams C.2 and D.2. We have $y, y + e_3 \in S_3(C_4, x_0)$, so contracting these points with $t_0$ must be by 1 or 2, so we immediately get $\rho(y + e_1) \leq \lambda(44C_3 + 6C_4 + K)\mu < \mu$.

Therefore, we can only have diagrams B.1, C.1, D.1, or a diagram symmetric to B.1, which we shall refer to as B.2. Suppose now that $y$ with $\rho(y) \leq 7C_3\mu$, $d(y+e_1, y+e_2) \leq 7\lambda C_3\mu$ had diagram C.1 or D.1. Also, assume $\rho(y+e_3) \leq 7C_3\mu$, $d(y+e_3+e_1, y+e_3+e_2) \leq \lambda C_3\mu$, thus $y+e_3$ itself has one of the above diagrams. Suppose that it had diagram B.1 or B.2. Without loss of generality, it is B.1, since the other case is symmetric.

Suppose $y$ has diagram C.1. Then given any point $z$ with $d(z, y) \leq 2\rho(y)$, suppose $d(y+e_1, z+e_1), d(y+e_2, z+e_2) > 5\lambda\rho(y)$. Then $z \overset{3}{\frown} y$ and so $y+e_1 \overset{2}{\frown} z$, $y+e_2 \overset{1}{\frown} z$. However, we can apply Proposition 6.4 to $(y; y+e_2, y+e_1, y+e_3; y+2e_3)$ with constant $42C_3$ to see that $\operatorname{diam} N(z) \leq 800\lambda C_3\mu$, so after contracting $y, z$ we obtain $\rho(z) < 12C_3\mu$ and applying Proposition 6.3 gives a contradiction, provided $\lambda < 1/(32800C_1C_3)$. So whenever $d(z, y) \leq 2\rho(y)$, we must have $d(y+e_1, z+e_1) \leq 5\lambda\rho(y)$ or $d(y+e_2, z+e_2) \leq 5\lambda\rho(y)$. But contract $z$ with $y+e_2$ in the former case and with $y+e_1$ in the latter to see that for some choice of distinct $i, j \in [3]$ we must have $d(z+e_i, y+e_1), d(z+e_j, y+e_1) \leq 20\lambda\rho(y)$, so $d(z+e_i, y), d(z+e_j, y) \leq 2\rho(y)$, thus we can repeat these arguments for the points $z+e_i, z+e_j$. Doing so, we obtain a 2-way set of diameter at most $280\lambda C_3\mu$ by considering the distance from $y+e_1$, if the point $z$ is removed. But, by Lemma 4.2, we get such a 2-way set in every 3-way set, which is a contradiction by Proposition 5.10, since $\lambda < 1/(840C_3)$. Similarly we argue if $y$ had diagram D.1.

We conclude that if $y$ is as described and has diagram C.1 or D.1, then $y+e_3$ also has one of these two diagrams. Now, start from a point $y_0$ with $d(y_0+e_1, y_0+e_2) \leq 3\lambda C_3\mu$, $\rho(y_0) \leq 3C_3\mu$ and diagram C.1 or D.1, provided such a point exists. Define the sequence $y_n = y_0 + ne_3$ for all $n \geq 0$; we aim to show that it is Cauchy. By induction on $n$ we shall show that $\rho(y_n) \leq 7C_3\mu$, $d(y_n+e_1, y_{n+1}+e_1), d(y_n+e_2, y_{n+1}+e_2) \leq (n+3)\lambda^{n+1}C_3\mu$, $d(y_n+e_1, y_n+e_2) \leq 3C_3\lambda^{n+1}\mu$ and $y_n$ has either diagram C.1 or diagram D.1, which is true for $n = 0$.

Suppose the claim holds for all $m$ not greater than some $n \geq 0$. By Proposition 6.4 applied to $(y_0; p_1, p_2, p_3; y_n)$ with constant $18C_3$ with suitable $\{p_1, p_2, p_3\} = N(y_0)$ we get $d(y_1, y_{n+1}) \leq 288\lambda C_3\mu$, so we infer that

$$\begin{aligned}\rho(y_{n+1}) &\leq d(y_{n+1}, y_{n+1}+e_1) + d(y_{n+1}+e_1, y_{n+1}+e_2) \\ &\leq d(y_{n+1}, y_1) + d(y_1, y_0+e_2) + d(y_0+e_2, y_1+e_2) \\ &\quad + d(y_1+e_2, y_2+e_2) + \cdots + d(y_n+e_2, y_{n+1}+e_2) \leq 7C_3\mu\end{aligned}$$

and $y_n+e_1 \overset{3}{\frown} y_n+e_2$, so $d(y_{n+1}+e_1, y_{n+1}+e_2) \leq 3\lambda^{n+2}C_3$, therefore $y_{n+1}$ must itself have diagram C.1 or D.1. If $y_n$ and $y_{n+1}$ have the same diagram, then we can see that $y_n + e_1 \overset{3}{\frown} y_{n+1} + e_2$ and $y_{n+1} + e_1 \overset{3}{\frown} y_n + e_2$, which is sufficient to establish the claim, as we obtain

$$\begin{aligned}
d(y_{n+1} + e_1, y_{n+2} + e_1) &\leq d(y_{n+1} + e_1, y_{n+2} + e_2) + d(y_{n+2} + e_2, y_{n+2} + e_1) \\
&\leq \lambda(d(y_n + e_1, y_{n+1} + e_1) + d(y_{n+1} + e_1, y_{n+1} + e_2)) \\
&\quad + \lambda d(y_{n+1} + e_1, y_{n+1} + e_2) \\
&\leq \lambda d(y_n + e_1, y_{n+1} + e_2) + 6\lambda^{n+3} C_3 \mu \\
&\leq (n+3)\lambda^{n+2} C_3 \mu + \lambda^{n+2} C_3 \mu \\
&\leq (n+4)\lambda^{n+2} C_3 \mu.
\end{aligned}$$

Likewise, we get the bound on $d(y_{n+1} + e_2, y_{n+2} + e_2)$. If the diagrams are different, it must be the case that $y_n + e_1 \overset{3}{\frown} y_{n+1} + e_1$ and $y_n + e_2 \overset{3}{\frown} y_{n+1} + e_2$, once again proving the claim; this time this is immediate.

