FUNDAMENTA
MATHEMATICAE
232 (2016)

On exposed points and extremal points of convex sets in $\mathbb{R}^n$ and Hilbert space

by

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Abstract. Let $\mathbb{V}$ be a Euclidean space or the Hilbert space ℓ^2, let $k \in \mathbb{N}$ with $k < \dim \mathbb{V}$, and let B be convex and closed in $\mathbb{V}$. Let $\mathcal{P}$ be a collection of linear k-subspaces of $\mathbb{V}$. A set $C \subset \mathbb{V}$ is called a $\mathcal{P}$-*imitation* of B if B and C have identical orthogonal projections along every $P \in \mathcal{P}$. An *extremal* point of B with respect to the projections under $\mathcal{P}$ is a point that all closed subsets of B that are $\mathcal{P}$-imitations of B have in common. A point x of B is called *exposed* by $\mathcal{P}$ if there is a $P \in \mathcal{P}$ such that $(x + P) \cap B = \{x\}$. In the present paper we show that all extremal points are limits of sequences of exposed points whenever $\mathcal{P}$ is open. In addition, we discuss the question whether the exposed points form a G_δ-set.

1. Introduction. Throughout this paper $\mathbb{V}$ stands for a separable real Hilbert space. Thus $\mathbb{V}$ is isomorphic to either an $\mathbb{R}^n$ or ℓ^2. Let B be convex and closed in $\mathbb{V}$, and let $\mathcal{G}_k(\mathbb{V})$ consist of all k-dimensional linear subspaces of $\mathbb{V}$ with the natural topology; see Definition 1. Let $\mathcal{P} \subset \mathcal{G}_k(\mathbb{V})$. We say that $x \in B$ *is exposed by* $\mathcal{P}$ if there is a $P \in \mathcal{P}$ such that $(x + P) \cap B = \{x\}$. We denote by $\mathcal{X}^k_{\mathrm{p}}(B, \mathcal{P})$ the set of all points of B exposed by $\mathcal{P}$. This definition generalizes the concept of an exposed point as defined in [6], that is, a point of $B \subset \mathbb{R}^n$ that is exposed by $\mathcal{G}_{n-1}(\mathbb{R}^n)$. We call $C \subset \mathbb{V}$ a $\mathcal{P}$-*imitation* of B if $C + P = B + P$ for every $P \in \mathcal{P}$, that is, B and C have identical projections along each element of $\mathcal{P}$. The set of *extremal* points of B with respect to $\mathcal{P}$ is denoted by $\mathcal{X}^k_{\mathrm{t}}(B, \mathcal{P})$ and is defined as the intersection of all closed subsets of B that are $\mathcal{P}$-imitations of B. Clearly, every exposed point is extremal as well. In [5, Theorem 14] we proved

Theorem 1. *For closed and convex sets $B \subset \ell^2$ with empty geometric interior B° we have $\mathcal{X}^k_{\mathrm{p}}(B, \mathcal{P}) = \mathcal{X}^k_{\mathrm{t}}(B, \mathcal{P}) = B$ for any $k \in \mathbb{N}$ and nonempty open $\mathcal{P} \subset \mathcal{G}_k(\ell^2)$.*

2010 *Mathematics Subject Classification*: Primary 52A07, 52A20.
Key words and phrases: exposed point, extremal point, separable Hilbert space, imitation.

DOI: 10.4064/fm232-2-2

Absent the condition on B° it is easy to see that exposed points and extremal points do not coincide in general; see [1, Example 3] for a simple example. The main purpose of this paper is to establish the following connection between exposed and extremal points in the general setting.

THEOREM 2. *Let $k \in \mathbb{N}$ with $k < \dim \mathbb{V}$, let $B \subset \mathbb{V}$ be closed and convex, and let $\mathcal{P} \subset \mathcal{G}_k(\mathbb{V})$ be open. Then $\mathcal{X}_{\mathrm{p}}^k(B, \mathcal{P})$ is dense in $\mathcal{X}_{\mathrm{t}}^k(B, \mathcal{P})$.*

We say that *generic* elements of a space have a certain property if the space has a dense G_δ-subset all of whose elements have that property.

THEOREM 3. *Let B be closed and convex in $\mathbb{R}^n$ and let $n \in \mathbb{N}$ with $n \geq 2$. Then $\mathcal{X}_{\mathrm{p}}^1(B, \mathcal{G}_1(\mathbb{R}^n))$ is a G_δ-set in $\mathcal{X}_{\mathrm{t}}^1(B, \mathcal{G}_1(\mathbb{R}^n))$. Consequently, in this case generic extremal points are exposed.*

Choquet, Corson, and Klee [6] investigated the space $\mathcal{X}_{\mathrm{p}}^{n-1}(B, \mathcal{G}_{n-1}(\mathbb{R}^n))$ and proved this theorem for the case $n = 2$. Surprisingly, Theorem 3 fails to hold for $\mathcal{X}_{\mathrm{p}}^1(B, \mathcal{P})$ when $\mathcal{P}$ is an open proper subset of $\mathcal{G}_1(\mathbb{R}^n)$ (see Example 2). Moreover, Corson [7] gives an example of a convex compactum $B \subset \mathbb{R}^3$ such that $\mathcal{X}_{\mathrm{p}}^2(B, \mathcal{G}_2(\mathbb{R}^3))$ is of the first category and hence does not contain a dense G_δ-subset of $\mathcal{X}_{\mathrm{t}}^2(B, \mathcal{G}_2(\mathbb{R}^3))$. This is generalized to higher dimensions in Example 3.

2. Definitions and preliminaries. The inner product in $\mathbb{V}$ is denoted by $x \cdot y$ and $\mathbf{0}$ stands for the zero vector. By a projection of a point onto a plane we always mean the orthogonal projection. If $\varepsilon > 0$ and $x \in \mathbb{V}$ then the open ball centred at x and with radius ε is denoted by $U_\varepsilon(x)$. The norm on $\mathbb{V}$ is given by $\|u\| = \sqrt{u \cdot u}$ and the metric d is given by $d(u, v) = \|v - u\|$. We also define, for $A \subset \mathbb{V}$,

$$A^\perp = \{v \in \mathbb{V} : v \cdot x = v \cdot y \text{ for all } x, y \in A\}$$

and

$$\operatorname{codim} A = \dim A^\perp \in \{0, 1, \ldots, \infty\}.$$

A *plane* in $\mathbb{V}$ is a closed affine subspace of $\mathbb{V}$; a *k-plane* in $\mathbb{V}$ is a k-dimensional affine subspace of $\mathbb{V}$. Let H be a *hyperplane* in $\mathbb{V}$, that is, a plane with codimension 1. The two components of $\mathbb{V} \setminus H$ are called the *sides* of H. We say that H *cuts* a subset A of $\mathbb{V}$ if A contains points on both sides of H. We say that a hyperplane H in $\mathbb{V}$ is *supporting to* A at x if $x \in H$ and H does not cut A. A closed subset L of $\mathbb{V}$ is called a *halfspace* of $\mathbb{V}$ if it is the union of a hyperplane and one of its sides.

