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On the first homology of Peano continua

by

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Abstract. We show that the first homology group of a locally connected compact metric space is either uncountable or finitely generated. This is related to Shelah's well-known result (1988) which shows that the fundamental group of such a space satisfies a similar condition. We give an example of such a space whose fundamental group is uncountable but whose first homology is trivial, showing that our result does not follow from Shelah's. We clarify a claim made by Pawlikowski (1998) and offer a proof of the clarification.

1. Introduction. The classical Hahn–Mazurkiewicz Theorem states that a connected, locally connected compact metric space is the continuous image of an arc—a *Peano continuum*. Shelah showed [S] that if the first homotopy group, the *fundamental group*, of a Peano continuum is countable then it is finitely generated. This can be compared to our result:

Theorem 1.1. *The first homology group of a compact locally connected metric space is either uncountable or isomorphic to a direct sum of finitely many cyclic groups.*

If X is a locally connected compact metric space then X has only finitely many connected components, each of which is a Peano continuum and therefore path connected. Then $H_1(X)$ is the direct sum of the first homology of each of the (path) components of X. Thus it suffices to prove Theorem 1.1 for a Peano continuum. For the sake of generality, we note that *is countable* may be replaced by *has cardinality less than the continuum* in both the result of Shelah and our result.

Example 1.2. We construct a space whose existence testifies that our result cannot follow immediately from Shelah's. Let X be a simplicial complex whose fundamental group is the alternating group A_5. Recall that each

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element of A_5 is a commutator and thus $X^{\mathbb{N}}$, endowed with the product topology, is a Peano continuum whose fundamental group, $A_5^{\mathbb{N}}$, has the property that each of its elements is a commutator. Hence $H_1(X^{\mathbb{N}})$, which is the abelianization of $A_5^{\mathbb{N}}$ by the classical Hurewicz Theorem, is trivial.

2. A construction.

We begin with the following lemma.

LEMMA 2.1. *Let X and Y be Peano continua and $f : X \to Y$ be a mapping. If there exists $\epsilon > 0$ such that each loop of diameter less than ϵ is mapped under f to a nulhomogolous loop, then $f_*(H_1(X))$ is a finite sum of cyclic groups.*

Proof. Consider the composition of the Hurewicz map (which is surjective by the Hurewicz Theorem) and the map f_*, and apply Lemma 7.6 of [CC] which states that:

If X is a connected, locally path connected separable metric space which is locally trivial with respect to $g : \pi_1(X) \to K$ then $g(\pi_1(X))$ is countable, and furthermore is finitely generated if X is compact ([1]).

To finish, we need only remark that a finitely generated Abelian group is a finite sum of cyclic groups. ∎

Consider a Peano continuum X with metric d such that $H_1(X)$ is not finitely generated. By Lemma 2.1, letting $f : X \to X$ be identity, there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of continuous mappings of S^1 to X, with $\mathrm{diam}(f_n) \leq 1/2^n$, none of which is nulhomologous. By compactness of X, there exists a point $x \in X$ and a subsequence (without loss of generality, let the subsequence be the original sequence) such that the sequence of images $f_n(S^1)$ converges to x in the Hausdorff metric. By local path connectedness, we may assume that $x \in f_n(S^1)$ for all $n \in \mathbb{N}$, and by passing to a subsequence, we may assume again that $\mathrm{diam}(f_n) \leq 1/2^n$ for all $n \in \mathbb{N}$. Thus, each f_k can be thought of as a loop $f_k : [0, 1] \to X$ such that $f(0) = f(1) = x$ (this allows us to concatenate such mappings together).

Recall that the product space $\{0, 1\}^{\mathbb{N}}$ is the Cantor set, where each factor $\{0, 1\}$ is given the discrete topology.