Hence, if $y$ is such that $\rho(y) \leq 3C_3\mu$, $d(y+e_1, y+e_2) \leq 3\lambda C_3\mu$, then it can only have diagram B.1 or B.2. In the light of this, pick $y_0$ with $\rho(y_0) \leq C_3\mu$, $d(y_0 + e_1, y_0 + e_2) \leq \lambda C_3 \mu$, whose existence is provided by Proposition 7.2, so it has diagram B.1, without loss of generality. Set $y_1 = y_0 + e_1$ and so $\operatorname{diam}\{y_1, y_1 + e_1, y_1 + e_2\} \leq 3\lambda\rho(y_0)$ for the diagram for $y_0$. Also, by Proposition 6.4 applied to $(y_0; y_0 + e_3, y_0 + e_2, y_0 + e_1; y_1)$ with constant $6C_3$ we get $\rho(y_1) \leq 3C_3\mu$, so $y_1$ has diagram B.1 or B.2. If it is B.1 define $y_2$ to be $y_1 + e_1$, otherwise set $y_2 = y_1 + e_2$. Continuing, if $y_k$ is defined and has one of these diagrams, define $y_{k+1} = y_k + e_1$ when $y_k$ has diagram B.1, and $y_{k+1} = y_k + e_2$ if it has diagram B.2. We now claim that $y_k$ is defined, $\rho(y_k) \leq 3C_3\mu$ and $\operatorname{diam}\{y_k, y_k + e_1, y_k + e_2\} \leq 3(3\lambda)^k C_3 \mu$. This is clear for $k = 0$.

Suppose the claim holds for some $k \geq 0$. Then $y_k$ has diagram B.1 or B.2, say the former; we argue in the same way for the other option. Firstly, $y_{k+1}$ is defined. Then, from contractions implied by diagram B.1, we get $\operatorname{diam}\{y_{k+1}, y_{k+1} + e_1, y_{k+1} + e_2\} \leq 3\lambda \operatorname{diam}\{y_k, y_k + e_1, y_k + e_2\}$. Finally, as $d(y_0, y_k) \leq (\rho(y_0) + \rho(y_k))/(1-\lambda) < 5C_3\mu$, we may apply Proposition 6.4 to $(y_0; y_0 + e_3, y_0 + e_2, y_0 + e_1; y_{k+1})$ with constant $6C_3$ to obtain

$$\begin{aligned}
\rho(y_{k+1}) &= d(y_{k+1}, y_{k+1} + e_3) \\
&\leq d(y_{k+1}, y_0) + d(y_0, y_0 + e_3) + d(y_0 + e_3, y_{k+1} + e_3) \\
&\leq d(y_{k+1}, y_k) + d(y_k, y_{k-1}) + \cdots + d(y_1, y_0) + \rho(y_0) + 96\lambda C_3 \mu \\
&\leq 9\lambda C_3\mu/(1-3\lambda) + 2C_3\mu + 96\lambda C_3\mu \leq 3C_3\mu,
\end{aligned}$$

which proves the claim.

This brings us to the conclusion that $(y_k)_{k\geq 0}$ is a 1-way Cauchy sequence, yielding a contradiction. ■

*Proof of Proposition 8.4.* During the course of our argument, we shall prove a few auxiliary lemmata, the last one being Lemma 8.9, allowing us to conclude the proof. It suffices to prove the claim for $i = 3$, $j = 2$, $k = 1$. Suppose there is $t_0$ with $d(t_0 + e_1, t_0 + e_2) > 10\lambda C_5\mu$, $d(t_0, x_0) \le 3C_5\mu$, and whenever $s \in C_3(C_5, x_0)$ we must have either $s \overset{1}{\frown} t_0$ or $s \overset{2}{\frown} t_0$.

Set $C_{5,1} = 100C_3$ and consider the points $y$ with $\rho(y) \le C_{5,1}\mu$ and $d(y + e_1, y + e_2) \le \lambda C_{5,1}\mu$. Note that such a point exists by Proposition 7.2. The possible diagrams of contractions are shown in Figure 9, and the ar-

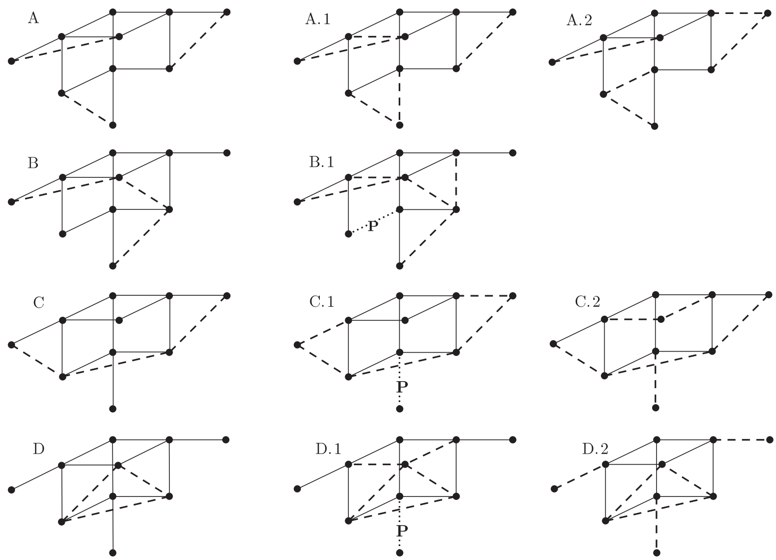


Fig. 9. Possible diagrams of points $p$ with $d(p + e_1, p + e_2) \le \lambda C_{5,1}\mu$, $\rho(p) \le C_{5,1}\mu$

guments to justify these are provided in the Appendix. These are precisely the same diagrams as in the previous proposition. Using $d(t_0 + e_1, t_0 + e_2) > 10\lambda C_5\mu$, we reject most of these.

