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Uniformly recurrent sequences and minimal Cantor omega-limit sets

by

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Abstract. We investigate the structure of kneading sequences that belong to unimodal maps for which the omega-limit set of the turning point is a minimal Cantor set. We define a scheme that can be used to generate uniformly recurrent and regularly recurrent infinite sequences over a finite alphabet. It is then shown that if the kneading sequence of a unimodal map can be generated from one of these schemes, then the omega-limit set of the turning point must be a minimal Cantor set.

1. Introduction. One of the simplest examples of a dynamical system is a continuous map of an interval to itself, and one subcollection of such maps that has received a great deal of interest in the literature are the unimodal maps [3, 4, 5, 6]. There has been recent interest in locating unimodal maps f where $\omega(c)$, the omega-limit set of the turning point, is a minimal Cantor set and $f|_{\omega(c)}$ is a homeomorphism. Many sufficient conditions exist that can be placed on a unimodal map to guarantee such a homeomorphic restriction, although there is no known combinatoric characterization of this behavior. In order to better study these homeomorphic restrictions, we first investigate conditions that can be placed on the kneading sequence of a unimodal map to guarantee $\omega(c)$ will be a minimal Cantor set.

It is well-known that if $\lim_{k\to\infty} Q(k) = \infty$, where $Q(k)$ is the kneading map of a unimodal map f, then $\omega(c)$ is a minimal Cantor set. However, there exist many unimodal maps for which $\lim_{k\to\infty} Q(k) \neq \infty$ but $\omega(c)$ is still a minimal Cantor set. Further, it is well-established that if the kneading sequence $\mathcal{K}(f)$ for a unimodal map f is infinite and c is recurrent, then $\omega(c)$ is a minimal Cantor set if and only if $\mathcal{K}(f)$ is a uniformly recurrent sequence. We thus study the structure of uniformly recurrent sequences and

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then provide a sufficient condition for a kneading sequence to belong to a unimodal map f with $\omega(c)$ a minimal Cantor set.

2. Background

2.1. Unimodal maps. A *unimodal map* is a continuous map $f : [0,1] \to [0,1]$ for which there exists a point $c \in (0,1)$ such that $f|_{[0,c)}$ is strictly increasing and $f|_{(c,1]}$ is strictly decreasing. We call the point c the *turning point* of the map and for each $n \in \mathbb{N}$ we set $c_n = f^n(c)$. Examples of unimodal maps include symmetric tent maps and logistic maps. The *symmetric tent map* $T_a : [0,1] \to [0,1]$ with $a \in [0,2]$ is given by $T_a(x) = \min\{ax, a(1-x)\}$, whereas the *logistic map* $g_a : [0,1] \to [0,1]$ with $a \in [0,4]$ is defined by $g_a(x) = ax(1-x)$. In this paper, unless otherwise stated, we assume that all maps are unimodal, have no wandering intervals, and have no attracting periodic orbits; thus we may assume the map is from either the symmetric tent family or the logistic family. We further suppose $c_2 < c < c_1$ and $c_2 \le c_3$, as otherwise the asymptotic dynamics will be uninteresting.

Let f^n be an iterate of f and let J be a maximal subinterval on which $f^n|_J$ is monotone. If $c \in \partial J$, then $f^n : J \to [0,1]$ is called a *central branch* of f^n. An iterate n is called a *cutting time* if the image of the central branch of f^n contains c. We denote the cutting times $S_0, S_1, S_2, \ldots$, where $S_0 = 1$ and $S_1 = 2$. As the difference between two consecutive cutting times is again a cutting time, we define a function $Q : \mathbb{N} \to \mathbb{N} \cup \{0\}$, called the *kneading map*, by $S_{Q(k)} = S_k - S_{k-1}$. A function Q is the kneading map for some unimodal map f if and only if $\{Q(k+j)\}_{j \ge 1} \geqq \{Q(Q^2(k)+j)\}_{j \ge 1}$ and $Q(k) < k$ for all $k \ge 1$, where $\geqq$ represents the lexicographical ordering [7].

Given a unimodal map f and a point $x \in [0,1]$, the *itinerary of x under f* is given by $I(x) = I_0 I_1 I_2 \cdots$ where $I_j = 1$ if $f^j(x) > c$, $I_j = 0$ if $f^j(x) < c$, and $I_j = *$ if $f^j(x) = c$. The *kneading sequence* of f, denoted $\mathcal{K}(f)$, is the itinerary $I(c_1)$. We make the convention that the kneading sequence stops the first time an $*$ appears, as the turning point is thus periodic; if $c_n \neq c$ for all $n \in \mathbb{N}$, then $\mathcal{K}(f)$ is infinite.

