FUNDAMENTA
MATHEMATICAE
232 (2016)

Weak Rudin–Keisler reductions on projective ideals

by

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Abstract. We consider a slightly modified form of the standard Rudin–Keisler order on ideals and demonstrate the existence of complete (with respect to this order) ideals in various projective classes. Using our methods, we obtain a simple proof of Hjorth's theorem on the existence of a complete $\mathbf{\Pi}^1_1$ equivalence relation. This proof enables us (under PD) to generalize Hjorth's result to the classes of $\mathbf{\Pi}^1_{2n+1}$ equivalence relations.

1. Introduction. An *ideal* on ω is a family $\mathcal{I}$ of subsets of ω such that, for any $x, y \subseteq \omega$, one has $x, y \in \mathcal{I} \Rightarrow x \cup y \in \mathcal{I}$ and $y \subseteq x \in \mathcal{I} \Rightarrow y \in \mathcal{I}$. For the present purposes, we identify the power set of the natural numbers with the Cantor space 2^ω and consider definable ideals on ω as subsets of 2^ω.

One defines the *Rudin–Keisler order*, $\leq_{\mathrm{RK}}$, for families of subsets of ω by letting $\mathcal{I} \leq_{\mathrm{RK}} \mathcal{J}$ (for families $\mathcal{I}, \mathcal{J} \subseteq \mathcal{P}(\omega)$) iff there exists a function $f : \omega \to \omega$ such that

$$x \in \mathcal{I} \;\Leftrightarrow\; f^{-1}[x] \in \mathcal{J}$$

for each $x \in 2^\omega$. (Here and throughout this paper, we use the notation $f[x]$ (respectively, $f^{-1}[x]$) to denote the f-image (respectively, the f-preimage) of the set $x \subseteq \omega$.) The map f is called a *Rudin–Keisler reduction* of $\mathcal{I}$ to $\mathcal{J}$.

The Rudin–Keisler order was first described by Mary Ellen Rudin and has been studied extensively since its introduction. Subsequently, several variants of the Rudin–Keisler order have been considered. Many of these are discussed in the survey [4] by Hrušák or in Chapter 3 of Kanovei [5]. In some cases, these variations of the Rudin–Keisler order coincide with the ordinary Rudin–Keisler order when restricted to the class of ultrafilters (or maximal ideals). For instance, the Katětov order, $\leq_{\mathrm{K}}$, has this property, where $\mathcal{I} \leq_{\mathrm{K}} \mathcal{J}$ iff there is $f : \omega \to \omega$ such that $x \in \mathcal{I} \Rightarrow f^{-1}[x] \in \mathcal{J}$ for each $x \subseteq \omega$. In what follows, we consider another modification of the

2010 *Mathematics Subject Classification*: 03E15, 03E60, 03E05, 28A05.
Key words and phrases: projective ideals, weak Rudin–Keisler reductions, projective determinacy, co-analytic equivalence relations.

DOI: 10.4064/fm232-1-5

Rudin–Keisler order which again agrees with the usual Rudin–Keisler order on the set of ultrafilters. We will, however, be only interested in its restriction to the class of projective (hence nonmaximal) ideals on ω.

1.1. The wRK-order. The following is our principal definition, which we introduced in [1].

DEFINITION 1.1. If $\mathcal{I}$ and $\mathcal{J}$ are ideals on ω, we say that $\mathcal{I}$ is *weak Rudin–Keisler reducible* to $\mathcal{J}$ if there is an infinite set $A \subseteq \omega$ and a function $f : A \to \omega$ such that $x \in \mathcal{I} \Leftrightarrow f^{-1}[x] \in \mathcal{J}$ for each $x \in 2^\omega$. We write $\mathcal{I} \leq_{\mathrm{wRK}} \mathcal{J}$ and call the map f a *weak Rudin–Keisler reduction* of $\mathcal{I}$ to $\mathcal{J}$.

If $\mathcal{C}$ is a family of ideals and $\mathcal{J} \in \mathcal{C}$ is such that $\mathcal{I} \leq_{\mathrm{wRK}} \mathcal{J}$ for each $\mathcal{I} \in \mathcal{C}$, then we say that $\mathcal{J}$ is a *wRK-complete* $\mathcal{C}$ ideal on ω.

Note that, as with the usual Rudin–Keisler order, if $\mathcal{U} \leq_{\mathrm{wRK}} \mathcal{V}$ and $\mathcal{V}$ is an ultrafilter, then so is $\mathcal{U}$. For the sake of completeness, we verify that $\leq_{\mathrm{RK}}$ and $\leq_{\mathrm{wRK}}$ coincide on the set of ultrafilters.

PROPOSITION 1.2. *If $\mathcal{U}$ and $\mathcal{V}$ are ultrafilters on ω, then $\mathcal{U} \leq_{\mathrm{wRK}} \mathcal{V}$ iff $\mathcal{U} \leq_{\mathrm{RK}} \mathcal{V}$.*

Proof. One half of the claim follows from the definitions. For the other half, suppose that $\mathcal{U} \leq_{\mathrm{wRK}} \mathcal{V}$ via $f : A \to \omega$. We wish to define a Rudin–Keisler reduction of $\mathcal{U}$ to $\mathcal{V}$. Let $y = f[A]$ and note that $y \in \mathcal{U}$, since $f^{-1}[y] = A = f^{-1}[\omega] \in \mathcal{V}$. Let $z \subseteq y$ be such that $z \in \mathcal{U}$ and $y \setminus z$ is infinite. Such a z exists by the maximality of $\mathcal{U}$. Let $B = f^{-1}[z]$ and define $h : \omega \to \omega$ by letting $h{\restriction}B = f{\restriction}B$ and defining $h{\restriction}(\omega \setminus B)$ to be any fixed bijection between $\omega \setminus B$ and $y \setminus z$. Since $\mathcal{U}$ and $\mathcal{V}$ are ultrafilters, to verify that h is a Rudin–Keisler reduction of $\mathcal{U}$ to $\mathcal{V}$ it will suffice to check that $h^{-1}[x] \in \mathcal{V}$ whenever $x \in \mathcal{U}$. Indeed, suppose that $x \in \mathcal{U}$. We have

$$h^{-1}[x] = h^{-1}[x \cap z] \cup h^{-1}[x \setminus z] \supseteq h^{-1}[x \cap z] = f^{-1}[x \cap z] \in \mathcal{V},$$

since $x, z \in \mathcal{U}$ and f is a weak Rudin–Keisler reduction of $\mathcal{U}$ to $\mathcal{V}$. It follows from the upward closure of $\mathcal{V}$ that $h^{-1}[x] \in \mathcal{V}$. ∎