Let A be a subset of $\mathbb{V}$. We denote by $\overline{A}$ the closure and by $\operatorname{int} A$ the interior of A in $\mathbb{V}$. Furthermore, $\langle A \rangle$ stands for the convex hull of A and $\operatorname{aff} A$ is the intersection of all planes that contain A. Note that $\operatorname{codim} A = \operatorname{codim}(\operatorname{aff} A)$. We write ∂A for the geometric boundary of A, that is, the boundary with respect to $\operatorname{aff} A$, and we let $A^\circ = A \setminus \partial A$ denote the geometric interior. Note that if A is convex and $A^\circ \neq \emptyset$ then A° is dense in A and $A^\circ \neq \emptyset$ if A is finite-dimensional.

Definition 1. Consider the closed unit ball $\mathbb{B} = \{v \in \mathbb{V} : \|v\| \leq 1\}$. Let $\mathcal{K}(\mathbb{B})$ stand for the hyperspace of all nonempty compact subsets of $\mathbb{B}$. Recall that the *Hausdorff metric* d_{H} on $\mathcal{K}(\mathbb{B})$ associated with d is defined as follows:

$$d_{\mathrm{H}}(A, B) = \sup\{d(x, A), d(y, B) : x \in B \text{ and } y \in A\}.$$

By [9, Theorem 1.11.3], $\mathcal{K}(\mathbb{B})$ is compact for $\mathbb{V} = \mathbb{R}^n$ and complete for $\mathbb{V} = \ell^2$.

We let $\mathcal{G}_m(\mathbb{V})$ stand for the collection of all m-dimensional linear subspaces of $\mathbb{V}$. We topologize $\mathcal{G}_m(\mathbb{V})$ by defining a metric ρ on $\mathcal{G}_m(\mathbb{V})$:

$$\rho(L_1, L_2) = d_{\mathrm{H}}(L_1 \cap \mathbb{B}, L_2 \cap \mathbb{B}).$$

It is readily seen that $\mathcal{G}_m(\mathbb{V})$ corresponds to a closed subset of $\mathcal{K}(\mathbb{B})$ and is therefore also compact for $\mathbb{V} = \mathbb{R}^n$ and complete for $\mathbb{V} = \ell^2$. When $\mathbb{V}$ is finite-dimensional, $\mathcal{G}_m(\mathbb{V})$ is known as a *Grassmann manifold*.

Remark 1. We also allow the degenerate cases $\mathcal{G}_0(\mathbb{V}) = \{\{\mathbf{0}\}\}$ and $\mathcal{G}_m(\mathbb{R}^m) = \{\mathbb{R}^m\}$. Note that $\mathcal{X}^0_{\mathrm{p}}(B, \mathcal{G}_0(\mathbb{V})) = \mathcal{X}^0_{\mathrm{t}}(B, \mathcal{G}_0(\mathbb{V})) = B$, and if B is not a singleton then

$$\mathcal{X}^m_{\mathrm{p}}(B, \mathcal{G}_m(\mathbb{R}^m)) = \mathcal{X}^m_{\mathrm{t}}(B, \mathcal{G}_m(\mathbb{R}^m)) = \emptyset.$$

Definition 2. Let B be a closed and convex set in $\mathbb{V}$. A subset F of B is called a *face* of B if there is a hyperplane H of $\operatorname{aff} B$ that does not cut B with the property $F = B \cap H$. Note that F is also closed and convex and that $\operatorname{codim} F > \operatorname{codim} B$. If F is a face of B we write $F \prec B$. We say that a subset F of B is a *derived face* of B if $F = B$ or there exists a sequence $F = F_1 \prec \cdots \prec F_m = B$ for some m.

Definition 3. Let $\mathcal{P}$ be a collection of linear subspaces of a vector space $\mathbb{V}$. We say that an affine subspace H of $\mathbb{V}$ is *consistent with* $\mathcal{P}$ if there is a $P \in \mathcal{P}$ such that $z + P \subset H$ for some $z \in H$. Let B be a convex and closed subset of $\mathbb{V}$. A subset F of B is called a $\mathcal{P}$*-face* of B if $F = B \cap H$ for some hyperplane H of $\mathbb{V}$ that does not cut B and that is consistent with $\mathcal{P}$. A *derived* $\mathcal{P}$*-face* is a derived face of a $\mathcal{P}$-face. If $k \in \mathbb{N}$ and $k < \dim \mathbb{V}$ then we set

$$\mathcal{F}_k(B, \mathcal{P}) = \{F : F \text{ is a derived } \mathcal{P}\text{-face of } B \text{ with} \operatorname{codim} F > k\},$$

and we let $\mathcal{E}^k(B, \mathcal{P})$ be the closure of $\bigcup \mathcal{F}_k(B, \mathcal{P})$.

To determine which points are extremal we will rely on the following result from [4, Theorems 15 and 16] and [5, Theorems 3 and 19].

THEOREM 4. *Let $k \in \mathbb{N}$, B be closed and convex in $\mathbb{V}$, and $\mathcal{P} \subset \mathcal{G}_k(\mathbb{V})$ be such that $\mathcal{P} \subset \operatorname{int} \overline{\mathcal{P}}$. If* $\operatorname{codim} B \neq k$ *then $\mathcal{E}^k(B,\mathcal{P}) = \mathcal{X}_{\mathrm{t}}^k(B,\mathcal{P})$. If* $\operatorname{codim} B = k$ *then $\mathcal{E}^k(B,\mathcal{P}) \subset \mathcal{X}_{\mathrm{t}}^k(B,\mathcal{P})$. If $B^\circ = \emptyset$ then $B = \mathcal{E}^k(B,\mathcal{P}) = \mathcal{X}_{\mathrm{t}}^k(B,\mathcal{P})$.*

REMARK 2. Let $\mathcal{P}$ be *somewhere dense*, that is, $\operatorname{int} \overline{\mathcal{P}} \neq \emptyset$, and let $\operatorname{codim} B \geq k$. If $P \in \operatorname{int} \overline{\mathcal{P}}$ then P can be approximated by a $P' \in \mathcal{P}$ such that $P' \cap \operatorname{aff} B$ is a singleton (cf. [3, Lemma 13]) and hence $B = \mathcal{X}_{\mathrm{p}}(B,\mathcal{P}) = \mathcal{X}_{\mathrm{t}}(B,\mathcal{P})$.

