FUNDAMENTA
MATHEMATICAE
232 (2016)

Calibres, compacta and diagonals

by

Paul Gartside and **Jeremiah Morgan** (Pittsburgh, PA)

Abstract. For a space Z let $\mathcal{K}(Z)$ denote the partially ordered set of all compact subspaces of Z under set inclusion. If X is a compact space, $\varDelta$ is the diagonal in X^2, and $\mathcal{K}(X^2 \setminus \varDelta)$ has calibre (ω_1, ω), then X is metrizable. There is a compact space X such that $X^2 \setminus \varDelta$ has relative calibre (ω_1, ω) in $\mathcal{K}(X^2 \setminus \varDelta)$, but which is not metrizable. Questions of Cascales et al. (2011) concerning order constraints on $\mathcal{K}(A)$ for every subspace of a space X are answered.

Introduction. Let Z be a topological space and let $\mathcal{K}(Z)$ be the collection of all compact subsets of Z. Then $\mathcal{K}(Z)$ is partially ordered by set inclusion, $\subseteq$. The purpose of this paper is to investigate the order properties of $\mathcal{K}(Z)$, especially in the case when Z is the complement of $\varDelta$, the diagonal in the square of a compact space X. This is motivated by Schneider's theorem that if a space X is compact and $\mathcal{K}(X^2 \setminus \varDelta)$ has countable cofinality, then X is metrizable. (Schneider's theorem is normally stated as: 'a compact space with G_δ-diagonal is metrizable', but taking complements and using compactness easily gives our formulation. Here and below, all spaces are assumed to be T_1 and regular.)

Let P be a partially ordered set. For cardinals $\kappa \geq \lambda \geq \mu$ we say that P has *calibre* (κ, λ, μ) if for every subset S of P with cardinality $\geq \kappa$, there is a λ-sized subset S_1 of S such that every subset of S_1 with cardinality $\leq \mu$ has an upper bound in P. Following convention, 'calibre $(\kappa, \lambda, \lambda)$' is abbreviated to 'calibre (κ, λ)', and 'calibre (κ, κ)' to 'calibre κ'. Note that if P has countable cofinality, then it has calibre ω_1 and hence calibre (ω_1, ω).

Our main positive result (Theorem 2.1) is the following strengthening of Schneider's theorem: if X is a compact space such that $\mathcal{K}(X^2 \setminus \varDelta)$ has calibre (ω_1, ω), then X is metrizable. The proof is direct and topological.

2010 *Mathematics Subject Classification*: Primary 54E35; Secondary 54D30, 54G20.
Key words and phrases: (relative) calibres, space of compact subsets, metrization, Tukey quotient, compact spaces, cosmic spaces, $\aleph_0$-spaces, domination by a separable metric space, domination by irrationals.

Theorem 2.1 is also a natural strengthening of results of Cascales, Orihuela and Tkachuk [2, 4]. To state these results, and to make the connection between our theorem and theirs, we remind the reader of the concept of a Tukey quotient which comes from order theory.

For partially ordered sets P and Q, a function $\phi : P \to Q$ is a *Tukey quotient*, and we write $P \geq_T Q$, provided ϕ maps cofinal sets to cofinal sets. The existence, and non-existence, of Tukey quotients between natural partial orders occurring in topology, analysis and set theory has been heavily studied (see [9, 15], for example). In our context, where P and Q are both of the form $\mathcal{K}(Z)$ for some space Z, and so both P and Q are Dedekind complete, we have $P \geq_T Q$ if and only if there is a function $\phi : P \to Q$ which is order-preserving and such that $\phi(P)$ is cofinal in Q.

In terms of Tukey quotients, Schneider's theorem essentially says that if X is compact and $\mathcal{K}(\mathbb{N}) \geq_T \mathcal{K}(X^2 \setminus \Delta)$, then X is metrizable, while the relevant theorem from [2] is that if X is compact and $\mathcal{K}(\mathbb{N}^{\mathbb{N}}) \geq_T \mathcal{K}(X^2 \setminus \Delta)$, then X is metrizable. In [4] it is more generally shown that if X is compact and $\mathcal{K}(M) \geq_T \mathcal{K}(X^2 \setminus \Delta)$, where M is some separable metrizable space, then X is metrizable. (For a version of this theorem valid for compact spaces of uncountable weight see [1], and see [3] for a recent survey of these and related results.) The proofs given of the latter two results are elegant but indirect. The proof in [2] passes through consideration of the space $C(X)$ of all continuous real-valued functions on X, with the supremum norm, and an application of the Arzelà–Ascoli theorem. The proof in [4] utilizes $C_p(X)$ (i.e. $C(X)$ with the pointwise topology), implicitly reproves part of the Arzelà–Ascoli theorem and appeals to a non-trivial result of Baturov.

It is easy to see that if P has calibre (κ, λ, μ) and $P \geq_T Q$, then Q also has calibre (κ, λ, μ). Let M be separable metrizable. Using the usual (Vietoris) topology on $\mathcal{K}(M)$, it is straightforward (Lemma 1.6) to verify that $\mathcal{K}(M)$ has calibre (ω_1, ω). Hence if $\mathcal{K}(M) \geq_T \mathcal{K}(Z)$, then $\mathcal{K}(Z)$ has calibre (ω_1, ω), and thus the results of [2, 4] follow from Theorem 2.1.

Natural weakenings of the hypothesis in our main positive theorem lead to additional results, open problems and further connections with the work in [4]. We generalize calibres of partially ordered sets as follows. Let P be a partially ordered set and P' a subset of P. For cardinals $\kappa \geq \lambda \geq \mu$, it is said that P' has *relative calibre* (κ, λ, μ) (*in* P) if for every subset S of P' with cardinality $\geq \kappa$, there is a λ-sized subset S_1 of S such that every subset of S_1 with cardinality $\leq \mu$ has an upper bound in P.

We say that a space Z has *relative calibre* (κ, λ, μ) if it has relative calibre (κ, λ, μ) in $\mathcal{K}(Z)$. Clearly we have the following equivalence: a space Z has relative calibre (κ, λ, μ) if and only if for any subset S of Z with $|S| \geq \kappa$, there is a λ-sized subset S_1 contained in S such that for any $S_2 \subseteq S_1$ with $|S_2| \leq \mu$,

$\overline{S_2}$ is compact. For compact X we now see that $X^2 \setminus \Delta$ has relative calibre ω_1 if and only if X has a small diagonal. Whether or not every compact space with a small diagonal is metrizable is a deep open problem, but positive results follow from the continuum hypothesis [13] and the proper forcing axiom (PFA) (see [7]). So, consistently at least, a compact space X with $X^2 \setminus \Delta$ of relative calibre ω_1 is metrizable. In our main negative result, we show that—in contrast to the non-relative case—we cannot weaken 'relative calibre ω_1' to 'relative calibre (ω_1, ω)'. Indeed, Theorem 3.1 shows that, in ZFC, there is a compact first countable, non-metrizable space X such that $X^2 \setminus \Delta$ has relative calibre (ω_1, ω).

Let P and Q be partially ordered sets, and let Q' be a subset of Q. A function $\phi : P \to Q$ is a *Tukey quotient relative to* Q', denoted $P \geq_T (Q', Q)$, provided ϕ is such that for each cofinal set C of P, the set $\phi(C)$ is cofinal for Q' (for every q in Q' there is a p from C such that $q \leq \phi(p)$). For our purposes, Q' will be a space Y considered as a subset of $\mathcal{K}(Y)$. In this case it suffices for ϕ to be order-preserving and to map P to a set cofinal for Y, in other words: ϕ is order-preserving and $\phi(P)$ is a (compact) cover of Y. In this case we abbreviate '$P \geq_T (Y, \mathcal{K}(Y))$' to '$P \geq_T Y$'. Note that when $Y = \mathcal{K}(X)$, we apparently have two meanings for '$P \geq_T \mathcal{K}(X)$' depending on whether we treat $\mathcal{K}(X)$ as a space or a partial order, but in fact they coincide (see Lemma 1.1).

