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Singular homology groups of one-dimensional Peano continua

by

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Abstract. Let X be a one-dimensional Peano continuum. Then the singular homology group $H_1(X)$ is isomorphic to a free abelian group of finite rank or the singular homology group of the Hawaiian earring.

1. Introduction and main result. The study of singular homology of one-dimensional spaces goes back to Curtis and Fort [3]. They proved that for every one-dimensional separable metric space X the singular homology groups $H_k(X)$ are $\{0\}$ for $k \geq 2$.

A *Peano continuum* is a locally connected, connected, compact metric space. As we have proved previously, the fundamental groups of one-dimensional Peano continua determine their homotopy types [8], and in particular the fundamental groups of one-dimensional Peano continua which are not semi-locally simply connected everywhere determine their homeomorphism types [7]. Consequently, the fundamental groups of one-dimensional Peano continua are abundant. We recall that the *Hawaiian earring* is the plane compactum

$$\mathbb{H} = \bigcup_{1 \leq n < w} \{(x, y) : (x - 1/n)^2 + y^2 = 1/n^2\}.$$

It is known that the singular homology group of the Hawaiian earring is isomorphic to the abelian group

$$\mathbb{Z}^\omega \oplus \bigoplus_{\mathbf{c}} \mathbb{Q} \oplus \prod_{p:\,\text{prime}} A_p,$$

where ω is the least infinite ordinal, $\mathbf{c}$ is the cardinality of the continuum, and A_p is the p-adic completion of the free abelian group of rank $\mathbf{c}$ [11, Theorem 3.1] (see Remark 1.3).

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In contrast to the case of the fundamental groups, we have

THEOREM 1.1. *Let X be a one-dimensional Peano continuum. Then the singular homology group $H_1(X)$ is isomorphic to a free abelian group of finite rank or the singular homology group of the Hawaiian earring.*

The proof shows:

COROLLARY 1.2. *Let X be a one-dimensional Peano continuum. If X is semi-locally simply connected, then $H_1(X)$ is isomorphic to a free abelian group of finite rank. Otherwise, $H_1(X)$ is isomorphic to the singular homology group of the Hawaiian earring.*

The result is somewhat unexpected, because the classification is the same as those of the Čech homology groups and the shape groups (Čech homotopy groups) of one-dimensional Peano continua, while that of the fundamental groups is different, as mentioned above. Though proofs for the classifications of the Čech homology groups and the shape groups are rather geometric, the proof for the singular homology groups is highly group-theoretic as we show in what follows.

As is well-known, M. G. Barratt and J. Milnor [1] proved that the three-dimensional singular homology group of the two-dimensional Hawaiian earring is non-trivial, which shows a counter-intuitive behavior of singular homology. Our result is another counter-intuitive one even in dimension one.

REMARK 1.3. The proof of [11, Theorem 3.1] depends on [6, Lemma 4.11]. However, there is a gap in the proof of the latter. Hence we prove Lemma 3.6 of the present paper, which permits us to give another proof and generalize [11, Theorem 3.1].

2. Sequences and abelian groups. To express finite or infinite sequences of paths and elements of groups, we introduce some notation, which we have used in [6, 5, 9]. Let Seq be the set of all finite sequences of non-negative integers and denote the length of $s \in \mathrm{Seq}$ by $\mathrm{lh}(s)$. The empty sequence is denoted by $(\ )$. For $s, t \in \mathrm{Seq}$, let $s * t$ be the concatenation of s and t, i.e. $\mathrm{lh}(s * t) = \mathrm{lh}(s) + \mathrm{lh}(t)$ and $(s * t)_i = s_i$ for $1 \le i \le \mathrm{lh}(s)$ and $(s * t)_i = t_{i-\mathrm{lh}(s)}$ for $\mathrm{lh}(s) + 1 \le i \le \mathrm{lh}(s) + \mathrm{lh}(t)$. In general, $s \in \mathrm{Seq}$ is denoted by $(s_1, \ldots, s_n)$ where s_k $(1 \le k \le n)$ are non-negative integers and $n = \mathrm{lh}(s)$. The lexicographical ordering is denoted by $\preceq$, i.e. for $s, t \in \mathrm{Seq}$, $s \preceq t$ if $s_i < t_i$ for the minimal i with $s_i \ne t_i$ or t extends s. For a non-empty sequence $s \in \mathrm{Seq}$, let $s^+ \in \mathrm{Seq}$ be the sequence such that $\mathrm{lh}(s^+) = \mathrm{lh}(s)$ and $s^+_i = s_i$ for $i < \mathrm{lh}(s)$ and $s^+_i = s_i + 1$ for $i = \mathrm{lh}(s)$.

We recall some notions for abelian groups (in this section a *group* means an abelian group). For a group A, the *Ulm subgroup* $U(A)$ of A is $\bigcap\{n!A : n < \omega\}$. If A is torsionfree, $U(A)$ becomes the divisible subgroup $D(A)$ of A.

The divisible subgroup is a direct summand of A. A torsionfree divisible group is a direct sum of copies of the rational group $\mathbb{Q}$.

A group A is called *complete mod-U* if $A/U(A)$ is complete [16, VII 39], i.e. for a given $a_n \in A$ ($n \in \mathbb{N}$) such that $n! \mid a_{n+1} - a_n$, there exists an element a such that $n! \mid a - a_n$ for every $n \in \mathbb{N}$.

It is known that a group A is *algebraically compact* if and only if A is complete mod-U and $U(U(A)) = U(A)$ [4]. If A is torsionfree, then $U(A) = U(U(A)) = D(A)$. Hence, a torsionfree, complete mod-U group is algebraically compact. The structure of a torsionfree algebraically compact group is well-known and determined by cardinalities depending on primes [16, p. 169]. Let $\widehat{\mathbb{Z}}$ be the $\mathbb{Z}$-completion of $\mathbb{Z}$ [16, p. 164]. Then $\widehat{\mathbb{Z}} \cong \prod_{p:\,\mathrm{prime}} \mathbb{J}_p$, where $\mathbb{J}_p$ is the p-adic integer group.

A subgroup S of a group A is *pure* if, for $a \in S$, $n \mid a$ in A implies $n \mid a$ in S. It is known that a group A is algebraically compact if and only if A is *pure-injective*, i.e. whenever A is a pure subgroup of a group B, then A is a direct summand of B.

For a group A, $R_{\mathbb{Z}}(A)$ is the subgroup $\bigcap\{\mathrm{Ker}(h) : h \in \mathrm{Hom}(A, \mathbb{Z})\}$, which is a radical, i.e. $R_{\mathbb{Z}}(A/R_{\mathbb{Z}}(A)) = \{0\}$. It is easy to see that $A/R_{\mathbb{Z}}(A)$ is a subgroup of a direct product of copies of $\mathbb{Z}$. For undefined notions for abelian groups, we refer the reader to [16].

3. Paths in one-dimensional metric spaces and group-theoretic properties. To investigate the divisibility in $H_1(X)$ we recall reduced paths following [7].