Let f_k^1 be f_k, and f_k^0 be the constant loop at x. For $\alpha \in \{0, 1\}^{\mathbb{N}}$, let $f^\alpha = f_0^{\alpha(0)} * f_1^{\alpha(1)} * \cdots$. This can be thought of as the (pointwise) limit of a Cauchy sequence in the complete metric space $\mathcal{C}([0, 1], X)$, the metric being

([1]) The referee has pointed out that there was a minor flaw in the proof of Lemma 7.6, which we correct here. Select open covers $\{U_x\}$, C_1 and C_2 as in the original proof. Choose C to be an open refinement of C_2 which consists of open path connected sets. If X is compact we may take C to be finite, and if X is merely separable we may take C to be countable (since separable metric spaces are Lindelöf). Now apply Theorem 7.3(2) of [CC] to conclude that the image of g is finitely generated (resp. countable) in case X is compact (resp. separable).

the sup metric. Thus, each concatenation gives a continuous function from S^1 to X. We define an equivalence relation on the Cantor set $\{0,1\}^{\mathbb{N}}$ as follows: $\alpha \sim \beta$ iff f^α is homologous to f^β.

DEFINITION 2.2. We say a space Y is *Polish* if it is completely metrizable and separable (the Cantor set $\{0,1\}^{\mathbb{N}}$ is an example). If Y is a Polish space, we say that $A \subseteq Y$ is *analytic* if there exists a Polish space Z and a closed set $D \subseteq Y \times Z$ such that A is the projection of D in Y. Analytic spaces are preserved under continuous images and preimages, products, and under countable unions and intersections. Closed sets are analytic. If $\sim$ is an analytic subset of $X \times X$ we say that $\sim$ is an *analytic relation* on X. A subspace of a topological space is *perfect* if it is closed, non-empty and has no isolated points. A perfect subspace of a Polish space has the cardinality of the continuum.

LEMMA 2.3. *The space $\sim$ is an analytic equivalence relation on the Cantor set which has the property that if α and β differ at exactly one point then $\alpha \nsim \beta$.*

Proof. Let H be the space of all continuous maps of $[0,1]$ to X such that $\{0,1\} \mapsto x$, with the metric on H being the sup metric. For each $n \in \mathbb{N}$ let $C_n : H^{2n} \to H$ be the map defined by mapping $(l_1, l_2, \ldots, l_{2n})$ to

$$l_1 * l_2 * (l_1)^{-1} * (l_2)^{-1} * l_3 * l_4 * (l_3)^{-1} * (l_4)^{-1} * \cdots * l_{2n-1} * l_{2n} * (l_{2n-1})^{-1} * (l_{2n})^{-1}.$$

Each such map C_n is clearly continuous, and so the image $C_n(H^{2n})$ is an analytic subset of H. Let $\mathcal{H}$ be the space of homotopies between loops, also under the sup metric. Let $D \subseteq H^3 \times \mathcal{H}$ be defined by $D = \{(l_1, l_2, l_3, h) : h$ homotopes l_1 to $l_2 * l_3\}$. The set D is obviously closed in $H^3 \times \mathcal{H}$. Then the set $D_n = (H \times H \times C_n(H^{2n}) \times \mathcal{H}) \cap D$ is analytic as an intersection of two analytic sets. Letting $\sim_n$ be the projection of D_n to $H \times H$, we see that $\sim_n$ is analytic. Then $\bigcup_{n=0}^{\infty} \sim_n$ is also analytic. Letting $F : \{0,1\}^{\mathbb{N}} \to H$ be given by $F(\alpha) = f^\alpha$, it is clear that F is continuous. Finally, $\sim \, = F^{-1}(\bigcup_{n=0}^{\infty} \sim_n)$, and so $\sim$ is analytic.

If α and β differ exactly at $n \in \mathbb{N}$, then we know that $f_0^{\alpha(0)} * \cdots * f_{n-1}^{\alpha(n-1)}$ is homotopic to $f_0^{\beta(0)} * \cdots * f_{n-1}^{\beta(n-1)}$, and $f_{n+1}^{\alpha(n+1)} * f_{n+2}^{\alpha(n+2)} * \cdots$ is homotopic to $f_{n+1}^{\beta(n+1)} * f_{n+2}^{\beta(n+2)} * \cdots$ (the loops are in fact the same). Supposing that $\alpha \sim \beta$, we apply cancellations on the right and left so that $f_n^{\alpha(n)}$ is homologous to $f_n^{\beta(n)}$, hence f_n is nullhomologous, a contradiction. ∎

3. Proving Theorem 1.1. Following the literature, there are two possible paths on which we might proceed to finish the proof of our result. Shelah [S] and later Pawlikowski [P] both argue that any equivalance relation on the Cantor set satisfying the conclusion of the previous lemma must contain a perfect set, which necessarily has the cardinality of the continuum. However, this final portion of Shelah's seminal article is extremely terse and apparently uses a sophisticated technique related to forcing not generally available to the naive topologist. For the sake of clarity and completeness we conclude by offering a complete and simplified discussion of Pawlikowski's method, which has the advantage of using terminology covered in a basic topology course.