The paper [4] is mainly devoted to the study of spaces X for which there is a separable metrizable space M such that $\mathcal{K}(M) \geq_T X$. One interesting result obtained in that paper is that *consistently*, if X is compact and $\mathcal{K}(\mathbb{N}^{\mathbb{N}}) \geq_T X^2 \setminus \Delta$, then X is metrizable. An intriguing open problem is whether a compact space X must be metrizable if $\mathcal{K}(M) \geq_T X^2 \setminus \Delta$ for some separable metrizable space M. It is easy to verify that if a partially ordered set P has calibre (κ, λ, μ) and $P \geq_T Y$, then Y has relative calibre (κ, λ, μ) (in $\mathcal{K}(Y)$). Hence if $\mathcal{K}(M) \geq_T Y$, where M is separable metrizable, then Y has relative calibre (ω_1, ω). Further, consistently (see Section 2.2 below for details), $\mathcal{K}(\mathbb{N}^{\mathbb{N}})$ has calibre ω_1, hence if $\mathcal{K}(\mathbb{N}^{\mathbb{N}}) \geq_T Y$, then Y has relative calibre ω_1, and the result of [4] mentioned immediately above is a consequence of Corollary 2.5.

The recent book [14] provides numerous applications for the metrizability problem of compact spaces arising in functional analysis. Indeed many results presented therein can usefully be rephrased in terms of Tukey quotients and relative Tukey quotients.

The results above have mostly focused on when a *specific* subspace, $X^2 \setminus \Delta$, of a square, X^2, has some order condition on its compact subsets. The third section of this paper considers what occurs if *all* subspaces of a space have conditions imposed on the order structure of their compact subsets. We answer a number of problems from [4]. We also show that, for example, it is consistent and independent that if X is compact and *all* subspaces of X^2 have relative calibre (ω_1, ω), then X is metrizable. This should be compared with our ZFC example, Theorem 3.1.

1. Preliminary lemmas. Here we collect together some miscellaneous lemmas.

LEMMA 1.1. *Let P be a partial order and X a space. Then $P \geq_T (\mathcal{K}(X), \subseteq)$ if and only if $P \geq_T \mathcal{K}(X)$ (where $\mathcal{K}(X)$ is considered as a topological space).*

Proof. Suppose first that $\phi : P \to (\mathcal{K}(X), \subseteq)$ is a Tukey quotient. For each p in P, let $K_p = \mathcal{K}(\phi(p))$. Then K_p is a compact subset of $\mathcal{K}(X)$, and if $p \leq p'$ then clearly $K_p \subseteq K_{p'}$. Further if L is in $\mathcal{K}(X)$, then as $\phi(P)$ is cofinal in $\mathcal{K}(X)$, there is a p in P such that $\phi(p) \supseteq L$. Then L is in $\mathcal{K}(\phi(p)) = K_p$, and so the K_p's cover $\mathcal{K}(X)$.

Conversely, suppose $\phi : P \to \mathcal{K}(\mathcal{K}(X))$ is order-preserving and $\phi(P)$ covers $\mathcal{K}(X)$. Define $\hat{\phi} : P \to \mathcal{K}(X)$ by $\hat{\phi}(p) = \bigcup \phi(p)$. Then $\hat{\phi}$ is order-preserving, and if L is a compact subset of X, then for some p in P, we know $L \in \phi(p)$, and so $L \subseteq \hat{\phi}(p)$. ∎

LEMMA 1.2. *If $\mathcal{K}(X)$ has relative calibre (κ, λ, μ) in $\mathcal{K}(\mathcal{K}(X))$, then $\mathcal{K}(X)$ has calibre (κ, λ, μ).*

Proof. This follows from the fact that if $\mathcal{S}$ is a subset of $\mathcal{K}(X)$ and K is an upper bound for S in $\mathcal{K}(\mathcal{K}(X))$ (i.e., $\{F\} \subseteq K$ for each $F \in \mathcal{S}$), then $\hat{K} = \bigcup K$ is an upper bound for $\mathcal{S}$ in $\mathcal{K}(X)$ (i.e., $F \subseteq \hat{K}$ for each $F \in \mathcal{S}$). ∎

LEMMA 1.3. *If X has relative calibre (ω_1, ω), then X has countable extent.*

Proof. If X does not have countable extent, then there is an uncountable closed discrete subspace S of X. Since any subset of S is also discrete and closed in X, then no infinite subset of S can have compact closure. Hence, X does not have relative calibre (ω_1, ω). ∎

LEMMA 1.4. *If X is Fréchet–Urysohn and has countable extent, then X has relative calibre (ω_1, ω).*

Proof. Let S be any uncountable subset of X. Then S is not a closed discrete subspace, so there is an $x \in X$ such that $x \in \overline{S \setminus \{x\}}$. Hence, there is an infinite sequence S_1 contained in $S \setminus \{x\}$ and converging to x. Then S_1 has compact closure $\overline{S_1} = S_1 \cup \{x\}$. ∎

Combining Lemmas 1.2 and 1.4 gives:

Lemma 1.5. *If $\mathcal{K}(X)$ is Fréchet–Urysohn and has countable extent, then $\mathcal{K}(X)$ has calibre (ω_1, ω).*

In the next result, $\mathfrak{b}$ is the minimal size of an unbounded subset of $(\mathbb{N}^{\mathbb{N}}, <^*)$, where for f and g in $\mathbb{N}^{\mathbb{N}}$, we say $f <^* g$ if $f(n) < g(n)$ for all but finitely many n. Part (i) follows from the preceding lemma (if M is separable metrizable then so is $\mathcal{K}(M)$). See [10] for a proof of the second part.

Lemma 1.6.

(i) *If M is separable metrizable, then $\mathcal{K}(M)$ has calibre (ω_1, ω).*
(ii) *$\mathcal{K}(\mathbb{N}^{\mathbb{N}})$ has calibre ω_1 if and only if $\omega_1 < \mathfrak{b}$.*

The following is similar to the well-known fact that the product of a partial order with calibre ω_1 and a partial order with calibre (ω_1, ω) has calibre (ω_1, ω); verification is straightforward.

Lemma 1.7. *If X has relative calibre ω_1 and Y has relative calibre (ω_1, ω), then $X \times Y$ has relative calibre (ω_1, ω).*

Lemma 1.8. *Suppose $X = \bigcup_{n \in \mathbb{N}} X_n$ and $\lambda \leq \omega_1$. If each X_n has relative calibre (ω_1, λ), then so does X.*

Proof. If S is an uncountable subset of X, then there is an $n \in \mathbb{N}$ such that $S_n = S \cap X_n$ is uncountable. So there is a λ-sized subset S' of S_n with compact closure in X_n. But then $\overline{S'}^X = \overline{S'}^{X_n}$, so S' has compact closure in X. ∎

Lemma 1.9. *Suppose Y is an F_σ subset of X and $\lambda \leq \omega_1$. If X has relative calibre (ω_1, λ), then so does Y.*

Proof. By Lemma 1.8, it suffices to consider the case when Y is a closed subset of X. But then the result follows immediately from the fact that $\overline{S}^Y = \overline{S}^X$ for any $S \subseteq Y$. ∎

2. Positive results for $X^2 \setminus \Delta$

2.1. Calibres

Theorem 2.1. *Let X be a compact space. If $\mathcal{K}(X^2 \setminus \Delta)$ has calibre (ω_1, ω), then X is metrizable.*

Proof. Our first goal will be to show that X is first countable. It suffices to show X is hereditarily Lindelöf because then the points of X are G_δ, and compactness of X then implies first countability.

If X is not hereditarily Lindelöf, then it contains an uncountable right-separated sequence $\{y_\alpha : \alpha \in \omega_1\}$. So each y_α has an open neighborhood U_α such that $y_\beta \notin U_\alpha$ for all $\beta > \alpha$. For each $\alpha \in \omega_1$, let $V_\alpha = X \setminus \{y_\alpha\}$ and $N_\alpha = U_\alpha^2 \cup V_\alpha^2$, which is an open neighborhood of the diagonal. Then

$K_\alpha = X^2 \setminus N_\alpha$ is in $\mathcal{K}(X^2 \setminus \Delta)$. The family $\mathcal{K} = \{K_\alpha : \alpha \in \omega_1\}$ is uncountable, so there is an infinite $A \subseteq \omega_1$ and a $K \in \mathcal{K}(X^2 \setminus \Delta)$ such that $K_\alpha \subseteq K$ for all $\alpha \in A$. Hence, $N = X^2 \setminus K$ is an open neighborhood of Δ such that $N \subseteq \bigcap_{\alpha \in A} N_\alpha$. Without loss of generality, we may assume $N = \bigcup_{W \in \mathcal{W}} W^2$ for some open cover $\mathcal{W}$ of X.

