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On Borel reducibility in generalized Baire space

by

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Abstract. We study the Borel reducibility of Borel equivalence relations on the generalized Baire space κ^κ for an uncountable κ with $\kappa^{<\kappa} = \kappa$. The theory looks quite different from its classical counterpart where $\kappa = \omega$, although some basic theorems do generalize.

1. Introduction. Classical descriptive set theory deals in particular with classification problems by translating them into questions of Borel reducibility. The space ω^ω with the standard product topology is called *the Baire space*. A *standard Borel space* is a space equipped with a Borel structure which arises from some Polish topology on that space. For equivalence relations E and E' on standard Borel spaces B and B' respectively, we say that E is *Borel reducible* E' if there is a Borel map $f\colon B \to B'$ which is injective on the equivalence classes, i.e. induces a one-to-one map from B/E to B'/E'. As shown below, these notions naturally generalize to the setting where ω is replaced by an uncountable regular cardinal κ satisfying $\kappa^{<\kappa} = \kappa$. The notion of a standard Borel space also generalizes, along with the generalization of Borel sets.

The results of this paper can be split into four main themes. First we show that every equivalence relation induced by a Borel action of a "small" group (of size $\leq \kappa$) is Borel reducible to a generalized counterpart of the equivalence relation known as E_0 in the classical case (Theorem 2). In the classical case this result holds for hyperfinite equivalence relations, but not for general countable ones. Also in the generalized context not all Borel equivalence relations with classes of size $\leq \kappa$ have to be induced by a Borel action of such a small group, even if we restrict ourselves to smooth (reducible to identity) equivalence relations with classes of size 2 (Theorem 3).

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Second, we show that the counterparts of E_0 and E_1 are Borel bireducible (Theorem 5), which is not the case in the classical setting where E_1 is not reducible to any equivalence relation induced by a Polish group action while E_0 is.

Third, E_0 is strictly below id^+ with respect to Borel reducibility, where id is the identity relation on 2^κ and $E \mapsto E^+$ is the analogue of H. Friedman–Stanley jump operator [FS89] (Theorem 7). This is also true for the classical counterpart, but as follows from the above, this is also true with E_0 replaced by E_1, which is not true in the classical setting.

Finally, if $\mathcal{M}$ is a Borel class of structures of size κ such that $\cong_{\mathcal{M}}$, the isomorphism relation on $\mathcal{M}$, is Borel (in particular if $\mathcal{M}$ is the class of models with domain κ of a classifiable complete first order theory [FHK11]), then $\cong_{\mathcal{M}}$ is Borel reducible to the equivalence on 2^κ modulo the non-stationary ideal restricted to a regular cardinal $\mu < \kappa$ (Corollaries 14 and 15).

2. Basic notions. The generalized Baire space κ^κ consists of all functions from κ to κ where κ is an uncountable cardinal which satisfies $\kappa^{<\kappa} = \kappa$. The topology on this space is generated by the open sets

$$[p] = \{\eta \in \kappa^\kappa \mid \eta \supset p\}$$

where $p \in \kappa^{<\kappa}$. The resulting collection of open sets is closed under intersections of length $< \kappa$. The class of *κ-Borel sets* in this space is the smallest class containing the basic open sets and closed under taking unions and intersections of length κ (apart from definitions, we always drop the prefix "κ-"). More generally, for a topological space X, by X^κ we mean the space equipped with the *κ-product topology*, i.e. the least topology containing the standard Tychonoff topology and closed under intersections of size $< \kappa$.

In this generalized setting, let us define a space equipped with a *κ-algebra*, i.e. a set of subsets which is closed under κ-unions, complements and κ-intersections, to be a *standard Borel space* if it is Borel isomorphic to a Borel subset of κ^κ. In this paper the κ-algebra will always be generated by the open sets in the topology of X. A subset of a standard Borel space B is said to be *analytic*, or Σ^1_1, if it is a projection of a Borel subset $B' \subset B \times \kappa^\kappa$. This has various equivalent definitions, as in the classical case (see e.g. [FHK11]).

In this paper we often work with spaces of the form $(2^\alpha)^\beta$ for some ordinals $\alpha, \beta \le \kappa$. If $x \in (2^\alpha)^\beta$, then technically x is a function $\beta \to 2^\alpha$ and we denote by $x_\gamma = x(\gamma)$ the value at $\gamma < \beta$. Thus x_γ is a function $\alpha \to 2$ for each γ and we denote its value at $\delta < \alpha$ by $x_\gamma(\delta)$ or $x(\gamma)(\delta)$. The lengthier notation for $x \in (2^\alpha)^\beta$ is $(x_\gamma)_{\gamma<\beta}$ as a β-sequence of functions $\alpha \to 2$. For $\alpha, \beta < \kappa$ and $p \in (2^\alpha)^\beta$ define a basic open set of $(2^\kappa)^\kappa$ by

$$[p] = \{\eta \in (2^\kappa)^\kappa \mid \forall\gamma < \beta\ \forall\delta < \alpha\ (\eta(\gamma)(\delta) = p(\gamma)(\delta))\}.$$

We say that a topological space is *κ-Baire* if the intersection of κ-many dense open sets is never empty. The generalized Baire space is κ-Baire [MV93]. If X is a topological space, we say that $A \subseteq X$ is *κ-meager* if its complement contains an intersection of κ-many dense open sets. Thus, X is κ-Baire if and only if X is itself not meager. The complement of a meager set is called *co-meager*. A set $A \subseteq X$ has the *Baire property* if there exists an open set U such that the symmetric difference $U \triangle A$ is meager. As in the classical setup, Borel sets have the Baire property [HS01].

A function is *Borel* if the inverse image of every Borel set is Borel. As in the classical setup, a Borel function is Baire and is continuous on a co-meager set [FHK11]. An equivalence relation E on a standard Borel space B is said to be *Borel reducible* to an equivalence relation E' on a standard Borel space B' if there is a Borel map $f\colon B \to B'$ which is injective on the equivalence classes, i.e. induces an injection from B/E to B'/E'. Two equivalence relations are *Borel bireducible* if one is reducible to the other and vice versa.

3. Equivalence relations induced by a group action

DEFINITION 1. Suppose G is a topological group which is also a standard Borel space. All groups considered in this paper are such. Let X be a Borel subset of κ^κ. An action $\rho\colon G \times X \to X$ is *Borel* if it is Borel as a function. This action induces an equivalence relation on X in which two elements x and y are equivalent if there exists $g \in G$ such that $\rho(g,x) = y$. For example, if the action is Borel, then it is easy to see that this equivalence relation is Σ^1_1. This equivalence relation is denoted by $E^X_{G,\rho}$, or just E^X_G if the action is clear from the context.