DEFINITION 3.1. Recall that a set A in a topological space Y is *nowhere dense* if its closure has empty interior, is *meager* if it can be written as a countable union of nowhere dense subsets of Y, and *comeager* if $Y - A$ is meager. The Baire Category Theorem states that a complete metric space is not meager as a subset of itself. A set $A \subseteq Y$ has the *property of Baire* if there exists an open set $O \subseteq Y$ such that $A \triangle O = (A \cup O) - (A \cap O)$ is meager (we say that A is *comeager in* O). Analytic spaces have the property of Baire.

To finish, we shall use the following consequence of the Kuratowski–Ulam Theorem:

LEMMA 3.2. *If Y is a Polish space and $A \subseteq Y \times Y$ is comeager in $U \times V$, with U and V open sets in Y, then $\{y \in U : \{z \in V : (y,z) \in A\}$ is comeager in $V\}$ is comeager in U. Also, if $A \subseteq Y \times Y$ is meager, then $\{y \in Y : \{z \in Y : (y,z) \in A\}$ is meager in $Y\}$ is comeager in Y.*

For the uncountability of the number of classes of $\sim$ we offer a lemma similar to one in [P] but whose proof is elementary.

LEMMA 3.3. *If $\sim$ is an equivalence relation on the Cantor set satisfying the conclusion of Lemma* 2.3 *then $\sim$ is meager and has uncountably many classes.*

Proof. We show first that $\sim$ is meager. Since $\sim$ is analytic it satisfies the property of Baire. If $\sim$ were non-meager, then it would be comeager in a neighborhood in $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$ of the form $U \times V$, with U and V basic open sets in $\{0,1\}^{\mathbb{N}}$. By Lemma 3.2,

$$A = \big\{\alpha \in U : \{\beta \in V : \alpha \sim \beta\} \text{ is comeager in } V\big\}$$

is comeager in U. Let $n \in \mathbb{N}$ be greater than the length of a generator of U. Define $\Phi : U \to U$ by $\Phi(\alpha)(n) = 1 - \alpha(n)$ and $\Phi(\alpha)(i) = \alpha(i)$ for $i \neq n$. Notice that Φ is a homeomorphism of U to itself. Thus we can choose $\alpha \in A \cap \Phi(A)$, as $A \cap \Phi(A)$ is comeager. Letting $\gamma = \Phi(\alpha)$, we see that γ and α differ only at n, so that $\alpha \nsim \gamma$. From the definition of A, we have comeagerly many $\beta \in V$ such that $\alpha \sim \beta$. Since $\gamma \in A$, the same can be said of γ. Then for some β, we have $\beta \sim \alpha$ and $\beta \sim \gamma$, implying $\gamma \sim \alpha$, a contradiction.

To see that $H_1(X)$ is uncountable we construct a set $Y \subseteq \{0,1\}^{\mathbb{N}}$ of cardinality $\aleph_1$ such that for distinct α and β in Y we have $\alpha \nsim \beta$. For $\alpha \in \{0,1\}^{\mathbb{N}}$ we write $\sim^{\alpha}$ for the equivalence class of α. As $\sim$ is meager, we deduce by Lemma 3.2 that $J = \{\alpha \in \{0,1\}^{\mathbb{N}} : \sim^{\alpha} \text{ is meager}\}$ is comeager in $\{0,1\}^{\mathbb{N}}$. By transfinite induction define a sequence $\{\alpha_i\}_{i<\omega_1}$ by picking α_j such that $\alpha_j \in J - \bigcup_{i<j} \sim^{\alpha_i}$, the Baire Category Theorem guaranteeing that this choice is always possible. Let $Y = \{\alpha_i\}_{i<\omega_1}$. ∎

To see that there is in fact a continuum of pairwise non-homologous loops, we invoke the following theorem of Mycielski [M]:

THEOREM 3.4. *Every meager relation on a perfect Polish space admits a perfect, pairwise non-related set.*

4. A clarification in the non-compact setting. In [P], a claim is stated without proof, which we recall after first giving some definitions.