We will show that each $W \in \mathcal{W}$ contains at most one point of the infinite set $\{y_\alpha : \alpha \in A\}$. Indeed, if $y_\alpha, y_\beta \in W$ for some $\alpha, \beta \in A$, then we see that

$$(y_\alpha, y_\beta) \in W^2 \subseteq N \subseteq \bigcap_{\gamma \in A} N_\gamma \subseteq N_\alpha = U_\alpha^2 \cup V_\alpha^2.$$

Now since $y_\alpha \notin V_\alpha$, we must have $y_\beta \in U_\alpha$, which gives $\beta \le \alpha$. Similarly, we have $(y_\alpha, y_\beta) \in U_\beta^2 \cup V_\beta^2$, which implies that $\alpha \le \beta$, and so $\alpha = \beta$.

Hence, $\mathcal{W}$ has no finite subcover, which contradicts the compactness of X. So X is hereditarily Lindelöf, and thus first countable, as claimed.

Now, for each pair (U_0, U_1) of open sets of X with disjoint closures, pick open sets $V_\ell = V_\ell(U_0, U_1)$ for $\ell = 0, 1$ such that $\overline{U_\ell} \subseteq V_\ell$ and $\overline{V_0} \cap \overline{V_1} = \emptyset$. Also, let $C_\ell = C_\ell(U_0, U_1) = X \setminus V_\ell$ for $\ell = 0, 1$.

Let (U_0', U_1') be another pair of open sets with disjoint closures, and let $V_\ell' = V_\ell(U_0', U_1')$ for $\ell = 0, 1$. We say the pairs (U_0, U_1) and (U_0', U_1') are *comparable* if $U_\ell \subseteq V_\ell'$ and $U_\ell' \subseteq V_\ell$ for $\ell = 0, 1$. By an easy Zorn's lemma argument, there is a maximal incomparable collection $\mathcal{S}$ of such pairs.

For any subsets $A, B \subseteq X$, let $R(A, B) = (A \times B) \cup (B \times A) \subseteq X^2$. Then for any $(U_0, U_1) \in \mathcal{S}$, define

$$K(U_0, U_1) = (\overline{V_0} \times \overline{V_1}) \cup R(\overline{U_0}, C_0) \cup R(\overline{U_1}, C_1).$$

Note that $K(U_0, U_1) \in \mathcal{K}(X^2 \setminus \Delta)$ since $\overline{V_0} \cap \overline{V_1} = \emptyset$ and $\overline{U_\ell} \cap C_\ell = \emptyset$ for $\ell = 0, 1$. Let $\mathcal{K} = \{K(U_0, U_1) : (U_0, U_1) \in \mathcal{S}\}$.

We will be done after proving the following two facts:

(A) $\mathcal{K}$ covers $X^2 \setminus \Delta$, and
(B) no infinite subset of $\mathcal{K}$ has an upper bound in $\mathcal{K}(X^2 \setminus \Delta)$.

Indeed, (B) shows that $\mathcal{K}$ must be countable since $\mathcal{K}(X^2 \setminus \Delta)$ has calibre (ω_1, ω). But then (A) shows that X has a G_δ diagonal, so Schneider's theorem implies X is metrizable.

Proof of (A). Take any (x_0, x_1) in $X^2 \setminus \Delta$. We can find open sets U_0, U_1 with disjoint closures and with $x_\ell \in U_\ell$ for $\ell = 0, 1$. If $(U_0, U_1) \in \mathcal{S}$, then $x_\ell \in U_\ell \subseteq V_\ell$ shows that $(x_0, x_1) \in \overline{V_0} \times \overline{V_1} \subseteq K(U_0, U_1)$. Otherwise, (U_0, U_1) is not in $\mathcal{S}$, so by the maximality of $\mathcal{S}$, we cannot add (U_0, U_1) to $\mathcal{S}$. Thus, there is a (U_0', U_1') in $\mathcal{S}$, with corresponding $V_\ell' = V_\ell(U_0', U_1')$ for $\ell = 0, 1$, that is comparable with (U_0, U_1). In particular, $U_\ell \subseteq V_\ell'$, and so $x_\ell \in V_\ell'$ for $\ell = 0, 1$, which shows that $(x_0, x_1) \in \overline{V_0'} \times \overline{V_1'} \subseteq K(U_0', U_1')$. Hence, (A) is proved.

Proof of (B). Suppose $\{(U_0^i, U_1^i) : i \in \mathbb{N}\}$ is an infinite subset of $\mathcal{S}$, and the compact sets $K_i = K(U_0^i, U_1^i)$ are all distinct. We have to show that the K_i's do not have a common upper bound in $\mathcal{K}(X^2 \setminus \Delta)$, i.e. that there is no open neighborhood of the diagonal disjoint from $\bigcup_i K_i$. So it suffices to show that there is a point (x_∞, x_∞) on the diagonal which is in the closure of $\bigcup_i K_i$.

For each $\ell = 0, 1$ and $i \in \mathbb{N}$, let $V_\ell^i = V_\ell(U_0^i, U_1^i)$ and $C_\ell^i = C_\ell(U_0^i, U_1^i)$. Since the elements of $\mathcal{S}$ are incomparable, for any $i < j$ there exists a point $x_{i,j} \in X$ witnessing one of the following four conditions:

$$\text{(i) } U_0^i \not\subseteq V_0^j, \quad \text{(ii) } U_1^i \not\subseteq V_1^j,$$
$$\text{(iii) } U_0^j \not\subseteq V_0^i, \quad \text{(iv) } U_1^j \not\subseteq V_1^i.$$

Applying Ramsey's theorem, we find an infinite subset M of $\mathbb{N}$ such that one of the four conditions is witnessed by all $x_{i,j}$ with $i, j \in M$ and $i < j$. Without loss of generality, we can then assume that $M = \mathbb{N}$, so there is *one* of the four conditions which is witnessed by *all* of the $x_{i,j}$. The following construction does not depend on which of the conditions is satisfied.

Recall that X is first countable, so if a point is in the closure of a set, it is the limit of a sequence on that set. Also X is compact, so every infinite subset has an accumulation point (and so a proper limit point).

Hence, we may inductively construct a descending sequence of infinite subsets $\mathbb{N} \supseteq S_1 \supseteq S_2 \supseteq \cdots$ such that $\{x_{i,j}\}_{j \in S_i}$ converges to some point $x_{i,\infty} \in X$ for each $i \in \mathbb{N}$.

For each $m \in \mathbb{N}$, fix a $j_m \in S_m$ such that $j_1 < j_2 < \cdots$. Let $J = \{j_m : m \in \mathbb{N}\}$. Then we can find an infinite subset $J' \subseteq J$ such that $\{x_{j,\infty}\}_{j \in J'}$ converges to some limit point x_∞.

Take any open neighborhood W of x_∞. Pick an $i \in J'$ such that $x_{i,\infty} \in W$. Note that for each $m \geq i$, we have $j_m \in S_m \subseteq S_i$, and so $J' \cap S_i$ is infinite. As $\{x_{j,\infty}\}_{j \in J'}$ converges to $x_\infty \in W$ and $\{x_{i,j}\}_{j \in S_i}$ converges to $x_{i,\infty} \in W$, we can find a $j \in J' \cap S_i$ with $j > i$ such that $x_{j,\infty} \in W$ and $x_{i,j} \in W$. There must then be a $k \in S_j$ such that $k > j$ and $x_{j,k} \in W$.

So we have found $i < j < k$ such that $x_{i,j}$ and $x_{j,k}$ are both in W, and as noted above, one of the four conditions (i)–(iv) is witnessed by both $x_{i,j}$ and $x_{j,k}$. If it is (i) or (ii), then for some $\ell \in \{0, 1\}$, we have $x_{i,j} \in U_\ell^i \setminus V_\ell^j \subseteq C_\ell^j$ and $x_{j,k} \in U_\ell^j \setminus V_\ell^k \subseteq U_\ell^j$. If condition (iii) or (iv) is witnessed instead, then for some $\ell \in \{0, 1\}$, we have $x_{i,j} \in U_\ell^j \setminus V_\ell^i \subseteq U_\ell^j$ and $x_{j,k} \in U_\ell^k \setminus V_\ell^j \subseteq C_\ell^j$. In any case, $(x_{i,j}, x_{j,k}) \in R(\overline{U_\ell^j}, C_\ell^j) \subseteq K_j$. Therefore $(x_{i,j}, x_{j,k}) \in K_j \cap (W \times W)$.