B.1: Suppose that $y$ as above has diagram B.1. First of all, as $\lambda < 1/(4920C_1C_{5,1})$, apply Proposition 6.4 to $(y; y + e_3, y + e_2, y + e_1; y)$ with constant $6C_{5,1}$ to see that in particular $y, y + e_1, y + e_2, y + e_3$ are all in $C_3(x_0, C_5)$, as $d(y, x_0) \le (C_{5,1} + 2)\mu/(1 - \lambda)$, $\rho(y) \le C_{5,1}\mu$, $d(y, y + e_3) \le C_{5,1}\mu$, $d(y + e_1, y + e_1 + e_3) \le (2 + 96\lambda)C_{5,1}\mu$, $d(y + e_2, y + e_2 + e_3) \le \lambda C_{5,1}\mu$ and $d(y + e_3, y + 2e_3) \le (2 + 3\lambda)C_{5,1}\mu$.

If $t_0 \overset{1}{\frown} y$, then contract $t_0, y + e_3$ to get $\rho(y + e_1) < 6\lambda(C_{5,1} + C_5)\mu < \mu$ when $t_0 \overset{1}{\frown} y + e_3$, or

$$
\begin{aligned}
&d(t_0 + e_1, t_0 + e_2) \\
&\qquad \le d(t_0 + e_1, y + e_1) + d(y + e_1, y + e_3 + e_2) + d(y + e_3 + e_2, t_0 + e_2) \\
&\qquad \le \lambda(3C_{5,1} + 3C_5)\mu + 3\lambda C_{5,1}\mu + (3C_{5,1} + 3C_5)\mu < 10\lambda C_5\mu
\end{aligned}
$$

otherwise, both of which are not permissible.

C.1: Suppose $y$ has diagram C.1. Then $d(y,y+e_3) \leq C_{5,1}\mu$, $d(y+e_1, y+e_1+e_3), d(y+e_2,y+e_2+e_3) \leq 3\lambda C_{5,1}\mu$, $d(y+e_3,y+2e_3) \leq 96\lambda C_{5,1}\mu$. Also $d(y,x_0) \leq (C_{5,1}+2)\mu/(1-\lambda)$, $\rho(y) \leq C_{5,1}\mu$, so $y, y+e_1, y+e_2, y+e_3 \in S_3(x_0, C_5)$. Without loss of generality $y \overset{1}{\frown} t_0$. But if $y+e_2 \overset{1}{\frown} t_0$, then $\rho(y+e_1) = d(y+e_1, y+e_1+e_2) \leq \lambda d(y,t_0) + \lambda d(y+e_2,t_0) \leq 6\lambda(C_{5,1}+C_5)\mu < \mu$. However, $y+e_2 \overset{2}{\frown} t_0$ is impossible as well, for it implies $d(t_0+e_1, t_0+e_2) \leq d(t_0+e_1, y+e_1) + d(y+e_1, y+2e_2) + d(y+2e_2, t_0+e_2) \leq 6\lambda(C_{5,1}+C_5)\mu + 7\lambda C_{5,1} < 10\lambda C_5\mu$.

C.2: Assume that $y$ has diagram C.2. First of all apply Proposition 6.2 to $y$ (we have $\lambda < 1/(78C_{5,1})$) to see that $d(y+e_1, y+e_1+e_3), d(y+e_2, y+e_2+e_3) \leq 9C_{5,1}\mu$. Also $d(y+e_3, y+2e_3) \leq \lambda C_{5,1}\mu$, $\rho(y) \leq C_{5,1}\mu$, $d(y,x_0) \leq (C_{5,1}+2)\mu/(1-\lambda)$, so $y, y+e_1, y+e_2, y+e_3 \in S_3(x_0,C_5)$. Without loss of generality $y \overset{1}{\frown} t_0$. If $y+e_1 \overset{1}{\frown} t_0$ then $\rho(y+e_1) \leq d(y+e_1, y+2e_1) + d(y+2e_1, y+e_1+e_2) \leq \lambda(d(y,t_0) + d(y+e_1,t_0)) + 2\lambda C_{5,1}\mu \leq 6\lambda(C_{5,1}+C_5)\mu + 2\lambda C_{5,1} < \mu$. So, we must have $y+e_1 \overset{2}{\frown} t_0$, but this also implies a contradiction as

$$\begin{aligned} d(t_0+e_1, t_0+e_2) &\leq d(t_0+e_1, y+e_1) + d(y+e_1, y+y_1+e_2) \\ &\quad + d(y+e_1+e_2, t_0+e_2) \\ &\leq \lambda d(y,t_0) + \lambda C_{5,1}\mu + \lambda d(y+e_1,t_0) \\ &\leq 6\lambda(C_5+C_{5,1})\mu + \lambda C_{5,1}\mu < 10\lambda C_5\mu. \end{aligned}$$

D.1: Let $y$ have diagram D.1. Then $d(y,y+e_3) \leq C_{5,1}\mu$, $d(y+e_1, y+e_1+e_3)$, $d(y+e_2,y+e_2+e_3) \leq 3\lambda C_{5,1}\mu$, $d(y+e_3,y+2e_3) \leq 96\lambda C_{5,1}\mu$. Also $d(y,x_0) \leq (C_{5,1}+2)\mu/(1-\lambda)$, $\rho(y) \leq C_{5,1}\mu$, so $y, y+e_1, y+e_2, y+e_3 \in S_3(x_0,C_5)$. Without loss of generality $y \overset{1}{\frown} t_0$. If $t_0 \overset{1}{\frown} y+e_1$, then $\rho(y+e_1) = d(y+e_1, y+2e_1) \leq d(y+e_1, t_0+e_1) + d(t_0+e_1, y+2e_1) \leq \lambda(d(y,t_0) + d(t_0, y+e_1)) \leq 6\lambda(C_5+C_{5,1}) < \mu$. On the other hand, $t_0 \overset{2}{\frown} y+e_1$ implies $d(t_0+e_1, t_0+e_2) \leq d(t_0+e_1, y+e_1) + d(y+e_1, y+e_1+e_2) + d(y+e_1+e_2, t_0+e_2) \leq \lambda(6C_5 + 7C_{5,1})\mu < 10\lambda C_5\mu$. Thus, $y$ cannot have diagram D.1.