For $a \le b$, a continuous map $f : [a,b] \to X$ is called a *path* from $f(a)$ to $f(b)$. The points $f(a)$ and $f(b)$ are called the *initial point* and the *terminal point* of f respectively. When $a = b$, the path f is said to be *degenerate.* A *loop* f is a path with $f(a) = f(b)$. For a path $f : [a,b] \to X$, f^- denotes the path such that $f^-(s) = f(a+b-s)$ for $a \le s \le b$. Two paths $f : [a,b] \to X$ and $g : [c,d] \to X$ are *equivalent*, denoted $f \equiv g$, if there exists a homeomorphism $\varphi : [a,b] \to [c,d]$ such that $\varphi(a) = c$, $\varphi(b) = d$ and $f = g \cdot \varphi$. Two paths $f : [a,b] \to X$ and $g : [c,d] \to X$ are *homotopic* if there exists a continuous map H whose domain is the quadrangle in the plane with vertices $(a,0), (b,0), (c,1)$ and $(d,1)$ such that

$$\begin{cases} H(s,0) = f(s) & \text{for } a \le s \le b, \\ H(s,1) = g(s) & \text{for } c \le s \le d, \\ H((1-t)a + tc, t) = f(a) = g(c) & \text{for } 0 \le t \le 1, \\ H((1-t)b + td, t) = f(b) = g(d) & \text{for } 0 \le t \le 1. \end{cases}$$

The homotopy class containing the path f is denoted by $[f]$. The homotopy defined above is usually called "homotopy relative to end points". Since all homotopies that appear in this paper have this property, we drop the phrase "relative to end points" for simplicity.

A path $f : [a, b] \to X$ is *reduced* if no subloop of f is null-homotopic, that is, for each pair $u < v$ with $f(u) = f(v)$, $f\restriction[u, v]$ is not null-homotopic. Note that a constant map is reduced if and only if it is degenerate. For paths $f : [a, b] \to X$ and $g : [c, d] \to X$ with $f(b) = g(c)$, fg denotes the concatenation of f and g, that is, the path from $[a, b+d-c]$ to X such that $fg(s) = f(s)$ for $a \le s \le b$ and $fg(s) = g(s - b + c)$ for $b \le s \le b + d - c$. A loop f is *cyclically reduced* if ff is reduced. An *arc* A between points x and y is a subspace of X which is homeomorphic to the unit interval $[0, 1]$ whose end points are x and y.

LEMMA 3.1 ([7, Lemma 2.4]). *Let X be a one-dimensional normal space. Then every path is homotopic to a reduced path, and the reduced path is unique up to equivalence.*

LEMMA 3.2 ([7, Lemma 2.5]). *For a reduced loop f, there exist a unique reduced path g and a unique reduced loop h up to equivalence such that $f \equiv g^- hg$ and h is cyclically reduced.*

LEMMA 3.3 ([7, Lemma 2.6]). *Let X be a one-dimensional space. For reduced paths $f : [a, b] \to X$ and $g : [c, d] \to X$ with $f(b) = g(c)$, there exist unique paths h, f' and g' up to equivalence such that*

- *$f \equiv f'h^-$ and $g \equiv hg'$;*
- *$f'g'$ is a reduced path.*

Though any path in a one-dimensional space X is homotopic to a reduced path (Lemma 3.1), there is no effective reduction procedure in general (see Example 3.9). However, if $f_1 \cdots f_n$ is a path in X and each f_i is a reduced path, we obtain the reduced path of $f_1 \cdots f_n$ by cancellations using Lemma 3.3 at most $n - 1$ times, i.e. we have a finite step reduction. For a loop f in a space we denote the homotopy class of f by $[f]$ and the singular homology class of f by $[f]_h$.

DEFINITION 3.4. A sequence of non-degenerate reduced paths $f_1, \ldots, f_{2N}$ is a 0-*form* if its concatenation $f_1 \cdots f_{2N}$ is a loop and the indices $\{1, \ldots, 2N\}$ can be paired up into $\{i_k, j_k\}$ $(1 \le k \le N)$ so that $f_{i_k} \equiv f_{j_k}^-$ for $1 \le k \le N$.

The word 0-form means that the concatenated loop represents the trivial element in the singular homology group. We remark that the empty sequence is a 0-form.

DEFINITION 3.5. The *length* of a 0-form $f_1, \ldots, f_{2N}$ is N and its *rank* is the cardinality of the set $\{1 \le i \le 2N - 1 : f_i f_{i+1}$ is not reduced$\}$.

LEMMA 3.6. *Let l_0 be a reduced loop in a one-dimensional space X. Then $[l_0]_h = 0$ in $H_1(X)$ if and only if l_0 is a degenerate loop or there exists a 0-form $f_1, \ldots, f_{2N}$ such that $l_0 \equiv f_1 \cdots f_{2N}$.*

Proof. The "if" part is clear; we show the other direction. Since by Lemma 3.1 any loop is homotopic to a unique reduced loop up to equivalence, and the homotopy class of a 0-homologous loop belongs to the commutator subgroup of the fundamental group by the Poincaré–Hurewicz theorem, it suffices to show that any 0-homologous loop is homotopic to a reduced loop of a 0-form.

We prove the lemma by induction on the rank r and the length N where the ordering of the pairs (r, N) is lexicographical. We remark that this ordering is a wellordering, which ensures our induction works. If $r = 0$, then the loop of a 0-form is reduced and we have the conclusion. On the other hand, if $N = 1$, then $f_1 f_2$ is homotopic to a degenerate loop. Hence we proceed to the inductive steps.

We introduce a *basic reduction* of a 0-form $f_1, \ldots, f_{2N_0}$. Suppose that $f_{i+1} \cdots f_{2N_0}$ is reduced and $f_i \cdots f_{2N_0}$ is not reduced. Let r_0 be the rank of $f_1, \ldots, f_{2N_0}$. By Lemma 3.3 we have $f_i \equiv f'_i h$ and $f_{i+1} \cdots f_{2N_0} \equiv h^- f'_{i+1}$ with $f'_i f'_{i+1}$ reduced. A basic reduction of $f_1, \ldots, f_{2N_0}$ is the following 0-form $f^*_1, \ldots, f^*_{2N_1}$.

CASE 1: *f'_i and f'_{i+1} are not empty.* We cancel hh^-, replace f_i and f_{i+1} by f'_i and f'_{i+1} respectively and write the 0-form $f_1, \ldots, f_{i-1}, f'_i, f'_{i+1}, f_{i+2}, \ldots, f_{2N_0}$ as $f^*_1, \ldots, f^*_{2N_1}$, whose rank is $r_0 - 1$ and $N_1 = N_0 + 1$.

CASE 2: *f'_i or f'_{i+1} is empty.*

SUBCASE 2.1: *f'_i is empty and $f_{i-1} f'_{i+1}$ is reduced, or f'_{i+1} is empty and $f'_i f_{i+2}$ is reduced.* We cancel hh^-, rearrange pairings if necessary and get a 0-form $f^*_1, \ldots, f^*_{2N_1}$. Then in the former case $N_1 = N_0 - 1$ or the rank is $r_0 - 1$ according to whether f'_{i+1} is empty or not, and in the latter case $N_1 = N_0 - 1$ or the rank is $r_0 - 1$ according to whether f'_i is empty or not.