Hence, every basic open neighborhood $W \times W$ of (x_∞, x_∞) meets some K_i, and (x_∞, x_∞) is in the closure of the union of all K_i, as required to complete the proof of (B). ∎

Let M be separable metrizable. By Lemma 1.5, $\mathcal{K}(M)$ has calibre (ω_1, ω), and hence if $\mathcal{K}(M) \geq_T \mathcal{K}(Z)$, then $\mathcal{K}(Z)$ has calibre (ω_1, ω). We deduce:

COROLLARY 2.2 (Cascales et al. [2, 4]). *Let X be a compact space. If $\mathcal{K}(M) \geq_T \mathcal{K}(X^2 \setminus \Delta)$, where M is separable metrizable, then X is metrizable.*

We give an application of Theorem 2.1 in the spirit in which Cascales and Orihuela originally proved their result [2]. Here $C_k(X)$ denotes $C(X)$ with the compact-open topology.

THEOREM 2.3. *If $\mathcal{K}(X)$ has calibre (ω_1, ω), then every compact subset of $C_k(X)$ is metrizable.*

Proof. For any space X, the family of all sets $B(\mathbf{0}, K, 1/n) = \{f \in C(X) : |f(x)| < 1/n \text{ for all } x \in K\}$, where $K \in \mathcal{K}(X)$ and $n \in \mathbb{N}$, is a local base at the constant zero function, $\mathbf{0}$. Note that $\mathcal{K}(X) \times \mathbb{N} \geq_T (\mathcal{T}_{\mathbf{0}}, \supseteq)$, where $\mathcal{T}_{\mathbf{0}} = \{T : \mathbf{0} \in T \text{ and } T \text{ is open in } C_k(X)\}$, under the natural map $(K, n) \mapsto B(\mathbf{0}, K, 1/n)$.

Suppose $\mathcal{K}(X)$ has calibre (ω_1, ω). Then from the previous paragraph, we see that so does $\mathcal{T}_{\mathbf{0}}$ under reverse inclusion. For any T in $\mathcal{T}_{\mathbf{0}}$, the set $U_T = \bigcup\{(T + f) \times (T + f) : f \in C(X)\}$ is an open neighborhood of the diagonal in $C_k(X)^2$. For a compact subset K of $C_k(X)$, the family $\{U_T \cap (K \times K) : T \in \mathcal{T}_{\mathbf{0}}\}$ is easily seen to be a base around the diagonal, Δ, of K^2. Hence, taking complements, we deduce that $\mathcal{K}(K^2 \setminus \Delta)$ has calibre (ω_1, ω), and so Theorem 2.1 shows that K is indeed metrizable. ∎

As a simple consequence, we observe that for any cardinal κ, every compact subset of $C_k(\kappa)$ is metrizable. Note that the previous results [2, 4] are restricted to cardinals with cofinality no more than the continuum. So we have a small example of the value of our calibre result over Tukey quotients from $\mathcal{K}(M)$, where M is separable metrizable.

2.2. Relative calibres. A space X is said to have a *small diagonal* if every uncountable S contained in $X^2 \setminus \Delta$ contains an uncountable subset S_1 whose closure misses the diagonal. If X is compact then $\overline{S_1}$ is a compact subset of $X^2 \setminus \Delta$. The next lemma is then immediate.

LEMMA 2.4. *Let X be compact. Then $X^2 \setminus \Delta$ has relative calibre ω_1 if and only if X has a small diagonal.*

It is known that, under various set-theoretic hypotheses including PFA [7], a compact space with a small diagonal is metrizable.

COROLLARY 2.5. *Consistently, if X is compact and $X^2 \setminus \Delta$ has relative calibre ω_1, then X is metrizable.*

We can now recover a result from [4], albeit with a stronger set-theoretic assumption, PFA rather than MA + $\neg$CH.

THEOREM 2.6. (PFA) *If X is compact and $\mathcal{K}(M) \geq_T X^2 \setminus \Delta$, where $\mathcal{K}(M)$ has calibre ω_1, then X is metrizable. In particular, this holds when M is the space of irrationals, $\mathbb{N}^{\mathbb{N}}$.*

This is because under PFA we have $\omega_1 < \mathfrak{b}$, so $\mathcal{K}(\mathbb{N}^{\mathbb{N}})$ has calibre ω_1 (see Lemma 1.6(ii)), and thus $X^2 \setminus \Delta$ has relative calibre ω_1. Note that in [10] it is shown that consistently (in particular, under PFA) there are separable metrizable spaces M that are *not* Polish but for which $\mathcal{K}(M)$ has calibre ω_1.

For completeness, note that if X is compact, M is separable metrizable and $\mathcal{K}(M) \geq_T X^2 \setminus \Delta$ then X is metrizable without any additional set-theoretic hypothesis *provided* X is countably tight. See [3, Theorem 2.9] for the case when $M = \mathbb{N}^{\mathbb{N}}$, and [4] for general separable metrizable M.

3. A counterexample for $X^2 \setminus \Delta$

THEOREM 3.1. *There is a first countable, compact space X that is not metrizable, but for which $X^2 \setminus \Delta$ has relative calibre (ω_1, ω) in $\mathcal{K}(X^2 \setminus \Delta)$.*

In Section 3.1, we will construct the above space X while assuming the existence of a topology τ on the closed unit interval $I = [0, 1]$ with certain desirable properties. We will then define such a τ in Section 3.2 by slightly modifying van Douwen's construction of the space Λ in [6].

3.1. Constructing the counterexample. If τ is any topology on the closed unit interval I, then define X_τ to be the space with underlying set $I \times \{0, 1\}$ and with the topology generated by basic open sets of the form $U \times \{1\}$, where $U \in \tau$, and $(V \times \{0, 1\}) \setminus (K \times \{1\})$, where V is open in the usual topology on I and K is a τ-compact subset of I. As an aside, notice that if τ is the discrete topology, then X_τ is the so-called Alexandrov duplicate of the unit interval.

It is straightforward to verify the following lemma.

LEMMA 3.2. *If τ is a first countable, locally countable, locally compact topology on I refining the usual topology, then X_τ is compact, first countable and not metrizable.*

For any set Z, a subset Y of $Z \times Z$ is called *small* if there is a countable subset $C \subseteq Z$ such that $Y \subseteq (C \times Z) \cup (Z \times C)$. In other words, Y is contained in the union of a countable family of 'horizontal' and 'vertical' lines. A subset Y is called *big* if it is not small.

In addition to the hypotheses of Lemma 3.2, τ will be made to also satisfy the following two properties, the second of which follows from the first (see Lemma 3.3):

(λ_1^2) if $F \subseteq I^2$ and $\overline{F}^{I\times I}$ is big, then $\overline{F}^{\tau\times\tau}$ is uncountable;

(λ_1) if $F \subseteq I$ and $\overline{F}^{I}$ is uncountable, then $\overline{F}^{\tau}$ is uncountable.

Lemma 3.3. *If τ satisfies (λ_1^2), then it also satisfies (λ_1).*

Proof. For any $F \subseteq I$, consider $F_\Delta = \{(x,x) : x \in F\}$. If $\overline{F}^I$ is uncountable, then $\overline{F_\Delta}^{I\times I}$ is an uncountable subset of Δ, and so $\overline{F_\Delta}^{I\times I}$ is big since any horizontal or vertical line meets Δ in at most one point. Then by (λ_1^2), $\overline{F_\Delta}^{\tau\times\tau}$ is uncountable. It follows that $\overline{F}^\tau$ is also uncountable. ■

Proof of Theorem 3.1. Assuming τ has already been constructed to satisfy (λ_1^2) and the hypotheses of Lemma 3.2, we are now ready to show that X_τ proves the theorem.

Let $X = X_\tau$. Then by Lemma 3.2, it remains to show $X^2 \setminus \Delta$ has relative calibre (ω_1, ω).

Let A denote the interval I with the new topology τ. Notice that $X^2 \setminus \Delta$ is the union of four subspaces which are naturally homeomorphic to $I^2 \setminus \Delta$, $I \times A$, $A \times I$ and $A^2 \setminus \Delta$. By Lemma 1.8, it is sufficient to show that these four spaces all have relative calibre (ω_1, ω).