Here are some examples of equivalence relations which are, up to Borel bireducibility, induced by a Borel action on a standard Borel space:

- id, the identity relation.
- id^+, the jump of identity. This is an equivalence relation on $(2^\kappa)^\kappa$ where $(x_\alpha)_{\alpha<\kappa}$ and $(y_\alpha)_{\alpha<\kappa}$ are equivalent if the sets $\{x_\alpha \mid \alpha < \kappa\}$ and $\{y_\alpha \mid \alpha < \kappa\}$ are equal (cf. Definition 6). This is not defined as an equivalence relation induced by a Borel action, but is easily seen to be Borel bireducible with id^+_* which is an equivalence relation on $(2^\kappa)^\kappa$ where $(x_\alpha)_{\alpha<\kappa}$ and $(y_\alpha)_{\alpha<\kappa}$ are equivalent if there exists a permutation $s \in S_\kappa$ (S_κ is the group of all permutations of κ) such that $x_\alpha = y_{s(\alpha)}$ for all α. The latter is induced by a Borel action of S_κ.
- E_0, an equivalence relation on 2^κ, where $(\eta,\xi) \in E_0$ if there exists $\alpha < \kappa$ such that for all $\beta > \alpha$ we have $\eta(\beta) = \xi(\beta)$.

- E_1, an equivalence relation on $(2^\kappa)^\kappa$, where $(x_\alpha)_{\alpha<\kappa}$ and $(y_\alpha)_{\alpha<\kappa}$ are equivalent if there exists $\alpha < \kappa$ such that for all $\beta > \alpha$ we have $x_\beta = y_\beta$. This is induced by the action of the Borel group
$$G = \{(x_\alpha)_{\alpha<\kappa} \in (2^\kappa)^\kappa \mid \exists \gamma < \kappa\ \forall \alpha > \gamma\ \forall \beta < \kappa\ (x_\alpha(\beta) = 0)\}.$$
The addition of the group is coordinatewise addition modulo 2 and it acts on the ambient space $(2^\kappa)^\kappa$ by translation.

Since all the topologies in this paper are closed under intersections of length $< \kappa$, we replace "finite" by "less than κ" when referring to product topologies below.

Theorem 2. *Let G be a discrete group of cardinality $\leq \kappa$ and let it act in a Borel way on a Borel subset $X \subseteq 2^\kappa$. Let E_G^X be the* (*Borel*) *equivalence relation induced by this action. Then $E_G^X \leq_B E_0$.*

Proof. Let $\mathcal{P}(G)$ be the set of all subsets of G. Each element $A \in \mathcal{P}(G)$ can be identified with a function $\eta \in 2^G$ such that for all $g \in G$, $g \in A \Leftrightarrow \eta(g) = 1$. Then 2^G can be identified with 2^μ where μ is the cardinality of G via a bijection $G \to \mu$. If $\mu = \kappa$, then the topology on $\mathcal{P}(G)$ is in this way induced from 2^κ. If $\mu < \kappa$, then the topology on $\mathcal{P}(G)$ is discrete. Further, equip $\mathcal{P}(G)^\kappa$ with the "κ-product topology", i.e. the open sets are of the form $\prod_{i<\alpha} U_i \times \prod_{\alpha \leq j < \kappa} \mathcal{P}(G)$ where $\alpha < \kappa$ and U_i are open subsets of $\mathcal{P}(G)$.

The group G acts on $\mathcal{P}(G)^\kappa$ coordinatewise by multiplication on the right, $g \cdot (X_i)_{i<\kappa} = (X_i g)_{i<\kappa}$. This gives rise to the equivalence relation $E_G^{P(G)^\kappa}$.

Claim 2.1. $E_G^X \leq_B E_G^{\mathcal{P}(G)^\kappa}$.

Proof. Let $\pi\colon \kappa \to 2^{<\kappa}$ be a bijection. Let $x \in X$ and for each $\alpha < \kappa$ let
$$Z_\alpha(x) = \{g \in G \mid gx \in [\pi(\alpha)]\}.$$
This defines a reduction: an element $x \in X$ is mapped to $(Z_\alpha(x))_{\alpha<\kappa}$. Suppose there is $g_0 \in G$ such that $y = g_0 x$ for some $x, y \in X$. Then
$$Z_\alpha(x) = \{g \in G \mid gx \in [\pi(\alpha)]\} = \{gg_0 \in G \mid gy \in [\pi(\alpha)]\} = Z_\alpha(y)g_0.$$
On the other hand, suppose that there exists $g \in G$ such that $Z_\alpha(x) = Z_\alpha(y)g$ for all $\alpha < \kappa$. It is enough to show that $g^{-1}y \in [p]$ for all basic open neighborhoods $[p]$ of x. So suppose $U = [p]$ is a basic neighborhood containing x and let $\alpha = \pi^{-1}(p)$. Now obviously $1_G \in Z_\alpha(x)$, so $1_G \in Z_\alpha(y)g$ and thus $g^{-1} \in Z_\alpha(y)$, i.e. $g^{-1}y \in [p]$. $\blacksquare_{2.1}$

For a set S, F_S is the free group generated by elements of S. $F_\emptyset = F_0$ is the trivial group.

Claim 2.2. $E_G^{\mathcal{P}(G)^\kappa} \leq_B E_{F_\kappa}^{\mathcal{P}(F_\kappa)^\kappa}$.

Proof. Since G has size $\leq \kappa$ and F_κ is a free group on κ generators, there is a normal subgroup $N \subseteq F_\kappa$ such that $G \cong F_\kappa/N$ (see e.g. [Rot95, Chapter 11]). Assume without loss of generality that $G = F_\kappa/N$. Let pr be the canonical projection map $F_\kappa \to F_\kappa/N$. For $(A_\alpha)_{\alpha<\kappa} \in \mathcal{P}(G)^\kappa$, let

$$F((A_\alpha)_{\alpha<\kappa}) = (\mathrm{pr}^{-1}[A_\alpha])_{\alpha<\kappa}.$$

This is clearly a continuous reduction. $\blacksquare_{2.2}$

CLAIM 2.3. $E_{F_\kappa}^{\mathcal{P}(F_\kappa)^\kappa} \leq_B E_0$.