DEFINITION 4. Let $B \subset \mathbb{V}$ be closed and convex. Let $k \in \mathbb{N}$, $k < \dim \mathbb{V}$, and $\mathcal{P} \subset \mathcal{G}_k(\mathbb{V})$. For the proof of Theorem 2 we will work with the following subspace of $\mathcal{G}_k(\mathbb{V}) \times B$:

$$\mathcal{T}_k(B,\mathcal{P}) = \{(P,x) \in \mathcal{P} \times B : x \in F \text{ for some } F \in \mathcal{F}_k(B,\{P\})\}.$$

3. Some lemmas. In this section we prove the lemmas that we need to prove our main results.

LEMMA 5. *Let $k \in \mathbb{N}$ with $k < \dim \mathbb{V}$, and let B be closed and convex in $\mathbb{V}$. Let $(w_i)_{i\in\mathbb{N}}$ be a sequence of points in B that converges to w. Let $(P_i)_{i\in\mathbb{N}}$ be a sequence that converges to some P in $\mathcal{G}_k(\mathbb{V})$. If $(w+P) \cap B$ is bounded then for every neighbourhood U of $(w+P) \cap B$ there is a $j \in \mathbb{N}$ such that $(w_i + P_i) \cap B \subset U$ for every $i > j$.*

Proof. Let U be an open neighbourhood of $(w+P)\cap B$. We may assume that U is bounded. Striving for a contradiction, and without loss of generality, we may assume that $(w_i + P_i) \cap B \setminus U \neq \emptyset$ for every $i \in \mathbb{N}$. Choose $a_i \in (w_i+P_i)\cap B\setminus U$, and note that since P_i is connected we can pick a_i in the boundary of U. Thus $\{a_i : i \in \mathbb{N}\}$ is bounded and so is $A = \{a_i - w_i : i \in \mathbb{N}\}$ because $(w_i)_i$ converges. Let M be such that $A \subset \mathbb{B}_M = \{x \in \mathbb{V} : \|x\| \leq M\}$. Since $a_i - w_i \in P_i$ we may select, by the definition of ρ, a point y_i in $P \cap \mathbb{B}_M$ such that $d(a_i - w_i, y_i) \leq M\rho(P_i, P)$ and hence $\lim_{i\to\infty} d(a_i - w_i, y_i) = 0$. Since $P \cap \mathbb{B}_M$ is compact we may assume (by passing to a subsequence) that $\lim_{i\to\infty} y_i = y \in P \cap \mathbb{B}_M$. Consequently, we also have $\lim_{i\to\infty}(a_i - w_i) = y$ and $\lim_{i\to\infty} a_i = w + y$. In conclusion, $w + y \in B \setminus U$, which contradicts the assumption that $(w+P) \cap B \subset U$. ∎

LEMMA 6. *Let $k \in \mathbb{N}$ with $k < \dim \mathbb{V}$, $\varepsilon > 0$, and let B be closed and convex in $\mathbb{V}$. Then the set $S_\varepsilon = \{(P,x) \in \mathcal{G}_k(\mathbb{V}) \times B : \operatorname{diam}((x+P)\cap B) \geq \varepsilon\}$ is closed.*

Proof. Let $(P,x) \in \mathcal{G}_k(\mathbb{V}) \times B$ be the limit of a sequence $(P_i,x_i)_i$ in S_ε. If $\operatorname{diam}((x+P)\cap B) < \varepsilon$ then choose a neighbourhood U of $(x+P)\cap B$ with $\operatorname{diam} U < \varepsilon$. Apply Lemma 5 to find a $(P_i,x_i) \in S_\varepsilon$ such that $(x_i + P_i) \cap B$ is contained in U, which implies that $\operatorname{diam} U \geq \varepsilon$. Thus $(P,x) \in S_\varepsilon$. ∎

LEMMA 7. *Let $k \in \mathbb{N}$ with $k < \dim \mathbb{V}$, let B be closed and convex in $\mathbb{V}$, and let $P \in \mathcal{G}_k(\mathbb{V})$. Let w be a point in a $\{P\}$-face of B such that $(w+P) \cap B$ is line-free. Then for every $\varepsilon > 0$ there is a $w^* \in B$ such that (P, w^*) is in $\mathcal{T}_k(B, \{P\})$ and $U_\varepsilon(w^*) \cap (w+P) \cap B \neq \emptyset$.*

Proof. Let F^* be the $\{P\}$-face that contains w, so there is a supporting hyperplane H to B at w such that $w + P \subset H$ and $F^* = B \cap H$. Consider

$$\mathcal{F} = \{F : F \text{ is a derived face of } F^* \text{ such that } F \cap (w+P) \neq \emptyset\}.$$

The collection $\mathcal{F}$ is nonempty because it contains F^*.

Let us first assume that there is an $F_1 \in \mathcal{F}$ such that $F_1^\circ = \emptyset$. Consider $A = F_1 \cap (w+P)$. According to [2, Lemma 5] the set $\bigcup\{F : F \text{ is a derived face of } F_1 \text{ and } \operatorname{codim} F > k\}$ is dense in F_1. So, we can choose a derived face $\hat{F}$ of F_1 and a $w^* \in \hat{F}$ such that $\operatorname{codim} \hat{F} > k$ and $U_\varepsilon(w^*) \cap A \neq \emptyset$. Consequently, $\hat{F} \in \mathcal{F}_k(B, \{P\})$ and $(P, w^*) \in \mathcal{T}_k(B, \{P\})$, as required.

Next, we may assume that $F^\circ \neq \emptyset$ for every $F \in \mathcal{F}$. Striving for a contradiction, assume that $\operatorname{codim} F \leq k$ for every $F \in \mathcal{F}$. Then we may select an $F \in \mathcal{F}$ with maximal codimension. Note that H contains both $w + P$ and F and $\operatorname{codim} H = 1$. Thus $\operatorname{codim}(\operatorname{aff} F)$ in H is at most $k - 1$. Since $\dim P = k$ we have $\dim((w+P) \cap \operatorname{aff} F) \geq 1$. Let ℓ be a line contained in $(w+P) \cap \operatorname{aff} F$ such that $\ell \cap F \neq \emptyset$. Since $(w+P) \cap B$ is line-free, there is a $y \in \ell \cap \partial F$. Since $F^\circ \neq \emptyset$, the point y is contained in some face G of F by Hahn–Banach. Now we see that $\operatorname{codim} G > \operatorname{codim} F$ and $G \in \mathcal{F}$, contrary to the choice of F. Thus, we can conclude that there is an $\hat{F} \in \mathcal{F}$ with $\operatorname{codim} \hat{F} > k$. In this case, for w^* we simply take any point in $\hat{F} \cap (w+P)$. ∎

LEMMA 8. *Let $k \in \mathbb{N}$ with $k < \dim \mathbb{V}$ and let B be closed and convex in $\mathbb{V}$. Let $\varepsilon > 0$, let $\mathcal{P}$ be a subset of $\mathcal{G}_k(\mathbb{V})$ such that $\mathcal{P} \subset \operatorname{int} \overline{\mathcal{P}}$, and let $(L, w) \in \mathcal{T}_k(B, \operatorname{int} \overline{\mathcal{P}})$. Then there is an $(L^*, w^*) \in \mathcal{T}_k(B, \mathcal{P})$ such that $\|w - w^*\| < \varepsilon$, $\rho(L, L^*) < \varepsilon$, and $\operatorname{diam}((w^* + L^*) \cap B) < \varepsilon$.*

Proof. Note that $\mathcal{P} \neq \emptyset$ since $L \in \operatorname{int} \overline{\mathcal{P}}$. Define the open nonempty set

$$\mathcal{U} = \{L' \in \operatorname{int} \overline{\mathcal{P}} : \rho(L', L) < \varepsilon/2\}$$

of $\mathcal{G}_k(\mathbb{V})$. We consider three cases.

CASE I: $\operatorname{codim} B > k$. According to Remark 2 we can approximate L with an $L^* \in \mathcal{P}$ such that $L^* \cap \operatorname{aff} B$ is a singleton and hence $(w + L^*) \cap B = \{w\}$. Note that $\operatorname{codim}(L^* + \operatorname{aff} B) \geq \operatorname{codim} B - \dim L^* \geq 1$, thus $B + L^*$ is contained in some hyperplane of $\mathbb{V}$. Consequently, $B \in \mathcal{F}_k(B, \{L^*\})$ and $(L^*, w) \in \mathcal{T}_k(B, \mathcal{P})$.