In fact, we will be done if we can prove both A and A^2 have relative calibre (ω_1, ω). Indeed, I has every relative calibre, by compactness, so if A has relative calibre (ω_1, ω), then $I \times A$ and $A \times I$ have relative calibre (ω_1, ω) by Lemma 1.7. Also, Δ is a G_δ subset of I^2, which is compact, so Lemma 1.9 implies that $I^2 \setminus \Delta$ has relative calibre (ω_1, ω). Similarly, as τ refines the usual topology on I, also Δ is a G_δ subset of A^2, so if we can show A^2 has relative calibre (ω_1, ω), then $A^2 \setminus \Delta$ will have relative calibre (ω_1, ω) as well.

We will first verify that A has relative calibre (ω_1, ω). Fix an uncountable subset S of A. As I is hereditarily separable, we can find a countable subset $C \subseteq S$ such that $S \subseteq \overline{C}^I$. Then $\overline{C}^I$ is uncountable, so (λ_1) implies that $\overline{C}^A$ is uncountable. Hence, we can find an infinite sequence $S_1 \subseteq C$ converging in A to a point $x \in \overline{C}^A \setminus C$. The A-closure of S_1 is therefore $S_1 \cup \{x\}$, which is compact.

Now we will check that A^2 has relative calibre (ω_1, ω). Suppose $S \subseteq A^2$ is uncountable. If S is small, then we may assume that, without loss of generality, S is contained in a horizontal or vertical line. Hence, we are done since this line is just a homeomorphic copy of A, which has relative calibre (ω_1, ω). So we will instead assume that S is big. Choose a countable subset $C \subseteq S$ such that $S \subseteq \overline{C}^{I\times I}$. Then $\overline{C}^{I\times I}$ is also big, so (λ_1^2) implies that $\overline{C}^{A\times A}$ is uncountable. Hence, as in the proof for A, we can then find an infinite sequence S_1 in C converging in A^2 to a point outside of C, so that S_1 has compact A^2-closure. ■

3.2. Construction of τ. Now we will construct the topology τ on I which was used in the previous section. It will be first countable, locally countable, locally compact, will refine the usual topology on I, and satisfy (λ_1^2). We point out that (λ_1^2) is the natural dimension 2 'upgrade' of (λ_1). Our construction is similar to that of the space $\varLambda$ by van Douwen [6]. Indeed, van Douwen introduces the condition (λ_1) and a clearly stronger condition (λ_ω). Further, he proves that his space $\varLambda$ satisfies a condition (λ_ω^2), which is an 'upgrade' of (λ_ω) to dimension 2. Unfortunately, (λ_ω^2) is a weak upgrade and does not naturally imply (λ_1^2). So we cannot simply adopt van Douwen's $\varLambda$, but instead opt to prove directly the existence of τ satisfying (λ_1^2). In doing so we note that there is a gap in van Douwen's construction. We explain below how the gap arises, why it cannot be bridged, and detail how to detour around the gap.

Consider the family $\mathcal{C}$ of all countable subsets $C \subset I^2$ such that $\overline{C}^{I\times I}$ is big. Note that $|\mathcal{C}| \le \mathfrak{c}$, so we can enumerate $\mathcal{C} = \{C_\gamma : \gamma < \mathfrak{c}\}$ in such a way that each member of $\mathcal{C}$ is listed $\mathfrak{c}$ times. Also, we may enumerate $I = \{x_\alpha : \alpha < \mathfrak{c}\}$ with $x_\alpha \ne x_\beta$ when $\alpha \ne \beta$. Then define $X_\alpha = \{x_\beta : \beta < \alpha\}$ for each $\alpha < \mathfrak{c}$.

Let $\pi_1, \pi_2 : I^2 \to I$ be the natural projections. We will next construct injections $\psi_1, \psi_2 : \mathfrak{c} \to \mathfrak{c} \setminus \omega$ satisfying the following conditions for each $\gamma \in \mathfrak{c}$:

$$\pi_1[C_\gamma] \cup \pi_2[C_\gamma] \subseteq X_{\psi_1(\gamma)} \cap X_{\psi_2(\gamma)}, \tag{1}$$

$$(x_{\psi_1(\gamma)}, x_{\psi_2(\gamma)}) \in \overline{C_\gamma}^{I\times I}, \tag{2}$$

$$\psi_i(\gamma) \ne \psi_j(\delta) \quad \text{for any } i, j \text{ and } \delta \ne \gamma. \tag{3}$$

Fix $\gamma \in \mathfrak{c}$ and suppose we have already defined $\psi_1(\delta)$ and $\psi_2(\delta)$ satisfying the above conditions for every $\delta < \gamma$. Since $D_\gamma = \pi_1[C_\gamma] \cup \pi_2[C_\gamma]$ is countable, we may write $D_\gamma = \{x_{\beta_n} : n \in \mathbb{N}\}$. Since $\mathfrak{c}$ has uncountable cofinality, there is an $\alpha < \mathfrak{c}$ such that $\beta_n < \alpha$ for all $n \in \mathbb{N}$, i.e. $D_\gamma \subseteq X_\alpha$. Let $\alpha_\gamma = \min\{\alpha < \mathfrak{c} : D_\gamma \subseteq X_\alpha\}$. So $D_\gamma \subseteq X_\alpha$ for all $\alpha \ge \alpha_\gamma$. Note that $\alpha_\gamma \ge \omega$ since D_γ is infinite.

Consider $\varPsi_\gamma = \{\psi_i(\delta) : \delta < \gamma, i = 1, 2\}$ and $S_\gamma = \{x_\beta : \beta \in \alpha_\gamma \cup \varPsi_\gamma\}$. Then $|S_\gamma| < \mathfrak{c}$. The following lemma, which is proved in [6], shows that there is a point (y_1, y_2) in $\overline{C_\gamma}^{I\times I}$ such that $y_1, y_2 \notin S_\gamma$.

LEMMA 3.4. *For any big closed $Y \subseteq \mathbb{R}^2$ and for any $S \subset \mathbb{R}$ with $|S| < \mathfrak{c}$, there is a $(y_1, y_2) \in Y$ such that $\{y_1, y_2\} \cap S = \emptyset$.*

Now for $i = 1, 2$, define $\psi_i(\gamma)$ to be the unique element of $\mathfrak{c}$ such that $y_i = x_{\psi_i(\gamma)}$. Then (2) is satisfied immediately. Also, since $y_i \notin S_\gamma$, we have $\psi_i(\gamma) \notin \alpha_\gamma \cup \varPsi_\gamma$. Hence, $\psi_i(\gamma) \ge \alpha_\gamma$, which implies that $D_\gamma \subseteq X_{\psi_i(\gamma)}$ for $i = 1, 2$. So (1) is satisfied. And of course, $\psi_i(\gamma) \ge \alpha_\gamma \ge \omega$ shows that each $\psi_i(\gamma)$ is really in $\mathfrak{c} \setminus \omega$, as desired.

So by transfinite induction, we have constructed $\psi_1, \psi_2 : \mathfrak{c} \to \mathfrak{c} \setminus \omega$ satisfying (1) and (2). Also, (3) is satisfied—and in particular, the ψ_i are injections—since $\psi_i(\gamma) \notin \varPsi_\gamma$ for each i.

Choose a countable base $\{B_i : i \in \mathbb{N}\}$ for I with $B_1 = I$, and then define $E_j(x) = \bigcap\{B_i : i \le j \text{ and } x \in B_i\}$ for each $x \in I$ and $j \in \mathbb{N}$. Then $\{E_j(x) : j \in \mathbb{N}\}$ is a neighborhood base for x in I such that

(4) if $y \in E_j(x)$ and $i \ge j$, then $E_i(y) \subseteq E_j(x)$.

Let $\Psi \subseteq \mathfrak{c} \setminus \omega$ be the union of the images of ψ_1 and ψ_2. By (2) and (3), there are well-defined sequences $s_\alpha = (s^i_\alpha)_{i\in\mathbb{N}}$ for each $\alpha \in \Psi$ satisfying:

(5) $s^i_\alpha \in E_i(x_\alpha)$ for each $\alpha \in \Psi$ and $i \in \mathbb{N}$,

(6) if $\psi_1(\gamma) \ne \psi_2(\gamma)$, then $(s^i_{\psi_1(\gamma)}, s^i_{\psi_2(\gamma)}) \in C_\gamma$ for each $i \in \mathbb{N}$,

(6′) if $\alpha = \psi_1(\gamma) = \psi_2(\gamma)$ [1], then $(s^{2i-1}_\alpha, s^{2i}_\alpha) \in C_\gamma$ for each $i \in \mathbb{N}$.