D.2: Suppose that $y$ as above has diagram D.2. First of all, apply Proposition 6.2 to $y$ to see that $d(y+e_1, y+e_1+e_3), d(y+e_2, y+e_2+e_3) \leq 9C_{5,1}\mu$. Also $d(y+e_3, y+2e_3) \leq \lambda C_{5,1}\mu$, $\rho(y) \leq C_{5,1}\mu$, $d(y,x_0) \leq (C_{5,1}+2)\mu/(1-\lambda)$, so $y, y+e_1, y+e_2, y+e_3 \in S_3(x_0, C_5)$. Without loss of generality $y \overset{1}{\frown} t_0$. Now contract $y+e_2, t_0$. If these are contracted by 1, then $\rho(y+e_1) \leq d(y+e_1, y+e_1+e_2) + d(y+e_1+e_2, y+e_1+e_3) \leq \lambda(6C_5 + 8C_{5,1})\mu < \mu$, a contradiction. Therefore $t_0 \overset{2}{\frown} y+e_2$, which gives $d(t_0+e_1, t_0+e_2) \leq d(t_0+e_1, y+e_1) + d(y+e_1, y+2e_2) + d(y+2e_2, t_0+e_2) \leq \lambda(6C_5+8C_{5,1})\mu < 10\lambda C_5\mu$.

Thus, we are only left with diagrams A.1 and A.2. Let A.1′ and A.2′ be symmetric to these after swapping $e_1$ and $e_2$. Let $y$ be the same point as before. We now distinguish the possibilities for contractions with $t_0$.

If $y$ has diagram A.1, then $y, y+e_1, y+e_3 \in S_3(x_0, C_5)$, and it is easy to see that $t_0, y$ and $t_0, y+e_1$ are contracted in the same direction, while $t_0, y+e_3$ is contracted in the other. Similarly, we see the possible contractions with $t_0$ for diagram A.1′.

If $y$ has diagram A.2, all the points in $\{y\} \cup N(y)$ are in $S_3(x_0, C_5)$, and pairs $t_0, y$ and $t_0, y+e_1$ must be contracted in different directions (otherwise $\rho(y+e_2) < \mu$). The same holds for the pairs $y+e_1, t_0$ and $y+e_3, t_0$. From this we see that $t_0 \overset{2}{\frown} y$, $t_0 \overset{2}{\frown} y+e_3$, $t_0 \overset{1}{\frown} y+e_1$. Analogously, we classify the contractions for A.2′.

LEMMA 8.6. *Let $K \le C_{5,1}$. There is no sequence $(y_k)_{k\in I}$ for suitable index set $I \subset \mathbb{N}_0$, with the following properties:*

1. *$y_0$ is defined, has $\rho(y_0) \le K/(2+6\lambda)$, $d(y_0+e_1, y_0+e_2) \le \lambda K/(2+\lambda)\mu$.*
2. *If $y_k$ is defined and satisfies $\rho(y_k) \le K\mu$, $d(y_0+e_1, y_0+e_2) \le \lambda K\mu$, then $y_k$ has diagram* A.1 *or* A.1′, *and we define $y_{k+1} = y_k + e_i$, with $i = 1$ when the diagram of $y_k$ is* A.1 *and $i = 2$ otherwise.*

*Proof.* We claim that $y_k$ is defined and $\operatorname{diam}\{y_k, y_k+e_1, y_k+e_2\} \le (3\lambda)^k K\mu/(2+6\lambda)$. This trivially holds for $k = 0$. Also, without loss of generality $y_0$ has diagram A.1.

Suppose that the claim holds for all $k' \le k$, where $k \ge 0$. Observe that

$$\begin{aligned} d(y_0, y_k) &\le d(y_0, y_1) + d(y_1, y_2) + \cdots + d(y_{k-1}, y_k) \\ &\le (1 + 3\lambda + \cdots + (3\lambda)^{k-1})\frac{K\mu}{2+6\lambda} < \frac{1}{(1-3\lambda)(2+6\lambda)} K\mu. \end{aligned}$$

Now, contract $y_0+e_3, y_k$. It is contracted neither by 1 nor by 2, since we get either $\rho(y_0+e_1) < \mu$ or $\rho(y_0+e_2) < \mu$. Hence $y_k \overset{3}{\frown} y_0+e_3$, so

$$\begin{aligned} d(y_k+e_3, y_0+e_3) &\le d(y_k+e_3, y_0+2e_3) + d(y_0+2e_3, y_0+e_3) \\ &\le \frac{\lambda(3-6\lambda)}{(1-3\lambda)(2+6\lambda)} K\mu < 2\lambda K\mu. \end{aligned}$$

Finally, we establish $\rho(y_k) \le (2+6\lambda)K\mu/(2+6\lambda) = K\mu$, which combined with $d(y_k+e_1, y_k+e_2) \le \lambda K\mu$ gives that $y_k$ itself has diagram A.1 or A.1′. Hence $y_{k+1}$ is defined, and $\operatorname{diam}\{y_{k+1}, y_{k+1}+e_1, y_{k+1}+e_2\} \le 3\lambda \operatorname{diam}\{y_k, y_k+e_1, y_k+e_2\}$, as desired. However, this shows that $(y_k)_{k\ge 0}$ is a 1-way Cauchy sequence, which is not allowed. ■

COROLLARY 8.7. *There exists a point $y$ with $\rho(y) \le 3C_3\mu$, $d(y+e_1, y+e_2) \le 3\lambda C_3\mu$ with diagram* A.2 *or* A.2′.

*Proof.* Suppose the contrary, and let $y_0$ be a point with $\rho(y_0) \leq C_3\mu$ and $d(y_0 + e_1, y_0 + e_2) \leq \lambda C_3\mu$, given by Proposition 7.2. We shall now define a sequence $(y_k)$ inductively, as long as we can. The starting point $y_0$ is as above. Given $y_k$, provided it satisfies $\rho(y_k) \leq 3C_3\mu$, $d(y_k + e_1, y_k + e_2) \leq 3\lambda C_3\mu$, define $y_{k+1}$ to be $y_k + e_1$ when $y_k$ has diagram A.1, and $y_k + e_2$ if $y_k$ has diagram A.1′ (note that by assumption these two are the only permissible diagrams). But this gives a contradiction by Lemma 8.6 with $K = 3C_3$. ∎

Corollary 8.8. *We have $d(t_0 + e_1, t_0 + e_2) > 5C_3\mu$.*

*Proof.* Suppose the contrary. In order to reach a contradiction, we shall obtain a Cauchy sequence as in the previous proof. Consider a point $y$ with $\rho(y) \leq 36C_3\mu$, $d(y + e_1, y + e_2) \leq 36\lambda C_3\mu$. Assume that this point has diagram A.2. Recall that $t_0 \overset{1}{\frown} y + e_1$, $t_0 \overset{2}{\frown} y$. This gives $\rho(y + e_1) \leq C_{5,1}\mu$, and from contractions of $\{y\} \cup N(y)$ we get $d(y+e_1+e_1, y+e_1+e_2) \leq \lambda C_{5,1}\mu$ as well, so $y + e_1$ has one of the four diagrams considered so far. However, we immediately see that it is not possible for $y + e_1$ to have diagram A.2, for $t_0 \overset{1}{\frown} y + e_1$.