1.2. Factor algebras. Observe that if $\mathcal{I} \leq_{\mathrm{wRK}} \mathcal{J}$ via f, there is an embedding of the Boolean algebra $2^\omega/\mathcal{I}$ into $2^\omega/\mathcal{J}$. To see this, let $\pi_{\mathcal{I}}$ and $\pi_{\mathcal{J}}$ be the quotient maps onto $2^\omega/\mathcal{I}$ and $2^\omega/\mathcal{J}$, respectively. One may define an injective homomorphism $\tilde{f} : 2^\omega/\mathcal{I} \to 2^\omega/\mathcal{J}$ by

$$\tilde{f}(\pi_{\mathcal{I}}(x)) = \pi_{\mathcal{J}}(f^{-1}[x]).$$

Thus, if $\mathcal{J}$ is a wRK-complete ideal for a class $\mathcal{C}$ of ideals, then $2^\omega/\mathcal{I}$ embeds in $2^\omega/\mathcal{J}$ for each $\mathcal{I} \in \mathcal{C}$. In other words, the algebra $2^\omega/\mathcal{J}$ is injectively universal for algebras of the form $2^\omega/\mathcal{I}$ with $\mathcal{I} \in \mathcal{C}$.

In [7], Kechris discusses the "Borel cardinality" of a factor algebra of the form $2^\omega/\mathcal{I}$. He regards $2^\omega/\mathcal{J}$ as having greater Borel cardinality than $2^\omega/\mathcal{I}$

if there is an embedding $\varphi : 2^\omega/\mathcal{I} \to 2^\omega/\mathcal{J}$ which has a Borel-measurable lifting to 2^ω. That is, there exists a Borel-measurable map $\varphi^* : 2^\omega \to 2^\omega$ such that the diagram

$$\begin{array}{ccc} 2^\omega & \xrightarrow{\varphi^*} & 2^\omega \\ \big\downarrow{\scriptstyle \pi_I} & & \big\downarrow{\scriptstyle \pi_J} \\ 2^\omega/\mathcal{I} & \xrightarrow{\varphi} & 2^\omega/\mathcal{J} \end{array}$$

commutes. Equivalently, $2^\omega/\mathcal{J}$ has greater Borel cardinality than $2^\omega/\mathcal{I}$ iff there is a Borel-measurable map $h : 2^\omega \to 2^\omega$ such that $x \vartriangle y \in \mathcal{I} \Leftrightarrow h(x) \vartriangle h(y) \in \mathcal{J}$ for all $x, y \in 2^\omega$, where "$x \vartriangle y$" denotes the symmetric difference of x and y.

Since all maps of the form $x \mapsto f^{-1}[x]$ (for $f : \omega \to \omega$) are continuous on 2^ω (in particular, Borel-measurable), our notion of $\mathcal{I} \leq_{\mathrm{wRK}} \mathcal{J}$ implies that $2^\omega/\mathcal{J}$ has greater Borel cardinality than $2^\omega/\mathcal{I}$. It follows that if $\mathcal{J}$ is a wRK-complete ideal for some class $\mathcal{C}$ of ideals, then $2^\omega/\mathcal{J}$ has maximum Borel cardinality among factor algebras by ideals in $\mathcal{C}$.

1.3. wRK-complete ideals. In Section 3, we prove our principal results. These establish the existence of wRK-complete ideals for the classes of $\mathbf{\Sigma}^1_n$ ideals and $\mathbf{\Pi}^1_1$ ideals.

Theorem 1.3. *For each natural number $n > 0$, there is a wRK-complete $\mathbf{\Sigma}^1_n$ ideal on ω.*

Theorem 1.4. *There is a wRK-complete $\mathbf{\Pi}^1_1$ ideal on ω.*

The proof of Theorem 1.4 uses the prewellordering property of the pointclass $\mathbf{\Pi}^1_1$ as well as closure under universal quantification over the reals and closure under countable unions and intersections. As such, the proof carries over to any other ranked pointclass with similar closure properties. In particular, assuming projective determinacy (PD), we obtain the following corollary to the proof of Theorem 1.4.

Corollary 1.5. (PD) *For each n, there is a wRK-complete $\mathbf{\Pi}^1_{2n+1}$ ideal on ω.*

In Section 4, we prove similar results for ideals which are "nontrivial" in a certain sense.

Definition 1.6. An ideal $\mathcal{I} \subseteq 2^\omega$ is *proper* if $\mathcal{I} \neq 2^\omega$ and $\mathcal{I}$ contains all finite subsets of ω.

Theorem 1.7. *There is a wRK-complete proper uncountable $\mathbf{\Pi}^1_1$ ideal on ω.*

1.4. Projective equivalence relations. There are analogies between the theory of equivalence relations and that of ideals. In many cases, arguments in the context of ideals carry over to equivalence relations. The proof of Theorem 1.4 is an example of this.

We begin by establishing some terminology.

Suppose that E and F are equivalence relations on a Polish space X. A map $f : X \to X$ is a *reduction* of E to F if, for each pair $x, y \in X$, one has $xEy \Leftrightarrow f(x)Ff(y)$. In general, one restricts attention to those reductions which are at least Borel-measurable. In the case that there is a Borel-measurable reduction of E to F, we say that E is *Borel-reducible* to F and write $E \leq_{\mathrm{B}} F$. If $\mathcal{C}$ is a class of equivalence relations and $F \in \mathcal{C}$ is such that $E \leq_{\mathrm{B}} F$ for each $E \in \mathcal{C}$, then we say that F is a *complete* $\mathcal{C}$ equivalence relation.

Hjorth [3] proved that there is a Π^1_1 (effectively co-analytic) equivalence relation on 2^ω which is a complete co-analytic equivalence relation. The key step in his proof was the following parameterization theorem for co-analytic equivalence relations.

THEOREM 1.8 (Hjorth, 1996). *There is a universal* Π^1_1 *set for* $\boldsymbol{\Pi}^1_1$ *equivalence relations, i.e., there is a* (*lightface*) Π^1_1 *set* $\mathcal{E} \subseteq 2^\omega \times 2^\omega \times 2^\omega$ *such that the cross-sections* $\mathcal{E}_\tau$ *are exactly the* $\boldsymbol{\Pi}^1_1$ *equivalence relations on* 2^ω.

Hjorth's proof involves admissible ordinals and critical use of the Luzin–Sierpiński Theorem (see Moschovakis [8, Theorem 4A.4]) and is thus intimately connected with the effective theory of Π^1_1 sets. Even under PD, this theory is not known to generalize to the classes Π^1_{2n+1}. By combining elements of the proof of our Theorem 1.4 with an *s-n-m* Theorem argument, we obtain Hjorth's theorem without use of any effective theory specific to the class Π^1_1. In fact, our argument generalizes, assuming PD, to yield a universal Π^1_{2n+1} set for $\boldsymbol{\Pi}^1_{2n+1}$ equivalence relations. In Section 5, we prove the following result.