SUBCASE 2.2: *Otherwise, i.e. f'_i is empty and $f_{i-1} f'_{i+1}$ is not reduced, or f'_{i+1} is empty and $f'_i f_{i+2}$ is not reduced.* We get a 0-form $f^*_1, \ldots, f^*_{2N_1}$ as in Case 2.1, whose rank is at most r_0 and $N_1 = N_0$ (actually the rank is r_0 but this is not necessary for our argument).

Starting from a given loop l of a 0-form, we iterate basic reductions. If the cases other than Subcase 2.2 appear, we have the conclusion by induction hypothesis. We will show that Subcase 2.2 does not occur infinitely many times, which completes the proof of Lemma 3.6. For contradiction, suppose that Subcase 2.2 occurs infinitely many times starting from a loop l of a 0-form. Then we have an infinite sequence of 0-forms σ_n and $0 < a_{n+1} < a_n < \cdots < a_1 = b_1 < \cdots < b_n < b_{n+1} < 1$ such that:

(1) the rank and the length of σ_n are the same as those of σ_0;
(2) $(l\restriction[0,a_n])(l\restriction[b_n,1])$ is a concatenation of paths in σ_n.

We remark that $(l\restriction[a_n,a_1])^- \equiv l\restriction[b_1,b_n])$. Let $a_\infty = \inf\{a_n : n < \infty\}$ and $b_\infty = \sup\{b_n : n < \infty\}$.

In step m_0 we have N pairings. If the two intervals of a pair are in $[0,a_\infty] \cup [b_\infty,1]$, then this pair is not changed in any step m for $m \geq m_0$. For intervals appearing in some steps, we call an interval *outside* if it is contained in $[0,a_\infty] \cup [b_\infty,1]$, and *inside* if it is contained in $[a_\infty,b_\infty]$. We call an interval $[c,d]$ *overlapping* if $c < a_\infty < d < b_\infty$ or $a_\infty < c < b_\infty < d$. First we claim that an outside interval is never paired with an overlapping one.

For contradiction, suppose an outside interval $[c_0,d_0]$ is paired with an overlapping interval $[c_1,d_1]$. We assume $c_1 < a_\infty < d_1$, since the other case is symmetric. Once $[c_0,d_0]$ and $[c_1,d_1]$ are paired, infinitely many $[u,d_0]$ are paired with some overlapping interval $[c_1,v]$ in some steps. This implies that there are more than N pairs in some step one of whose intervals are subintervals of $[c_0,d_0]$, which is a contradiction.

Next we show that after some steps, all outside intervals are paired with other outside intervals. If an outside interval I is paired with an inside interval, then I is possibly partitioned. But such partitionings for I occur only finitely many times, since this procedure fixes the number N_0 of the pairs. Now we consider a non-degenerate subinterval I_0 of I, which will not be partitioned. We claim that I_0 will be paired with an outside interval. Otherwise, I_0 is paired with infinitely many inside intervals, which implies that I_0 is the degenerate path $l(a_\infty) = l(b_\infty)$, a contradiction. Hence we conclude that after some steps every outside interval is paired with another outside one.

We remark that if an overlapping interval does not appear in some step, then it does not appear in further steps, and if an overlapping interval is paired with another overlapping interval in some step, then in further steps two overlapping intervals are paired.

Next we show that after some steps all overlapping intervals are paired with other overlapping intervals. For contradiction, suppose that an overlapping interval $[c_0,d_0]$ with $c_0 < a_\infty < d_0 < b_\infty$ is paired with an inside interval, and in further steps its overlapping subintervals are paired with inside intervals. Then as in the case of outside intervals there appear only finitely many subintervals of $[c_0,a_\infty]$ in further steps, and hence we have an overlapping interval $[c_1,d_1]$ with $c_0 \leq c_1 < a_\infty < d_1 < d_0$ such that in further steps an overlapping interval containing a_∞ is of the form $[c_1,d]$ for some $d \leq d_1$. Since $l\restriction[c_1,a_\infty]$ is not degenerate, we have a contradiction as in the case of outside intervals. The case $a_\infty < c_0 < b_\infty < d_0$ is symmetric and we omit its proof.

This implies that after some steps every inside interval is paired with another inside interval. Now choose two points u_0, u_1 from an inside interval so that $l(u_1) \neq l(u_2)$. Then we have their copies in some inside interval at any further steps and we have a contradiction $l(u_1) = l(a_\infty) = l(b_\infty) = l(u_2)$.

Now we have completed the proof of Lemma 3.6. We remark that our proof implies that the basic reductions stop after finitely many steps, since Subcase 2.2 never occurs infinitely many times and other cases decrease the order of the pair (r, N). ∎

A family $\mathcal{U}$ of open subsets of a space X is *of order* 2 if $U \cap V \cap W = \emptyset$ for any distinct $U, V, W \in \mathcal{U}$. If a space X is one-dimensional, then every finite open cover has a refinement of order 2 [15].

There is a natural homomorphism from singular homology to Čech homology. Though we shall use a result of [12] in principle, we need to investigate the homomorphism more precisely and we present a direct presentation of the homomorphism according to [10].

For a loop l in a one-dimensional space X, we define a loop $f_\mathcal{U}$ in the nerve $X_\mathcal{U}$ as follows [14].

We take a sequence $0 = t_0 < \cdots < t_n = 1$ and elements $U_0, \ldots, U_n \in \mathcal{U}$ with the following properties:

- $l(t_i) \in U_i$ for each $0 \leq i \leq n$ and $U_0 = U_n = x_\mathcal{U}$;
- $l([t_i, t_{i+1}]) \subset U_i \cup U_{i+1}$ for $0 \leq i < m$.

Define $l_\mathcal{U} : [0,1] \to X_\mathcal{U}$ by setting $l_\mathcal{U}(t_i) = U_i$ and extending linearly on each $[t_i, t_{i+1}]$. Then $l_\mathcal{U}$ is unique up to homotopy:

(1) Take another sequence $0 = t'_0 < \cdots < t'_n = 1$ and elements $U'_1, \ldots, U'_n \in \mathcal{U}$ and define a loop $l'_\mathcal{U}$ in $X_\mathcal{U}$ so as to satisfy the above two conditions. Then $l_\mathcal{U}$ and $l'_\mathcal{U}$ are homotopic.

(2) If m is a loop in X homotopic to l, then $m_\mathcal{U}$ and $l_\mathcal{U}$ are also homotopic.

The natural homomorphism $\sigma : H_1(X) \to \check{H}_1(X)$ for a path-connected space X is defined by $\rho_\mathcal{U}(\sigma([l]_h)) = [l_\mathcal{U}]_h$, where $\rho_\mathcal{U}$ is the projection from $\check{H}_1(X)$ to $H_1(X_\mathcal{U})$, $[l]_h$ is the homology class containing l, and $[l_\mathcal{U}]_h$ is the homology class containing $l_\mathcal{U}$.

For the following construction we suppose that X is a locally path-connected metric space and $\mathcal{U}$ is an open cover of X consisting of path-connected sets and is of order 2. Since we use this construction for locally path-connected spaces, we always use covers consisting of path-connected sets.