We compare itineraries using the *parity lexicographical ordering*, or plo for short. Suppose that v and w are two itineraries such that $v \neq w$. Find the first position in which v and w differ and compare that position using the ordering $0 \prec * \prec 1$ if the number of 1's preceding that position is even (*even parity*); else (*odd parity*) use the ordering $0 \succ * \succ 1$. Let e be either an infinite sequence of 1's and 0's or a finite sequence of 1's and 0's ending in $*$; then e is *shift maximal* if $\sigma^k(e) \preceq e$ for all $k \in \mathbb{N}$, where σ represents the shift map. Every unimodal map has a shift maximal kneading sequence, and if e is a shift maximal sequence of 0's, 1's and $*$'s, then there exists a unimodal map f such that $\mathcal{K}(f) = e$ (see [2]). Further, the sequence of cutting times, kneading map, and kneading sequence all completely determine the associated unimodal map up to conjugacy.

A unimodal map f is *renormalizable* of period $n \geq 2$ provided there exists an interval $J \ni c$ such that $f^n(J) \subset J$ and $f^n|_J$ is again a unimodal map, relaxing the definition of unimodal to allow for decreasing to the left of the turning point and increasing to the right. If $f^n|_J$ is again renormalizable, then we say f is twice renormalizable. Similarly, if we can continue this process forever, then f is *infinitely renormalizable.* As we assume all unimodal maps have no wandering intervals and no attracting periodic orbits, if the unimodal map f is non-renormalizable, then we may take f to be from the symmetric tent family; in the case where f is renormalizable, we may assume f is logistic [3].

We conclude this section by noting there is a natural relationship between renormalization and kneading sequences that is well-understood. A unimodal map is renormalizable if and only if its kneading sequence may be written as a star product [6, Chapter II.2].

Let $n, m \in \mathbb{N}$, $P = P_1 \cdots P_n \in \{0,1\}^n$, and $Q = Q_1 \cdots Q_m \in \{0,1\}^m$. The *star product* (*$\star$-product*) of P and Q is defined by

$$P \star Q = \begin{cases} P\widetilde{Q}_1 \cdots P\widetilde{Q}_m P & \text{if } P \text{ has odd parity,} \\ PQ_1 \cdots PQ_m P & \text{if } P \text{ has even parity,} \end{cases}$$

where $\widetilde{Q}_i = 1 - Q_i$. If $P*$ and $Q*$ are both shift maximal, then $(P \star Q)*$ is also shift maximal. Further, these definitions and results extend to sequences P and Q where P has finite length and $Q \in \{0,1\}^{\mathbb{N}}$.

2.2. Omega-limit sets and recurrence. The *omega-limit set* of a point $x \in [0,1]$ under f is defined by $\omega(x, f) = \omega(x) = \{y \in [0,1] \mid \exists n_1 < n_2 < \cdots \text{ with } f^{n_i}(x) \to y\}$. A point $x \in [0,1]$ is *recurrent* if for every open set $U \ni x$, there exists an $m \in \mathbb{N}$ such that $f^m(x) \in U$; equivalently, x is recurrent if and only if $x \in \omega(x)$. The point $x \in [0,1]$ is *uniformly recurrent* if for every open set $U \ni x$, there exists an $M \in \mathbb{N}$ such that whenever $f^j(x) \in U$ for $j \geq 0$, then $f^{j+k}(x) \in U$ for some $0 < k \leq M$. We say $x \in [0,1]$ is *regularly recurrent* if for every open set $U \ni x$, there exists an $M \in \mathbb{N}$ such that $f^{M \cdot n}(x) \in U$ for all $n \in \mathbb{N}$. Note that every regularly recurrent point is uniformly recurrent and every uniformly recurrent point is recurrent, but the converses do not hold.

Given a continuous map $f : E \to E$ of a compact metric space, a set $F \subseteq E$ is *minimal* provided F is non-empty, closed, invariant, and no proper subset of F has these properties. As we are interested in determining which unimodal maps f are such that $\omega(c)$ is a minimal Cantor set, we now consider the concepts of recurrence and minimality in terms of infinite sequences. Given a finite alphabet $\mathcal{A}$, the sequence $w \in \mathcal{A}^{\mathbb{N}}$ is *uniformly recurrent* or *minimal* if for any word u appearing in w, there exists an M such that every word of length M in w contains at least one occurrence of u. A sequence $w = w_1w_2\cdots \in \mathcal{A}^{\mathbb{N}}$ is called a *Toeplitz sequence* if for every $i \in \mathbb{N}$, there exists a $p_i > 1$ such that $w_i = w_{i+np_i}$ for all $n \in \mathbb{N}$. A point $x \in [0,1]$ is regularly recurrent under f if and only if its itinerary is a Toeplitz sequence [1]; we will refer to Toeplitz sequences as *regularly recurrent* sequences.

We note the following well-known results relating omega-limit sets, minimality, and recurrence. For a more thorough discussion see [3].