THEOREM 1.9. (PD) *There is a universal* Π^1_{2n+1} *set for* $\boldsymbol{\Pi}^1_{2n+1}$ *equivalence relations on* 2^ω.

As in Hjorth [3], the existence of complete $\boldsymbol{\Pi}^1_{2n+1}$ equivalence relation follows immediately from Theorem 1.9. Indeed, if $\mathcal{E}$ is a universal set for $\boldsymbol{\Pi}^1_{2n+1}$ equivalence relations on 2^ω, one may define a complete $\boldsymbol{\Pi}^1_{2n+1}$ equivalence relation F on $2^\omega \times 2^\omega \approx 2^\omega$ by

$$(\sigma, x)F(\tau, y) \;\Leftrightarrow\; \tau = \sigma \;\&\; (x, y) \in \mathcal{E}_\tau.$$

If E is any $\boldsymbol{\Pi}^1_{2n+1}$ equivalence relation on 2^ω, with $E = \mathcal{E}_\tau$, the (continuous) map $x \mapsto (\tau, x)$ reduces E to F.

2. Preliminaries and notation. Our principal references are Kechris [6] and Moschovakis [8]. Our notation is largely the same as theirs. We review some key facts and terminology below.

For sets X, Y, and $A \subseteq X \times Y$, if $x \in X$, let A_x denote the vertical cross-section, $\{y \in Y : (x, y) \in A\}$, of A. As mentioned above, if $f : X \to Y$ is any function and $x \subseteq X$, we let $f[x] = \{f(a) : a \in x\}$ and, if $y \subseteq Y$, we let $f^{-1}[y] = \{a : f(a) \in y\}$. We will freely identify $\mathcal{P}(\omega)$ with the Cantor space 2^ω. For example, we regard $\emptyset$ and ω as the elements $\bar{0}$ and $\bar{1}$ in the Cantor space.

2.1. Classical notions. If Γ is a pointclass, we say that Γ is 2^ω-*parameterized* if, for each Polish space X, there is a Γ-set $\mathcal{U} \subseteq 2^\omega \times X$ such that, for each $A \subseteq X$ with $A \in \Gamma$, there exists $\tau \in 2^\omega$ with $A = \mathcal{U}_\tau$. Such a set $\mathcal{U}$ is called a *universal* Γ-*set for* X. Each of the projective classes $\boldsymbol{\Sigma}^1_n$ and $\boldsymbol{\Pi}^1_n$ is 2^ω-parameterized (see [6, Theorem 37.7]).

Suppose Γ is a pointclass; then Γ is *ranked* if, for each Polish space X and each $A \subseteq X$, with $A \in \Gamma$, there is a Γ-*rank* $\varphi : A \to \mathsf{ORD}$. That is, there are relations $S, P \subseteq X^2$ in Γ and $\bar{\Gamma}$, respectively, such that, for each $y \in A$,

$$x \in A \ \&\ \varphi(x) \le \varphi(y) \ \Leftrightarrow\ S(x, y) \ \Leftrightarrow\ P(x, y),$$

for each $x \in X$. (Here $\bar{\Gamma}$ denotes the class of complements of Γ-sets.) In other words, the predicate "$x \in A$ & $\varphi(x) \le \varphi(y)$" is uniformly in $\Gamma \cap \bar{\Gamma}$, provided that $y \in A$. See [6, §34] for an in-depth treatment of Γ-ranks. It is a fundamental result that $\boldsymbol{\Pi}^1_1$ is a ranked pointclass. Assuming PD, the classes $\boldsymbol{\Pi}^1_{2n+1}$ are also ranked. See [6, Theorems 34.4 and 39.2] for proofs of these facts. The notation "$x \le_\varphi y$" is a shorthand to indicate "$\varphi(x) \le \varphi(y)$" when φ is a Γ-rank on a set A and $x, y \in A$.

2.2. Effective theory. For each classical, or "boldface", pointclass $\boldsymbol{\Gamma}$ (e.g., $\boldsymbol{\Sigma}^1_1$, $\boldsymbol{\Pi}^1_1$, etc.) on a recursively presented Polish space, we let Γ denote its effective, or "lightface", counterpart, (e.g., Σ^1_1, Π^1_1, etc.). The effective pointclasses share many of the properties of their classical brethren. For instance, the pointclass Π^1_1 is also ranked. (See [8, Theorem 4B.2].) For a Π^1_1-rank φ, the predicates S and P as above will, in fact, be in the (lightface) classes Σ^1_1 and Π^1_1, respectively. As in the classical case, PD guarantees the lightface classes Π^1_{2n+1} are ranked as well. (See [8, Theorem 6B.1].)

The other tools we require from the effective theory are the existence of *good universal systems* and the resulting *s-n-m* Theorem. We say that a Polish space X is a *product space* if X is a product of at most countably many copies of 2^ω. (This is a restricted definition compared to that in [8].)

Definition 2.1. Let Γ be a lightface pointclass and $\boldsymbol{\Gamma}$ the corresponding boldface pointclass. A family $\{\mathcal{U}^X \subseteq 2^\omega \times X : X$ is a product space$\}$ of sets is a *good universal system* for (boldface) $\boldsymbol{\Gamma}$ if the following hold:

(1) Each $\mathcal{U}^X$ is a (lightface) Γ-set.
(2) For each product space X, the set $\mathcal{U}^X$ is universal for $\mathbf{\Gamma}$-subsets of X, i.e., the cross-sections $\mathcal{U}^X_\tau$ are exactly the $\mathbf{\Gamma}$-subsets of X.
(3) If $A \subseteq X$ is a (lightface) Γ-set, there is a recursive $\varepsilon \in 2^\omega$ such that $A = \mathcal{U}^X_\varepsilon$.
(4) For each pair X_1, X_2 of product spaces, there is a recursive function $s : 2^\omega \times X_1 \to 2^\omega$ such that, for each $\alpha \in 2^\omega$ and $x \in X_1$,
$$(\alpha, x, y) \in \mathcal{U}^{X_1 \times X_2} \Leftrightarrow (s(\alpha, x), y) \in \mathcal{U}^{X_2}.$$

Property (4) is also known as the *s-n-m Theorem.*

It follows from [8, Theorem 3H.1] that each of the projective classes $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$ has a good universal system. For illustrative purposes, we now describe a typical application of good universal systems.