We use the preceding notation for a loop l in X and a cover of X. Let $\mathcal{U}_0 = \{U_i : 0 \leq i \leq n\} \subseteq \mathcal{U}$ be a finite cover of $\mathrm{Im}(l)$ and $p_{U_0} = l(0)$. Choose $p_U \in U$ for $U \in \mathcal{U}_0$ with $U \neq U_0$. Then using the path-connectivity of U

and V we inductively define an arc $A_{UV} = A_{VU} \subseteq U \cup V$ between p_U and p_V for $U, V \in \mathcal{U}_0$ with $U \cap V \neq \emptyset$ so that A_{UV} is the unique arc between p_U and p_V in $(U \cup V) \cap \bigcup\{A_{UV} : U, V \in \mathcal{U}_0\}$. Then $\bigcup\{A_{UV} : U, V \in \mathcal{U}_0\}$ is homeomorphic to a finite graph and $(U \cup V) \cap \bigcup\{A_{UV} : U, V \in \mathcal{U}_0\}$ is simply connected for any $U, V \in \mathcal{U}_0$. We remark that p_U may not be a branching point in this finite graph and A_{UU} is the one-point set $\{p_U\}$. Since $\mathcal{U}$ is infinite, to avoid a tedious argument we do not construct a graph in X for the nerve $X_{\mathcal{U}}$.

Next we construct a loop $\bar{l}$ in the finite graph $\bigcup\{A_{UV} : U, V \in \mathcal{U}\}$ for a loop l with base point U_0 in the nerve $X_{\mathcal{U}_0}$, which is a finite graph, so that a path in the edge UV corresponds to a path from p_U to p_V on the arc A_{UV}.

We apply this construction to the above loop $l_{\mathcal{U}}$. Then $\overline{l_{\mathcal{U}}} \restriction [t_i, t_{i+1}]$ is a path from p_{U_i} to $p_{U_{i+1}}$ on the arc $A_{U_i U_{i+1}}$ and $\overline{l_{\mathcal{U}}}(0) = l(0) = l(1) = l_{\mathcal{U}}(1)$.

Lemma 3.7. *Let X be a one-dimensional locally path-connected metric space. If l is a loop such that $[l]_h \in \mathrm{Ker}(\sigma)$, then l is homologous to a sum of arbitrarily small cycles. In addition, the cycles can be chosen in the image of l.*

Proof. Let l be a loop with $[l]_h \in \mathrm{Ker}(\sigma)$. For a given cover $\mathcal{V}$, by the paracompactness of X we have a locally finite refinement $\mathcal{V}_0$ of $\mathcal{V}$. By Dowker's theorem [15, 7.2.4], we have an open 2-cover $\mathcal{V}_1$ which refines $\mathcal{V}_0$. Let $\mathcal{U}$ be the set of all path-connected components of sets $V \in \mathcal{V}_1$. Then $\mathcal{U}$ is a 2-cover consisting of path-connected open sets. Hence, for a given $\varepsilon > 0$ we can choose an open 2-cover $\mathcal{U}$ of X which consists of path-connected open sets with size less than $\varepsilon/2$. Taking sufficiently large n, according to the preceding construction we obtain $0 = t_0 < t_1 < \cdots < t_n = 1$, $U_i \in \mathcal{U}$, $\mathcal{U}_0$, p_U for $U \in \mathcal{U}_0$, $l_{\mathcal{U}}$ and $\overline{l_{\mathcal{U}}}$.

Let q_i be a path from p_{U_i} to $l(t_i)$. Since $[l_{\mathcal{U}}]_h = 0$, we have a partition of the index set $\{0, 1, \ldots, n-1\} = \{i_k, j_k : 1 \leq k \leq m\} \cup S$ such that $n = 2m + |S|$ and $l \restriction [t_{j_k}, t_{j_k+1}] = (l \restriction [t_{i_k}, t_{i_k+1}])^-$ and $U_i = U_{i+1}$ for each $i \in S$. We remark that this is the edge-path version of the 0-form in Lemma 3.6. Hence $\overline{l_{\mathcal{U}}}$ is a null-homologous loop in X. We have

$$\begin{aligned} [l]_h - [\overline{l_{\mathcal{U}}}]_h &= [l]_h - [\overline{l_{\mathcal{U}}}]_h + \sum_{i=1}^{n-1} [q_i (q_i)^-]_h \\ &= \big[(l \restriction [t_0, t_1]) q_1 (\overline{l_{\mathcal{U}}} \restriction [t_0, t_1])^-\big]_h \\ &\quad + \sum_{i=2}^{n-2} \big[(l \restriction [t_i, t_{i+1}]) q_{i+1} (\overline{l_{\mathcal{U}}} \restriction [t_i, t_{i+1}])^- (q_i)^-\big]_h \\ &\quad + \big[(l \restriction [t_{n-1}, t_n]) (\overline{l_{\mathcal{U}}} \restriction [t_{n-1}, t_n])^- q_n^-\big]_h. \end{aligned}$$

Since the homology classes of cycles in the last summations are of size less than ε and $[\overline{l_{\mathcal{U}}}]_h = 0$, we have the conclusion.

For the additional statement, we remark that $\mathrm{Im}(l)$ is a Peano continuum and every path in X is homotopic to the reduced path in its image. Thus, the preceding proof can be done in $\mathrm{Im}(l)$. ∎

LEMMA 3.8. *Let X be a one-dimensional locally path-connected metric space. Then $R_{\mathbb{Z}}(H_1(X)) \le \mathrm{Ker}(\sigma)$.*

Proof. Decompose X into path-connected components X_i ($i \in I$). Then $H_1(X) = \bigoplus_{i\in I} H_1(X_i)$ and $R_{\mathbb{Z}}(H_1(X)) = \bigoplus_{i\in I} R_{\mathbb{Z}}(H_1(X_i))$. Hence, without loss of generality we can assume that X is path-connected. To prove $R_{\mathbb{Z}}(H_1(X)) \le \mathrm{Ker}(\sigma)$, for contradiction suppose that $\sigma([l]_h) \neq 0$ and $[l]_h \in R_{\mathbb{Z}}(H_1(X))$ for a loop l. According to the proof of Lemma 3.7, there is a 2-cover $\mathcal{U}$ consisting of path-connected open sets such that $0 \neq [l_{\mathcal{U}}]_h \in H_1(X_{\mathcal{U}})$. Since $H_1(X_{\mathcal{U}})$ is a free abelian group, we conclude that $[l]_h \notin R_{\mathbb{Z}}(H_1(X))$, a contradiction. ∎

EXAMPLE 3.9. We show the existence of a loop l which is homotopic to the constant loop, but does not contain a non-degenerate subloop of the form ff^-. We denote the clockwise winding onto the ith circle of the Hawaiian earring $\mathbb{H}$ by a_i. Let Seq(2) be the subset of Seq consisting of sequences of $0, 1$. We define a loop as an infinite concatenation of loops whose sizes converge to zero. Let $\bar{l} = \mathrm{Seq}(2) \setminus \{(\ )\}$ and l be the loop obtained by concatenating a_i and a_i^- according to the lexicographical ordering of $\bar{l}$, i.e.

$$l\restriction\Big[\sum_{i=1}^{n-1} 2^{-2i} + \sum_{i=1}^{n} s(i)2^{-2i+1}, \sum_{i=1}^{n-1} 2^{-2i} + \sum_{i=1}^{n} s(i)2^{-2i+1} + 2^{-2n}\Big] \equiv a_i$$

if $s_n = 0$, and

$$l\restriction\Big[\sum_{i=1}^{n-1} 2^{-2i} + \sum_{i=1}^{n} s(i)2^{-2i+1}, \sum_{i=1}^{n-1} 2^{-2i} + \sum_{i=1}^{n} s(i)2^{-2i+1} + 2^{-2n}\Big] \equiv a_i^-$$

if $s_n = 1$, where $n = \mathrm{lh}(s)$.