THEOREM 2.1. *Let $f : E \to E$ be a continuous map of a compact metric space and $x \in E$. If $x \in \omega(x,f)$, then $\omega(x,f)$ is minimal if and only if x is uniformly recurrent.*

LEMMA 2.2. *Let $f : I \to I$ be a continuous map of a closed interval. If $F \subset I$ is infinite and minimal, then F is a Cantor set.*

COROLLARY 2.3. *Let $\mathcal{K}(f)$ be an infinite kneading sequence. If $c \in \omega(c)$, then $\omega(c)$ is a minimal Cantor set if and only if c is uniformly recurrent.*

COROLLARY 2.4. *Let $\mathcal{K}(f)$ be an infinite kneading sequence. Then c is uniformly recurrent if and only if $\mathcal{K}(f)$ is uniformly recurrent. Similarly, a point $x \in [0,1]$ (with $f^n(x) \neq c$ for all $n \in \mathbb{N}$) is uniformly recurrent if and only if its itinerary $I(x)$ is uniformly recurrent.*

PROPOSITION 2.5. *Let f be a unimodal map and suppose that $\lim_{k\to\infty} Q(k) = \infty$. Then c is uniformly recurrent and $\omega(c)$ is a minimal Cantor set.*

We now make some observations about $\omega(c,f)$, where f is a unimodal map. If f has an attracting periodic orbit in $[c_2, c_1]$, then $\omega(c)$ is precisely that orbit. Further, if f is infinitely renormalizable, then c is regularly recurrent and $\omega(c)$ is a minimal Cantor set. We note that if f is finitely renormalizable, then there exists a restrictive interval J and an integer k such that $f^k|_J$ is topologically conjugate to a non-renormalizable map. As we are interested in unimodal maps with no wandering intervals and no attracting periodic orbits, if we further restrict our attention to non-renormalizable maps, then we may assume f is a symmetric tent map.

For a symmetric tent map f and turning point c, if the orbit of c is infinite but not dense in the core, then $\omega(c)$ is a nowhere dense, totally disconnected set. In this case either all points in the orbit of c are isolated with respect to the orbit, or at most finitely many points in the orbit of c are isolated with respect to the orbit. In the former case, either $\omega(c)$ is countable or $\omega(c)$ is the union of a Cantor set and a (possibly empty) countable set; in the latter case, $\omega(c)$ is exactly a Cantor set [3, Section 10.2].

We focus on the case where at most finitely many c_i are isolated with respect to the orbit of c, as $\omega(c)$ will be a Cantor set and there will exist an M such that $c_n \in \omega(c)$ for all $n \geq M$. It follows that $\omega(c) = \omega(c_M)$

will be minimal if and only if c_M is uniformly recurrent. We thus study the structure of uniformly recurrent sequences in order to begin to characterize those unimodal maps f for which $\omega(c)$ is a minimal Cantor set. It is still unknown whether $\omega(c)$ can be an infinite minimal Cantor set if every point in the orbit of c is isolated with respect to the orbit.

3. Uniform recurrence in sequences. In this section we explore the occurrences of words that appear in uniformly recurrent sequences. We observe that if x is a word occurring at a position congruent to $q \bmod k$ in a uniformly recurrent sequence w, then x will again appear in a position congruent to $q \bmod k$ within a fixed number of positions.

Let $w = w_1 w_2 \cdots$ be a uniformly recurrent sequence. The function $R_w : \mathbb{N} \to \mathbb{N}$ is a *recurrence return function* of w if every word of length n that occurs in w occurs in each block of length $R_w(n)$ in w. Note that because w is uniformly recurrent, such a function does exist. That is, consider the list of all words of length n that appear in w. Each of these words will have a first occurrence in w, so let $w_1 \cdots w_N$ be the smallest initial block of w that contains every word of length n. As w is uniformly recurrent, there exists an M_n such that every block of length M_n contains $w_1 \cdots w_N$ as a subword. Let $R_w(n) = M_n$.

LEMMA 3.1. *Suppose that $w = w_1 w_2 \cdots$ is a uniformly recurrent sequence. Fix $k \in \mathbb{N}$ and let $x = w_i w_{i+1} \cdots w_j$ be a word that appears in w. If $i \equiv q \bmod k$, then x appears in w at $x = w_{i'} \cdots w_{j'}$ for some $i' \equiv q \bmod k$ with $i < i' < R^{k-1}(R_w(|x|)) + i$, where $R(n) = R_w(n+1)$.*

Proof. Let $x_0 = w_i w_{i+1} \cdots w_j$ be a word that appears in w such that $i \equiv q \bmod k$. Let $w_{i_0} \cdots w_{j_0}$ be the first occurrence of x_0 in w such that $i_0 > i$. Then $j_0 \le R_w(|x_0|) + i$. Suppose that $i_0 \equiv q_0 \bmod k$. If $q_0 = q$, then the conclusion holds. Thus suppose $q_0 \ne q$.