Example 2.2. Let $\mathbf{\Gamma}$ be a pointclass, closed under finite intersections, with an associated good universal system $\mathcal{U}^X$. There is a recursive function $f : (2^\omega)^2 \to 2^\omega$ such that, for $\sigma, \tau \in 2^\omega$, we have $\mathcal{U}^{2^\omega}_{f(\sigma,\tau)} = \mathcal{U}^{2^\omega}_\sigma \cap \mathcal{U}^{2^\omega}_\tau$.

To find such an f, let $s : (2^\omega)^3 \to 2^\omega$ be the recursive function from property (4) above, for the product spaces $X_1 = 2^\omega \times 2^\omega$ and $X_2 = 2^\omega$, and let $\varepsilon \in 2^\omega$ be recursive such that, for $\sigma, \tau, x \in 2^\omega$,
$$\begin{aligned}(\sigma, x) \in \mathcal{U}^{2^\omega} \ \&\ (\tau, x) \in \mathcal{U}^{2^\omega} &\Leftrightarrow (\varepsilon, \sigma, \tau, x) = \mathcal{U}^{2^\omega \times 2^\omega \times 2^\omega} \\ &\Leftrightarrow (s(\varepsilon, \sigma, \tau), x) \in \mathcal{U}^{2^\omega}.\end{aligned}$$
Define $f(\sigma, \tau) = s(\varepsilon, \sigma, \tau)$. This is the desired recursive function.

3. Projective ideals on ω

3.1. $\mathbf{\Sigma}^1_n$ ideals.

We now prove Theorem 1.3.

Proof of Theorem 1.3. Fix $n \in \omega$. The proof is the same for each n. First of all, let $\{A_p : p \in 2^{<\omega}\}$ be a partition of ω into infinite sets and let $h_p : \omega \to A_p$ be fixed bijections. Let $\mathcal{U} \subseteq 2^\omega \times 2^\omega$ be a universal $\mathbf{\Sigma}^1_n$ set for 2^ω. As a point of notation, if $p \in 2^{<\omega}$ and $\tau \in 2^\omega$, we write $p \prec \tau$ if p is an initial segment of τ. Define a $\mathbf{\Sigma}^1_n$ set $F \subseteq 2^\omega$ by letting $x \in F$ iff there exists $\tau \in 2^\omega$ such that

(1) $(\forall p \in 2^{<\omega})(p \not\prec \tau \Rightarrow x \cap A_p = \emptyset)$,
(2) $h_\emptyset^{-1}[x] \in \mathcal{U}_\tau$, and
(3) $(\forall p \in 2^{<\omega})(p \prec \tau \Rightarrow h_p^{-1}[x] = h_\emptyset^{-1}[x])$.

Let $\mathcal{J}$ be the ideal generated by F. Since the class $\mathbf{\Sigma}^1_n$ is closed under continuous images and countable unions, it follows that $\mathcal{J}$ is also $\mathbf{\Sigma}^1_n$.

To see that $\mathcal{J}$ is a wRK-complete $\mathbf{\Sigma}^1_n$ ideal, suppose that $\mathcal{I}$ is an arbitrary $\mathbf{\Sigma}^1_n$ ideal and $\tau \in 2^\omega$ is such that $\mathcal{I} = \mathcal{U}_\tau$. Define a map $f : \bigcup_{p \prec \tau} A_p \to \omega$ by $f \restriction A_p = h_p^{-1}$. This map is well-defined because the A_p are pairwise disjoint.

Suppose first that $x \in \mathcal{I}$. To see that $y = f^{-1}[x] \in \mathcal{J}$, it will suffice to verify conditions (1)–(3) above and conclude that $y \in F$. That y satisfies condition (1) follows from the observation that, for each $p \not\prec \tau$, since $\mathrm{dom}(f) \cap A_p = \emptyset$, we have $y \cap A_p = \emptyset$. Also, for each $p \prec \tau$, we have $y \cap A_p = h_p[x]$ and therefore

$$h_\emptyset^{-1}[y] = h_\emptyset^{-1}[y \cap A_\emptyset] = h_\emptyset^{-1}[h_\emptyset[x]] = x \in \mathcal{I} = \mathcal{U}_\tau.$$

This shows that y satisfies condition (2). Finally, the definition of f guarantees that y satisfies condition (3).

Suppose now that $y = f^{-1}[x] \in \mathcal{J}$. We wish to see that $x \in \mathcal{I}$. Let $y_0, \ldots, y_k \in F$ be such that $y \subseteq y_0 \cup \cdots \cup y_k$. Let $\tau_0, \ldots, \tau_k \in 2^\omega$ all be as in the definition of F and such that each τ_i witnesses the membership of y_i in F. Let $m \in \omega$ be large enough that $\tau_i{\restriction}m \neq \tau{\restriction}m$ for each $\tau_i \neq \tau$. Now define $x_0, \ldots, x_k$ by letting $x_i = h_{\tau\restriction m}^{-1}[y_i]$ for each $i \leq k$. Note that $x_i = \emptyset$ for i with $\tau_i \neq \tau$, and that $x_i \in \mathcal{U}_\tau = \mathcal{I}$ for each $i \leq k$. Finally, observe that

$$x = h_{\tau\restriction m}^{-1}[y] \subseteq \bigcup_{i \leq k} h_{\tau\restriction m}^{-1}[y_i] = x_0 \cup \cdots \cup x_k.$$

As $\mathcal{I}$ is an ideal, we conclude that $x \in \mathcal{I}$.

We have shown that f is a weak Rudin–Keisler reduction of $\mathcal{I}$ to $\mathcal{J}$. This concludes the proof. ∎

3.2. Co-analytic ideals. We now prove our main result for co-analytic ideals on ω, Theorem 1.4. The key lemma in the proof is a parameterization of the $\mathbf{\Pi}^1_1$ ideals on ω.

LEMMA 3.1. *There is a universal set for $\mathbf{\Pi}^1_1$ ideals on ω, i.e., there is a $\mathbf{\Pi}^1_1$ set $\mathcal{A} \subseteq 2^\omega \times 2^\omega$ such that the cross-sections $\mathcal{A}_\tau$ are exactly the $\mathbf{\Pi}^1_1$ ideals on ω.*

Proof. We proceed inductively to define $\mathbf{\Pi}^1_1$ sets $\mathcal{A}^{(0)} \supseteq \mathcal{A}^{(1)} \supseteq \cdots$ such that $\mathcal{A} = \bigcap_n \mathcal{A}^{(n)}$ is a universal set for $\mathbf{\Pi}^1_1$ ideals on ω. Along the way, we will also select $\mathbf{\Pi}^1_1$-ranks $\varphi_n : \mathcal{A}^{(n)} \to \omega_1$.