To show that l is homotopic to the constant loop, let p_n be the projection of $\mathbb{H}$ to the bouquet B_n consisting of the first n circles. Then $p_n \circ l$ is a loop in B_n and it is easy to see that $p_n \circ l$ is null-homotopic. Then l itself is null-homotopic [10, Thm. 1]. The reason for the non-existence of a subloop of l of the form ff^- is that in l both a_i and a_i^- have immediate successors, but neither has an immediate predecessor.

The next example shows that we cannot replace the notion of reducedness of a loop in a space X with a sequence of reduced loops in the nerves of X.

EXAMPLE 3.10. We construct a reduced loop l in $\mathbb{H}$ such that no projection of l to B_n is reduced for $1 \le n < \omega$. The construction is similar to the above. Let $\bar{l} = \mathrm{Seq}(2) \setminus \{\langle\ \rangle\}$ and concatenate $a_i a_i$ and a_i^- according to the lexicographical ordering on $\bar{l}$ instead of concatenating a_i and a_i^-.

That $p_n \circ l$ is not reduced can be seen as follows. Consider the appearance of $a_n a_n$ in $p_n \circ l$. Then a_n^- follows immediately, i.e. there is a subloop $a_n a_n a_n^-$ of $p_n \circ l$ and hence $p_n \circ l$ is not reduced. To see that l is reduced, for contradiction suppose that a non-degenerate subloop l' of l is null-homotopic. Without loss of generality we may assume that the base point of l' is o. Then l' should be an infinite concatenation of a_i. Let n be minimal such that a_n or a_n^- appears in l'. Since l' is null-homotopic, the times of appearances of a_n and a_n^- are the same. In the subloop between neighboring a_n and a_n^-, or a_n^- and a_n, a_{n+1} appears one more time than a_{n+1}^-, and hence l' is not null-homotopic. Thus, l is reduced.

4. Construction of loops. For our construction of loops and cycles we prepare some notions which have been used in [6, 5, 9], but some modification is necessary, since we need to deal with loops with different base points. Though this has been done by J. Cannon and G. Conner in the proof of [2, Theorem 6.7], their presentation is not sufficiently precise to prove the next lemma. An exact presentation as in the previous section is preferable, and we follow [6, 5, 9].

Suppose that natural numbers k_i are given. Let

$$S = \{s \in \mathrm{Seq} : 0 \le s_i < k_i \text{ for } 1 \le i \le \mathrm{lh}(s)\},$$

and for $s \in S$ let $a_s = \sum_{i=1}^{\mathrm{lh}(s)} s_i / \prod_{j=1}^{i} k_j$. Next let

$$T = \{t \in \mathrm{Seq} : 0 \le t_i < (i+1)k_i \text{ for } 1 \le i \le \mathrm{lh}(t)\}.$$

Let $S_m = \{s \in S : \mathrm{lh}(s) = m\}$ and $T_m = \{t \in T : \mathrm{lh}(t) = m\}$. For $t \in \mathrm{Seq}$ with $0 \le t_i < (i+1)k_i$, define $s_t, c_t \in \mathrm{Seq}$ with $\mathrm{lh}(s_t) = \mathrm{lh}(c_t) = \mathrm{lh}(t)$ by

$$(i+1)(s_t)_i + (c_t)_i = t_i, \quad 0 \le (s_t)_i < k_i, \quad 0 \le (c_t)_i < i+1.$$

Let

$$\begin{aligned} b_t &= \sum_{i=1}^{\mathrm{lh}(t)} ((3i+2)(s_t)_i + (c_t)_i + 1) / \prod_{j=1}^{i} (3j+2)k_j \\ &= \sum_{i=1}^{\mathrm{lh}(t)} (3t_i - (s_t)_i + 1) / \prod_{j=1}^{i} (3j+2)k_j \end{aligned}$$

and $\varepsilon_m = 1/\prod_{i=1}^{m}(3i+2)k_i$. If $(c_t)_{\mathrm{lh}(t)} < \mathrm{lh}(t) = m$ for $t \in T$, then $t^+ \in T$ and $b_{t^+} = b_t + 3\varepsilon_m$. But if $(c_t)_{\mathrm{lh}(t)} = \mathrm{lh}(t) = m$, then $b_t + 3\varepsilon_m$ is not equal to any $b_{t'}$ for $t' \in T$. We remark that $a_s \le a_{s'}$ if and only if $s \preceq s'$ for $s, s' \in S$, and $b_t \le b_{t'}$ if and only if $t \preceq t'$ for $t, t' \in T$.

Let $f : [0,1] \to X$ be a path.

(∗) Suppose that we are given finite open covers $\mathcal{U}_n$ of $\mathrm{Im}(f)$ such that each $U \in \mathcal{U}_n$ is path-connected, the diameter of each $U \in \mathcal{U}_n$ is less than $1/n$, and $\mathcal{U}_{n+1}$ is a refinement of $\mathcal{U}_n$, and also suppose that U_s in $\mathcal{U}_{\mathrm{lh}(s)}$ and k_n are chosen so that $f([a_s, a_{s^+}]) \subseteq U_s$ and $U_t \subseteq U_s$ for $s \prec t$.

Let l_s be a loop in $U_s \in \mathcal{U}_{\mathrm{lh}(s)}$ with base point $f(a_s)$ for $s \in S$ with $\mathrm{lh}(s) = n$. Let $\alpha_{m+1} = \sum_{i=1}^{m} \sum_{s \in S_i} (i+1)![l_s]_h + \alpha_1$ in $H_1(X)$ for $m \geq 1$. Our purpose is to define a path g along f so that $g \cdot f^-$ is a loop and

$$(m+1)! \mid [g \cdot f^-]_h + \alpha_1 - \alpha_m \quad \text{for each } m \in \mathbb{N}.$$

For $t \in T_m$, define $g{\restriction}[b_t, b_t - \varepsilon_m] \equiv l_{s_t}$ and for $t \in T$ with $\mathrm{lh}(t) = m$ and $0 \leq (c_t)_m < m$, define $g{\restriction}[b_t + \varepsilon_m, b_t + 2\varepsilon_m] \equiv (f{\restriction}[a_{s_t}, a_{s_t^+}])^-$. If we define these for $t \in T$ with $\mathrm{lh}(t) \leq m$, the parts in $[0,1]$ where g is not defined are $\bigcup_{t \in T_m} (b_t, b_t + \varepsilon_m) \cup \{1\}$. For t satisfying $t_i = (i+1)(k_i - 1) + i$ (for $1 \leq i \leq m = \mathrm{lh}(t)$), we have $b_t + \varepsilon_m = 1$. If $g(x)$ is defined for $x \in (b_t, b_t + \varepsilon_m)$, then $g(x) \in U_{s_t}$. Hence g uniquely extends to a continuous map on $[0,1]$, which we also denote by g. Now g is a path from $f(0)$ to $f(1)$, and hence gf^- is a loop. We shall show that

$$[gf^-]_h - \sum_{i=1}^{m-1} \sum_{s \in S_i} (i+1)![l_s]_h$$

is divisible by $(m+1)!$.