CASE II: $B^\circ = \emptyset$. By Theorem 1 we find a $P \in \mathcal{U}$ such that $(w+P) \cap B = \{w\}$. For $i \in \mathbb{N}$ define the nonempty set

$$\mathcal{P}_i = \{P' \in \mathcal{P} : \rho(P', P) < 1/i\}$$

and note that $\mathcal{P}_i \subset \operatorname{int} \overline{\mathcal{P}}_i$. By Theorem 4 we have $B = \mathcal{E}^k(B, \mathcal{P}_i)$ and we can choose a $w_i \in B$ such that $\|w_i - w\| < 1/i$ and $w_i \in F$ for some $F \in \mathcal{F}_k(B, \mathcal{P}_i)$. So there is a $P_i \in \mathcal{P}_i$ with $F \in \mathcal{F}_k(B, \{P_i\})$ and hence $(P_i, w_i) \in \mathcal{T}_k(B, \mathcal{P})$. Let O be a neighbourhood of w such that $\operatorname{diam} O < \varepsilon$. By Lemma 5 there exists an $i \in \mathbb{N}$ such that $\rho(P_i, P) < \varepsilon/2$, $\|w_i - w\| < \varepsilon$, and $(w_i + P_i) \cap B \subset O$. Taking $w^* = w_i$ and $L^* = P_i$ we see that the proof for this case is complete.

Case III: $\operatorname{codim} B \leq k$ and $B^\circ \neq \emptyset$. Let $F \in \mathcal{F}_k(B, \{L\})$ be such that $w \in F$. Since $\operatorname{codim} F > k$ we can find by [2, Remark 1] a sequence of affine spaces $H_{k+1} \subset H_k \subset \cdots \subset H_0 = \mathbb{V}$ such that $w + L \subset H_1$, $F \subset H_{k+1}$, and H_i is a hyperplane in H_{i-1} that does not cut $H_{i-1} \cap B$ for $i \in \{1, \ldots, k+1\}$. Note that $\operatorname{codim} H_i = i$ for each i and that $B \not\subset H_{k+1}$ because $\operatorname{codim} B \leq k$. Let i be such that $B \not\subset H_i$ but $B \subset H_{i-1}$. Choose a coordinate system such that $\mathbf{0} \in B \setminus H_i$ and $\|w\| < \varepsilon/3$. Set

$$U = U_{\varepsilon/3}(w) \cap H_i, \qquad \mathcal{Z} = (0,1]U, \qquad C = B \setminus \mathcal{Z}.$$

Since $\mathcal{Z}$ is open in the halfspace of H_{i-1} that has H_i as its boundary and that contains B, we see that C is closed. Also $\operatorname{diam} \mathcal{Z} < \varepsilon$ and C is a cone with vertex $\mathbf{0}$, that is, $[0,1]C = C$. Since $w \in F \in \mathcal{F}_k(B, \{L\})$ and $L \in \mathcal{U}$ we find that $w \in \mathcal{E}^k(B, \mathcal{U})$. Hence $w \in \mathcal{E}^k(B, \mathcal{U}) \setminus C \subset \mathcal{X}_{\mathrm{t}}(B, \mathcal{U}) \setminus C$ by Theorem 4. Thus C is not a $\mathcal{U}$-imitation of B, and there are an $\hat{L} \in \mathcal{U}$ and a $\hat{w} \in B \setminus C$ such that $(\hat{w} + \hat{L}) \cap C = \emptyset$. This means that $(\hat{w} + \hat{L}) \cap B$ is a subset of $\mathcal{Z}$ and thus $\operatorname{diam}((\hat{w} + \hat{L}) \cap B) < \varepsilon$. Let $\lim_{n\to\infty} \hat{L}_n = \hat{L}$ with $\hat{L}_n \in \mathcal{P}$ for every n. By Lemma 5 we can choose an $i \in \mathbb{N}$ such that $\rho(\hat{L}, \hat{L}_i) < \varepsilon/2$ and $(\hat{w} + \hat{L}_i) \cap B \subset \mathbb{V} \setminus C$, thus $(\hat{w} + \hat{L}_i) \cap C = \emptyset$. Set $L^* = \hat{L}_i$ and observe that $\rho(L, L^*) < \varepsilon$.

Claim 1. *$(t\hat{w} + L^*) \cap C = \emptyset$ for every $t \geq 1$.*

Proof. Suppose that there are $t \in \mathbb{R}$, $t \geq 1$, and $v \in L^*$ such that $t\hat{w} + v \in C$. Consider the point $z = \frac{1}{t}(t\hat{w} + v) = \hat{w} + \frac{1}{t}v$. Observe that $z \in (\hat{w} + L^*) \cap C$, a contradiction. ∎

Note that $\hat{w} \notin L^*$ because otherwise we would have had $\mathbf{0} \in (\hat{w} + L^*) \cap C$. Thus $\mathbb{R}\hat{w} \cap L^* = \{\mathbf{0}\}$ and the natural map from $\mathbb{R} \times L^*$ to $\mathbb{R}\hat{w} + L^*$ is a homeomorphism. By Claim 1 we see that $([1,\infty)\hat{w} + L^*) \cap B$ is a subset of the bounded set $\mathcal{Z}$ and is therefore compact. We may now define

$$s = \max\{t \geq 1 : (t\hat{w} + L^*) \cap B \neq \emptyset\}.$$

Choose a $v \in (s\hat{w} + L^*) \cap B$ and note that $s\hat{w} + L^* = v + L^*$ and $(v + L^*) \cap C = \emptyset$ by Claim 1. Also, $(v + L^*) \cap B$ is compact because the set is contained in $\mathcal{Z}$. Observe that $(v + L^*) \cap B^\circ = \emptyset$, otherwise s would not be maximal. By Hahn–Banach and the assumption $B^\circ \neq \emptyset$ there exists a hyperplane H in $\mathbb{V}$ such that $(v + L^*) \subset H$ and H does not cut B. Note that v is in the $\{L^*\}$-face

$B\cap H$. By Lemma 7, we construct a sequence $(w_n^*)_n$ of points in B such that $(L^*,w_n^*)\in\mathcal{T}_k(B,\mathcal{P})$ and $U_{1/n}(w_n^*)\cap(v+L^*)\cap B\neq\emptyset$. Since $(v+L^*)\cap B$ is compact, we may assume that $(w_n^*)_n$ converges to a point in $(v+L^*)\cap B$. Now, we apply Lemma 5 to find some $j\in\mathbb{N}$ such that $(w_j^*+L^*)\cap C=\emptyset$ and hence $(w_j^*+L^*)\cap B\subset\mathcal{Z}$ and $\operatorname{diam}((w^*+L^*)\cap B)<\varepsilon$. We observe that w and w_j^* are both in $\mathcal{Z}$ and thus $\|w_j^*-w\|<\varepsilon$. Taking w_j^* for w^* we conclude that (L^*,w^*) is as required. ∎