Let $\mathcal{U} \subseteq 2^\omega \times 2^\omega$ be a universal $\mathbf{\Pi}^1_1$ set. As the base case of our induction, let $\mathcal{A}^{(0)} = \mathcal{U} \cup (2^\omega \times \{\emptyset\})$. Given the $\mathbf{\Pi}^1_1$ set $\mathcal{A}^{(n)}$, with $\varphi_n : \mathcal{A}^{(n)} \to \omega_1$ a $\mathbf{\Pi}^1_1$-rank, we describe how to define $\mathcal{A}^{(n+1)}$. Let $\mathcal{A}^{(n+1)} \subseteq 2^\omega \times 2^\omega$ be the set of all (τ, x) such that either $x = \emptyset$ or the following hold:

(1) $x \in \mathcal{A}^{(n)}_\tau$,
(2) $(\forall y)(y \subseteq x \Rightarrow y \in \mathcal{A}^{(n)}_\tau)$, and
(3) $(\forall y)((y \in \mathcal{A}^{(n)}_\tau \ \&\ (\tau, y) \leq_{\varphi_n} (\tau, x)) \Rightarrow x \cup y \in \mathcal{A}^{(n)})$.

In essence, we are defining $\mathcal{A}^{(n+1)}_\tau$ by removing from $\mathcal{A}^{(n)}_\tau$ those x for which there exist lower ranked witnesses to the failure of closure under finite unions. It follows from the definability properties of $\boldsymbol{\Pi}^1_1$-ranks that $\mathcal{A}^{(n+1)}$ is $\boldsymbol{\Pi}^1_1$. Finally, we choose φ_{n+1} to be any $\boldsymbol{\Pi}^1_1$-rank on $\mathcal{A}^{(n+1)}$.

Let $\mathcal{A} = \bigcap_n \mathcal{A}^{(n)}$. Since each $\mathcal{A}^{(n)}$ is $\boldsymbol{\Pi}^1_1$, so is $\mathcal{A}$. First observe that, if $\mathcal{U}_\tau$ is already an ideal, then $\mathcal{A}^{(n)}_\tau = \mathcal{U}_\tau$ for each n, since the process of producing $\mathcal{A}^{(n+1)}$ from $\mathcal{A}^{(n)}$ removes no elements from $\mathcal{A}^{(n)}_\tau$ in this case. It follows that each $\boldsymbol{\Pi}^1_1$ ideal on ω appears as some cross-section $\mathcal{A}_\tau$.

It now only remains to see that every cross-section $\mathcal{A}_\tau$ is an ideal on ω. Indeed, suppose that $x, y \in \mathcal{A}_\tau$. Fix $n \in \omega$. We know that $x, y \in \mathcal{A}^{(n+1)}_\tau$ and we may assume that $(\tau, x) \leq_{\varphi_n} (\tau, y)$. Condition (3) in the definition of $\mathcal{A}^{(n+1)}$ thus implies that $x \cup y \in \mathcal{A}^{(n)}_\tau$. Similarly, the membership of x in $\mathcal{A}^{(n+1)}$ implies that every subset of x is a member of $\mathcal{A}^{(n)}_\tau$, by condition (2). As n was arbitrary, it follows that $x \cup y \in \mathcal{A}_\tau$ and that every subset of x is a member of $\mathcal{A}_\tau$. In short, $\mathcal{A}_\tau$ is an ideal. ∎

Before proceeding with the proof of Theorem 1.4, we remark that an argument analogous to the one used to produce a complete $\boldsymbol{\Pi}^1_1$ equivalence relation from a universal set for $\boldsymbol{\Pi}^1_1$ equivalence relations does not work for ideals, as the object one would obtain need not itself be an ideal. We thus require "coding" all $\boldsymbol{\Pi}^1_1$ ideals into a single $\boldsymbol{\Pi}^1_1$ ideal.

Proof of Theorem 1.4. Let $\{A_p : p \in 2^{<\omega}\}$ be a partition of ω into infinite sets and let $h_p : \omega \to A_p$ be fixed bijections. Take $\mathcal{A} \subseteq 2^\omega \times 2^\omega$ to be a universal set for $\boldsymbol{\Pi}^1_1$ ideals, the existence of which is guaranteed by Lemma 3.1. Define

$$\mathcal{J} = \{x \in 2^\omega : (\forall \tau \in 2^\omega)(\forall^\infty p \prec \tau)(h_p^{-1}[x] \in \mathcal{A}_\tau)\}.$$

First, we wish to see that $\mathcal{J}$ is an ideal. That $\mathcal{J}$ is downward closed follows since each $\mathcal{A}_\tau$ is downward closed and preimages preserve containment. To verify closure under unions, fix $x, y \in \mathcal{J}$ and $\tau \in 2^\omega$, and let $p_0 \prec \tau$ be such that $h_p^{-1}[x], h_p^{-1}[y] \in \mathcal{A}_\tau$ for each $p \prec \tau$ with $p \supseteq p_0$. For each $p \prec \tau$, if $p \supseteq p_0$, we have

$$h_p^{-1}[x \cup y] = h_p^{-1}[x] \cup h_p^{-1}[y] \in \mathcal{A}_\tau,$$

since $\mathcal{A}_\tau$ is an ideal. In other words, $h_p^{-1}[x \cup y] \in \mathcal{A}_\tau$, for all but finitely many $p \prec \tau$. Since τ was arbitrary, we conclude that $x \cup y \in \mathcal{J}$.

To see that $\mathcal{J}$ is a wRK-complete $\boldsymbol{\Pi}^1_1$ ideal, suppose that $\mathcal{I} = \mathcal{A}_\tau$ is a fixed $\boldsymbol{\Pi}^1_1$ ideal on ω. Define $f : \bigcup_{p \prec \tau} A_p \to \omega$ by $f \restriction A_p = h_p^{-1}$. As before, f is well-defined by the disjointness of the A_p. Also, note that $f^{-1}[x] \cap A_p = h_p[x]$, for each $p \prec \tau$. Thus,

$$h_p^{-1}[f^{-1}[x]] = h_p^{-1}[f^{-1}[x] \cap A_p] = h_p^{-1}[h_p[x]] = x$$

and hence $x \in \mathcal{I}$ iff $h_p^{-1}[f^{-1}[x]] \in \mathcal{A}_\tau$ for each $p \prec \tau$, since $\mathcal{I} = \mathcal{A}_\tau$. This, combined with the observation that $f^{-1}[x] \cap A_p = \emptyset$ for each $p \not\prec \tau$, shows that $x \in \mathcal{I}$ iff $f^{-1}[x] \in \mathcal{J}$. ∎

3.3. Free abelian subgroups. It is a theorem of Nöbeling [9] that the group $\mathcal{B}$ of bounded sequences in $\mathbb{Z}^\omega$ is free. This result, however, requires an essential use of the axiom of choice. Indeed, Blass [2] has shown (under ZFC) that there are only countably many Borel-measurable homomorphisms $\varphi : \mathcal{B} \to \mathbb{Z}$. (In this case, Borel-measurability is equivalent to the statement that $\varphi^{-1}[\{n\}]$ is Borel for each $n \in \omega$.) Blass further showed that (under AD) there are only countably many homomorphisms $\varphi : \mathcal{B} \to \mathbb{Z}$ (Borel-measurable or otherwise). On the other hand, for any free abelian group G, there must be $\mathfrak{c}$-many homomorphisms $\varphi : \mathcal{B} \to \mathbb{Z}$. Thus, Blass' result shows that $\mathcal{B}$ is not free under AD. In essence, his result reveals that there is no explicit free basis of $\mathcal{B}$.