Let $x_1 = w_i w_{i+1} \cdots w_{j_0}$ and suppose $w_{i_1} \cdots w_{j_1}$ is the first occurrence of x_1 in w such that $i_1 > i$. Thus $j_1 \le R_w(R_w(|x_0|) + 1) + i = R(R_w(|x_0|)) + i$. Let $i_1 + i_0 - i \equiv q_1 \bmod k$. If $q_1 = q$, then $w_{i_1+i_0+i} \cdots w_{j_1} = x_0$ and the conclusion holds. Similarly, if $q_1 = q_0$, then $w_{i_1} \cdots w_{i_1+j-i} = x_0$ and the conclusion holds. Hence we suppose $q_1 \ne q$ and $q_1 \ne q_0$.

Let $x_2 = w_i w_{i+1} \cdots w_{j_1}$ and suppose $w_{i_2} \cdots w_{j_2}$ is the first occurrence of x_2 in w such that $i_2 > i$. Then $j_2 \le R_w(R_w(R_w(|x_0|) + 1) + 1) + i = R^2(R_w(|x_0|)) + i$. Let $i_2 + i_1 + i_0 - 2i \equiv q_2 \bmod k$. If $q_2 = q$, then $w_{i_2+i_1+i_0-2i} \cdots w_{j_2} = x_0$; if $q_2 = q_0$, then $w_{i_2+i_1-i} \cdots w_{i_2+i_1+j-2i} = x_0$; if $q_2 = q_1$, then $w_{i_2} \cdots w_{i_2+j-i} = x_0$. In each case the conclusion is satisfied, so we assume $q_2 \ne q$, $q_2 \ne q_0$ and $q_2 \ne q_1$.

Continue this process inductively for all $n < k - 1$. That is, set $x_n = w_i w_{i+1} \cdots w_{j_{n-1}}$ and suppose $w_{i_n} \cdots w_{j_n}$ is the first occurrence of x_n in w

such that $i_n > i$. Then $j_n \le R^n(R_w(|x_0|)) + i$. Let $i_n + i_{n-1} + \cdots + i_0 - ni \equiv q_n \bmod k$ where $q_n \in \{0, 1, \ldots, k-1\}$. If $q_n = q_i$ for some $0 \le i \le n-1$ or $q_n = q$, then we have an occurrence of x_0 starting at a position which is congruent to q mod k, and the result holds. Thus assume $q_n \ne q_i$ for all $0 \le i \le n-1$ and $q_n \ne q$.

As there are only k possible words in the set $\{0, 1, \ldots, k-1\}$, it follows that if $x_{k-1} = w_i \cdots w_{j_{k-2}}$ and $w_{i_{k-1}} \cdots w_{j_{k-1}}$ is the first occurrence of x_{k-1} in w such that $i_{k-1} > i$, then if $i_{k-1} + i_{k-2} + \cdots + i_0 - (k-1)i \equiv q_{k-1} \bmod k$, it must be that $q_{k-1} = q_i$ for some $i < k-1$ or $q_{k-1} = q$. We thus have $x_0 = w_{i'} \cdots w_{j'}$ with $i' \equiv q \bmod k$, $i' > i$, and $j' \le R^{k-1}(R_w(|x_0|)) + i$. Hence the result holds. ■

4. $\mathcal{A}$-uniform schemes. In this section we define a scheme that can be used to construct all uniformly recurrent one-sided sequences of symbols from the finite alphabet $\mathcal{A}$. We then modify this definition in order to construct all regularly recurrent sequences in $\mathcal{A}^{\mathbb{N}}$. Connections are made to kneading sequences of unimodal maps, and it is shown that a sequence $A \star B$ is uniformly recurrent if and only if B is uniformly recurrent. The definitions and techniques used in this section are motivated by computer science results in [8] and [9].

Definition 4.1. An *$\mathcal{A}$-uniform scheme* is a sequence of pairs $\langle l_n, A_n \rangle$ such that

(1) $\{l_n\}$ is an increasing sequence of positive integers.
(2) Each word in A_n is in $\mathcal{A}^{l_n}$.
(3) For each n and for each $u \in A_{n+1}$, $u = v_1 \cdots v_k$ where $v_i \in A_n$ for each i, and for each $w \in A_n$ there exists an $i < k$ such that $w = v_i$.

We say a sequence $w \in \mathcal{A}^{\mathbb{N}}$ is *generated by an $\mathcal{A}$-uniform scheme* $\langle l_n, A_n \rangle$ if for every $i, n \in \mathbb{N}$, $w_{il_n+1} w_{il_n+2} \cdots w_{(i+1)l_n} \in A_n$.