Suppose that $y + e_1$ has diagram A.2′. Firstly, suppose that $y + e_1 \overset{3}{\frown} y + e_2 + e_3$. Then contract $y, y + 2e_3$. If it is by 3, we have $\rho(y + 2e_3) < \mu$, otherwise $\rho(y + e_3) < \mu$. Hence $y + e_1 \overset{2}{\frown} y + e_2 + e_3$. This further implies $y+e_1 \overset{2}{\frown} y+2e_2$ (or otherwise $\rho(y+2e_1) < \mu$). However, $y+2e_2 \in S_3(x_0, C_5)$, so contract $y + e_2, t_0$ to get a contradiction.

Suppose now that $y$ has diagram A.1 and $\rho(y) \leq 17C_3\mu$, $d(y+e_1, y+e_2) \leq 17\lambda C_3\mu$. If $y + e_1$ has diagram A.2, then $y + e_1 \overset{2}{\frown} t_0$, $y + e_1 + e_3 \overset{2}{\frown} t_0$, $y + 2e_1 \overset{1}{\frown} t_0$. But $y$ has diagram A.1, so $t_0$ contracts with $y + e_1, y$ in the same direction, thus in $e_2$, and with $t_0, y + e_3$ in the other, i.e. $e_1$. However, then $\operatorname{diam} N_1(x + 2e_1) < 10\lambda C_5\mu$, contrary to Proposition 6.3 used with constant $10C_5$ after contracting $y, y + 2e_1$.

Assume that $y + e_1$ has diagram A.2′. Thus $t_0 \overset{1}{\frown} y + e_1$, $t_0 \overset{1}{\frown} y + e_1 + e_3$ and $t_0 \overset{2}{\frown} y + e_1 + e_2$. As $y$ has diagram A.1, we have $t_0 \overset{1}{\frown} y$ and $t_0 \overset{2}{\frown} y + e_3$. But as $d(t_0 + e_1, t_0 + e_2) \leq 5C_3\mu$, $y + e_1 + e_2$ is $100C_3$-good, so by the previous discussion $y + e_1 + e_2$ can only have diagram A.1 or A.1′ (as $y + e_1$ is $36C_3$-good).

If $y+e_1+e_2$ has diagram A.1 then $t_0 \overset{1}{\frown} y+e_1+e_2+e_3$, so $\rho(y+e_1) < \mu$, so we may assume $y+e_1+e_2$ has diagram A.1′, which implies $t_0 \overset{1}{\frown} y+e_1+e_2+e_3$. Look at pairs $y+2e_1, y+2e_1+e_3$ and $y+2e_1+e_2, y+2e_1+e_3$; both have length at most $6C_3\mu$, so cannot be contracted by 2, as otherwise $d(t_0+e_1, t_0+e_2) < 10C_5\mu$. Suppose that at least one of these pairs is contracted by 1. Then apply Proposition 6.4 to $(y; y + 3e_1, y + 2e_1, y + e_1; y + e_2)$ with constant

$10C_5$ (since $\lambda < 1/(8200C_1C_5)$) to see that $\rho(y+3e_3) < \mu$. Hence, the two pairs considered are contracted by 3. But contract $y+e_2, y+2e_1+e_3$ to get $\rho(y+2e_1+e_3) < \mu$ or $d(t_0+e_1, t_0+e_2) < 200\lambda C_5\mu$, giving $\rho(y+e_2) < \mu$. Now, start from $y_0'$ with $\rho(y_0') \le C_3\mu$, $d(y_0'+e_1, y_0'+e_2) \le \lambda C_3\mu$, given by Proposition 7.2. If $y_0'$ has diagram A.1 or A.1′ set $y_0 = y_0'$, otherwise set $y_0 = y_0' + e_1$ if the diagram is A.2, and $y_0 = y_0' + e_2$ if the diagram is A.2′. Hence, $\rho(y_0) \le 6C_3\mu$, $d(y_0+e_1, y_0+e_2) \le 6\lambda C_3\mu$, and defining the sequence as in Lemma 8.6 gives a contradiction for $K = 17C_3$ by the discussion above. ∎

LEMMA 8.9. *Suppose that $y_1, y_2$ are two points with $\rho(y_1), \rho(y_2) \le C_3\mu$. Then $d(y_1+e_3, y_2+e_3) \le 40\lambda C_3\mu$.*

*Proof.* Recall that we have a point $y_0$ with $\rho(y_0) \le 6C_3\mu$, $d(y_0+e_1, y_0+e_2) \le 6\lambda C_3\mu$, with diagram A.2 or A.2′, given by Corollary 8.7. Without loss of generality the diagram is A.2.

Let $z$ be any point with $\rho(z) \le C_3\mu$. We shall prove $d(y_0+e_3, z+e_3) \le 20\lambda C_3\mu$, which is clearly sufficient. Note that $d(z, x_0) \le (C_3+2)\mu/(1-\lambda) \le C_5\mu$,$d(z, t_0) \le d(z, x_0) + d(x_0, t_0) \le (C_3+2)\mu/(1-\lambda) + 3C_5\mu \le 4C_5\mu$, and similarly $d(y_0, z) \le 4C_5\mu$ and $y_0, z \in S_3(x_0, C_5)$.