Conditions (1), (6) and (6′) imply that the sequence s_α lies entirely in X_α for each $\alpha \in \Psi$, so the following construction by transfinite recursion makes sense. For each $\alpha \in \mathfrak{c} \setminus \Psi$, define $L_j(x_\alpha) = \{x_\alpha\}$ for all $j \in \mathbb{N}$. Now for any $\alpha \in \Psi$ (so $\alpha \ge \omega$), we may assume $L_i(x)$ has been defined for all $x \in X_\alpha$ and $i \in \mathbb{N}$, and so we define $L_j(x_\alpha) = \{x_\alpha\} \cup \bigcup_{i\ge j} L_i(s^i_\alpha)$ for each $j \in \mathbb{N}$. The next facts easily follow by transfinite induction on α and by (4):

(7) each $L_j(x)$ is countable,

(8) $L_j(x) \subseteq E_j(x)$,

(9) if $y \in L_j(x)$, then $L_i(y) \subseteq L_j(x)$ for some $i \in \mathbb{N}$.

Since $L_{j+1}(x) \subseteq L_j(x)$ for all $x \in I$ and $j \in \mathbb{N}$, we deduce from (9) that $\{L_j(x) : x \in I, j \in \mathbb{N}\}$ is a base generating a new topology on I. This new topology τ refines the usual topology on I because of (8), and since $\{L_j(x) : j \in \mathbb{N}\}$ is a neighborhood base at x, we see that τ is first countable. Also, τ is locally countable by (7). Additionally, it is easy to check that each $L_j(x_\alpha)$ is compact by transfinite induction, so τ is locally compact.

Note that the sequence s_α converges (with respect to τ) to x_α for each $\alpha \in \Psi$, and so by (6) and (6′), we have $(x_{\psi_1(\gamma)}, x_{\psi_2(\gamma)}) \in \overline{C_\gamma}^{\tau\times\tau}$ for all $\gamma \in \mathfrak{c}$. Since each member of $\mathcal{C}$ appears $\mathfrak{c}$ times in the enumeration $\{C_\gamma : \gamma < \mathfrak{c}\}$, and since $(x_{\psi_1(\gamma)}, x_{\psi_2(\gamma)}) \ne (x_{\psi_1(\delta)}, x_{\psi_2(\delta)})$ for $\gamma \ne \delta$ (as the ψ_i are injections), it follows that $\overline{C}^{\tau\times\tau}$ has cardinality $\mathfrak{c}$ for each $C \in \mathcal{C}$. This implies (λ^2_1), so τ has all the desired properties.

---

[1] In [6], it was asserted that we could always make $\psi_1(\gamma) < \psi_2(\gamma)$, in which case (6′) would be unnecessary. However, there is a γ such that $C_\gamma \subseteq \mathbb{Q}^2 \setminus \Delta$ and $\overline{C_\gamma}^{I\times I} = C_\gamma \cup \Delta$. The rationals in [6] are each of the form x_α for some $\alpha \in \omega$, and since $\psi_1, \psi_2 : \mathfrak{c} \to \mathfrak{c} \setminus \omega$, it is seen $x_{\psi_i(\gamma)} \notin \mathbb{Q}$ so $(x_{\psi_1(\gamma)}, x_{\psi_2(\gamma)}) \notin C_\gamma$. Hence, (2) implies $(x_{\psi_1(\gamma)}, x_{\psi_2(\gamma)}) \in \Delta$, so we are forced to have $\psi_1(\gamma) = \psi_2(\gamma)$. As $C_\gamma \cap \Delta = \emptyset$, we cannot have $(s^i_{\psi_1(\gamma)}, s^i_{\psi_2(\gamma)}) \in C_\gamma$, which shows why a modified condition like (6′) is necessary.

4. All subspaces of X and X^2

4.1. Conditions on all subspaces of a space. Proposition 2.6 from [4] states:

PROPOSITION 4.1. *If $\mathcal{K}(M) \geq_T X$, where M is a separable metric space, then there is a cover $\mathcal{C}$ of X and a countable collection $\mathcal{N}$ of subsets of X such that:*

(i) *every element of $\mathcal{C}$ is ω-bounded (every countable subset has compact closure),*
(ii) *$\mathcal{N}$ is a network for X modulo $\mathcal{C}$ (if an open set U contains some $C \in \mathcal{C}$, then there is an $N \in \mathcal{N}$ such that $C \subseteq N \subseteq U$).*

The next result answers Problems 4.14–4.16 from [4].

THEOREM 4.2. *Let X be a space.*

(i) *If for every subspace A of X, there is some separable metric space M_A such that $\mathcal{K}(M_A) \geq_T \mathcal{K}(A)$, then X is an $\aleph_0$-space.*
(ii) *If for every subspace A of X, there is some separable metric space M_A such that $\mathcal{K}(M_A) \geq_T A$, then X is a cosmic space.*

Proof. We can deduce (i) from (ii) as follows. Let X be as in (i). From (ii), we certainly know X is cosmic. So X has a coarser second countable topology, and hence $\mathcal{K}(X)$ has a coarser second countable topology. From the proposition above, $\mathcal{K}(X)$ has a cover $\mathcal{C}$ of ω-bounded sets and a countable network $\mathcal{N}$ modulo $\mathcal{C}$. As $\mathcal{K}(X)$ has a coarser second countable topology, all the elements of $\mathcal{C}$ must be compact. Thus $\mathcal{K}(X)$ is Lindelöf Σ and has a coarser second countable topology, so it is cosmic. However, $\mathcal{K}(X)$ is cosmic if and only if X is an $\aleph_0$-space.

We now prove (ii). Let X be as in its statement. Every subspace of X has countable extent. Hence X is hereditarily ccc. If X is hereditarily Lindelöf, then from the proposition above, it is hereditarily Lindelöf Σ. It follows that X is cosmic [12]. Otherwise X contains a right-separated subspace A. Since A is hereditarily ccc, it must also be hereditarily separable. We now work inside A. By the proposition above, there is a collection $\mathcal{C}$ of ω-bounded subsets of A and a countable network $\mathcal{N}$ for A modulo $\mathcal{C}$. Take any C from $\mathcal{C}$ and pick a countable dense subset D of C. Then $C = \overline{D}$ is compact. Hence A is Lindelöf Σ. In particular, it is Lindelöf, which contradicts A being right-separated. ∎

Instead of a general separable metrizable space controlling the compact subsets of a subspace of X, as in the preceding theorem, we can restrict the M_A to be the irrationals, $\mathbb{N}^{\mathbb{N}}$. For metrizable X, we have a complete characterization of such spaces.

Proposition 4.3. *Let X be a metrizable space. Then the following are equivalent:*

(i) *for every subspace A of X, we have $\mathcal{K}(\mathbb{N}^{\mathbb{N}}) \geq_T \mathcal{K}(A)$,*
(ii) *X is countable and Polish,*
(iii) *X is countable and scattered.*

The proof of Proposition 4.3 is deferred until after Theorem 4.8.

Proposition 4.4. *Let X be a metrizable space. Then X is countable if and only if $\mathcal{K}(\mathbb{N}^{\mathbb{N}}) \geq_T A$ for each subspace A of X.*

This result is part of Theorem 4.8 below.

Call a space X *hereditarily relative calibre* (κ, λ, μ) if each subspace of X is relative calibre (κ, λ, μ). Note that X is hereditarily relative calibre (ω_1, ω) (respectively, ω_1) if and only if for each uncountable $S \subseteq X$, there is an infinite (respectively, uncountable) $S_1 \subseteq S$ with $\overline{S_1}^S = \overline{S_1} \cap S$ compact.

Observe that if X is a space such that 'for every subspace A of X, there is some separable metric space M_A such that $\mathcal{K}(M_A) \geq_T A$', then it is also true that 'X is hereditarily relative calibre (ω_1, ω)'. Similarly, observe that consistently (precisely when $\omega_1 < \mathfrak{b}$), if X is a space such that 'for every subspace A of X, we have $\mathcal{K}(\mathbb{N}^{\mathbb{N}}) \geq_T A$', then it is also true that 'X is hereditarily relative calibre ω_1'.

Call a space X *hereditarily calibre* (κ, λ, μ) if, for each subspace A of X, the partial order $\mathcal{K}(A)$ has calibre (κ, λ, μ). Observe that if X is a space such that 'for every subspace A of X, there is some separable metric space M_A such that $\mathcal{K}(M_A) \geq_T \mathcal{K}(A)$', then it is also true that 'X is hereditarily calibre (ω_1, ω)'. Similarly, observe that consistently (precisely when $\omega_1 < \mathfrak{b}$), if X is a space such that 'for every subspace A of X, we have $\mathcal{K}(\mathbb{N}^{\mathbb{N}}) \geq_T \mathcal{K}(A)$', then it is also true that 'X is hereditarily calibre ω_1'.