Theorem 4.2. *A sequence $w \in \mathcal{A}^{\mathbb{N}}$ is uniformly recurrent if and only if w is generated by an $\mathcal{A}$-uniform scheme.*

Proof. Suppose a sequence w is generated by an $\mathcal{A}$-uniform scheme $\langle l_n, A_n \rangle$. Let x be a word in w such that $x = w_{m+1} \cdots w_{m+|x|}$ is the first appearance of x in w. Choose an n such that $l_n \ge m + |x|$. Then $w_1 \cdots w_{l_n} \in A_n$ and $w_1 \cdots w_{l_n}$ contains x as a subword. As every word in A_{n+1} contains a copy of $w_1 \cdots w_{l_n}$, every word in A_{n+1} contains x as a subword. Hence, every block of length $2l_{n+1}$ in w contains a full copy of at least one word from A_{n+1} and thus contains x as a subword. It follows w is uniformly recurrent.

On the other hand, suppose that w is uniformly recurrent. We will construct a uniform scheme $\langle l_n, A_n \rangle$ that generates w. We say the occurrence $w_{i+1} \cdots w_{i+|x|}$ of the word x is a *good occurrence* of $x \in A_n$ if $l_n \mid i$.

Let $A_n = \{u \in \mathcal{A}^{l_n} \mid u \text{ has infinitely many good occurrences in } w\}$. First of all note that $A_n \neq \emptyset$ as there are only finitely many words of length l_n and w is an infinite sequence. We define the sequence $\{l_n\}$ by induction. Suppose l_n is given; for completeness, define $l_0 = 1$.

Let M be chosen such that $w_1 \cdots w_M$ contains a good occurrence of each word in A_n. By Lemma 3.1 there exists a good occurrence of $w_1 \cdots w_M$ within every block of length $R^{l_n-1}(R_w(M))$. Thus let l_{n+1} be the least number with $l_{n+1} \geq R^{l_n-1}(R_w(M))$ and $l_n \mid l_{n+1}$. It follows that every block of length l_{n+1} contains a good occurrence of each word in A_n.

It remains to show that $\langle l_n, A_n \rangle$ is an $\mathcal{A}$-uniform scheme generating w. Note that conditions (1) and (2) of Definition 4.1 are clearly met. Fix $n \in \mathbb{N}$ and let $u \in A_{n+1}$. Then u has infinitely many good occurrences in w, including $u = w_{jl_{n+1}+1} w_{jl_{n+1}+2} \cdots w_{(j+1)l_{n+1}}$ for some $j \in \mathbb{N}$. Note that because $|u| = l_{n+1}$ and $l_n \mid l_{n+1}$, we have $u = v_1 \cdots v_k$ where each $|v_i| = l_n$ and $k = l_{n+1}/l_n$. Hence there are infinitely many good occurrences of each v_i in w, and thus we have $v_i \in A_n$ for each i. Now, because each block of length l_{n+1} contains a good occurrence of each word in A_n, $w_{jl_{n+1}} w_{jl_{n+1}+1} \cdots w_{(j+1)l_{n+1}-1}$ contains a good occurrence of each word in A_n. Thus $v_1 \cdots v_{k-1}$ contains each word in A_n. Hence condition (3) from Definition 4.1 is satisfied. Lastly, for all $i, n \in \mathbb{N}$ the word $x = w_{il_n+1} w_{il_n+2} \cdots w_{(i+1)l_n}$ starts at a position congruent to 1 mod l_n. By Lemma 3.1, x occurs infinitely often in w starting at positions congruent to 1 mod l_n. That is, $x \in A_n$, as desired. ∎

Using Theorem 4.2 and the results stated in Section 2.2, we make the following observations.

Corollary 4.3. *Let $\mathcal{K}(f)$ be an infinite kneading sequence. Then c is uniformly recurrent if and only if $\mathcal{K}(f)$ is generated by a $\{0,1\}$-uniform scheme.*

Corollary 4.4. *Let $\mathcal{K}(f)$ be an infinite kneading sequence and suppose that at most finitely many points in the orbit of c are isolated with respect to the orbit of c. Then $\omega(c)$ is a minimal Cantor set if and only if $\sigma^M(\mathcal{K}(f))$ is generated by a $\{0,1\}$-uniform block scheme for some $M \geq 0$, where σ is the shift map.*

We now add one additional condition to the uniform scheme to gain a stronger result involving regular recurrence.

Theorem 4.5. *A sequence $w \in \mathcal{A}^{\mathbb{N}}$ is regularly recurrent if and only if w may be generated by an $\mathcal{A}$-uniform scheme $\langle l_n, A_n \rangle$ with the additional property that each word in A_{n+1} begins with $w_1 \cdots w_{l_n}$.*

Proof. Let w be generated by an $\mathcal{A}$-uniform scheme with the above additional property. Fix $N \in \mathbb{N}$ and choose $m \in \mathbb{N}$ such that $1 \leq N \leq l_m$. Hence

$w_1 \cdots w_N \cdots w_{l_m} \in A_m$ and $w_{i(l_{m+1})+1} \cdots w_{i(l_{m+1})+l_m} = w_1 \cdots w_N \cdots w_{l_m}$ for all $i \in \mathbb{N}$. Therefore $w_1 \cdots w_N = w_{i(l_{m+1})+1} \cdots w_{i(l_{m+1})+N}$ for all $i \in \mathbb{N}$. It follows that w is regularly recurrent.