Assume $t_0 \overset{1}{\frown} z$. Recall that $t_0 \overset{2}{\frown} y_0$. If $y_0 \overset{1}{\frown} z$, then

$$\begin{aligned} d(t_0+e_1, t_0+e_2) &\le d(t_0+e_1, z+e_1) + d(z+e_1, y_0+e_1) \\ &\qquad + d(y_0+e_1, y_0+e_2) + d(y_0+e_2, t_0+e_2) \\ &\le \lambda 4C_5\mu + \lambda 7C_3/(1-\lambda) + 6\lambda C_3\mu + 4\lambda C_5\mu \\ &< 5C_3\mu < d(t_0+e_1, t_0+e_2), \end{aligned}$$

a contradiction. Similarly we discard the case $y_0 \overset{2}{\frown} z$, as then $d(t_0+e_1, t_0+e_2) \le d(t_0+e_1, z+e_1) + d(z+e_1, z+e_2) + d(z+e_2, y_0+e_2) + d(y_0+e_2, t_0+e_2) \le 5C_3\mu$. Therefore, $y_0 \overset{3}{\frown} z$, so $d(y_0+e_3, z+e_3) \le \lambda 7C_3\mu/(1-\lambda) < 8\lambda C_3\mu$.

Thus, we must have $z \overset{2}{\frown} t_0$. But we can have neither $y_0+e_1 \overset{1}{\frown} z$ nor $y_0+e_1 \overset{2}{\frown} z$, for otherwise we obtain

$$\begin{aligned} &d(t_0+e_1, t_0+e_2) \\ &\quad \le d(t_0+e_1, y_0+2e_1) + d(y_0+2e_1, z+e_2) + d(z+e_2, t_0+e_2) \\ &\quad \le \lambda d(t_0, y_0+e_1) + d(y_0+2e_1, y_0+e_1+e_2) \\ &\qquad + \lambda d(y+e_1, z) + 2\rho(z) + \lambda d(z, t_0) \le 5C_3\mu. \end{aligned}$$

Hence, $y_0+e_1 \overset{3}{\frown} z$, so $d(y_0+e_3, z+e_3) \le \lambda d(y_0+e_1, z) + d(y_0+e_1+e_3, y_0+e_3) \le 14\lambda C_3\mu + 6\lambda C_3\mu = 20\lambda C_3\mu$, as desired. ∎

We are now ready to establish the final contradiction. By Proposition 7.2, we have points $x_1, x_2, x_3$ such that whenever $\{i, j, k\} = [3]$, we have $\rho(x_i) \le$

$C_3\mu, d(x_i+e_j, x_i+e_k) \leq \lambda C_3\mu$. First of all, $x_1, x_2, x_3$ all belong to $S_3(x_0, C_5)$, since $d(x_0, x_i) \leq (C_3+2)\mu/(1-\lambda)$. Suppose that for some $i, j$ we have $t_0 \overset{1}{\frown} x_i$ and $t_0 \overset{2}{\frown} x_j$. Then, by the triangle inequality and FNI,

$$\begin{aligned} &d(t_0 + e_1, t_0 + e_2) \\ &\quad \leq d(t_0 + e_1, x_i + e_1) + d(x_i + e_1, x_i) + d(x_i, x_j) + d(x_j, x_j + e_2) \\ &\qquad + d(x_j + e_2, t_0 + e_2) \\ &\quad \leq \lambda(d(t_0, x_0) + d(x_0, x_i)) + \rho(x_i) + (\rho(x_i) + \rho(x_j))/(1 - \lambda) \\ &\qquad + \rho(x_j) + \lambda(d(x_j, x_0) + d(x_0, t_0)) \\ &\quad \leq \lambda(3C_5\mu + (\rho(x_0) + \rho(x_i))/(1 - \lambda)) + C_3\mu + 2C_3\mu/(1 - \lambda) \\ &\qquad + C_3\mu + \lambda((\rho(x_j) + \rho(x_0))/(1 - \lambda) + 3C_5\mu) \\ &\quad \leq 5C_3\mu, \end{aligned}$$

which is not possible, hence $t_0$ contracts with $x_1, x_2, x_3$ in the same direction, $e_1$ say. But also Lemma 8.9 gives $\operatorname{diam}\{x_1 + e_3, x_2 + e_3, x_3 + e_3\} \leq 40\lambda C_3\mu$, and $\operatorname{diam}\{x_1 + e_1, x_2 + e_1, x_3 + e_1\} \leq 8\lambda C_5\mu$, so $\operatorname{diam} N(x_1) \leq 9\lambda C_5\mu$, a contradiction by Proposition 6.3. ■

Now combine Corollary 8.2 with Propositions 8.3 and 8.4 to obtain a contradiction. ■

**9. Concluding remarks.** Let us restrict our attention once again to Austin's conjecture in its generality. Thus, we can formulate the following hypothesis, which captures its essence.

CONJECTURE 9.1. *Let $n$ be a positive integer and $\lambda$ a real with $0 \leq \lambda < 1$. Suppose $(\mathbb{N}_0^n, d)$ is an $n$-dimensional $\lambda$-contractive grid, that is, a pseudometric space with the property that given $x, y \in \mathbb{N}_0^n$ we have some $i \in [n]$ with $d(x + e_i, y + e_i) \leq \lambda d(x, y)$. Then there is a 1-way Cauchy sequence.*

Of course, having in mind the proof of Theorem 1.4, other similar versions of this hypothesis that can be formulated. Recall $\mu = \inf \rho(x)$, where $x$ ranges over all points in the grid, and set $\mu_\infty = \lim_{k\to\infty} \inf_{x\in S_k} \rho(x)$, where $S_k$ is the $n$-way set generated by $(k, \ldots, k)$. Also, say that a pseudometric space is a *contractive grid* if it is an $n$-dimensional $\lambda$-contractive grid for some $0 \leq \lambda < 1$ and a positive integer $n$.

QUESTION 9.2. *Can $\mu > 0$ occur in a contractive grid?*

QUESTION 9.3. *Can $\mu_\infty = \infty$ occur in a contractive grid?*

Note that the proofs we give here are of combinatorial nature, with the flavor of Ramsey theory, and it seems that the complete proof of Conjecture 9.1 should be based on a similar approach. Note that the proof of the Generalized Banach Theorem rests on the Ramsey Theorem.

QUESTION 9.4. *What is the combinatorial principle behind Conjecture 9.1? Is it related to Ramsey theory?*

Finally, we consider some points of the proof of Theorem 1.4, and pose the following questions and conjectures.