LEMMA 9. *Let $k\in\mathbb{N}$ with $k<\dim\mathbb{V}$, let B be closed and convex in $\mathbb{V}$, and let $\mathcal{P}$ be a subset of $\mathcal{G}_k(\mathbb{V})$ such that $\mathcal{P}\subset\operatorname{int}\overline{\mathcal{P}}$. Then $\mathcal{E}^k(B,\mathcal{P})=\mathcal{E}^k(B,\operatorname{int}\overline{\mathcal{P}})$ and $\mathcal{X}_{\mathrm{t}}^k(B,\mathcal{P})=\mathcal{X}_{\mathrm{t}}^k(B,\operatorname{int}\overline{\mathcal{P}})$.*

Proof. By Theorem 4 and Remark 2, $\mathcal{X}_{\mathrm{t}}$ follows from $\mathcal{E}$. Clearly, it suffices to show that $\mathcal{E}^k(B,\operatorname{int}\overline{\mathcal{P}})\subset\mathcal{E}^k(B,\mathcal{P})$, which means that $\bigcup\mathcal{F}_k(B,\operatorname{int}\overline{\mathcal{P}})\subset\mathcal{E}^k(B,\mathcal{P})$. So let $w\in F\in\mathcal{F}_k(B,\operatorname{int}\overline{\mathcal{P}})$ and $\varepsilon>0$. Then there is an L such that $(L,w)\in\mathcal{T}_k(B,\operatorname{int}\overline{\mathcal{P}})$. By Lemma 8 there is an $(L^*,w^*)\in\mathcal{T}_k(B,\mathcal{P})$ such that $\|w-w^*\|<\varepsilon$ and hence $w^*\in\mathcal{E}^k(B,\mathcal{P})$. Thus $w\in\mathcal{E}^k(B,\mathcal{P})$ because that set is closed. ∎

4. Proofs and examples. Theorem 2 follows immediately from the following stronger theorem.

THEOREM 10. *Let $k\in\mathbb{N}$ with $k<\dim\mathbb{V}$, let $B\subset\mathbb{V}$ be closed and convex, and let $\mathcal{P}$ be a G_δ-subset of $\mathcal{G}_k(\mathbb{V})$ such that $\mathcal{P}\subset\operatorname{int}\overline{\mathcal{P}}$. Then $\mathcal{X}_{\mathrm{p}}^k(B,\mathcal{P})$ is dense in $\mathcal{X}_{\mathrm{t}}^k(B,\mathcal{P})$.*

Proof. First of all, we may assume that $\mathcal{P}\neq\emptyset$ since otherwise the theorem is trivial. Now, if $\operatorname{codim}B\geq k$ then we are done by Remark 2.

Thus we may assume that $\operatorname{codim}B<k$. By Theorem 4 we know that $\mathcal{E}^k(B,\mathcal{P})=\mathcal{X}_{\mathrm{t}}^k(B,\mathcal{P})$. By the definition of $\mathcal{E}^k(B,\mathcal{P})$ it suffices to show that $\bigcup\mathcal{F}_k(B,\mathcal{P})\subset\overline{\mathcal{X}_{\mathrm{p}}^k(B,\mathcal{P})}$. Let M denote the closure of $\mathcal{T}_k(B,\mathcal{P})$ in $\mathcal{P}\times B$ and notice that M is topologically complete because B and $\mathcal{G}_k(\mathbb{V})$ are complete and $\mathcal{P}$ is topologically complete. For $n\in\mathbb{N}$, define

$$S_n=\{(P,x)\in M:\operatorname{diam}((x+P)\cap B)\geq 1/n\},$$

and note that this set is closed by Lemma 6. According to Lemma 8 every element of $\mathcal{T}_k(B,\mathcal{P})$ can be approximated by an element of $\mathcal{T}_k(B,\mathcal{P})\setminus S_n$, thus S_n is nowhere dense in M. By the Baire Category Theorem we deduce that $M\setminus\bigcup_{n=1}^\infty S_n$ is dense in M. Let $F\in\mathcal{F}_k(B,\mathcal{P})$, let $x\in F$, and let $\varepsilon>0$. Then there is a $P\in\mathcal{P}$ such that F is a derived $\{P\}$-face of B. Thus $(P,x)\in\mathcal{T}_k(B,\mathcal{P})$ and there is a $(P',x')\in M\setminus\bigcup_{n=1}^\infty S_n$ such that $\|x'-x\|<\varepsilon$ and $P'\in\mathcal{P}$ of course. Note that $\operatorname{diam}((x'+P')\cap B)<1/n$ for all n, and hence $(x'+P')\cap B=\{x'\}$ and $x'\in\mathcal{X}_{\mathrm{p}}^k(B,\mathcal{P})$. Consequently, $\bigcup\mathcal{F}_k(B,\mathcal{P})\subset\overline{\mathcal{X}_{\mathrm{p}}^k(B,\mathcal{P})}$ and the theorem is proved. ∎

With Lemma 9 we have

Corollary 11. *Let $k \in \mathbb{N}$ with $k < \dim \mathbb{V}$, let $B \subset \mathbb{V}$ be closed and convex, and let $\mathcal{P} \subset \mathcal{G}_k(\mathbb{V})$ be such that $\mathcal{P} \subset \operatorname{int} \overline{\mathcal{P}}$. Then*

$$\overline{\mathcal{X}_{\mathrm{p}}^k(B, \operatorname{int} \overline{\mathcal{P}})} = \mathcal{X}_{\mathrm{t}}^k(B, \mathcal{P}).$$

A natural question is whether we can replace the left hand side of the equation in Corollary 11 by $\overline{\mathcal{X}_{\mathrm{p}}^k(B, \mathcal{P})}$, that is: Is Theorem 10 valid without the G_δ-condition on $\mathcal{P}$ as in Theorem 4? The following example shows that the answer is no.

Example 1. We construct a convex compactum $B \subset \mathbb{R}^2$ and a $\mathcal{P} \subset \mathcal{G}_1(\mathbb{R}^2)$ with $\mathcal{P} \subset \operatorname{int} \overline{\mathcal{P}}$ such that $\mathcal{X}_{\mathrm{p}}^1(B, \mathcal{P})$ is not dense in $\mathcal{X}_{\mathrm{t}}^1(B, \mathcal{P})$. This example can easily be generalized to higher dimensions using the method of Example 3 below.

Let C be a Cantor set in $\mathbb{I} = [0, 1]$ such that $0, 1 \in C$ and every nonempty open subset has positive Lebesgue measure λ. Let (a_n, b_n), $n \in \mathbb{N}$, list the gaps of C, and set $U = \bigcup_{n=1}^{\infty} [a_n, b_n]$. Let χ be the characteristic function on C and define $f(x) = \int_0^x \chi(t)\, dt$ for $x \in \mathbb{I}$. Note that f is a nondecreasing continuous function from $\mathbb{I}$ onto $[0, \lambda(C)]$. Moreover, f is constant on the intervals $[a_n, b_n]$ and we set $\{m_n\} = f([a_n, b_n])$. If $s < x < t$ in $\mathbb{I}$ and $x \in C \setminus U$ then $\lambda(C \cap (s, x)) > 0$ and $\lambda(C \cap (x, t)) > 0$, thus $f(s) < f(x) < f(t)$. Since the union U is dense in $\mathbb{I}$, we see that $M = \{m_n : n \in \mathbb{N}\}$ is dense in $[0, \lambda(C)]$. Let $\mathcal{P}$ consist of the lines in $\mathcal{G}_1(\mathbb{R}^2)$ that have a slope in M and note that $\mathcal{P} \subset \operatorname{int} \overline{\mathcal{P}}$. Define $F(x) = \int_0^x f(t)\, dt$ for $x \in \mathbb{I}$ and note that the graph G of F is concave up, so the convex hull B of that graph is a compactum in the plane that is bounded below by G and above by the line segment that connects the origin $(0, F(0))$ with the point $(1, F(1))$. Note that the part of G above $[a_n, b_n]$ is a straight line segment L_n.