Conversely, let w be regularly recurrent. We construct a uniform scheme with the additional property that for all $n \in \mathbb{N}$, every word in A_{n+1} begins with $w_1 \cdots w_{l_n}$.

Let $l_0 = 1$ and A_0 be the smallest subset of $\mathcal{A}$ containing every letter that appears in w. Let N_1 be the smallest integer such that $w_1 \cdots w_{N_1-1}$ contains each word from A_0. As w is regularly recurrent, there exists a T_1 such that $w_1 \cdots w_{N_1} = w_{iT_1+1} \cdots w_{iT_1+N_1}$ for all $i \in \mathbb{N}$. Set $l_1 = T_1$ and let $A_1 = \{w_{il_1+1} w_{il_1+2} \cdots w_{(i+1)l_1} \mid i \in \mathbb{N}\}$.

Now let N_2 be the smallest integer such that $w_1 \cdots w_{N_2-1}$ contains a good occurrence of each word in A_1. As w is regularly recurrent, there exists a $T_2 > N_2$ such that $T_1 \,|\, T_2$ and $w_1 \cdots w_{N_2} = w_{iT_2+1} \cdots w_{iT_2+N_2}$ for all $i \in \mathbb{N}$. Set $l_2 = T_2$ and let $A_2 = \{w_{il_2+1} w_{il_2+2} \cdots w_{(i+1)l_2} \mid i \in \mathbb{N}\}$.

Continue this process inductively for each $n \in \mathbb{N}$: Let N_n be the smallest integer for which $w_1 \cdots w_{N_n-1}$ contains a good occurrence of each word from A_{n-1}; let $T_n > N_n$ be such that $T_{n-1} \,|\, T_n$ and $w_1 \cdots w_{N_n} = w_{iT_n+1} \cdots w_{iT_n+N_n}$ for all $i \in \mathbb{N}$; set $l_n = T_n$ and then set $A_n = \{w_{il_n+1} w_{il_n+2} \cdots w_{(i+1)l_n} \mid i \in \mathbb{N}\}$.

We now show $\langle l_n, A_n \rangle$ is a uniform scheme with the additional property that A_n begins with $w_1 \cdots w_{l_{n-1}}$ for all $n \in \mathbb{N}$. Note that conditions (1) and (2) of Definition 4.1 are clearly met. Further, since $l_{n-1} < N_n < l_n$ and $l_{n-1} \,|\, l_n$, each word $u \in A_n$ is such that $u = v_1 \cdots v_k$ where $v_i \in A_{n-1}$ for each i, and each word $w \in A_{n-1}$ is such that $w = v_i$ for some $i < k$. Finally, based on the construction of $\langle l_n, A_n \rangle$, the additional property that every word of A_n begins with $w_1 \cdots w_{l_{n-1}}$ is satisfied. ∎

We refer to an $\mathcal{A}$-uniform scheme that is given the additional property that for each $n \in \mathbb{N}$ there is a word $u \in A_{n-1}$ such that each word in A_n begins with u as an *$\mathcal{A}$-regular scheme.*

COROLLARY 4.6. *Let $\mathcal{K}(f)$ be an infinite kneading sequence. Then c is regularly recurrent if and only if $\mathcal{K}(f)$ is generated by a $\{0,1\}$-regular scheme.*

Note that in the above theorems and corollaries it does not matter if f is a renormalizable unimodal map or if it is non-renormalizable. If f is infinitely renormalizable, then c is regularly recurrent, $c \in \omega(c)$, $\omega(c)$ is a minimal Cantor set, and $\mathcal{K}(f)$ can be generated from a $\{0,1\}$-regular scheme. If f is finitely renormalizable, then $\mathcal{K}(f) = A \star B$, where B is the kneading sequence of a non-renormalizable unimodal map. We now show:

LEMMA 4.7. *The infinite sequence $A\star B \in \{0,1\}^{\mathbb{N}}$ is uniformly recurrent if and only if the infinite sequence $B \in \{0,1\}^{\mathbb{N}}$ is uniformly recurrent.*

Proof. Let $A = a_1 \cdots a_n \in \{0,1\}^n$ and $B = b_1 b_2 \cdots \in \{0,1\}^{\mathbb{N}}$. Without loss of generality we assume that A has even parity; thus $A\star B = Ab_1Ab_2\cdots$.

First suppose that B is uniformly recurrent. Then for each M there exists an N such that if $b_1 \cdots b_M = b_{i+1} \cdots b_{i+M}$, then $b_1 \cdots b_M = b_{i+1+j} \cdots b_{i+M+j}$ for some $j \le N$. Thus whenever $Ab_1 \cdots Ab_M = Ab_{i+1} \cdots Ab_{i+M}$, then $Ab_{i+1+j} \cdots Ab_{i+M+j} = Ab_1 \cdots Ab_M$ for some $j \le N$. It follows that $A \star B$ is uniformly recurrent.