DEFINITION 4.1. Let κ be an infinite cardinal less than continuum. We say a topological space Y is *κ-separable* if Y has a dense subset of cardinality less than or equal to κ, and that Y is *κ-Polish* if Y is completely metrizable and κ-separable.

DEFINITION 4.2 ([Mo]). If Y is a Polish space, we say that $A \subseteq Y$ is *κ-Suslin* if there exists a closed set $D \subseteq Y \times \kappa^{\mathbb{N}}$ such that A is the projection of D in Y (here κ is given the discrete topology).

OBSERVATION 4.3. Since any κ-Polish space is the continuous image of $\kappa^{\mathbb{N}}$, we may say $A \subseteq Y$ is κ-Suslin iff there exists a κ-Polish space Z and a closed set $D \subseteq Y \times Z$ such that A is the projection of D in Y. Thus $\aleph_0$-Polish spaces and $\aleph_0$-Suslin sets are precisely Polish spaces and analytic sets respectively. Also, a metric space is κ-separable if and only if every open cover contains a subcover of cardinality at most κ. This latter condition is often called *κ-Lindelöf.*

The aforementioned claim from [P] is the following:

CLAIM 4.4 (Pawlikowski). *Let κ be an infinite cardinal less than continuum. Suppose that X is a path connected, locally path connected metric space which is κ-Lindelöf. Then $\pi_1(X)$ is of cardinality $\le \kappa$ or of cardinality continuum.*

We know of no proof of this claim using the methods of [P]. We state and prove a theorem with more hypotheses but which is nonetheless of interest. Towards this, let $\mathrm{BP}(\kappa)$ be the statement that all κ-Suslin sets have the property of Baire. The claim $\mathrm{BP}(\aleph_0)$ is simply true, as we noted earlier. If one assumes extra set-theoretic assumptions, for example Martin's Axiom, then $\mathrm{BP}(\kappa)$ holds provided the successor cardinal κ^+ is less than continuum. This follows from [Mo, Theorem 2F.2] combined with the consequence of Martin's Axiom that in a Polish space the collection of sets with the property of Baire is closed under unions of index less than continuum (see [F]). Martin's Axiom is known to be consistent with the standard axioms of Zermelo–Fraenkel set theory with the axiom of choice [B].

Our theorem is the following:

THEOREM 4.5. *Assume* BP(κ). *Suppose that X is a connected, locally path connected κ-Polish space. Then $\pi_1(X)$ is either of cardinality $\leq \kappa$ or of cardinality continuum.*

Proof. There are two cases.

Suppose there exists a point x near which we have arbitrarily small essential (non-nulhomotopic) loops. By local path connectedness we have a sequence of essential loops $f_1, f_2, \ldots$ based at x with $\mathrm{diam}(f_n) \leq 2^{-n}$. Define a map from the Cantor set $\{0,1\}^{\mathbb{N}}$ to the space L of continuous loops based at x under the sup metric as in Section 2 by letting $\alpha \mapsto f^\alpha$. This map is continuous. Define an equivalence relation $\approx$ on the Cantor set by letting $\alpha \approx \beta$ if and only if f^α is homotopic to f^β.

Let D be the space of continuous mappings $H : [0,1] \times [0,1] \to X$ such that $H(s,0) = H(s,1) = x$, also under the sup metric. Letting $\mathcal{D} = \{(f,g,H) \in L \times L \times D : (\forall t \in [0,1])(f(t) = H(0,t) \text{ and } g(t) = H(1,t))\}$, we find that the spaces L, D and $L \times L \times D$ are all κ-Polish and $\mathcal{D}$ is closed in $L \times L \times D$. Letting Gr be the graph of the map $(\alpha,\beta) \mapsto (f^\alpha, f^\beta)$, we see that the set $\mathrm{Gr} \times D$ is closed in $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \times L \times L \times D$. The relation $\approx$ is the projection of the closed set $(\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}} \times \mathcal{D}) \cap (\mathrm{Gr} \times D)$ to $\{0,1\}^{\mathbb{N}} \times \{0,1\}^{\mathbb{N}}$, and so is κ-Suslin. By BP(κ) the relation $\approx$ has the property of Baire. If $\alpha, \beta \in \{0,1\}^{\mathbb{N}}$ differ at exactly one point, then $\alpha \not\approx \beta$ by the same proof as in Lemma 2.3. By Lemma 3.3 the set $\approx$ is meager, and so $\pi_1(X)$ is of cardinality continuum by Theorem 3.4.