By contrast, the method of proof of Theorem 1.3 enables us to produce, in a large class of Polish groups, free abelian K_σ subgroups with free bases of a very explicit nature. It will follow that the free subgroups we describe have $\mathfrak{c}$-many definable homomorphisms into $\mathbb{Z}$.

THEOREM 3.2. *Let G be a Polish group with an element of infinite order. The countable power G^ω contains a K_σ subgroup H which is free abelian on $\mathfrak{c}$-many generators and such that there exist $\mathfrak{c}$-many Borel-measurable homomorphisms $\varphi : H \to \mathbb{Z}$.*

Proof. Fix a Polish group G, with identity element e, and $g \in G$ such that the powers of g are all distinct. For notational reasons, we will work with the group $G^{2^{<\omega}}$ rather than G^ω. Of course, these groups are isomorphic and so nothing is changed by this assumption. Define a set $S \subseteq G^{2^{<\omega}}$ by letting $x \in S$ iff there exists $\tau \in 2^\omega$ such that

(1) $(\forall p \in 2^{<\omega})(p \not\prec \tau \Rightarrow x(p) = e)$, and
(2) $(\forall p \in 2^{<\omega})(p \prec \tau \Rightarrow x(p) = g)$.

For $\tau \in 2^\omega$, let x_τ denote the unique element of S whose membership in S is witnessed by τ. Take H to be the subgroup generated by S.

First, observe that S is the projection onto the second coordinate of the compact set

$$\{(\tau, x) \in 2^\omega \times \{e, g\}^{2^{<\omega}} : (\forall p \prec \tau)(x(n_p) = g) \wedge (\forall p \not\prec \tau)(x(n_p) = e)\}$$

and is itself compact. Hence, H is compactly generated and therefore K_σ.

To see that H is a free abelian group, it suffices to show that there are no relations among the elements of S, besides those dictated by commutativity. Indeed, suppose that $x_{\sigma_0}, \ldots, x_{\sigma_m}, x_{\tau_0}, \ldots, x_{\tau_n} \in S$, with $\{\sigma_0, \ldots, \sigma_m\}$ and $\{\tau_0, \ldots, \tau_n\}$ sets of pairwise distinct reals, and $i_0, \ldots, i_m, j_0, \ldots, j_n \in \mathbb{Z}$ (all nonzero) are such that

$$y = x_{\sigma_0}^{i_0} \cdot \ldots \cdot x_{\sigma_m}^{i_m} = x_{\tau_0}^{j_0} \cdot \ldots \cdot x_{\tau_n}^{j_m} = z.$$

Our first claim is that $\{\sigma_0, \ldots, \sigma_m\} = \{\tau_0, \ldots, \tau_n\}$. Suppose that this was not the case. For instance, if $\sigma_0 \notin \{\tau_0, \ldots, \tau_n\}$, then we may choose $k \in \omega$ large enough that

$$\sigma_0 \restriction k, \tau_0 \restriction k, \ldots, \tau_n \restriction k$$

are all distinct and

$$\sigma_0 \restriction k, \ldots, \sigma_m \restriction k$$

are distinct as well. Then $y(\sigma_0 \restriction k) = g^{i_0} \neq e$, but $z(\sigma_0 \restriction k) = e$, since $\sigma_0 \restriction k \not\prec \tau_p$ for each $p \leq n$. This is a contradiction.

We may also assume that $\sigma_p = \tau_p$ for each $p \leq n$. Observe now that if k is as above, we have, for each $p \leq n$,

$$g^{i_p} = y(\sigma_p \restriction k) = z(\sigma_p \restriction k) = g^{j_p}.$$

Since g has infinite order, we conclude that $i_p = j_p$ for each $p \leq n$. This shows that S is a free basis for H.

Finally, we exhibit $\mathfrak{c}$-many Borel-measurable homomorphisms $\varphi : H \to \mathbb{Z}$. Since S is a free basis for H, it will suffice to define these homomorphisms on S and then extend to H. For each $\alpha \in 2^\omega$, let $\varphi_\alpha \restriction S$ be given by

$$\varphi_\alpha(x_\tau) = \begin{cases} 1 & \text{if } \tau = \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

We see that $\{\varphi_\alpha : \alpha \in 2^\omega\}$ is a family of $\mathfrak{c}$-many distinct group homomorphisms of H into $\mathbb{Z}$. To verify that each φ_α is Borel-measurable, observe that, for each $n \in \mathbb{Z}$,

$$\varphi_\alpha^{-1}[\{n\}] = \{x \in H : (\exists k \in \omega)(\exists s \in \mathbb{Z}^{k+1})(\exists \tau_0, \ldots, \tau_k \in 2^\omega)(\forall i \leq k)(\tau_i \neq \alpha \\ \& \ x = x_\alpha^n \cdot x_{\tau_0}^{s(0)} \cdot \ldots \cdot x_{\tau_k}^{s(k)})\}.$$

In particular, $\varphi^{-1}[\{n\}]$ is analytic for each $n \in \omega$. On the other hand, every $\mathbf{\Sigma}_1^1$-measurable map between Borel spaces is Borel-measurable and hence each φ_α is Borel-measurable. We cannot, however, use Pettis' Theorem to conclude that the φ_α are continuous, since H is not a Baire group. ∎

4. Parameterizing co-analytic proper ideals. We turn our attention to *proper* ideals, that is, ideals $\mathcal{I}$ such that $\mathcal{I} \neq 2^\omega$ and $\mathcal{I}$ contains all finite subsets of ω (equivalently, $\bigcup \mathcal{I} = \omega$). Before proceeding, we recall some standard notation:

$$\mathsf{Fin} = \{x \in 2^\omega : (\forall^\infty n)(x(n) = 0)\}, \qquad \mathsf{Inf} = 2^\omega \setminus \mathsf{Fin},$$
$$\mathsf{Cof} = \{x \in 2^\omega : \omega \setminus x \in \mathsf{Fin}\}, \qquad \mathsf{Coinf} = 2^\omega \setminus \mathsf{Cof}.$$

Theorem 4.1 is an elaboration on the method of Lemma 3.1 which produces a universal set for proper uncountable co-analytic ideals.