Note also that 'hereditarily calibre (κ, λ, μ)' implies 'hereditarily relative calibre (κ, λ, μ)'. A further, and stronger, condition is that $\mathcal{K}(X)$ is hereditarily relative calibre (κ, λ, μ).

Lemma 4.5. *Let X be a space. If $\mathcal{K}(X)$ is hereditarily relative calibre (κ, λ, μ), then X is hereditarily calibre (κ, λ, μ).*

Proof. Take any subspace A of X. Since $\mathcal{K}(A)$ is a subspace of $\mathcal{K}(X)$, it follows that $\mathcal{K}(A)$ has relative calibre (κ, λ, μ). So by Lemma 1.2, $\mathcal{K}(A)$ has calibre (κ, λ, μ). ∎

We now compare and contrast the situation when for every subspace A of a space X, there is a separable metrizable M_A such that $\mathcal{K}(M_A) \geq_T A$, versus all subspaces having a (relative) calibre.

In the weakest case, there is a clear difference between the two scenarios. The second part of Theorem 4.2 says that if each subspace A of X has a separable metric space M_A such that $\mathcal{K}(M_A) \geq_T A$, then X is cosmic. Weakening the hypothesis on X to being hereditarily relative calibre (ω_1, ω) does not suffice to deduce cosmicity of X. For example, the Sorgenfrey line is not cosmic but is hereditarily ccc and first countable, and so it is hereditarily relative calibre (ω_1, ω) (since each subspace has countable extent and is first countable). Nor, consistently at least, is it sufficient to strengthen 'X hereditarily relative calibre (ω_1, ω)' to '$\mathcal{K}(X)$ hereditarily relative calibre (ω_1, ω)'.

Example 4.6. ($\mathfrak{b} = \omega_1$) There is an uncountable subspace X of the Sorgenfrey line such that $\mathcal{K}(X)$ is first countable and hereditarily ccc (see [8]). Hence $\mathcal{K}(X)$ is hereditarily relative calibre (ω_1, ω) (and X is hereditarily calibre (ω_1, ω)), but X is not cosmic.

In [8] it is shown that under the Open Coloring Axiom (OCA), if $\mathcal{K}(X)$ is first countable and hereditarily ccc, then X is cosmic. However, the argument given in that paper does not obviously show that OCA implies that 'if $\mathcal{K}(X)$ is hereditarily relative calibre (ω_1, ω), then X is cosmic'.

Moving from the relative calibre (ω_1, ω) case to relative calibre ω_1, however, we get equivalence between calibres and Tukey quotient, and equivalence to X being countable. We will use the following result:

Theorem 4.7 (Christensen [5]). *If M is a separable metrizable space, then $\mathcal{K}(\mathbb{N}^\mathbb{N}) \geq_T \mathcal{K}(M)$ if and only if M is Polish, and $\mathcal{K}(\mathbb{N}^\mathbb{N}) \geq_T M$ if and only if M is analytic.*

Theorem 4.8. *Let X be a space. Then the following are equivalent:*

(i) *X is hereditarily relative calibre ω_1,*
(ii) *for every subspace A of X, we have $\mathcal{K}(\mathbb{N}^\mathbb{N}) \geq_T A$,*
(iii) *X is countable.*

Proof. It is vacuously true that (iii) implies (i). Condition (iii) also implies (ii). Suppose X is countable and A is a subspace of X. Enumerate $A = \{a_n : n \in \mathbb{N}\}$. Define $\phi : \mathcal{K}(\mathbb{N}^\mathbb{N}) \to \mathcal{K}(A)$ by $\phi(K) = \{a_1, \ldots, a_{n(K)}\}$ where $n(K) = \max\{f(1) : f \in K\}$. Then ϕ is order-preserving and its image is a compact cover of A.

Next, we will prove that (ii) implies (iii). So assume, for a contradiction, that X satisfies (ii) but is uncountable. Then by Theorem 4.2, we know X is cosmic. Hence it has a coarser separable metrizable topology τ. Since any subset of X which is compact in the original topology is also compact in τ, we see that (X, τ) also satisfies (ii). In particular, $\mathcal{K}(\mathbb{N}^\mathbb{N}) \geq_T (X, \tau)$, so (X,τ) is analytic by Christensen's theorem quoted above. Hence (X,τ) contains a non-analytic subspace A (because an uncountable analytic space must contain a Cantor set, which contains non-analytic subspaces). But then we cannot have $\mathcal{K}(\mathbb{N}^{\mathbb{N}}) \geq_T A$.

We complete the proof by showing that the negation of (iii) implies the negation of (i). Suppose X is an uncountable space. We have to show it contains a subspace A which is not relative calibre ω_1. Note that it suffices to find a subspace A of X such that

($*$) A is uncountable, and all compact subsets of A are countable,

because then no uncountable subset of A can have compact closure in A.

If X itself satisfies ($*$), then we are, of course, done. If not, then X contains an uncountable compact subspace, and so, without loss of generality, we can assume X is compact.

If X contains a right-separated subspace A of size ω_1, then every compact subset of A is contained in an initial (countable) interval, so A satisfies ($*$), and so we are done. If not then X is hereditarily Lindelöf. Since X is also compact, we see that X is first countable. Hence, X has size the continuum, $\mathfrak{c}$, and weight no more than $\mathfrak{c}$. Applying the fact that X is hereditarily Lindelöf again, we see that X contains no more than $\mathfrak{c}$ open subsets. So the collection $\mathcal{K}$ of all uncountable compact subsets of X has $|\mathcal{K}| \leq \mathfrak{c}$. Observe that each member of $\mathcal{K}$ has cardinality exactly $\mathfrak{c}$.

Next, we will follow the construction of Bernstein's set to form an uncountable subspace $A \subseteq X$ that does not contain any element of $\mathcal{K}$. Enumerate $\mathcal{K} = \{K_\alpha : \alpha < \mathfrak{c}\}$, possibly with repetitions. Using transfinite induction, we will construct uncountable sequences $\{x_\alpha : \alpha < \mathfrak{c}\}$ and $\{y_\alpha : \alpha < \mathfrak{c}\}$ such that each $x_\alpha, y_\alpha \in K_\alpha$. We will also ensure that $x_\alpha \neq x_\beta$ and $y_\alpha \neq y_\beta$ whenever $\alpha \neq \beta$, and that $x_\alpha \neq y_\beta$ for any $\alpha, \beta < \mathfrak{c}$. Indeed, if $\beta < \mathfrak{c}$ and if we have already constructed x_α and y_α for each $\alpha < \beta$, then we can find distinct points $x_\beta, y_\beta \in K_\beta \setminus (\{x_\alpha : \alpha < \beta\} \cup \{y_\alpha : \alpha < \beta\})$ since $|K_\beta| = \mathfrak{c}$ and $\beta < \mathfrak{c}$. Now let $A = \{x_\alpha : \alpha < \mathfrak{c}\}$.

Then A is uncountable and does not contain any K_α since $y_\alpha \notin A$. Thus A satisfies ($*$), and the proof is complete. ∎

Proof of Proposition 4.3. That condition (ii) follows from (i) is immediate from Theorem 4.8 and Christensen's Theorem 4.7, while (iii) is a consequence of (ii) by a Baire category argument. If X is countable, metrizable and scattered, then a straightforward argument by induction on the scattered height shows that X is Polish, so (iii) implies (ii). And if X is countable and Polish, then every subspace A of X is a G_δ subspace, hence also Polish, so by Christensen's theorem $\mathcal{K}(\mathbb{N}^{\mathbb{N}}) \geq_T \mathcal{K}(A)$. ∎

THEOREM 4.9. *Let X be a metrizable space. Then the following are equivalent:*

(i) *X is hereditarily calibre ω_1,*
(ii) *either X is homeomorphic to the disjoint sum of a countable (possibly empty) disjoint sum of convergent sequences and a countable (possibly empty) discrete space, or $\omega_1 < \mathfrak{b}$ and X is countable and scattered.*

Proof. If X is homeomorphic to the disjoint sum of a countable (possibly empty) disjoint sum of convergent sequences and a countable (possibly empty) discrete space, then every subspace A of X is locally compact and countable, and so it is easily seen that $\mathcal{K}(A)$ has calibre ω_1.