Supposing that no x as above exists, select for each $x \in X$ an open neighborhood U_x such that any loop which lies in U_x is nulhomotopic in X. Since paracompactness follows from metrizability, select an open star refinement $\mathcal{U}_1$ of $\{U_x\}_{x \in X}$ that is locally finite. By local path connectedness, let $\mathcal{U}_2$ be the open cover consisting of all path components of elements of $\mathcal{U}_1$. As X is κ-separable and metric, we may pick a subcover $\mathcal{U} \subseteq \mathcal{U}_2$ whose cardinality is less than or equal to κ. Notice that the identity mapping $\pi_1(X) \to \pi_1(X)$ is 2-set simple relative to $\mathcal{U}$ (see [CC, Definition 7.1]). Letting $\mathcal{N}$ be the nerve of $\mathcal{U}$, we find that $\pi_1(X)$ is a factor group of $\pi_1(\mathcal{N})$ by [CC, Theorem 7.3(2)]. As $\mathcal{N}$ is a simplicial complex of at most κ vertices and 1-cells, we are done. ∎

The comparable statement for first homology also holds, via a similar argument:

Theorem 4.6. *Suppose* BP(κ). *Suppose that X is a connected, locally path connected κ-Polish space. Then $H_1(X)$ is either of cardinality $\leq \kappa$ or of cardinality continuum.*

Proof. The proof treats the two analogous cases to those in the previous theorem. Each element of $H_1(X)$ has a representative that is a mapping of a circle.

Suppose there exists a point x near which there exist arbitrarily small mappings of S^1 that are not nulhomologous. Then as in the proof of Theorem 1.1 we treat these mappings as paths that are based at x by local path connectedness, and select paths f_n at x that are not nulhomologous such that $\mathrm{diam}(f_n) \leq 1/2^n$, and define f^α for each $\alpha \in \{0,1\}^{\mathbb{N}}$ and relations $\sim_n$ in the same way. Each $\sim_n$ has the property of Baire since $\sim_n$ is κ-Suslin by a proof as in the previous theorem, and letting $\sim = \bigcup_{n\in\mathbb{N}} \sim_n$ we see that $\sim$ has the property of Baire, and $\alpha \sim \beta$ if and only if f^α is homologous to f^β. The relation $\sim$ also enjoys the property that if α and β differ at exactly one point then $\alpha \nsim \beta$. By Lemma 3.3 and Theorem 3.4 we get continuum many classes.

If no x as above exists, then for each $x \in X$ we have an open neighborhood U_x such that any loop in U_x is nulhomologous. This gives an open cover $\{U_x\}_{x\in X}$ of X, which we refine as in the previous theorem to a cover $\mathcal{U}$ of cardinality at most κ each of whose elements is path connected and such that any given mapping of S^1 to X whose image lies entirely in the union of two elements of $\mathcal{U}$ is nullhomologous. Let $g : \pi_1(X) \to H_1(X)$ be the Hurewicz map, which is onto. This map g is 2-set simple relative to $\mathcal{U}$ [CC, Definition 7.1], hence $g(\pi_1(X))$ is a factor group of the fundamental group of the nerve of $\mathcal{U}$, and so is at most of cardinality κ by Theorem 7.3(2) of [CC]. ∎

Since BP($\aleph_0$) is simply true, we may state special cases of the above two theorems:

Theorem 4.7. *Suppose that X is a connected, locally path connected Polish space. Then $\pi_1(X)$ is either countable or of cardinality continuum. Also, $H_1(X)$ is either countable or of cardinality continuum.*

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