Proof. Thinking of F_α as a subgroup of F_κ for all $\alpha < \kappa$, we see that the action of F_κ on $\mathcal{P}(F_\kappa)^\kappa$ induces a canonical action of F_α on $\mathcal{P}(F_\alpha)^\alpha$. Denote $X_\alpha = \mathcal{P}(F_\alpha)^\alpha$ for all $\alpha \leq \kappa$ and let $*$ denote the action of all the groups of the form F_α. After natural identifications we have $X_\alpha \subset X_\beta \subset X_\kappa$ for all $\alpha < \beta < \kappa$. Fix a well-ordering $<_\alpha$ of X_α for each $\alpha < \kappa$. For all $\alpha < \beta \leq \kappa$ and $x \in X_\beta$, denote by $x \restriction \alpha$ the element of X_α which is the canonical restriction of x to X_α.

Fix $x \in X_\kappa$. For each α, let $x(\alpha)$ be the $<_\alpha$-least element of

$$\{g * (x \restriction \alpha) \mid g \in F_\alpha\}$$

and let $H(x) = (x(\alpha))_{\alpha<\kappa}$. We claim that for all $x, y \in X_\kappa$, $y = g * x$ for some $g \in F_\kappa$ if and only if there exists $\beta < \kappa$ such that for all $\alpha > \beta$, $x(\alpha) = y(\alpha)$.

Assume first that such a $g \in F_\kappa$ exists. Then $g \in F_\beta$ for some $\beta < \kappa$, and $g \in F_\alpha$ for all $\alpha > \beta$. Thus, it is obvious that $x(\alpha) = y(\alpha)$ for $\alpha > \beta$, because for these α,

$$\{g * (x \restriction \alpha) \mid g \in F_\alpha\} = \{g * (y \restriction \alpha) \mid g \in F_\alpha\}.$$

Assume now that there exists $\beta < \kappa$ such that $x(\alpha) = y(\alpha)$ for all $\alpha > \beta$. Then for each $\alpha > \beta$ there exists $g_\alpha \in F_\alpha$ such that $x \restriction \alpha = g_\alpha * (y \restriction \alpha)$. For each $\alpha > \beta$, let $\gamma(\alpha)$ be the least ordinal such that $g_\alpha \in F_{\gamma(\alpha)}$. If α is a limit ordinal, then $\gamma(\alpha) < \alpha$ and so there is γ_0 and a stationary $S_0 \subseteq \lim \kappa$ such that for all $\alpha \in S_0$ we have $g_\alpha \in F_{\gamma_0}$. Since $|F_{\gamma_0}| < \kappa$, there is a stationary $S \subseteq S_0$ and $g_* \in F_{\gamma_0}$ such that for all $\alpha \in S$ we have $g_\alpha = g_*$. Since S is unbounded, this obviously implies that $y = g_* * x$.

Fix bijections $f_\alpha \colon X_\alpha \to \kappa$ and map each $x \in X_\kappa$ to the sequence $(f_\alpha(x(\alpha)))_{\alpha<\kappa}$; denote this mapping by G. By the above we have $x = g * y$ for some $g \in F_\kappa$ if and only if $(G(x), G(y)) \in E_0$. It remains to show that G is continuous.

Suppose $x \in X_\kappa$ and take an open neighborhood U of $G(x)$. Then there is β such that

$$\{\eta \in \kappa^\kappa \mid \forall \alpha < \beta \ (\eta(\alpha) = f_\alpha(x(\alpha)))\} \subseteq U.$$

Now, the set $\{y \in F_\kappa \mid y \restriction \beta = x \restriction \beta\}$ is mapped inside U and contains x, so it remains to show that this set is open; but this follows from the definition of the topology on $X_\kappa = \mathcal{P}(F_\kappa)^\kappa$, in particular from the fact that the collection of open sets is closed under intersections of length $< \kappa$. $\blacksquare_{2.3}$

This completes the proof of Theorem 2. $\blacksquare$

THEOREM 3. ($V = L$) *There is a Borel equivalence relation E whose classes have size* 2, *which is smooth* (*i.e. Borel reducible to* id) *yet not induced by a Borel action of a group of size $\leq \kappa$.*

Proof.

CLAIM 3.1. *There is an open dense set $O \subseteq 2^\kappa$ and a bijection $f\colon O \to 2^\kappa \setminus O$ such that the graph of f is Borel, but f is not Borel as a function on any non-meager Borel set. However, the inverse of f is Borel.*

Proof. We let O be the complement of a certain closed set of "master codes" for size κ initial segments of L. This is defined as follows. Let $\mathcal{L}$ be the language of set theory augmented by constant symbols $\bar{\alpha}$ for each ordinal $\alpha < \kappa$. Also let T_0 denote the theory ZFC$^-$ (ZFC minus the power set axiom) plus $V = L$ plus the statement "there are only boundedly many ordinals β such that L_β satisfies ZFC$^-$". We consider complete, consistent theories T which extend T_0 and which in addition satisfy the following:

(1) There is no ω-sequence of formulas $\varphi_n(x)$ (mentioning constants $\bar{\alpha}$ for $\alpha < \kappa$) such that for each n both the sentence "$\exists! x\ \varphi_n(x)$" and the sentence "$\exists x, y\ (\varphi_n(x) \wedge \varphi_{n+1}(y) \wedge y \in x)$" belong to T.
(2) For each $\beta < \kappa$ and formula $\varphi(x)$ (mentioning constants $\bar{\alpha}$ for $\alpha < \kappa$) if the sentences "$\exists! x\ \varphi(x)$" and "$\exists x\ (\varphi(x) \wedge x < \bar{\beta})$" both belong to T then so does the sentence "$\exists x\ (\varphi(x) \wedge x = \bar{\gamma})$" for some $\gamma < \beta$.

By identifying sentences of $\mathcal{L}$ with ordinals less than κ, we can regard theories in $\mathcal{L}$ as subsets of κ. Now let $C \subseteq 2^\kappa$ be the set of theories T as above. Then C is a closed set, because if T fails to satisfy any of the properties above, the failure is witnessed by $T \restriction \alpha$ for some α. On the other hand "inconsistency of a theory" is a dense property, so the complement of C is dense. Therefore the complement is dense and open, and so C is nowhere dense.