Conversely, suppose that $A\star B$ is uniformly recurrent. For ease we denote $A \star B = c_1c_2\cdots$. Fix M and consider $c_1 \cdots c_{(n+1)M} = Ab_1 \cdots Ab_M$. By Lemma 3.1, $Ab_1 \cdots Ab_M$ appears in $A \star B$ at $c_{i+1} \cdots c_{i+(n+1)M}$ for some $i \equiv 0 \bmod (n+1)M$ with $0 < i < R^{(n+1)M-1}(R_{A\star B}((n+1)M))$. In fact, whenever $Ab_1 \cdots Ab_M$ appears in an initial position congruent to 0 mod $(n+1)M$, it appears again in a position congruent to 0 mod $(n+1)M$ within a bound of $R^{(n+1)M-1}(R_{A\star B}((n+1)M))$. Thus, there exists a $K \in \mathbb{N}$ such that whenever $b_1 \cdots b_M = b_{i+1} \cdots b_{i+M}$, then $b_{i+j+1} \cdots b_{i+j+M} = b_1 \cdots b_M$ for some $j \le K$. That is, B is uniformly recurrent. ∎

COROLLARY 4.8. *The infinite sequence $A \star B$ is such that $\sigma^M(A \star B)$ is uniformly recurrent for some $M \ge 0$ if and only if $\sigma^k(B)$ is uniformly recurrent for some $k \ge 0$.*

We note that both Lemma 4.7 and Corollary 4.8 can be extended to the regularly recurrent case.

5. Constructing kneading sequences via uniform schemes. In this section we focus on generating kneading sequences from $\{0,1\}$-uniform schemes. This leads to our main result, Theorem 5.2, which provides sufficient conditions for a kneading sequence to belong to a unimodal map f for which $\omega(c,f)$ is a minimal Cantor set. Such a map is constructed in Example 5.3.

Suppose $w \in \{0,1\}^{\mathbb{N}}$ is generated from a $\{0,1\}$-uniform scheme. Then $w_1 \cdots w_{l_n} \in A_n$ for all n. Further,

$$w_1 \cdots w_{l_{n+1}} = (w_1 \cdots w_{l_n})(w_{l_n+1} \cdots w_{2l_n}) \cdots (w_{l_{n+1}-l_n+1} \cdots w_{l_{n+1}}),$$

where $w_{il_n+1} \cdots w_{(i+1)l_n} \in A_n$ for each i. Hence, we may denote the words in A_n as $A_n^1, \ldots, A_n^{k_n}$ where $k_n = |A_n|$ and $A_n^1 = w_1 \cdots w_{l_n}$ for all n.

LEMMA 5.1. *The sequence $w \in \{0,1\}^{\mathbb{N}}$ generated by a uniform scheme is shift maximal if and only if $A_n^1 \succeq \sigma^k(A_n^1)$ for all $k \ge 0$ and all $n \ge 1$.*

Proof. If w is shift maximal, then $w \succeq \sigma^k(w)$ for all $k \in \mathbb{N}$. As w is the concatenation of blocks from A_n beginning with A_n^1 (this can be done for each $n \geq 1$), $A_n^1 \succeq \sigma^k(A_n^1)$ for all $k \geq 0$ and all $n \geq 1$.

If w is not shift maximal, then there exists some $k \in \mathbb{N}$ such that $w \prec \sigma^k(w)$. If $m \in \mathbb{N}$ is the first position in w where w and $\sigma^k(w)$ differ, then choose n such that $l_n > k + m$. This implies $A_n^1 \prec \sigma^k(A_n^1)$. ∎

It thus follows that a sequence w generated by a $\{0,1\}$-uniform scheme $\langle l_n, A_n\rangle$ is the kneading sequence for a unimodal map if and only if $A_n = \{A_n^1, \dots, A_n^{k_n}\}$ where

(1) $A_{n+1}^1 = v_1 \cdots v_k$ where $v_i \in A_n$ for all n, $v_1 = A_n^1$, and each $x \in A_n$ is such that $x = v_i$ for some $i < k$.
(2) $A_n^1 \succeq \sigma^k(A_n^1)$ for all $n, k \in \mathbb{N}$.
(3) $w = \lim_{n\to\infty} A_n^1$.