Let $(x, F(x))$ be a point of G with $x \in (0, 1)$ and let ℓ be a supporting hyperplane (a line in this case) to B at that point. Since F is differentiable, ℓ must be the tangent line to G and the slope of ℓ is $f(x)$. If $x \in \mathbb{I} \setminus U$ then $f(x) \notin M$ because f is strictly increasing at x. Thus $(x, F(x)) \notin \mathcal{X}_{\mathrm{p}}^1(B, \mathcal{P})$. If $x \in U$ then $\ell \cap B = L_n$ for some n, so x is not exposed by any line. But the endpoints $(a_n, F(a_n))$ and $(b_n, F(b_n))$ of L_n are in $\mathcal{E}^1(B, \mathcal{P}) = \mathcal{X}_{\mathrm{t}}^1(B, \mathcal{P})$. Clearly, the closure of $\mathcal{X}_{\mathrm{p}}^1(B, \mathcal{P})$ does not equal $\mathcal{X}_{\mathrm{t}}^1(B, \mathcal{P})$.

We have the following improvement over Theorem 1.

Theorem 12. *For closed and convex sets $B \subset \ell^2$ with empty geometric interior B° we have $\mathcal{X}_{\mathrm{p}}^k(B, \mathcal{P}) = \mathcal{X}_{\mathrm{t}}^k(B, \mathcal{P}) = B$ for any $k \in \mathbb{N}$ and somewhere dense G_δ-set $\mathcal{P} \subset \mathcal{G}_k(\ell^2)$.*

We showed in [5, Example 3] that one cannot do without the G_δ-condition on $\mathcal{P}$ in this theorem.

Proof of Theorem 12. Obviously it suffices to prove that $B \subset \mathcal{X}_{\mathrm{p}}^k(B,\mathcal{P})$. Define $\mathcal{P}' = \mathcal{P} \cap \operatorname{int}\overline{\mathcal{P}}$ and note that $\mathcal{P}'$ is a nonempty G_δ-set with the property $\mathcal{P}' \subset \operatorname{int}\overline{\mathcal{P}'} = \mathcal{O}$. Let $w \in B$ and define, for $n \in \mathbb{N}$,

$$\mathcal{O}_n = \{P \in \mathcal{O} : \operatorname{diam}((w+P)\cap B) < 1/n\}.$$

By Lemma 6 every $\mathcal{O}_n$ is open. By Theorem 1 every nonempty open subset of $\mathcal{O}$ contains a P with $(w+P)\cap B = \{w\}$, thus $\bigcap_{n=1}^\infty \mathcal{O}_n$ is dense in $\mathcal{O}$. Since $\mathcal{O}$ is topologically complete and $\mathcal{P}'$ is also a dense G_δ-set in $\mathcal{O}$, we deduce according to Baire that there is an element P in $\mathcal{P}' \cap \bigcap_{n=1}^\infty \mathcal{O}_n$. Consequently, $(w+P)\cap B = \{w\}$ and $w \in \mathcal{X}_{\mathrm{p}}^k(B,\mathcal{P}') \subset \mathcal{X}_{\mathrm{p}}^k(B,\mathcal{P})$. ∎

Proof of Theorem 3. By Remark 2 we may assume that $\operatorname{codim} B = 0$, that is, $\dim B = n$. If $F_1 \prec F_2$ in $\mathbb{R}^n$ then we say that F_1 is a *facet* of F_2 if $\dim F_1 = \dim F_2 - 1$. Note that a facet F_1 has a nonempty interior in the $(\dim F_2 - 1)$-manifold ∂F_2. Also these interiors are disjoint for different facets of the same closed convex set. Consequently, by separability a closed convex set can have only countably many facets. A sequence of derived faces $F_m \prec F_{m-1} \prec \cdots \prec F_1 = B$ of B is called *regular* if every F_i is a facet of F_{i-1}. Also, we call every derived face of B for which a regular sequence exists a *regular* derived face. Note that B has countably many regular derived faces and one of them is B itself.

Claim 2. *If $x \in B\setminus\mathcal{X}_{\mathrm{p}}^1(B,\mathcal{G}_1(\mathbb{R}^n))$ then $x \in F^\circ$ for some regular derived face F of B.*

Proof. Let $\mathcal{F}$ be the set of all regular derived faces F of B such that $x \in F$. Since $\mathcal{F} \neq \emptyset$, we may select an $F \in \mathcal{F}$ with a minimal dimension. We show that $x \in F^\circ$. Indeed, suppose that $x \notin F^\circ$. Let H be a supporting hyperplane at x to F in $\operatorname{aff} F$. Then $\hat{F} = H \cap F$ is a face of F. Since F is a regular derived face with a minimal dimension, $\hat{F}$ cannot be a facet of F. Thus $\dim \hat{F} \le \dim F - 2$. Therefore, the codimension of $\hat{F}$ in H is at least 1. So we can select a line $\ell \in \mathcal{G}_1(\mathbb{R}^n)$ such that $x+\ell \subset H$ and ℓ is perpendicular to $\operatorname{aff}\hat{F}$. Then $(x+\ell)\cap B = \{x\}$ and x is exposed, a contradiction. We may conclude that $x \in F^\circ$. ∎

Claim 3. *Let F be a derived face of B. If there is an exposed point of B in F° then $F \subset \mathcal{X}_{\mathrm{p}}^1(B,\mathcal{G}_1(\mathbb{R}^n))$.*

Proof. Take a coordinate system such that $\mathbf{0} \in F^\circ$ and $\mathbf{0}$ is exposed, so there is an $\ell \in \mathcal{G}_1(\mathbb{R}^n)$ with $\ell \cap B = \{\mathbf{0}\}$. Consider an $x \in F$ and assume that $y \in (x+\ell)\cap B$ with $y \neq x$. Then $\ell = \mathbb{R}(y-x)$. Next, since $\mathbf{0} \in F^\circ$ we can choose a $t > 0$ such that $-tx \in F$. Since $-tx, y \in B$, we find by convexity that $w = \frac{t}{1+t}y + \frac{1}{1+t}(-tx)$ is in B. On the other hand, $w = \frac{t}{1+t}(y-x)$ and therefore $w \in \ell$. Hence $w \in B \cap \ell$ and $w \neq \mathbf{0}$, a contradiction. Therefore, $(x+\ell)\cap B = \{x\}$ and the claim is proved. ∎