The theories in C are exactly the first-order theories of models of the form L_β in which the constant symbol $\bar{\alpha}$ is interpreted as the ordinal α for each $\alpha < \kappa$ and in which the axioms of T_0 hold. For given a theory T in C, we can form a model of T out of terms built from T-definable functions applied to constants $\bar{\alpha}$, $\alpha < \kappa$, identifying two such terms when T proves them equal. We also define the $\in$-relation on terms using T. The result is a well-founded model by (1) above which can be identified with some L_β, with $\bar{\alpha}$

denoting α for each $\alpha < \kappa$ by (2) above. Conversely, given a model L_β of T_0 in which $\bar{\alpha}$ is interpreted as α for $\alpha < \kappa$, it is clear that $T =$ the theory of L_β with parameters less than κ belongs to C. For each T in C let $M(T)$ denote the model L_β of T described above, in which every element is definable from parameters less than κ, and $\bar{\alpha}$ is interpreted as α for each $\alpha<\kappa$.

We are ready to define the function $f : O \to C$ where O is the complement of C in 2^κ. List the elements of O in $<_L$-increasing order as $x_0, x_1, \ldots$ and list the elements of C in $<_L$-increasing order as $y_0, y_1, \ldots$; then set $f(x_i) = y_i$ for each $i < \kappa^+$. Note that f strictly increases L-rank because no model $M(T)$ for T in C is the limit of such models. It follows that the inverse of f is Borel: Given $y \in C$ we can identify $M(y)$ in a Borel way and then obtain $f^{-1}(y)$ (viewed as a subset of κ) as the set of $\gamma < \kappa$ such that $M(y)$ satisfies the sentence "$\bar{\gamma}$ belongs to the ith element in the $<_L$-increasing enumeration of $O = 2^\kappa \setminus C$ where i is the order type of the set of L-ranks of elements of C". Thus the graph of f is Borel. If B is a non-meager Borel set and g is a Borel function then we claim that g cannot agree with f on B: Indeed, let β_0 be so that L_{β_0} models ZFC^- and contains Borel codes for both B and g, and let $x \in B \cap O$ be κ-Cohen generic over L_{β_0}. Then $f(x) = T$ is the theory of a model $M(T) = L_\beta$ where β is greater than β_0. But $g(x)$ belongs to $L_{\beta_0}[x]$, which by the genericity of x is a model of ZFC^-, while $L_{\beta_0}[f(x)]$ does not satisfy ZFC^- as $f(x) = T$ codes the model L_β. ■3.1

Define xEy if and only if $x = y$, $y = f(x)$ or $x = f(y)$. Now E has a *Borel transversal*, i.e., a Borel function t such that $xEt(x)$ for all x and xEy if and only if $t(x) = t(y)$ for all x, y: Given $x \in 2^\kappa$, first decide in a Borel way if x is in O or not. If yes, then let $t(x) = x$, otherwise find $f^{-1}(x)$ in a Borel way (since f^{-1} is Borel) and let $t(x) = f^{-1}(x)$. This $t(x)$ is a Borel transversal. It follows that E is smooth.

Finally, suppose E is given by a Borel action of some group G of size at most κ. Then for each $x \in O$ choose $g_x \in G$ such that $f(x) = g_x \cdot x$; then for some fixed $g \in G$, $f(x) = g \cdot x$ for non-meager many $x \in O$, contradicting the fact that f is not Borel on any non-meager Borel set. ■

QUESTION 4. Is there a Borel equivalence relation with classes of size κ which is not reducible to E_0?

4. E_1 and E_{club}. Let E_1 be the equivalence relation on $(2^\kappa)^\kappa$ where $(x_\alpha)_{\alpha<\kappa}$ and $(y_\alpha)_{\alpha<\kappa}$ are equivalent if there exists $\beta < \kappa$ such that $x_\gamma = y_\gamma$ for all $\gamma > \beta$.

THEOREM 5. *E_1 and E_0 are bireducible.*

Proof. It is obvious that $E_0 \leq_B E_1$, so let us look at the other direction. To simplify notation, we think of E_0 on κ^κ: two functions η and ξ are

E_0-equivalent if the set $\{\alpha < \kappa \mid \eta(\alpha) \neq \xi(\alpha)\}$ is bounded. It is easy to see that E_0 on 2^κ is bireducible with this equivalence relation.

For all limit $\alpha < \kappa$, define E_1^α to be the equivalence relation on $(2^\alpha)^\alpha$ approximating E_1, i.e. $(x_i)_{i<\alpha} E_1^\alpha (y_i)_{i<\alpha}$ if for some $\beta < \alpha$, $x_i = y_i$ for all $i > \beta$. Now define the reduction $F\colon (2^\kappa)^\kappa \to \kappa^\kappa$ so that for all $(x_i)_{i<\kappa} \in (2^\kappa)^\kappa$, $F((x_i)_{i<\kappa})(\alpha) = 0$ if α is not limit, and otherwise it is a code for the E_1^α-equivalence class of $(x_i \restriction \alpha)_{i<\alpha}$.

Clearly F is continuous and if $(x_i)_{i<\kappa} E_1 (y_i)_{i<\kappa}$, then also $F((x_i)_{i<\kappa})$ and $F((y_i)_{i<\kappa})$ are E_0-equivalent (if $\beta < \kappa$ witnesses the first equivalence, it also witnesses the second).

Also if $(x_i)_{i<\kappa}$ and $(y_i)_{i<\kappa}$ are not E_1-equivalent, then for all $\alpha < \kappa$ there are $\gamma, \beta < \kappa$ such that $\beta > \alpha$ and $x_\beta(\gamma) \neq y_\beta(\gamma)$. Let $f(\alpha)$ be $\max\{\beta, \gamma\}$. Now if $\alpha^* < \kappa$ is limit and such that for all $\alpha < \alpha^*$, $f(\alpha) < \alpha^*$, then clearly $(x_i \restriction \alpha^*)_{i<\alpha^*}$ and $(y_i \restriction \alpha^*)_{i<\alpha^*}$ are not $E_1^{\alpha^*}$-equivalent, and thus $F((x_i)_{i<\kappa})(\alpha^*) \neq F((y_i)_{i<\kappa})(\alpha^*)$. Since the set of such α^* is unbounded, $F((x_i)_{i<\kappa})$ and $F((y_i)_{i<\kappa})$ are not E_0-equivalent. ■

Definition 6. If E is an equivalence relation on 2^κ, its *jump* is the equivalence relation on $(2^\kappa)^\kappa$ denoted by E^+, defined as follows. Two sequences $(x_\alpha)_{\alpha<\kappa}$ and $(y_\alpha)_{\alpha<\kappa}$ are E^+-equivalent if

$$\{[x_\alpha]_E \mid \alpha < \kappa\} = \{[y_\alpha]_E \mid \alpha < \kappa\}$$

where $[x]_E$ is the E-equivalence class of x. Since $(2^\kappa)^\kappa$ is homeomorphic to 2^κ we can assume without loss of generality that E^+ is also defined on 2^κ.