THEOREM 4.1. *There is a universal set for proper uncountable co-analytic ideals.*

Proof. For $\tau \in 2^\omega$, let $(\tau)_0 = \{n : 2n \in \tau\}$ and $(\tau)_1 = \{n : (2n+1) \in \tau\}$, and let $f : 2^\omega \to \mathsf{Inf} \setminus \mathsf{Cof}$ be a Borel surjection. Let $\mathcal{U} \subseteq 2^\omega \times 2^\omega$ be a universal co-analytic set. We define co-analytic sets $\mathcal{V}^{(0)} \supseteq \mathcal{V}^{(1)} \supseteq \cdots$, by induction, such that each $\mathcal{V}^{(n)}_\tau$ contains $\mathcal{P}(f((\tau)_0))$. Let $\mathcal{V}^{(0)} \subseteq 2^\omega \times 2^\omega$ be the set of $(\tau, x) \in 2^\omega \times 2^\omega$ such that

- $x \subseteq f((\tau)_0)$ or
- $x \in \mathcal{U}_{(\tau)_1}$ and x is co-infinite.

Given $\mathcal{V}^{(n)} \subseteq 2^\omega \times 2^\omega$, let $\varphi_n : \mathcal{V}^{(n)} \to \omega_1$ be a $\mathbf{\Pi}^1_1$-rank. Define $\mathcal{V}^{(n+1)} \subseteq 2^\omega \times 2^\omega$ by $(\tau, x) \in \mathcal{V}^{(n+1)}$ iff either $x \subseteq f((\tau)_0)$ or

(1) $x \in \mathcal{V}^{(n)}_\tau$,
(2) $(\forall y)(y \subseteq x \Rightarrow y \in \mathcal{V}^{(n)}_\tau)$,
(3) $(\forall y)(y \subseteq f((\tau)_0) \Rightarrow x \cup y \in \mathcal{V}^{(n)}_\tau)$, and
(4) $(\forall y)((y \in \mathcal{V}^{(n)}_\tau \ \&\ (\tau, y) \leq_{\varphi_n} (\tau, x)) \Rightarrow x \cup y \in \mathcal{V}^{(n)}_\tau)$.

By the properties of $\mathbf{\Pi}^1_1$-ranks, each $\mathcal{V}^{(n)}$ is $\mathbf{\Pi}^1_1$. Let

$$\mathcal{V} = \Big\{(\tau, x) : (\exists u \in \mathsf{Fin})(\forall n)\Big(u \mathbin{\triangle} x \in \bigcap_n \mathcal{V}^{(n)}_\tau\Big)\Big\}$$

and note that $\mathcal{V}$ is $\mathbf{\Pi}^1_1$.

What follows will establish that the cross-sections $\mathcal{V}_\tau$ are exactly the uncountable co-analytic proper ideals on ω.

First, we verify that each $\mathcal{V}_\tau$ is an ideal. Since $\mathcal{V}_\tau$ is the set of finite variants of members of $\bigcap_n \mathcal{V}^{(n)}_\tau$, it will suffice to show that $\bigcap_n \mathcal{V}^{(n)}_\tau$ is an ideal.

To verify downward closure, fix $y \subseteq x \in \bigcap_n \mathcal{V}^{(n)}_\tau$. In the case that $x \subseteq f((\tau)_0)$, we have $y \in \mathcal{P}(f((\tau)_0)) \subseteq \mathcal{V}^{(n)}_\tau$ for each n. Hence $y \in \bigcap_n \mathcal{V}^{(n)}_\tau$. In the case that $x \not\subseteq f((\tau)_0)$, part (2) of the definition of $\mathcal{V}^{(n+1)}$ from $\mathcal{V}^{(n)}$ implies that $y \in \mathcal{V}^{(n)}_\tau$ for each n.

Suppose that $x, y \in \bigcap_n \mathcal{V}^{(n)}_\tau$. We wish to see that $x \cup y \in \bigcap_n \mathcal{V}^{(n)}_\tau$. If $x, y \subseteq f((\tau)_0)$, then $x \cup y \in \mathcal{P}(f((\tau)_0)) \subseteq \bigcap_n \mathcal{V}^{(n)}_\tau$. If $y \subseteq f((\tau)_0)$ and $x \not\subseteq f((\tau)_0)$, then (3) implies that $x \cup y \in \mathcal{V}^{(n)}_\tau$. We argue similarly if $x \not\subseteq f((\tau)_0)$ and $y \subseteq f((\tau)_0)$. Finally, if $x, y \not\subseteq f((\tau)_0)$, then, for each n, we have $x, y \in V^{(n+1)}_\tau$ and, with no loss of generality, assume that $y \leq_{\varphi_n} x$. Then (4) implies that $x \cup y \in \mathcal{V}^{(n)}_\tau$. As this holds for each n, it follows that $x \cup y \in \bigcap_n \mathcal{V}^{(n)}_\tau$.

In order to check that each $\mathcal{V}_\tau$ is proper, we begin by noting that $\mathcal{V}_\tau$ is an ideal which contains all finite variants of its members and, in particular, must contain the ideal Fin. That $\mathcal{V}_\tau \subsetneq 2^\omega$ for each τ follows by observing that, for each $x \in \mathcal{V}_\tau$, there is $y \in \mathcal{V}_\tau^{(0)} \subseteq \mathsf{Coinf}$ such that x and y are equal mod finite. In particular, for each $x \in \mathcal{V}_\tau$, we have $x \neq \omega$.

Finally, if $\mathcal{I}$ is an uncountable proper ideal on ω, then $\mathcal{I} \nsubseteq \mathsf{Fin}$. Therefore let $\tau \in 2^\omega$ be such that $\mathcal{P}(f((\tau)_0)) \subseteq \mathcal{I}$ and $\mathcal{I} = \mathcal{U}_{(\tau)_1}$. We have $\mathcal{V}_\tau^{(0)} = \mathcal{I}$ and, in fact, $\mathcal{V}_\tau^{(n)} = V_\tau^{(0)}$ for each n, since $\mathcal{V}_\tau^{(0)}$ is already a proper ideal containing $\mathcal{P}(f((\tau)_0))$. This completes the proof. ∎

As discussed in the Introduction, Theorem 4.1 implies Theorem 1.7. The argument is essentially the same as that used to deduce Theorem 1.4 from Lemma 3.1.