Consider the countable set

$$\mathcal{L} = \{F^\circ : F \text{ is a regular derived face of } B \text{ with } F^\circ \cap \mathcal{X}^1_{\mathrm{p}}(B, \mathcal{G}_1(\mathbb{R}^n)) = \emptyset\}.$$

Obviously, $\bigcup \mathcal{L} \subset B \setminus \mathcal{X}^1_{\mathrm{p}}(B, \mathcal{G}_1(\mathbb{R}^n))$. If $x \in B \setminus \mathcal{X}^1_{\mathrm{p}}(B, \mathcal{G}_1(\mathbb{R}^n))$ then by Claim 2 there is a regular derived face F of B with $x \in F^\circ$. By Claim 3 no point of F° can be exposed, thus $F^\circ \in \mathcal{L}$. We observe that $\bigcup \mathcal{L} = B \setminus \mathcal{X}^1_{\mathrm{p}}(B, \mathcal{G}_1(\mathbb{R}^n))$. Every $F^\circ \in \mathcal{L}$ is an open subset of a closed set in $\mathbb{R}^n$, thus σ-compact. Since $\mathcal{L}$ is countable, $\bigcup \mathcal{L}$ is also σ-compact. Hence $\mathcal{X}^1_{\mathrm{p}}(B, \mathcal{G}_1(\mathbb{R}^n))$ is a G_δ-set in B and of course also in $\mathcal{X}^1_{\mathrm{t}}(B, \mathcal{G}_1(\mathbb{R}^n))$. ∎

The following example shows that in Theorem 3 we may not replace $\mathcal{G}_1(\mathbb{R}^n)$ by an open proper subset.

Example 2. We give an example for a compact and convex set K in $\mathbb{R}^3$ for which the set of points exposed by $\mathcal{G}_1(\mathbb{R}^3) \setminus \mathcal{G}_1(H)$, for some linear 2-space H of $\mathbb{R}^3$, is not a G_δ-set. In $\mathbb{R}^3$ take an xyz-coordinate system and set

$$K_1 = \{(x, 0, z) : (x+1)^2 + (z+1)^2 = 1\},$$
$$K_2 = \{(x, 0, z) : (x-1)^2 + (z+1)^2 = 1\}.$$

Let B be formed by revolving $\langle K_1 \cup K_2 \rangle$ around the z-axis. The intersection of B and the xy-plane is denoted by $D = \{(x, y, 0) : x^2 + y^2 \le 1\}$. Let $A = \{a_n : n \in \mathbb{N}\}$ be a dense countable subset of the circle ∂D. For $n \in \mathbb{N}$ let E_n be the hyperplane determined by the point $(0, 0, 1/n)$ and the tangent line r_n at a_n to ∂D in the xy-plane. Let V_n be the halfspace determined by E_n that contains the origin. Define a compact convex subset K of $\mathbb{R}^3$ by

$$K = B \cap \bigcap_{n=1}^{\infty} V_n.$$

Let $\mathcal{P} = \{P \in \mathcal{G}_1(\mathbb{R}^3) : P \not\subset xy\text{-plane}\}$. It suffices to prove that $\mathcal{X}^1_{\mathrm{p}}(K, \mathcal{P}) \cap \partial D = A$.

Note that if ℓ is a line through a_n and a point on the z-axis between 0 and $1/n$ then $\ell \cap K = \{a_n\}$. Thus $a_n \in \mathcal{X}^1_{\mathrm{p}}(K, \mathcal{P})$ for every $n \in \mathbb{N}$. Take w in $\partial D \setminus A$. We prove that $w \notin \mathcal{X}^1_{\mathrm{p}}(K, \mathcal{P})$. Let $\ell \in \mathcal{P}$ be arbitrary. Then $\ell = \mathbb{R}v$ for some nonzero vector $v = (v_x, v_y, v_z)$ with $v_z < 0$. Note that $w + \mathbb{R}v$ does not expose w in B (see [1, Example 3]). Thus, $(w + \mathbb{R}v) \cap B = w + [0, s]v$ for some $s > 0$.

Claim 4. *If $E_n \cap (w + \mathbb{R}^+ v) \neq \emptyset$ then $|v_z|/\sqrt{v_x^2 + v_y^2} \le 1/n$, where $\mathbb{R}^+ = (0, \infty)$.*

Proof. Suppose $u \in E_n \cap (w + \mathbb{R}^+ v)$. Let $\hat{u}$ be the projection of u onto the xy-plane. Note that u lies below the xy-plane because $v_z < 0$, and hence $\hat{u}$ and $\mathbf{0}$ lie on opposite sides of r_n in the xy-plane. Let p be the point of intersection of r_n with the line segment that connects $\hat{u}$ to $\mathbf{0}$. Since r_n is tangent to the unit circle ∂D we see that $\|p\| \geq 1$ and $\|p - \hat{u}\| \leq \|q - \hat{u}\|$ for any point $q \in \partial D$, in particular for w. Note that the slope of the line $w + \mathbb{R}v$ with respect to the xy-plane is

$$\frac{|v_z|}{\sqrt{v_x^2 + v_y^2}} = \frac{\|u - \hat{u}\|}{\|w - \hat{u}\|} \leq \frac{\|u - \hat{u}\|}{\|p - \hat{u}\|}.$$

Looking at the plane that contains the z-axis and u, we find by similarity of triangles that

$$\frac{\|u - \hat{u}\|}{\|p - \hat{u}\|} = \frac{1/n}{\|p\|} \leq \frac{1}{n}. \blacksquare$$

By Claim 4, only finitely many elements of $\{E_n : n \in \mathbb{N}\}$ intersect $w + \mathbb{R}^+ v$. Thus there exists a $t \in (0, s]$ such that $w + [0, t]v$ is contained in every halfspace V_n. Therefore, $w + [0, t]v \subset K$ and $w \notin \mathcal{X}_{\mathrm{p}}^1(K, \mathcal{P})$. Being countable and dense in ∂D the subset A is not a G_δ-set. Hence $\mathcal{X}_{\mathrm{p}}^1(K, \mathcal{P})$ is not a G_δ-set.

We turn to the question whether Theorem 3 is valid for $k > 1$. Klee [8, Example (6.10)] showed that there is a compact convex body B in $\mathbb{R}^3$ such that $\mathcal{X}_{\mathrm{p}}^2(B, \mathcal{G}_2(\mathbb{R}^3))$ is not a G_δ-set. Corson [7] constructed a more refined example: there is a compact convex body B in $\mathbb{R}^3$ such that $\mathcal{X}_{\mathrm{p}}^2(B, \mathcal{G}_2(\mathbb{R}^3))$ is of the first category and therefore does not contain a dense G_δ-subset of $\mathcal{X}_{\mathrm{t}}^2(B, \mathcal{G}_2(\mathbb{R}^3))$. The question here is whether there are similar statements for $\mathcal{X}_{\mathrm{p}}^k(B, \mathcal{G}_k(\mathbb{R}^n))$ for $1 < k < n$.

We need a proposition. If we assume that $\mathbf{0} \in B$ then $\operatorname{aff} B$ is a linear space and we can use $\mathcal{X}_{\mathrm{p}}^k(B)$ to denote $\mathcal{X}_{\mathrm{p}}^k(B, \mathcal{G}_k(\operatorname{aff} B))$.