For an ordinal $\alpha < \kappa^+$ define $E^{\alpha+}$ by transfinite induction. To begin, define $E^{0+} = E$. If $E^{\alpha+}$ is defined, then $E^{(\alpha+1)+} = (E^{\alpha+})^+$.

Suppose α is limit and $E^{\beta+}$ is defined as an equivalence relation on $\{\beta\} \times 2^\kappa$ for $\beta < \alpha$. Let $X = \alpha \times 2^\kappa$. Denote $X_\beta = \{\beta\} \times 2^\kappa$, thus $X = \bigcup_{\beta<\alpha} X_\beta$. Let h be a homeomorphism $X \to 2^\kappa$. Two functions η and ξ are defined to be $E^{\alpha+}$-equivalent if $h^{-1}(\eta)$ and $h^{-1}(\xi)$ both belong to the same X_β and are $E^{\beta+}$-equivalent. This is called the *join* of the equivalence relations $\{E^{\beta+} \mid \beta < \alpha\}$ and is denoted $\bigoplus_{\beta<\alpha} E^{\beta+}$.

Theorem 7. *$E_0 <_B \mathrm{id}^+$.*

Proof. Recall that, as in Definition 1, id^+ can be replaced by the equivalence relation on $(2^\kappa)^\kappa$ where $(x_\alpha)_{\alpha<\kappa}$ and $(y_\alpha)_{\alpha<\kappa}$ are equivalent if there exists a permutation $s \in S_\kappa$ such that $x_\alpha = y_{s(\alpha)}$ for all α. The reduction is defined by

$$E_0 \le_B \mathrm{id}^+\colon \eta \mapsto (p + \eta)_{p \in 2^{<\kappa}},$$

where $+$ is coordinatewise sum modulo 2.

To show that $\mathrm{id}^+ \not\le_B E_0$, we will show that $\mathrm{id}^+ \not\le_B E_1$, and the result will follow from Theorem 5. Suppose $f\colon (2^\kappa)^\kappa \to (2^\kappa)^\kappa$ is a Borel reduction from id^+ to E_1. There is a co-meager set D on which f is continuous. Without loss of generality assume that $D = \bigcap_{i<\kappa} D_i$ where D_i are dense open such that for all limit j we have $D_j = \bigcap_{i<j} D_i$.

For every $i < \kappa$ we will define ordinals γ_i together with sequences $x^i = (x^i_\alpha)_{\alpha<\gamma_i}$ and $y^i = (y^i_\alpha)_{\alpha<\gamma_i}$ where each $x^i_\alpha, y^i_\alpha \in 2^{\gamma_i}$ and permutations $\pi_i \in S_{\gamma_i}$. These will satisfy the following requirements for every $i < j < \kappa$:

(1) $\pi_i \subseteq \pi_j$.
(2) $\gamma_i < \gamma_j$,
(3) For all $\alpha < \gamma_i$ we have $x^i_\alpha \subseteq x^j_\alpha$ and $y^i_\alpha \subseteq y^j_\alpha$.
(4) For all $\alpha < \gamma_i$ we have $x^i_\alpha = y^i_{\pi_i(\alpha)}$.
(5) Let $[(x^i_\alpha)_{\alpha<\gamma_i}]$ be the set of all $x = (x_\alpha)_{\alpha<\kappa} \in (2^\kappa)^\kappa$ such that $x^i_\alpha \subseteq x_\alpha$ for all $\alpha < \gamma_i$ (recall the definition of $[p]$ in Section 2). If i is a successor, then there exist $\beta > i$, $\delta < \kappa$ and $p, q \in (2^{\delta+1})^{\beta+1}$ such that
$$f[[(x^i_\alpha)_{\alpha<\gamma_i}] \cap D] \subseteq [p], \qquad f[[(y^i_\alpha)_{\alpha<\gamma_i}] \cap D] \subseteq [q]$$
and $p_\beta \neq q_\beta$.
(6) $[(x^{i+1}_\alpha)_{\alpha<\gamma_{i+1}}] \subseteq D_i$ and $[(y^{i+1}_\alpha)_{\alpha<\gamma_{i+1}}] \subseteq D_i$.

This will lead to a contradiction as follows. Let $\tilde{x} = (\tilde{x}_\alpha)_{\alpha<\kappa}$ be such that for every α we have $\tilde{x}_\alpha \restriction \gamma_i = x^i_\alpha$ if $\gamma_i > \alpha$. This is possible by (2) and (3). Analogously define $\tilde{y}$. Now by (1) we can define $\pi = \bigcup_{i<\kappa} \pi_i$ which by (4) witnesses that $\tilde{x}$ and $\tilde{y}$ are id^+-equivalent. By (6) they are in D and by continuity on D and by (5) the images $f(\tilde{x})$ and $f(\tilde{y})$ cannot be E_1-equivalent.

Claim 7.1. *For every $\alpha, \beta < \kappa$ and $p \in (2^\alpha)^\beta$ there are $x^*, y^* \in [p] \cap D$ such that x^* is not id^+-equivalent to y^*.*