Now suppose $\omega_1 < \mathfrak{b}$ and X is countable and scattered. Take any subspace A of X. By the preceding theorem, $\mathcal{K}(\mathbb{N}^{\mathbb{N}}) \geq_T \mathcal{K}(A)$. It follows from $\omega_1 < \mathfrak{b}$ that $\mathcal{K}(\mathbb{N}^{\mathbb{N}})$ has calibre ω_1. Hence, $\mathcal{K}(A)$ also has calibre ω_1, and so (ii) implies (i).

For the converse, suppose that for every subspace A of X, the partial order $\mathcal{K}(A)$ has calibre ω_1. By Theorem 4.8, X is countable. Since $\mathcal{K}(\mathbb{Q})$ does not have calibre ω_1, the rationals $\mathbb{Q}$ do not embed in X. It follows that X is scattered. If X has scattered height 0, then it is discrete. If X has scattered height 1, then it is homeomorphic to the disjoint sum of a countable (non-empty) disjoint sum of convergent sequences and a countable (possibly empty) discrete space.

The remaining case is when X has scattered height at least 2. We have to show that $\omega_1 < \mathfrak{b}$. From the scattered height restriction on X, it follows that X contains a subspace A' which is homeomorphic to a convergent sequence of convergent sequences. Removing the limit points of the convergent sequences, but not the point that the sequence of convergent sequences conveges to, we obtain a subspace A of X which is homeomorphic to the metric fan, F. By hypothesis, $\mathcal{K}(F)$ has calibre ω_1. As shown in [10], $\mathcal{K}(F)$ and $\mathcal{K}(\mathbb{N}^{\mathbb{N}})$ are Tukey equivalent, and so share the same calibres. Specifically, $\mathcal{K}(F)$ has calibre ω_1 if and only if $\mathcal{K}(\mathbb{N}^{\mathbb{N}})$ has calibre ω_1, and this in turn holds if and only if $\omega_1 < \mathfrak{b}$. ∎

Define $\mathcal{K}^{(1)}(X) = \mathcal{K}(X)$, and inductively, $\mathcal{K}^{(n+1)}(X) = \mathcal{K}(\mathcal{K}^{(n)}(X))$.

THEOREM 4.10. *For any space X, the following are equivalent:*

(i) *$\mathcal{K}^{(n)}(X)$ is hereditarily calibre ω_1 for every natural number n,*
(ii) *$\mathcal{K}(X)$ is hereditarily relative calibre ω_1,*
(iii) *$\mathcal{K}(X)$ is countable,*
(iv) *$\mathcal{K}(X)$ is countable and all compact subspaces of $\mathcal{K}(X)$ are finite,*
(v) *X is countable and all compact subspaces of X are finite.*

Proof. As noted previously, 'hereditarily calibre ω_1' implies 'hereditarily relative calibre ω_1', so (ii) follows from (i). And by Theorem 4.8, (ii) is equivalent to (iii).

If (iii) holds, then X is also countable, and we will show that all compact subspaces of X must be finite as follows. Suppose X has an infinite compact subspace K. Since K is countably infinite and compact, it contains an infinite convergent sequence S. But then S has uncountably many compact subspaces, which contradicts $\mathcal{K}(X)$ being countable. Hence, (iii) implies (v).

If X is countable, then it has only countably many finite subsets, so (v) implies (iii). In fact, (v) implies (iv). To see this, assume (v) holds and K is a compact subset of $\mathcal{K}(X)$. Then $\hat{K} = \bigcup K$ is a compact subset of X, and so is finite. Hence, K is also finite since it is a subset of the power set of $\hat{K}$.

It remains to show that (iv) implies (i). Assume then that $\mathcal{K}(X)$ is countable and all its compact subspaces are finite. First note that (iv) trivially implies (iii), and hence (iv) and (v) are equivalent. Thus for every natural number n we see that $\mathcal{K}^{(n)}(X)$ is countable and all its compact subspaces are finite. So for a fixed n, every subspace of $\mathcal{K}^{(n)}(X)$ is hemicompact, and thus has calibre ω_1. ∎

4.2. Conditions on all subspaces of a compact square. We now apply our results above to the case when an order condition is imposed on the compact subsets of each subspace of the square of a compact space. Claim (4) below answers Problems 4.17–4.20 from [4].

Theorem 4.11. *Let X be compact.*

1. *If X^2 is hereditarily calibre (ω_1, ω), then X is metrizable.*
2. *It is consistent and independent that 'X^2 being hereditarily relative calibre (ω_1, ω) implies X is metrizable'.*
3. *If X (and a fortiori X^2) is hereditarily relative calibre ω_1, then X is countable and metrizable.*
4. *If for every subspace A of X (and a fortiori X^2) there is a separable metrizable space M_A such that $\mathcal{K}(M_A) \geq_T A$, then X is metrizable.*

Proof. If X^2 is hereditarily calibre (ω_1, ω), then in particular $\mathcal{K}(A)$ has calibre (ω_1, ω) when $A = X^2 \setminus \Delta$, so claim (1) follows from Theorem 2.1. Claim (3) follows immediately from Theorem 4.8, while Theorem 4.2(ii) gives claim (4).

Now suppose X^2 is hereditarily relative calibre (ω_1, ω). Then every subspace of X^2 has countable extent (Lemma 1.3), so X^2 is hereditarily ccc. Under PFA [16] it follows that X^2 is hereditarily Lindelöf, and X is metrizable. However, under the continuum hypothesis, Gruenhage [11] has constructed a compact, first countable, non-metrizable space X whose square is hereditarily ccc (hence hereditarily has countable extent). Combining the first countability and hereditary countable extent of X^2 with Lemma 1.4, we deduce that X^2 is hereditarily relative calibre (ω_1, ω). ■

References

[1] B. Cascales, M. Muñoz and J. Orihuela, *The number of K-determination of topological spaces*, Rev. Real Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. 106 (2012), 341–357.

[2] B. Cascales and J. Orihuela, *On compactness in locally convex spaces*, Math. Z. 195 (1987), 365–381.

[3] B. Cascales and J. Orihuela, *A biased view of topology as a tool in functional analysis*, in: K. P. Hart et al. (eds.), Recent Progress in General Topology III, Atlantis Press, Paris, 2014, 93–164.

[4] B. Cascales, J. Orihuela and V. V. Tkachuk, *Domination by second countable spaces and Lindelöf* Σ*-property*, Topology Appl. 158 (2011), 204–214.

[5] J. P. R. Christensen, *Topology and Borel Structure*, North-Holland, Amsterdam, 1974.

[6] E. K. van Douwen, *A technique for constructing honest locally compact submetrizable examples*, Topology Appl. 47 (1992), 179–201.

[7] A. Dow and O. Pavlov, *More about spaces with a small diagonal*, Fund. Math. 191 (2006), 67–80.

[8] S. Fisher, P. Gartside, T. Mizokami and N. Shimane, *Near metric properties of hyperspaces*, Topology Proc. 22 (1997), 197–211.

[9] D. H. Fremlin, *Measure Theory, Vol. 5*, Torres Fremlin, Colchester, 2008.

[10] P. Gartside and A. Mamatelashvili, *Notes on Tukey order*, preprint.

[11] G. Gruenhage and P. J. Nyikos, *Normality in* X^2 *for compact* X, Trans. Amer. Math. Soc. 340 (1993), 563–586.

[12] R. E. Hodel, *On a theorem of Arhangel'skiĭ concerning Lindelöf p-spaces*, Canad. J. Math. 27 (1975), 459–468.

[13] I. Juhász and Z. Szentmiklóssy, *Convergent free sequences in compact spaces*, Proc. Amer. Math. Soc. 116 (1992), 1153–1160.

[14] J. Kąkol, W. Kubiś and M. López-Pellicer, *Descriptive Topology in Selected Topics of Functional Analysis*, Developments Math. 24, Springer, New York, 2011.

[15] S. Todorčević, *Directed sets and cofinal types*, Trans. Amer. Math. Soc. 290 (1985), 711–723.

[16] S. Todorčević, *Partition Problems in Topology*, Contemp. Math. 84, Amer. Math. Soc., Providence, RI, 1989.

Paul Gartside, Jeremiah Morgan
Department of Mathematics
University of Pittsburgh
Pittsburgh, PA 15260, U.S.A.
E-mail: gartside@math.pitt.edu

Received 13 October 2013;
in revised form 11 April 2015