Proposition 13. *Let B be a closed and convex subset of $\mathbb{R}^n$. Then, for $2 \leq k \leq \dim B$,*

$$\mathcal{X}_{\mathrm{p}}^k(B \times \mathbb{I}) = (\mathcal{X}_{\mathrm{p}}^k(B) \times (0,1)) \cup (\mathcal{X}_{\mathrm{p}}^{k-1}(B) \times \{0,1\}).$$

Proof. Choose a coordinate system such that $\mathbf{0} \in B^\circ$. Let $\pi \colon \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be the projection. We identify $\mathbb{R}^n$ with $\mathbb{R}^n \times \{0\} \subset \mathbb{R}^{n+1}$. Fix $2 \leq k \leq \dim B$.

Let $x = (a, s) \in \mathcal{X}_{\mathrm{p}}^k(B \times \mathbb{I})$. Thus there is an $L \in \mathcal{G}_k(\operatorname{aff} B \times \mathbb{R})$ such that $(x + L) \cap (B \times \mathbb{I}) = \{x\}$. Consider the case $0 < s < 1$. Put $L' = \pi(L) \in \operatorname{aff} B$. Since $x + L$ does not contain the line $\{a\} \times \mathbb{R}$ we see that $\dim L' = \dim L = k$ and $L' \in \mathcal{G}_k(\operatorname{aff} B)$. If $b \in L'$ is such that $a + b \in B$ then there is a $t \in \mathbb{R}$ with $(b, t) \in L$. Because s is interior to $\mathbb{I}$ we can select an $r \in (0, 1)$ such that $q = s + rt \in \mathbb{I}$. Then $c = a + rb \in B$ and $(c, q) \in (x + L) \cap B$. Thus $c = a$ and hence $b = \mathbf{0}$. We have $(a + L') \cap B = \{a\}$ and $a \in \mathcal{X}_{\mathrm{p}}^k(B)$.

Now, let $s = 0$. Let $\hat{L} = L \cap (\mathbb{R}^n \times \{0\})$. Note that $\dim \hat{L} \geq k - 1$ and select a $(k-1)$-subspace P of $\hat{L}$. Then $(x + P) \cap B = \{a\}$ and $x \in \mathcal{X}_{\mathrm{p}}^{k-1}(B) \times \{0\}$. The case $s = 1$ is obviously the same.

For the other direction, let $(a,s) = x \in \mathcal{X}_{\mathrm{p}}^k(B) \times (0,1)$. Then there is an $L \in \mathcal{G}_k(\operatorname{aff} B)$ such that $(a+L) \cap B = \{a\}$. This implies $(x+L) \cap (B \times \mathbb{I}) = (x+L) \cap (B \times \{s\}) = \{x\}$. Hence $x \in \mathcal{X}_{\mathrm{p}}^k(B \times \mathbb{I})$.

Now, let $(a,0) = x \in \mathcal{X}_{\mathrm{p}}^{k-1}(B) \times \{0\}$. So, there is an $L \in \mathcal{G}_{k-1}(\operatorname{aff} B)$ such that $(a+L) \cap B = \{a\}$. Set $\hat{L} = L + \mathbb{R}(a,1) \in \mathcal{G}_k(\operatorname{aff} B \times \mathbb{R})$ and let $(b,t) \in (x+\hat{L}) \cap (B \times \mathbb{I})$. Thus $t \in \mathbb{I}$ and $b = a + p + ta \in B$ for some $p \in L$. Since $\mathbf{0} \in B^\circ$ we have

$$\frac{1}{1+t} b = a + \frac{1}{1+t} p \in (a+L) \cap B = \{a\}.$$

So $b = \frac{1}{1+t} a$ and $U = a + \frac{t}{1+t} B^\circ \subset B$ by convexity. If $t > 0$ then U is an open neighbourhood of a in $\operatorname{aff} B$ and hence $(a+L) \cap U$ is open in $a+L$. Thus $\dim((a+L) \cap B) = \dim L = k-1 \geq 1$, which contradicts $(a+L) \cap B = \{a\}$. Consequently, $t = 0$ and hence $x = (b,t)$, which means that $x \in \mathcal{X}_{\mathrm{p}}^k(B \times \mathbb{I})$. ■

Example 3. Let $B \subset \mathbb{R}^3$ be Corson's example as described above. Let $n \geq 3$ and $K_n = B \times \mathbb{I}^{n-3} \subset \mathbb{R}^n$. Since $\operatorname{aff} B = \mathbb{R}^3$ we have $\operatorname{aff} K_n = \mathbb{R}^n$. We show by induction with respect to n that whenever $2 \leq k < n$ the set $\mathcal{X}_{\mathrm{p}}^k(K_n)$ contains a nonempty open first category subspace. Consequently, $\mathcal{X}_{\mathrm{p}}^k(K_n)$ does not contain a dense G_δ-subset of the complete space $\mathcal{X}_{\mathrm{t}}^k(K_n)$.

For the base case of the induction we have $n = 3$ thus $k = 2$ with $K_3 = B$ obviously satisfying the hypothesis. Assume now that the hypothesis is valid for some $n \geq 3$, and consider K_{n+1} and $2 \leq k \leq n$. We apply Proposition 13 to get

$$\mathcal{X}_{\mathrm{p}}^k(K_{n+1}) = (\mathcal{X}_{\mathrm{p}}^k(K_n) \times (0,1)) \cup (\mathcal{X}_{\mathrm{p}}^{k-1}(K_n) \times \{0,1\}).$$

If $k < n$ then by the hypothesis $\mathcal{X}_{\mathrm{p}}^k(K_n)$ contains a nonempty open first category subspace O and hence $O \times (0,1)$ is a nonempty open first category subspace of $\mathcal{X}_{\mathrm{p}}^k(K_n)$. If $k = n$ then $\mathcal{X}_{\mathrm{p}}^k(K_n) = \emptyset$ (see Remark 1). So $\mathcal{X}_{\mathrm{p}}^k(K_{n+1}) = \mathcal{X}_{\mathrm{p}}^{k-1}(K_n) \times \{0,1\}$, the union of two clopen copies of $\mathcal{X}_{\mathrm{p}}^{k-1}(K_n)$. Since $k-1 = n-1 \geq 2$ the induction hypothesis applies to $\mathcal{X}_{\mathrm{p}}^{k-1}(K_n)$ and the proof is complete.

These examples show why we work in the space $\mathcal{G}_k(\mathbb{V}) \times B$ for the proof of Theorem 10: the Baire category argument cannot work in B.

We finish our paper with the following open problem.

Question 1. Let B be closed and convex in ℓ^2 with $\operatorname{int} B \neq \emptyset$. Is $\mathcal{X}_{\mathrm{p}}^1(B, \mathcal{G}_1(\ell^2))$ a G_δ-set in $\mathcal{X}_{\mathrm{t}}^1(B, \mathcal{G}_1(\ell^2))$?

Acknowledgements. The first author is pleased to thank the Vrije Universiteit Amsterdam for its hospitality and support. He was also supported in part by the Netherlands Organisation for Scientific Research (NWO), under grant 040.11.120.

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Received 28 December 2013;
in revised form 10 August 2015