Proof. We will define sequences $(\xi_k)_{k\leq\kappa}$ and $(\eta_k)_{k\leq\kappa}$ and ordinals ε_k such that for all $k < \kappa$ we have $\xi_k, \eta_k \in (2^{\varepsilon_k})^{\varepsilon_k}$, for $k_1 < k_2$ we have $\varepsilon_{k_1} < \varepsilon_{k_2}$, $\xi_{k_1} \subseteq \xi_{k_2}$ and $\eta_{k_1} \subseteq \eta_{k_2}$ (meaning that for all $\alpha < \varepsilon_{k_1}$ we have $\xi_{k_2}(\alpha) \restriction \varepsilon_{k_1} = \xi_{k_1}(\alpha)$ and $\xi_{k_1} \neq \xi_{k_2}$ and the same for $\eta_{k_1} \subsetneq \eta_{k_2}$), and the unions $\xi_\kappa = \bigcup_{k<\kappa} \xi_k$ and $\eta_\kappa = \bigcup_{k<\kappa} \eta_k$ are in D and not id^+-equivalent. Let $\varepsilon_0 = \max\{\alpha, \beta\}$ and extend p to q such that $q \in (2^{\varepsilon_0})^{\varepsilon_0}$ in an arbitrary way. Let $\xi_0 = \eta_0 = q$. If ξ_k and η_k are defined, first extend ξ_k to an element $\xi'_{k+1} \in (2^{\varepsilon'_{k+1}})^{\varepsilon'_{k+1}}$ (for suitable $\varepsilon'_{k+1} > \varepsilon_k$) such that $[\xi'_{k+1}] \subseteq D_k$. Then extend the first component of η_k so that it differs in a diagonal way from every component of ξ'_{k+1}. Next, extend the result to $\eta_{k+1} \in (2^{\varepsilon_{k+1}})^{\varepsilon_{k+1}}$ (for suitable $\varepsilon_{k+1} > \varepsilon'_{k+1}$) so that $[\eta_{k+1}] \subseteq D_k$ and $\varepsilon_{k+1} > \varepsilon'_{k+1}$. Finally extend ξ'_{k+1} to an element of $(2^{\varepsilon_{k+1}})^{\varepsilon_{k+1}}$ so that the first component of η_{k+1} is still diagonally different from every component of ξ_{k+1}; technically this means that $\eta_{k+1}(0)(\alpha) \neq \xi_{k+1}(\alpha)(\alpha)$. At limit k just take the natural limits of the sequences. In this way at the κth limit, ξ_κ and η_κ are as required, so we can define $x^* = \xi_\kappa$ and $y^* = \eta_\kappa$. $\blacksquare_{7.1}$

To start off the induction, let $\gamma_{-1} = 0$, let x^{-1} and y^{-1} be empty sequences and let the permutation π_{-1} be the empty function. Conditions (5) and (6) are not satisfied, but we are not concerned with that, because γ_{-1}, x^{-1}, y^{-1} and π_{-1} are not going to be included into our final sequence whose indexation will start from 0. Suppose that γ_i, x^i, y^i and π_i are defined for some i, and let x^* and y^* be sequences in $[x^i] \cap D$ and $[y^i] \cap D$ respectively which are not id^+-equivalent. This is possible by Claim 7.1.

Let β and δ be such that $f(x^*)(\beta)(\delta) \neq f(y^*)(\beta)(\delta)$; these exist because f is assumed to be a reduction and $f(x^*)$ and $f(y^*)$ are not E_1-equivalent. Now by continuity in D there is $\gamma^* > \max\{\gamma_i, \beta, \delta\}$ such that

$$f[[(x^*_\alpha \restriction \gamma^*)_{\alpha<\gamma^*}] \cap D] \subseteq [(f(x^*)_\alpha \restriction \delta + 1)_{\alpha<\beta+1}],$$
$$f[[(y^*_\alpha \restriction \gamma^*)_{\alpha<\gamma^*}] \cap D] \subseteq [(f(y^*)_\alpha \restriction \delta + 1)_{\alpha<\beta+1}]$$

and (6) is satisfied for $(x^*_\alpha \restriction \gamma^*)_{\alpha<\gamma^*}$ and $(y^*_\alpha \restriction \gamma^*)_{\alpha<\gamma^*}$. Now we want to glue a part of $(x^*_\alpha \restriction \gamma^*)_{\alpha<\gamma^*}$ to the end of $(y^*_\alpha \restriction \gamma^*)_{\alpha<\gamma^*}$ and vice versa: Let $\varepsilon = \gamma^* - \gamma_i$, the order type of $\gamma^* \setminus \gamma_i$, and let $\gamma_{i+1} = \gamma^* + \varepsilon$. Define x^{i+1}_α and y^{i+1}_α for all $\alpha < \gamma_{i+1}$ depending on α as follows. If $\alpha < \gamma^*$, let x^{i+1}_α be $x^*_\alpha \restriction \gamma_{i+1}$ and y^{i+1}_α be $y^*_\alpha \restriction \gamma_{i+1}$. If $\alpha = \gamma^* + \delta$ for some $\delta < \varepsilon$, then let x^{i+1}_α be $y^*_{\gamma_i+\delta}$ and y^{i+1}_α be $x^*_{\gamma_i+\delta}$. This gives us also the permutation π_{i+1} extending π_i.

If j is limit, then just define $\pi_j = \bigcup_{i<j} \pi_i$, $x^j_\alpha = \bigcup_{i'<i<j} x^i_\alpha$ and $y^j_\alpha = \bigcup_{i'<i<j} y^i_\alpha$ for some i' such that $\gamma_{i'} > \alpha$ and $\gamma_j = \sup_{i<j} \gamma_i$ and $\gamma_j = \sup_{i<j} \gamma_i$. Conditions (1)–(6) are easily seen to be satisfied. Note that (5) is not required at the limits. ∎

Definition 8. For a regular cardinal $\mu < \kappa$ and $\lambda \in \{2, \kappa\}$ let $E^\lambda_{\mu\text{-cub}}$ be the equivalence relation on λ^κ such that η and ξ are $E^\lambda_{\mu\text{-cub}}$-equivalent if the set $\{\alpha \mid \eta(\alpha) = \xi(\alpha)\}$ contains a *μ-cub*, i.e. an unbounded set which is closed under μ-cofinal limits. If T is a countable complete first-order theory, denote by $\cong^T_\kappa$ the isomorphism relation on the models of T of size κ.

In the following we show that

(1) The αth jump of identity for $\alpha < \kappa^+$ is reducible to $E^\kappa_{\mu\text{-cub}}$ for every regular $\mu < \kappa$,
(2) Every Borel isomorphism relation is reducible to $E^\kappa_{\mu\text{-cub}}$ for every regular $\mu < \kappa$,
(3) If T is a countable complete first-order classifiable (superstable with NDOP and NOTOP) and shallow theory, then $\cong^\kappa_T \le_B E^\kappa_{\mu\text{-cub}}$.

Definition 9. Fix a limit ordinal $\alpha \le \kappa$ and let t be a subtree of $\alpha^{<\omega}$ with no infinite branches. Let h be a function from the leaves of t to $2^{<\alpha}$. Then (t, h) determines the set $B_{(t,h)}$ as follows: $p \in 2^\alpha$ belongs to $B_{(t,h)}$ if player **II** has a winning strategy in the game $G(p, t, h)$: The players start at the root and then alternately, player **I** being first, choose a successor of the node they are at and then move to that successor. Eventually they reach a leaf l and player **II** wins if $h(l) \subset p$. We say that (t,h) is a *Borel code for* α.

If $\alpha = \kappa$, it is easy to see by induction on the rank of the tree that $B_{(t,h)}$ is a usual Borel set, and conversely, if $B \subset 2^\kappa$ is any Borel set, then there is a Borel code (t,h) for κ such that $B = B_{(t,h)}$.