THEOREM 5.2. *A sequence $w \in \{0,1\}^{\mathbb{N}}$ is the kneading sequence for a unimodal map f such that $\omega(c,f)$ is a minimal Cantor set if $w = BA$ (with B possibly the empty word) where A is generated by a $\{0,1\}$-uniform scheme $\langle l_n, A_n\rangle$ with $A_n = \{A_n^1, \dots, A_n^{k_n}\}$ satisfying:*

(1) *$A_{n+1}^1 = v_1 \cdots v_k$ where $v_i \in A_n$ for all n, $v_1 = A_n^1$, and each $u \in A_n$ is such that $u = v_i$ for some $i < k$.*
(2) *$A = \lim_{n\to\infty} A_n^1$.*
(3) *$BA_n^1 \succeq \sigma^k(BA_n^1)$ for all $n, k \in \mathbb{N}$.*

Proof. This follows directly from Corollary 4.4 and Lemma 5.1 ∎

We now provide an example of a unimodal map whose kneading sequence is generated by a $\{0,1\}$-uniform scheme.

EXAMPLE 5.3. For each n, set $l_n = 32 \cdot 4^{n-1}$. Let

$$A_1^1 = 10001010100110101010101010001011,$$
$$A_1^2 = 10001010100110101000101010001011.$$

For each $n \geq 1$, let

$$A_{n+1}^1 = A_n^1 A_n^2 A_n^1 A_n^1, \qquad A_{n+1}^2 = A_n^1 A_n^2 A_n^1 A_n^2.$$

Set $A = \lim_{n\to\infty} A_n^1$. We let g be the unimodal map such that $\mathcal{K}(g) = A$; this is precisely the map constructed in [1, Example 4.3]. Here $c \in \omega(c)$ and $\omega(c)$ is a minimal Cantor set. In fact, since $\langle l_n, A_n\rangle$ is also a regular scheme, c is regularly recurrent.

Now set $B = 10000$. Then BA is shift maximal and non-renormalizable. Thus $BA = \mathcal{K}(T)$ for some symmetric tent map T. Note that in this example c is not uniformly recurrent, but c_n is uniformly recurrent for all $n \geq 5$ (in fact, c_n is regularly recurrent for all $n \geq 5$). In this case $\omega(c)$ is a minimal Cantor set.

REMARK 5.4. Theorem 5.2 gives a sufficient condition for $\omega(c)$ to be a minimal Cantor set for a unimodal map f. At this time the author is uncertain whether it is possible to have a minimal Cantor set $\omega(c)$ when every point in the orbit of c is isolated with respect to the orbit (see Example 5.5 for a non-minimal Cantor set $\omega(c)$). If this is not possible, then Theorem 5.2 will provide a complete characterization for when a unimodal map f is such that $\omega(c)$ is a minimal Cantor set. Further, if it is possible, then such an example would be interesting.

We conclude by showing that there exists a unimodal map for which every point in the orbit of c is isolated with respect to the orbit and $\omega(c)$ is a Cantor set. In this case $\omega(c)$ is not minimal.

EXAMPLE 5.5. Let $A = 111$, $B = 101$, and $C = 010$. Let X_B denote the Cantor set of all concatenations of A and B and their shifts; similarly, let X_C denote the Cantor set of all concatenations of A and C and their shifts. Note that $X_B \cap X_C = \{1^\infty\}$.

Consider the sequence

$$\begin{aligned} e = 10^{12}&.A.B.1.A.C.1^2.AA.AB.BA.BB.1^3.AA.AC.CA.CC.1^4.AAA.AAB.\\ &ABA.ABB.BAA.BAB.BBA.BBB.1^5.AAA.AAC.ACA.ACC.CAA.\\ &CAC.CCA.CCC.1^6.AAAA.AAAB.AABA.AABB.ABAA.ABAB.\ldots \end{aligned}$$

where the dots are inserted only to clarify the pattern. Note that e is shift maximal and is the kneading sequence of a non-renormalizable symmetric tent map f. Let $I(\omega(c))$ denote the set of all itineraries of points in $\omega(c)$. We note that as f is a non-renormalizable symmetric tent map, each point in $\omega(c)$ has a unique itinerary and that itinerary corresponds to a limit point of $\{\sigma^k(e) \mid k \geq 0\}$.

Clearly $X_B \cup X_C \subseteq I(\omega(c))$. Suppose $y \in I(\omega(c)) \setminus (X_B \cup X_C)$; then y contains at least one copy of both B and C. It is not possible for B and C to appear consecutively in y, as there is no shift of e containing consecutive copies of B and C. Further, $B1^mC$ and $C1^mB$ can appear at most finitely many times in e by construction, so y cannot contain $B1^mC$ or $C1^mB$ for any $m \geq 1$. Hence $I(\omega(c)) \setminus (X_B \cup X_C) = \emptyset$.

It follows that $I(\omega(c)) = X_B \cup X_C$, and as $X_B \cap X_C = \{1^\infty\}$, $\omega(c)$ is a Cantor set. Further, no iterate of the turning point is recurrent; that is, every point in the orbit of c is isolated with respect to the orbit. Finally, $\omega(c)$ is not minimal, since if we let x be the point whose itinerary is 1^∞, then $\omega(c) \neq \omega(x)$.

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