For a fixed $1 \leq m < \omega$, we cut g into finitely many pieces and consider an element of the chain group:

$$\sum_{i=1}^{m-1} \sum_{t \in T_i} g{\restriction}[b_t - \varepsilon_i, b_t] + \sum_{i=1}^{m-1} \sum_{t \in T_i,\, 0 \leq (c_t)_i < i} g{\restriction}[b_t + \varepsilon_i, b_t + 2\varepsilon_i] + \sum_{t \in T_m} g{\restriction}[b_t, b_t + \varepsilon_m].$$

We see that $g{\restriction}[b_t - \varepsilon_i, b_t] \equiv l_{s_t}$ is a loop if $\mathrm{lh}(t) = i$, and $g{\restriction}[b_t, b_t + 2\varepsilon_i]$ is also a loop if $\mathrm{lh}(t) = i$ and $0 \leq (c_t)_i < i$.

For $s \in S_m$, let $T_{m,s} = \{t \in T_m : s_t = s\}$. For $t \in T_m$, define t^* so that $t = t^* * (t_{\mathrm{lh}(t^*)+1}, \ldots, t_m)$, $(c_t)_{\mathrm{lh}(t^*)} < \mathrm{lh}(t^*)$, and $(c_t)_i = i$ for $\mathrm{lh}(t^*) < i \leq m$. We remark that $t^* = t$ if and only if $(c_t)_m < m$, and $t^* = (\ )$ if and only if $(c_t)_i = i$ for $1 \leq i \leq m$.

Since $g{\restriction}[b_t, b_t + \varepsilon_m]$ is determined only by s_t, if $s_t = s_{t'}$ then $g{\restriction}[b_t, b_t + \varepsilon_m] \equiv g{\restriction}[b_{t'}, b_{t'} + \varepsilon_m]$ for $t, t' \in T_m$.

If $t^* = t'^*$ for distinct $t, t' \in T_m$, then $s_t \neq s_{t'}$. Hence the correspondence $t \to s_t$ on $\{t \in T_m : t^* = u\}$ is one-to-one for $u \in \bigcup_{i=1}^{m} T_i$ with $u(\mathrm{lh}(u)) < \mathrm{lh}(u)$ or for $u = (\ )$. In addition, for $u \in \bigcup_{i=1}^{m} T_i$ with $u(\mathrm{lh}(u)) < \mathrm{lh}(u)$ we have $g{\restriction}[b_u + \varepsilon_{\mathrm{lh}(u)}, b_u + 2\varepsilon_{\mathrm{lh}(u)}] \equiv (f{\restriction}[a_{s_u}, a_{s_u}^+])^-$, and for $t \in T_m$ with $t^* = (\ )$

we have a corresponding subpath in f^- with which $g\restriction[b_t, b_t+\varepsilon_m]$ forms a loop.

Let $C_m = \{t \in T_m : (c_t)_i = i \text{ for } 1 \le i \le m\}$. Since $|\{t \in T_m : s_t = s\}| = (m+1)!$ for $s \in S_m$, we have

$$[gf^-]_h = \sum_{i=1}^{m-1} \sum_{s\in S_i} (i+1)![l_s]_h + \sum_{s\in S_m} (m+1)!\beta_s,$$

where $\beta_s = [g\restriction[b_t, b_t+\varepsilon_m](f\restriction[a_s, a_s^+])^-]_h$ for $t \in C_m$ with $s_t = s$.

Hence, $[gf^-]_h + \alpha_1 - \alpha_m = \sum_{s\in S_m}(m+1)!\beta_s$ and $[gf^-]_h + \alpha_1$ is as desired.

Lemma 4.1. *Let X be a one-dimensional Peano continuum. Then* $\mathrm{Ker}(\sigma)$ *is complete mod-U.*

Proof. Let $\alpha_m \in \mathrm{Ker}(\sigma)$ and $(m+1)! \mid \alpha_{m+1} - \alpha_m$ in $\mathrm{Ker}(\sigma)$ for $1 \le m < \omega$. Then there exists $\gamma_m \in \mathrm{Ker}(\sigma)$ such that $(m+1)!\gamma_m = \alpha_{m+1} - \alpha_m$.

Let $f : [0,1] \to X$ be a path such that $\mathrm{Im}(f) = X$ and let $\mathcal{U}_m$ be finite open covers of X such that each $U \in \mathcal{U}_m$ is path-connected, the diameter of each $U \in \mathcal{U}_m$ is less than $1/m$, and $\mathcal{U}_{m+1}$ is a refinement of $\mathcal{U}_m$. To use the preceding construction, we inductively choose k_m in the following way. First, k_m should be so large that for each $s \in S$ with $\mathrm{lh}(s) = m$ there exists $U \in \mathcal{U}_m$ with $f([a_s, a_{s^+}]) \subseteq U$. By Lemma 3.7, γ_m can be expressed as the sum of the homology classes of arbitrarily small loops. We want loops in some $U \in \mathcal{U}_m$, hence the number of loops might be large. Second, k_m should be so large that γ_m can be expressed by k_m loops each of which is in some $U \in \mathcal{U}_m$.

We choose k_m which satisfies the two conditions. Since each $U \in \mathcal{U}_m$ is path-connected, a sum of the homology classes of loops in U can be replaced by a homologous loop in U. Hence there are $U_s \in \mathcal{U}_{\mathrm{lh}(s)}$ and loops l_s in U_s with base point $f(a_s)$ such that

$$\gamma_m = \sum_{\mathrm{lh}(s)=m} [l_s]_h.$$

Then $\alpha_{m+1} = \sum_{i=1}^m \sum_{s\in S_i} (i+1)![l_s]_h + \alpha_1$ in $H_1(X)$ for $m \ge 1$. Now, the assumptions for the preceding construction are satisfied and we have the desired element $[gf^-]_h + \alpha_1$. ■

Lemma 4.2 ([11, Theorem 2.1]). *Let X be a one-dimensional normal space. Then $H_1(X)$ is torsionfree.*

Now, according to the facts in Section 2, Lemmas 4.1 and 4.2 imply

Lemma 4.3. *Let X be a one-dimensional Peano continuum. Then* $\mathrm{Ker}(\sigma)$ *is algebraically compact.*

Lemma 4.4 (folklore). *Let X be a one-dimensional Peano continuum. If X is semi-locally simply connected, then the Čech homology group $\check{H}_1(X)$ is isomorphic to a free abelian group of finite rank. Otherwise, $\check{H}_1(X)$ is isomorphic to $\mathbb{Z}^\omega$.*

Next we construct loops whose homotopy classes are in $\mathrm{Ker}(\sigma)$ and the homology classes which generate pure subgroups of $H_1(X)$ when X is not semi-locally simply connected. Suppose that X is not semi-locally simply connected at $x_0 \in X$.