As before, PD yields a corresponding result for the projective classes $\mathbf{\Pi}^1_{2n+1}$.

By slightly modifying the proof of Theorem 1.3 one can also show that there is a wRK-complete proper uncountable $\mathbf{\Sigma}^1_n$ ideal for each $n \in \omega$.

5. Projective equivalence relations. We now give our proof of Hjorth's theorem on co-analytic equivalence relations (Theorem 1.8), as well as our generalization of it under PD (Theorem 1.9).

Proof of Theorems 1.8 and 1.9. Let $\mathcal{V} \subseteq (2^\omega)^4$ and $\mathcal{W} \subseteq (2^\omega)^5$ be part of a good universal system for $\mathbf{\Pi}^1_{2n+1}$. In particular, $\mathcal{V}$ and $\mathcal{W}$ are both Π^1_{2n+1}. Let $\varphi : \mathcal{V} \to \mathsf{ORD}$ be a Π^1_{2n+1}-rank. (For $n \geq 1$, the existence of such a φ follows from PD. If $n = 0$, then such a φ exists under ZFC.) Define $\mathcal{V}^* \subseteq (2^\omega)^4$ by $(\alpha, \tau, x, y) \in \mathcal{V}^*$ iff $x = y$ or

(1) $(x, y), (y, x) \in \mathcal{V}_{\alpha,\tau}$,
(2) $(\forall z)(((x, y), (y, z) \in \mathcal{V}_{\alpha,\tau} \;\&\; (\alpha, \tau, y, z) \leq_\varphi (\alpha, \tau, x, y)) \Rightarrow (x, z) \in \mathcal{V}_{\alpha,\tau})$, and
(3) $(\forall z)(((z, x), (x, y) \in \mathcal{V}_{\alpha,\tau} \;\&\; (\alpha, \tau, z, x) \leq_\varphi (\alpha, \tau, x, y)) \Rightarrow (z, y) \in \mathcal{V}_{\alpha,\tau})$.

Note that, by the definability properties of Π^1_{2n+1}-ranks, $\mathcal{V}^*$ is itself Π^1_{2n+1}.

We now require a recursive function $f : 2^\omega \to 2^\omega$ such that $\mathcal{V}^*_\alpha = \mathcal{V}_{f(\alpha)}$ for each $\alpha \in 2^\omega$. To produce such an f, let $\varepsilon \in 2^\omega$ be recursive and such that $\mathcal{V}^* = \mathcal{W}_\varepsilon$. Thus, for each $(\alpha, \tau, x, y) \in (2^\omega)^4$,

$$(\alpha, \tau, x, y) \in \mathcal{V}^* \;\Leftrightarrow\; (\varepsilon, \alpha, \tau, x, y) \in \mathcal{W} \;\Leftrightarrow\; (s(\varepsilon, \alpha), \tau, x, y) \in V,$$

where $s : (2^\omega)^2 \to 2^\omega$ is as in the definition of a good universal system. Define f by $f(\alpha) = s(\varepsilon, \alpha)$.

Let $\mathcal{U} \subseteq (2^\omega)^3$ be any universal Π^1_{2n+1} set for $\mathbf{\Pi}^1_{2n+1}$. The set $\mathcal{U} \cup (\{(\tau, x, x) : \tau, x \in 2^\omega\})$ is also Π^1_{2n+1} and hence, by the definition of a good universal system, there is a recursive $\alpha_0 \in 2^\omega$ such that $\mathcal{U} \cup (\{(\tau, x, x) : \tau, x \in 2^\omega\}) = \mathcal{V}_{\alpha_0}$. Now define $\alpha_n = f^n(\alpha_0)$, i.e., $\alpha_{n+1} = f(\alpha_n)$ for each n. As f is recursive, so is the map $n \mapsto \alpha_n$. Thus, the set $\mathcal{A} = \bigcap_n \mathcal{V}_{\alpha_n}$ is Π^1_{2n+1}.

First, each $\mathbf{\Pi}^1_{2n+1}$ equivalence relation on 2^ω appears as a cross-section $\mathcal{A}_\tau$. Indeed, if E is a $\mathbf{\Pi}^1_{2n+1}$ equivalence relation, with τ such that $\mathcal{U}_\tau = E$, then each $(x, y) \in \mathcal{V}_{\alpha_0,\tau}$ already satisfies conditions (1)–(3) above and hence

$$\mathcal{V}_{\alpha_0,\tau} = \mathcal{V}^*_{\alpha_0,\tau} = \mathcal{V}_{\alpha_1,\tau} = \mathcal{V}^*_{\alpha_1,\tau} = \mathcal{V}_{\alpha_2,\tau} = \cdots .$$

It follows that $\mathcal{A}_\tau = \mathcal{V}_{\alpha_0,\tau} = \mathcal{U}_\tau$.

The proof will be complete when we have verified that each $\mathcal{A}_\tau$ is an equivalence relation. It is a consequence of the choice of α_0 and the definition of $\mathcal{V}^*$ that $(x, x) \in \mathcal{A}_\tau$ for each $\tau, x \in 2^\omega$. Also, condition (1) above guarantees that $(x, y) \in \mathcal{A}_\tau$ iff $(y, x) \in \mathcal{A}_\tau$. To verify transitivity, suppose that $(x, y), (y, z) \in \mathcal{A}_\tau = \bigcap_n \mathcal{V}_{\alpha_n} = \bigcap_n \mathcal{V}^*_{\alpha_n}$. Fix n and assume that $(\alpha_n, \tau, x, y) \leq_\varphi (\alpha_n, \tau, y, z)$. Since $(y, z) \in \mathcal{V}^*_{\alpha_n,\tau}$, condition (3) guarantees that $(x, z) \in \mathcal{V}_{\alpha_n,\tau}$. If instead $(\alpha_n, \tau, y, z) \leq_\varphi (\alpha_n, \tau, x, y)$, then an analogous reasoning, using condition (2), shows that $(x, z) \in \mathcal{V}_{\alpha_n,\tau}$. In either case, since n was arbitrary, it follows that $(x, z) \in \bigcap \mathcal{V}_{\alpha_n,\tau} = \mathcal{A}_\tau$. We conclude that each $\mathcal{A}_\tau$ is an equivalence relation. ∎

Acknowledgements. The author acknowledges the US NSF grant DMS-0943870 for the support of his research. Also, he thanks his colleague and friend Anush Tserunyan for the discussion that led to the formulation of Theorem 3.2.

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Received 18 July 2014;
in revised form 18 June 2015