QUESTION 9.5. *What is the relation between $k$-way sets in higher-dimensional grids?*

CONJECTURE 9.6. *For each $k \geq 2$ there is a positive constant $C_k$ with the following property: Given a $k$-colouring of a countably infinite graph $G$, we can find sets of vertices $A_1, \ldots, A_{k-1}$ which cover the graph, and for suitable colours $c_1, \ldots, c_{k-1} \in [k]$ we have* $\operatorname{diam}_{c_i} G[A_i] \leq C_k$ *for all $i \in [k-1]$.*

QUESTION 9.7. *What conditions on a colouring $c$ of $\mathbb{N}_0^n$ ensure that the colouring is essentially trivial, that is, it is monochromatic on an $n$-way set?*

**Appendix. Discussion of the possible contraction diagrams.** In this appendix we discuss how we obtain the possible diagrams for contractions in the last part of the proof of Proposition 2.3. For this discussion we assume the propositions prior to Proposition 7.1 to hold.

Let us start with a point $x$ with $\rho(x) \leq K\mu$ for some $K \geq 1$. Consider first the contractions of the long edges, that is, those of the form $x+e_i, x+e_j$, where $i, j$ are distinct elements of $[3]$. If two such edges are contracted in the same direction, say $k$, then $\operatorname{diam} N(x+e_k) \leq 4\lambda K\mu$. Furthermore, we can contract $x, x+e_k$ to get $\rho(x_k) \leq (2+5\lambda)K\mu$, a contradiction by Proposition 6.3, provided $\lambda < 1/(164C_1K)$, which we shall assume. Thus, all three long edges must be contracted in different directions.

Contract now the short edges, i.e. those of the form $x, x+e_i$ for some $i \in [3]$. Given such an edge, there is a unique long edge $x+e_j, x+e_k$ such that $\{i,j,k\} = [3]$. We say that these edges are *orthogonal*. Suppose that a short edge $x+e_i$ is not contracted in the same direction as its orthogonal long edge. Then $x+e_i$ must be contracted in the same direction $e_l$ as $x+e_i, x+e_j$ for some $j \neq i$. Let $k$ be such that $\{i,j,k\} = [3]$. Then $x+e_k$ cannot be contracted in the same direction as $x+e_i$, as otherwise $\rho(x+e_l) \leq 3\lambda K\mu < \mu$, which is impossible. So, $x+e_k$ is contracted in the same direction as one of its non-orthogonal long edges. Hence $\operatorname{diam}\{x+e_l, x+e_l+e_i, x+e_l+e_j\}$, $\operatorname{diam}\{x+e_m, x+e_m+e_k, x+e_m+e_n\} \leq 3\lambda\rho(x)$ for some $m, n \in [3]$ where $m \neq l$ and $n \neq k$. From this we conclude that contractions in $\{x\} \cup N(x)$ can only give the diagrams shown in Figure 10. There, an edge shown as a dashed line has length at most $3\lambda\rho(x)$.

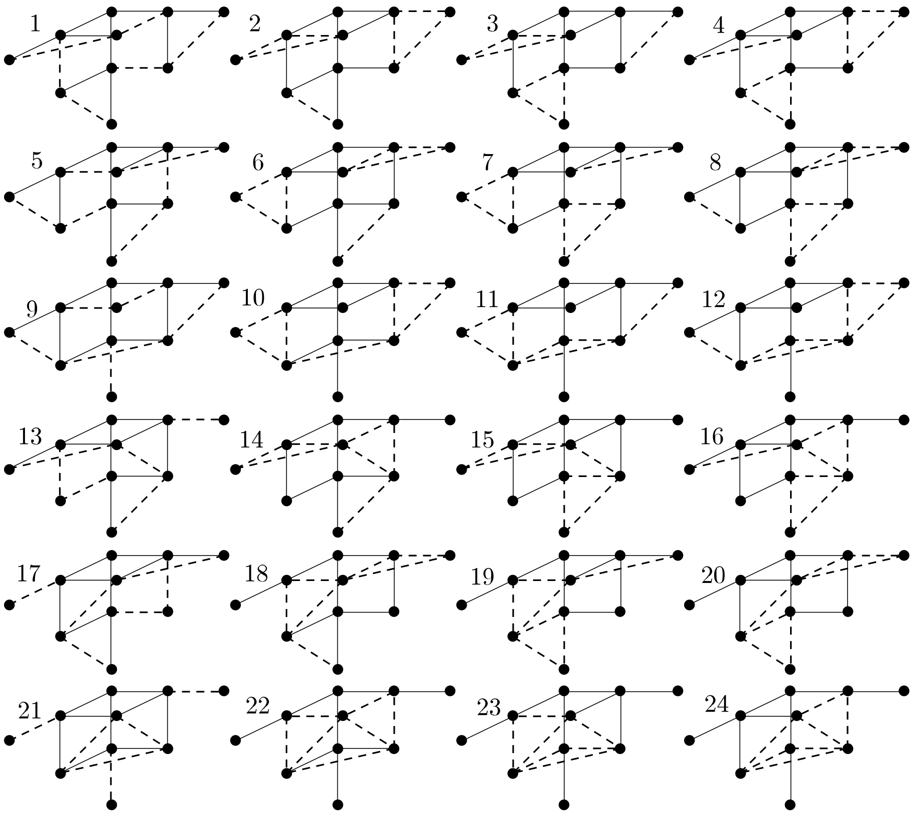


Fig. 10. All possible contraction diagrams

**A.1. Diagrams in the proof of Proposition 7.2.** As in the proof of Proposition 7.2, we consider a point $y$ with $d(y+e_1, y+e_3) \leq \lambda C_{3,1}\mu$ and $\rho(y) \leq C_{3,1}\mu$, that is, we set the previously considered $K$ to be $C_{3,1}$ instead, and so assume $\lambda < 1/(164C_1C_{3,1})$. Consider the possible diagrams of contractions of edges in $\{y\} \cup N(y)$. Recall that our assumption is that there is no point $x$ with $\rho(x) \leq C_3\mu$ and $d(x+e_1, x+e_2) \leq \lambda C_3\mu$. We now describe how to reject all diagrams except 2, 4, 6, 11, 15, 23.