The first lemma is well-known and it can be proved using arbitrarily small simple closed curves; we omit its proof.

Lemma 4.5. *Let X be a one-dimensional space Peano continuum which is not semi-locally simply connected at x_0. Then there exists a closed subspace Y such that (Y, x_0) is homotopy equivalent to the Hawaiian earring $(\mathbb{H}, o)$.*

Then we have a dendrite D in Y such that $Y \setminus D$ consists of countably many open arcs A_n which converge to x_0 by [7, Theorem 1.2 and its proof].

We construct certain reduced loops in Y. Let l_n be a reduced loop which starts from x_0, reaches one end of A_n in D, goes through A_n and goes back to x_0 in D. We call this direction of A_n to be plus and the reverse direction to be minus.

Let l_n^* be the reduced loop of $l_{2n}l_{2n+1}l_{2n}^- l_{2n+1}^-$, i.e l_n^* goes plus A_{2n}, plus A_{2n+1}, minus A_{2n} and minus A_{2n+1} when we disregard D. We call this last property $(*_n)$ for simplicity. Moreover, the reduced loops of $l_0^* \cdots l_m^*$ for $m \geq n$ also have property $(*_n)$. Let l^* be the reduced loop of the infinite concatenation $l_0^* \cdots l_n^* \cdots$. Then we see that, for each $\delta > 0$, $l^* \restriction [1-\delta, 1]$ has property $(*_n)$ for sufficiently large n, and for each n there exists $\delta > 0$ such that $l^* \restriction [0, 1-\delta]$ has property $(*_n)$. We remark that l^{*-} does not have property $(*_n)$.

For a non-degenerate path $f : [0,1] \to X$, a *tail* of f is a subpath $f \restriction [1-\delta, 1]$ for some $\delta > 0$. The following lemma is straightforward and we omit its proof.

Lemma 4.6. *Let $f_0 \cdots f_k$ be a reduced path. There exists a tail m_0 of l^* such that every subpath m in $f_0 \cdots f_k$ which is equivalent to m_0 or m_0^- is a subpath of some f_i.*

Lemma 4.7. *The homology class $[l^*]_h$ generates a pure subgroup of $H_1(X)$ which is isomorphic to $\mathbb{Z}$.*

Proof. Since $H_1(X)$ is torsionfree, it is sufficient to show that $[l^*]_h$ is not divisible by any $n \geq 2$. For contradiction, suppose that $[l^*]_h$ is divisible by some $n \geq 2$. Then we have a cyclically reduced loop l and a reduced path p

such that plp^- is reduced with base point x_0 and $l^*pl^np^-$ is a 0-form among paths in X. We consider several cases.

CASE 1: *p is degenerate and l^*l^n is reduced.* We have $l^*l^n \equiv f_1 \cdots f_k$ where $f_1, \ldots, f_k$ are paired forming a 0-form. By Lemma 4.6 we have a tail m_0 which has the property in the lemma for $l^*l \cdots l$ and $f_1 \cdots f_k$ under these presentations. Then the number of occurrences of m_0 in $f_1 \cdots f_k$ is the same as that of m_0^-. Let a^+ be the number of occurrences of m_0 in l and a^- be the number of occurrences of m_0 in l. Then $na^+ + 1 = na^-$ and hence $n(a^- - a^+) = 1$, which contradicts $n \geq 2$.

CASE 2: *p is non-degenerate and $l^*pl^np^-$ is reduced.* We choose m_0 similarly to Case 1 considering p and p^-. Since the number of occurrences of m_0 in p is the same as that of m_0^- in p^-, and that of m_0^- in p is the same as that of m_0 in p^-, we have a contradiction as in Case 1.

CASE 3: *p is degenerate and l^*l^n is not reduced.* Since there is a tail t of l^* such that t^- is a head of l, the reduced loop of l^*l^n is of the form $q_0q_2l^{n-1}$ where $q_0q_1 \equiv l^*$ and $q_1q_2 \equiv l$. Using the presentation $q_0q_2q_1 \cdots q_1q_2$ and the 0-form, we choose m_0. Let a^+ be the number of occurrences of m_0 in $l \equiv q_1q_2$ and a^- be the number of occurrences of m_0 in l as before. Since m_0^- occurs once in q_1 and m_0 does not, we have $n - 1 + n(a^+ - 1) = na^-$, and hence $n(a^+ - a^-) = 1$, which is a contradiction.

CASE 4: *p is non-degenerate and $l^*pl^np^-$ is not reduced.* For a sufficiently small tail m_0 of l^*, we have $q_0m_0 \equiv l^*$ and $m_0^-p_0 \equiv p$. Then in the reduction of $q_0p_0l^np_0^-m_0$ any tail of l or its inverse is canceled. Hence we have a contradiction as in Case 2. ∎

LEMMA 4.8. *Let X be a one-dimensional normal space. If Y is a path-connected subspace of X, then $H_1(Y)$ is a subgroup of $H_1(X)$.*

Proof. Since every element of $H_1(Y)$ is the homology class of a loop in Y, we let l be a reduced loop in Y. We only deal with the case that l is non-degenerate. Since the reduced loop of a loop is in the image of the original loop, the reducedness does not depend on whether we consider the loop in X or in Y. Suppose that the homotopy class of l belongs to the commutator subgroup of $\pi_1(X)$. Then l is equivalent to a 0-form where each path is in X, but Lemma 3.6 implies that each path is in Y. Therefore, $H_1(Y)$ is a subgroup of $H_1(X)$. ∎

Proof of Theorem 1.1. Let $h : H_1(X) \to \mathbb{Z}$ be a homomorphism. By Lemma 4.1 we have $h(\mathrm{Ker}(\sigma)) = \{0\}$, and consequently by Lemma 3.8 we have $\mathrm{Ker}(\sigma) = R_{\mathbb{Z}}(H_1(X))$. Therefore $H_1(X)/\mathrm{Ker}(\sigma)$ is a subgroup of the direct product of copies of $\mathbb{Z}$, which is obviously torsionfree. By Lemma 4.3 this implies that $\mathrm{Ker}(\sigma)$ is a direct summand. If X is semi-locally simply connected, then it is well-known that $H_1(X)$ is a free abelian group of finite rank. Otherwise, we have $\check{H}_1(X) \cong \mathbb{Z}^\omega$ and hence $H_1(X) \cong \mathrm{Ker}(\sigma) \oplus \mathbb{Z}^\omega$. Since there exists a subspace of X which is homotopy equivalent to the Hawaiian earring $\mathbb{H}$, the divisible part $D(H_1(X))$ contains $D(H_1(\mathbb{H})) \cong \bigoplus_{\mathbf{c}} \mathbb{Q}$ by Lemma 4.8. Since the cardinality of $H_1(X)$ is equal to or less than $\mathbf{c}$, we have $D(H_1(X)) \cong \bigoplus_{\mathbf{c}} \mathbb{Q}$. The remaining task is to determine the cardinality of the reduced algebraically compact group.