If t is replaced by a more general $\kappa^+\kappa$-tree (subtree of $\kappa^{<\kappa}$ without branches of length κ), then the sets obtained in this way are the so called Borel* sets (cf. [Bla81, MV93, Hal96, FHK11]).

Suppose (t,h) is a Borel code for κ and $\alpha < \kappa$. Say that α is *good* for (t,h) if all $p \in t$ that are leaves in $t \cap \alpha^{<\omega}$ are also leaves in t, and for all those leaves we have $h(p) \in 2^{<\alpha}$. It is standard to verify that the set of good α for a fixed (t,h) is a cub set.

For a good α, define the αth *approximation* of (t,h), denoted $(t,h)\restriction\alpha$, to be the pair $(t\restriction\alpha, h\restriction\alpha)$ where $t\restriction\alpha = t \cap \alpha^{<\omega}$ and $h\restriction\alpha = h\restriction(t\restriction\alpha)$. This is well defined by the definition of good ordinals for (t,h). It is obvious that if (t,h) is a Borel code for κ and $\alpha < \kappa$ is good for (t,h), then $(t,h)\restriction\alpha$ is a Borel code for α.

By replacing $2^{<\alpha}$ with $(2^{<\alpha})^2$ for the range of h and making necessary changes we can define Borel codes for subsets of $(2^\alpha)^2$. Note that the game $G(p,t,h)$ is determined for all $p \in 2^\alpha$ (this is not the case for general Borel* sets). Make a similar definition for codes of Borel subsets of $2^\kappa \times 2^\kappa$.

LEMMA 10. *Suppose that* $B = B_{(t,h)}$ *is a Borel subset of* $2^\kappa \times 2^\kappa$. *Then*

$$(\eta,\xi) \in B \iff (\eta\restriction\alpha, \xi\restriction\alpha) \in B_{(t,h)\restriction\alpha} \text{ for cub-many } \alpha,$$
$$(\eta,\xi) \notin B \iff (\eta\restriction\alpha, \xi\restriction\alpha) \notin B_{(t,h)\restriction\alpha} \text{ for cub-many } \alpha.$$

Proof. Suppose $(\eta,\xi) \in B$ and let σ be a winning strategy of player **II** in $G((\eta,\xi),t,h)$. Let C be the set of those limit α which are good for (t,h) and for which $t\restriction\alpha$ is closed under σ. Clearly $(\eta\restriction\alpha, \xi\restriction\alpha) \in B_{(t,h)\restriction\alpha}$ for all $\alpha \in C$ and C is cub.

Conversely, if $(\eta,\xi) \notin B$, then player **I** has a winning strategy τ in $G((\eta,\xi),t,h)$ and by closing under τ we obtain the needed cub set again. ∎

LEMMA 11. *Let S be the set of Borel equivalence relations E such that for some Borel code (t,h), $E = B_{(t,h)}$ and $B_{(t,h)\restriction\alpha}$ is an equivalence relation for cub-many good $\alpha < \kappa$. Then S contains* id *and is closed under jump and the join operation $\bigoplus$ as in the definition of iterated jump, Definition* 6.

Proof. Let $(p_{\beta,\gamma})_{\gamma<\kappa}$ be the enumeration of all $p \in 2^{<\kappa}$ with $\operatorname{dom} p > \beta$. Clearly $\mathrm{id} = \bigcap_{\beta<\kappa} \bigcup_{\gamma<\kappa} [(p_{\beta,\gamma}, p_{\beta,\gamma})]$. This can be translated into a Borel code as follows: Let $t = \kappa \cup \kappa^2$ be ordered as the subtree of $\kappa^{<\kappa}$ and $h(\beta,\gamma) = (p_{\beta,\gamma}, p_{\beta,\gamma})$ for all $\beta,\gamma < \kappa$. Informally, first player **I** chooses a length β and player **II** has to produce a sequence p of length at least β such that the payoff pair belongs to the basic open set determined by (p,p). The set of α such that for all $\beta,\gamma<\alpha$ we have $p_{\beta,\gamma}\in 2^{<\alpha}$ is cub, and for those α the set $B_{(t,h)\restriction\alpha}$ is the identity on 2^α.

Suppose E is an equivalence relation, without loss of generality on 2^κ, which is in S, and (t,h) is a code for E witnessing that $E=B_{(t,h)}$ and that C is a cub set of those α for which $B_{(t,h)\restriction\alpha}$ is an equivalence relation on 2^α. Note that C is a subset of the good ordinals for (t,h). Denote for simplicity $E_\alpha=B_{(t,h)\restriction\alpha}$ for $\alpha\in C$. Then E^+ can be defined as follows: $((x_i)_{i<\kappa},(y_i)_{i<\kappa})\in E^+$ if and only if

$$\forall i<\kappa\ \exists j<\kappa\ ((x_i,y_j)\in E)\wedge\forall j<\kappa\ \exists i<\kappa\ ((x_i,y_j)\in E). \tag{1}$$

Let us translate this into a Borel code of E^+ in a standard way. For $k\in\{1,2\}$ let t_k be the subtree of $\kappa^{<\omega}$ defined by

$$\{p\in\kappa^{<\omega}\mid p\subset(k,0,\alpha,\beta)^\frown q,\ (\alpha,\beta)\in\kappa^2,\ q\in t\}.$$

Each leaf l of t_k determines a branch η of t_k, which in turn determines ordinals $\alpha=\alpha(l)$ and $\beta=\beta(l)$ such that $\eta(2)=\alpha$ and $\eta(3)=\beta$. Let $\iota_{\alpha\beta,k}$ be the inclusion of t into t_k taking p to $(k,0,\alpha,\beta)^\frown p$. Let $h_k(l)$, $k=1,2$, be the open subsets of $((2^\kappa)^\kappa)^2$ defined by

$$\begin{aligned}h_1(l)&=\{((x_i)_{i<\kappa},(y_j)_{j<\kappa})\mid(x_{\alpha(l)},y_{\beta(l)})\in h(\iota^{-1}_{\alpha\beta,1}(l))\},\\ h_2(l)&=\{((x_i)_{i<\kappa},(y_j)_{j<\kappa})\mid(x_{\beta(l)},y_{\alpha(l)})\in h(\iota^{-1}_{\alpha\beta,2}(l))\}.\end{aligned}$$

Now let $t^+=t_1\cup t_2$; then $\emptyset$ is the root of t^+. Let $h^+(l)$ equal $h_i(l)$ if l is a leaf of t_i for $i\in\{1,2\}$. It is now easy to verify that player **II** has a winning strategy in $G((\eta,\xi),t^+,h^+)$ if and only if $(\eta,\xi)\in E^+$. The first move of **I** corresponds to the conjunction in formula (1). In the next move **II** has only one choice, namely to go to the node $(k,0)$ where (k) is the node picked by **I**. The following move of player **I** corresponds to one of the universal quantifiers in (1). The next move of player **II** corresponds to one of the existential quantifiers in the formula and after her move the players are essentially at the root of t, so the rest of the game corresponds to the atoms $(x_i,y_j)\in E$ of formula (1).