1. Immediately we get $\rho(y+e_3) \leq 4\lambda C_{3,1}\mu < \mu$.
3. We have $\rho(y+e_1) \leq 4\lambda C_{3,1}\mu < \mu$.
5. Similarly to previous ones, $\rho(y+e_1) \leq 7\lambda C_{3,1}\mu$.
7. We get $\rho(y+e_3) \leq 4\lambda C_{3,1}\mu < \mu$.
8. We have $\rho(y+e_2) \leq (2+3\lambda)C_{3,1}\mu, d(y+e_2+e_1, y+e_2+e_2) \leq 3\lambda C_{3,1}$, but we assume that there are no such points.
9. The diameter of $N(y)$ is at most $7\lambda C_{3,1}\mu$ and $\rho(y) \leq C_{3,1}\mu$, so apply Proposition 6.3, provided $\lambda < 1/(287C_1C_{3,1})$.

10. The diameter of $N(y)$ is at most $10\lambda C_{3,1}\mu$ and $\rho(y) \leq C_{3,1}\mu$, so apply Proposition 6.3, provided $\lambda < 1/(410 C_1 C_{3,1})$.
12. We apply Proposition 6.4 to $(y; y+e_2, y+e_1, y+e_3; y)$ with constant $9C_{3,1}$, so $\rho(y+e_2) \leq 144\lambda C_{3,1}\mu < \mu$, as long as $\lambda < 1/(7380 C_1 C_{3,1})$.
13. Use Proposition 6.2 to get $\rho(y+e_1) \leq (11+9\lambda)C_{3,1}\mu$ and $d(y+e_1+e_1, y+e_1+e_2) \leq 3\lambda C_{3,1}\mu$, as $\lambda < 1/(936 C_{3,1})$. This is a contradiction as $C_3 > 12 C_{3,1}$.
14. Apply Proposition 6.4 to $(y; y+e_3, y+e_2, y+e_1; y)$ with constant $9C_{3,1}$ to get $\rho(y+e_2) \leq 144\lambda C_{3,1}\mu$. Here we need $\lambda < 1/(7380 C_1 C_{3,1})$.
16. As 14.
17. As for 13, we get $\rho(y+e_2) \leq (11+9\lambda)C_{3,1}\mu$ and $d(y+e_2+e_1, y+e_2+e_2) \leq 3\lambda C_{3,1}\mu$.
18. Apply Proposition 6.4 to $(y; y+e_1, y+e_3, y+e_2; y)$ with constant $9C_{3,1}$ to get $\rho(y+e_2) \leq 144\lambda C_{3,1}\mu < \mu$.
19. Apply Proposition 6.4 to $(y; y+e_1, y+e_3, y+e_2; y+e_3)$ with constant $9C_{3,1}$ to get $\rho(y+e_2) \leq (2+6\lambda)C_{3,1}\mu$, $d(y+e_2+e_1, y+e_2+e_2) \leq 3\lambda C_{3,1}\mu$.
20. As 18.
21. Use Proposition 6.2 to get $\rho(y+e_3) \leq (9+3\lambda)C_{3,1}\mu$ and $d(y+e_3+e_1, y+e_3+e_2) \leq 3\lambda C_{3,1}\mu$, as $\lambda < 1/(78 C_{3,1})$.
22. We have $\operatorname{diam} N(y) \leq 7\lambda K\mu$, which contradicts Proposition 6.3 when $\lambda < 1/(287 C_1 C_{3,1})$.
24. Apply Proposition 6.4 to $(y; y+e_1, y+e_2, y+e_3; y+e_2)$ with constant $6C_{3,1}$ to get $\rho(y+e_2) \leq 96\lambda C_{3,1}\mu$.

Therefore, for $y$ given above, provided $\lambda < 1/(7380 C_1 C_{3,1})$, we can only have diagrams 2, 4, 6, 11, 15, 23. However, in all of these diagrams we can classify contractions more precisely.

2. Observe that we cannot have $y+e_1 \overset{2}{\frown} y$ or $y+e_1 \overset{3}{\frown} y$, as the first one of these gives $\rho(y+e_2) \leq 10\lambda C_{3,1}\mu < \mu$, while the latter implies $\rho(y+e_1) \leq 10 C_{3,1}\mu < \mu$. Hence $y+e_1 \overset{1}{\frown} y$. Similarly, we must have $y \overset{2}{\frown} y+e_3$, otherwise we get a point $p$ with $\rho(p) \leq 10\lambda C_{3,1}\mu < \mu$.
4. As in 2, if we do not have $y \overset{3}{\frown} y+e_1$ and $y \overset{2}{\frown} y+e_2$, we obtain a point $p$ with $\rho(p) \leq 10\lambda C_{3,1}\mu < \mu$.
6. As in 2, if we do not have $y \overset{2}{\frown} y+e_2$ and $y \overset{1}{\frown} y+e_3$, we obtain a point $p$ with $\rho(p) \leq 10\lambda C_{3,1}\mu < \mu$.
11. If $y \overset{3}{\frown} y+e_3$, then $\rho(y+e_3) \leq 10\lambda C_{3,1}\mu < \mu$. On the other hand, if $y \overset{2}{\frown} y+e_3$, then $\operatorname{diam} N(y) \leq 8\lambda C_{3,1}\mu$ and $\rho(y) \leq C_{3,1}\mu$, which is impossible by Proposition 6.3 if $\lambda < 1/(328 C_1 C_{3,1})$. Therefore, $y \overset{1}{\frown} y+e_3$, and in the same fashion $y \overset{3}{\frown} y+e_2$. Furthermore, apply

Proposition 6.4 to $(y; y+e_2, y+e_1, y+e_3; y)$ with constant $6\rho(y)/\mu$ to get $d(y+e_2, y+e_1+e_2) \le 96\lambda\rho(y)$.

15. As in 11, we obtain $y \overset{1}{\frown} y+e_1$ and $y \overset{3}{\frown} y+e_3$. Apply Proposition 6.4 to $(y; y+e_3, y+e_2, y+e_1; y)$ with constant $6\rho(y)/\mu$ to get $d(y+e_2, y+2e_2) \le 96\lambda\rho(y)$.
23. As in 11, we obtain $y \overset{3}{\frown} y+e_1$ and $y \overset{1}{\frown} y+e_3$. Apply Proposition 6.4 to $(y; y+e_1, y+e_2, y+e_3; y)$ with constant $6\rho(y)/\mu$ to get $d(y+e_2, y+2e_2) \le 96\lambda\rho(y)$.

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