Since $\sigma([l^*]_h) = 0$ for l^* in Lemma 4.7, we see that $[l^*]_h$ generates a pure subgroup of $\mathrm{Ker}(\sigma)$. To show that $\mathrm{Ker}(\sigma)$ contains a pure subgroup isomorphic to a free abelian group of continuum rank, we modify the construction of l^* as in the proof of [11, Lemma 3.5]. There exists an almost disjoint family consisting of infinite sets of integers, where S and T are *almost disjoint* if $S \cap T$ is finite. Let l^*_S be the reduced loop of $l^*_{i_0} \cdots l^*_{i_n} \cdots$, where $i_0 < \cdots < i_n < \cdots$ is the enumeration of S in the order of the integers. Now it suffices to show that $l^*_{S_1}, \ldots, l^*_{S_n}$ are linearly independent for an almost disjoint family $S_1, \ldots, S_n$. We have a finite set F of integers such that $S_i \cap S_j \subseteq F$ for distinct i, j. For a set S of integers let $r_S : Y \to Y$ be a retraction such that $r_S(A_n) \subseteq D$ for $n \notin S$ and $r_S \restriction A_n$ is the identity for $n \in S$. Let $\lambda_1 [l^*_{S_1}]_h + \cdots + \lambda_n [l^*_{S_n}]_h = 0$. By Lemma 4.8, we may work in Y. Let $S = S_i \setminus F$. Since $(r_S)_*([l^*_{S_j}]_h)$ is trivial for $j \neq i$ but $S \neq \emptyset$ and $H_1(X)$ is torsionfree, $(r_S)_*([l^*_{S_i}]_h)$ is non-zero and hence $\lambda_i = 0$. ∎

Remark 4.9. Here we show that the compactness of a space is essential for the algebraic compactness of $\mathrm{Ker}(\sigma)$ in Lemma 4.3. Let X be a subspace of the plane obtained by attaching copies of $\mathbb{H}$ to the half line $\{0\} \times [0, \infty)$:

$$X = \{0\} \times [1, \infty) \cup \bigcup_{\substack{3 \le n < \omega \\ 1 \le m < \omega}} \{(x, y) : (x - 1/n)^2 + (y - m)^2 = 1/n^2\}.$$

Then X is a locally path-connected, path-connected, separable metric space. In the mth copy of the Hawaiian earring, we have a non-trivial element α_m in $\mathrm{Ker}(\sigma)$ such that $\langle [\alpha_m]_h \rangle$ is a pure subgroup of $H_1(X)$, where σ is the natural homomorphism to the Čech homology group. Then

$$(m+1)! \mid \sum_{i=1}^{m+1} i! [\alpha_i]_h - \sum_{i=1}^{m} i! [\alpha_i]_h.$$

Suppose that $\mathrm{Ker}(\sigma)$ is algebraically compact. Then there is a loop l such that $(m+1)! \mid [l]_h - \sum_{i=1}^m i! [\alpha_i]_h$ for each $1 \le m < \omega$. Since the image of l is compact, there exists m_0 such that

$$\mathrm{Im}(l) \subseteq \{0\} \times [1, m_0 - 1] \cup \bigcup_{\substack{3 \le n < \omega \\ 1 \le m \le m_0 - 1}} \{(x, y) : (x - 1/n)^2 + (y - m)^2 = 1/n^2\}.$$

Considering the retraction of X to

$$\bigcup_{3\le n<\omega}\{(x,y):(x-1/n)^2+(y-m_0)^2=1/n^2\},$$

we conclude that $(m_0+1)!\mid -m_0![\alpha_{m_0}]_h$. Since $H_1(X)$ is torsionfree, we have $m_0+1\mid[\alpha_{m_0}]_h$, which contradicts $\langle[\alpha_{m_0}]_h\rangle$ being a pure subgroup.

Though $\mathrm{Ker}(\sigma)$ may not be algebraically compact for a non-compact space X, we have the following.

THEOREM 4.10. *Let X be a one-dimensional locally path-connected metric space. Then* $\mathrm{Ker}(\sigma)=R_{\mathbb{Z}}(H_1(X))$.

Proof. By Lemma 3.8 it suffices to show that $\mathrm{Ker}(\sigma)\le R_{\mathbb{Z}}(H_1(X))$. Since each path-connected component is open by local path-connectivity, the Čech homology group is the direct product of the Čech homology groups of the path-connected components. Hence without loss of generality we may assume that X is path-connected. Let l be a loop with $[l]_h\in\mathrm{Ker}(\sigma)$ and $h: H_1(X)\to\mathbb{Z}$ be a homomorphism. We define a map $\varphi:\widehat{\mathbb{Z}}\to\mathrm{Ker}(\sigma)$ such that $h\circ\varphi$ becomes to be a homomorphism. For $u\in\widehat{\mathbb{Z}}$, i.e. $u=\sum_{i=1}^{\infty}m!a_m$ where $0\le a_m\le m$, we define a loop l_u as follows. We modify the construction in the proof of Lemma 4.1. Replace f by l, and for each a_m express $a_m[l]_h$ as the sum of the homology classes of loops each of which is in some $U\in\mathcal{U}_m$. Then there is a loop l_u such that

$$(m+1)!\,\Big|\,[l_u]_h-\sum_{i=1}^{m}i!a_i[l]_h.$$

Let $\varphi(u)=[l_u]_h$. For $u,v\in\widehat{\mathbb{Z}}$, let $u=\sum_{i=1}^{\infty}i!a_i$, $v=\sum_{i=1}^{\infty}i!b_i$ and $u+v=\sum_{i=1}^{\infty}i!c_i$ where $0\le a_i,b_i,c_i\le i$. Since

$$(m+1)!\,\Big|\,\sum_{i=1}^{m}i!c_i-\Big(\sum_{i=1}^{m}i!a_i+\sum_{i=1}^{m}i!b_i\Big),$$

we have

$$(m+1)!\mid h([l_{u+v}]_h)-(h([l_u]_h+h([l_v]_h))$$

for every m, and so $h\circ\varphi(u+v)=h\circ\varphi(u)+h\circ\varphi(v)$. Since $\mathbb{Z}$ is cotorsionfree, $h\circ\varphi$ is a trivial homomorphism, which implies $h([l]_h)=h\circ\varphi(1)=0$. ∎

REMARK 4.11. According to Theorem 1.1, $H_1(X)/R_{\mathbb{Z}}(H_1(X))$ is isomorphic to a free abelian group of finite rank or $\mathbb{Z}^{\omega}$. Even for one-dimensional locally path-connected separable metric spaces X, $H_1(X)/R_{\mathbb{Z}}(H_1(X))$ is complicated. For this we refer the reader to [13, Section 6], where we defined a factor $H_n^T(X)$ of the singular homology group $H_n(X)$ and in our case $H_1^T(X)\cong H_1(X)/R_{\mathbb{Z}}(H_1(X))$. The spaces defined there are not metrizable, but by a standard method we get metrizable spaces X with the same $H_1(X)$ and $H_1^T(X)$.

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