Suppose α is in C and consider $(t^+,h^+)\restriction\alpha$. Then $B_{(t^+,h^+)\restriction\alpha}$ is the set

$$\forall i<\alpha\ \exists j<\alpha\ ((x_i,y_j)\in E_\alpha)\wedge\forall j<\alpha\ \exists i<\alpha\ ((x_i,y_j)\in E_\alpha),$$

which can be seen through the same translation procedure as above. This is by definition a code for the jump of E_α when $\alpha\in C$, and is therefore an equivalence relation.

Similarly suppose that $E_i\in S$ are equivalence relations for $i<\kappa$ and witnessing codes (t_i,h_i) are given together with C_i, cub sets of good ordinals for (t_i,h_i) such that $B_{(t_i,h_i)\restriction\alpha}$ is an equivalence relation for each $\alpha\in C_i$.

Denote this equivalence relation by E_i^α. Assume without loss of generality that the domain of E_i is $\{i\} \times 2^\kappa$. Let $t_\oplus$ be the tree defined by

$$\{p \in \kappa^{<\omega} \mid p \subset (0,i)^\frown q,\ i < \kappa,\ q \in t_i\}.$$

If l is a leaf of $t_\oplus$, then it is a leaf of $\{i\} \times t_i$ for some i. Denote by l' the corresponding leaf of t_i. Let $h_\oplus(l)$ be equal to $\{i\} \times h_i(l')$. The idea is that if $\eta(0) \neq \xi(0)$, then $(\eta,\xi) \notin B_{(t_\oplus,h_\oplus)}$, and otherwise $(\eta,\xi) \notin B_{(t_\oplus,h_\oplus)}$ holds if and only if $(\eta,\xi) \in B_{(t_i,h_i)}$ where $i = \eta(0) = \xi(0)$. This is easy to check and implies that $t_\oplus, h_\oplus$ is a code for $\bigoplus_{i<\kappa} E_i$. By $\triangle_{i<\kappa} C_i$ we mean the diagonal intersection of the cub sets:

$$\triangle_{i<\kappa} C_i = \{\alpha \mid \forall i < \alpha\ (\alpha \in C_i)\}.$$

If C_i are all cub sets, then $\triangle_{i<\kappa} C_i$ is also cub (cf. [Kun80]). If $\alpha \in \triangle_{i<\kappa} C_i$, then it is good for all (t_i,h_i) with $i < \alpha$, E_i^α is defined, and $B_{(t_\oplus,h_\oplus)\restriction\alpha}$ is therefore $\bigoplus_{i<\alpha} E_i^\alpha$. ■

It follows that S contains all iterates of the jump $\mathrm{id}^{+\beta}$, $\beta < \kappa^+$.

THEOREM 12. *Let E be an equivalence relation in S (as defined in the formulation of Lemma 11). Then E is reducible to $E^\kappa_{\mu\text{-cub}}$ for any regular $\mu < \kappa$ (cf. Definition 8).*

Proof. Let E be $B_{(t,h)}$ where (t,h) witnesses that E belongs to S. To each η assign the function f_η where $f_\eta(\alpha)$ is a code for the $B_{(t,h)\restriction\alpha}$ equivalence class of $\eta\restriction\alpha$ (if $\mathrm{cf}(\alpha) = \mu$ and $B_{(t,h)\restriction\alpha}$ is an equivalence relation, 0 otherwise). By Lemma 10, if $\eta E \xi$ then $f_\eta(\alpha) = f_\xi(\alpha)$ for μ-cub-many α, and if $\neg\eta E\xi$ then $f_\eta(\alpha) \neq f_\xi(\alpha)$ for μ-cub-many α. ■

COROLLARY 13. *The iterated jumps $\mathrm{id}^{\alpha+}$ of the identity are reducible to $E^\kappa_{\mu\text{-cub}}$ for each regular $\mu < \kappa$.* □

COROLLARY 14. *If $\mathcal{M}$ is a Borel class of models of size κ such that $\cong_\mathcal{M}$, the isomorphism relation on $\mathcal{M}$, is Borel, then $\cong_\mathcal{M}$ is Borel reducible to $E^\kappa_{\mu\text{-cub}}$ for all regular $\mu < \kappa$.*

Proof. Using similar techniques to those in classical descriptive set theory ([Fri00]; for a proof see also [Gao08, Lemma 12.2.7]) one can show that a Borel isomorphism can be reduced to an iterated jump of identity. ■

COROLLARY 15. *Suppose T is a countable complete first-order classifiable (superstable with NDOP and NOTOP) and shallow theory. Then $\cong^\kappa_T \leq_B E^\kappa_{\mu\text{-cub}}$.*

Proof. By [FHK11, Theorem 68] the isomorphism relation of a classifiable shallow theory is Borel, so we apply Corollary 14. ■

We have shown in [FHK11, Theorem 75] that under certain cardinality assumptions on κ, a complete countable first-order theory T is classifiable if and only if for all regular $\mu < \kappa$, $E^2_{\mu\text{-cub}} \not\leq_B \cong^\kappa_T$. Clearly $E^2_{\mu\text{-cub}} \leq_B E^\kappa_{\mu\text{-cub}}$.

QUESTION 16. Is $E^{\kappa}_{\mu\text{-cub}}$ reducible to $E^{2}_{\mu\text{-cub}}$?

If the answer to Question 16 is "yes", then using [FHK11, Theorem 75] we obtain: Suppose T_1 and T_2 are complete first-order theories with T_1 classifiable and shallow and T_2 non-classifiable. Also suppose that $\kappa = \lambda^+ = 2^\lambda > 2^\omega$ where $\lambda^{<\lambda} = \lambda$. Then $\cong^{\kappa}_{T_1}$ is Borel reducible to $\cong^{\kappa}_{T_2}$.

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