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On physical measures for Cherry flows

by

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Abstract. Studies of the physical measures for Cherry flows were initiated in Saghin and Vargas (2013). While the non-positive divergence case was resolved, the positive divergence case still lacked a complete description. Some conjectures were put forward. In this paper we make a contribution in this direction. Namely, under mild technical assumptions we solve some conjectures stated in Saghin and Vargas (2013) by providing a description of the physical measures for Cherry flows in the positive divergence case.

1. Introduction. One of the main goals of dynamical systems theory is to describe the typical behavior of orbits, especially when time goes to infinity, and understanding how this behavior is affected by small perturbations of the law that governs the system.

Such questions are especially difficult when the system is sensitive to initial conditions; that is, when a small change in the initial state results in a large variation in the long term behavior of the orbits. One way to address this problem is using the so-called physical measures. These are probability measures of a particular interest as they describe the statistical properties of a large set of orbits.

In general physical measures are still poorly understood. Even their existence has been established only for a narrow class of systems. In this paper we make a contribution to this area by studying the physical measures for Cherry flows.

We recall the classical construction of a Cherry flow given in [1]. It is a $\mathcal{C}^\infty$ flow on the two-dimensional torus without closed orbits and with two singularities, a sink and a saddle. In this case it is relatively easy to check that the only physical measure is the Dirac delta at the sink. The inverted flow, still called a Cherry flow, is a $\mathcal{C}^\infty$ flow on the torus $\mathbb{T}^2$, without closed orbits and with two singularities, a saddle point and a repelling point, both hyperbolic

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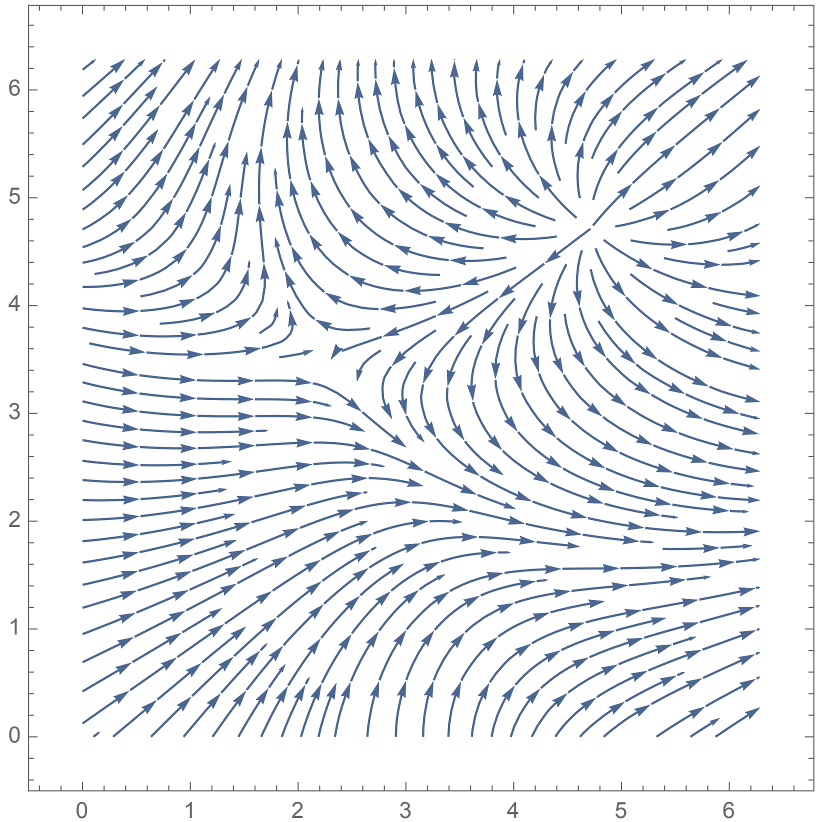

Fig. 1. Cherry flow

(see Figure 1). Describing the physical measures in this framework, which is the topic of this paper, is much more difficult and interesting.

Our results answer some open questions of [11].

Before we can state them formally, it is necessary to recall some basic definitions.

1.1. Basic definitions and general remarks

Physical measures. Let ϕ be a continuous flow on a compact manifold M. A probability measure ν on M is *invariant* under the flow if $\nu(\phi_t(A)) = \nu(A)$ for all $t \in \mathbb{R}$ and for any measurable set $A \subset M$.

DEFINITION 1.1. Let $t > 0$. We define a family of probability measures $m_t(z)$, $z \in M$, on M by

$$\int_M \alpha \, dm_t(z) = \frac{1}{t} \int_0^t \alpha(\phi_s(z)) \, ds$$

for each continuous function $\alpha : M \to \mathbb{R}$.

Definition 1.2. Let ν be an invariant probability measure. The *basin of attraction* $B(\nu) = B^{\phi}(\nu)$ of ν is the set of $z \in M$ such that

$$\lim_{t\to\infty} m_t(z) = \nu \quad \text{in the weak-}\star\text{ topology.}$$

The measure ν is said to be *physical* if its basin of attraction has strictly positive Lebesgue measure.

Cherry flows. In the following we provide some definitions and properties concerning Cherry flows. We will state them in compact form. For more details the reader can refer to [4]–[8].

Definition 1.3. A *Cherry flow* is a $\mathcal{C}^\infty$ flow on the torus $\mathbb{T}^2$ without closed orbits and with two singularities, a saddle point and a repelling point, both hyperbolic.

From now on, ϕ will denote a Cherry flow as in Definition 1.3. Moreover p_s will denote the saddle point of ϕ, and p_r its repelling point.

Proposition 1.4. *Let ϕ be a Cherry flow and let* $\mathrm{Sing}(\phi)$ *be the set of singularities of ϕ. There exists a closed $\mathcal{C}^\infty$ curve C on $\mathbb{T}^2 \setminus \mathrm{Sing}(\phi)$ with the following properties:*

- *C is everywhere transversal to the flow;*
- *C is not contractible to a point.*

Definition 1.5. The closed curve C constructed in Proposition 1.4 is called a *closed transversal.*

Fact 1.6. *Every Cherry flow admits a closed transversal C. The set $T^2 \setminus C$ is $\mathcal{C}^\infty$-equivalent to an annulus $\mathbb{S}^1 \times (0,1)$ and we can write $T^2 \cong \mathbb{S}^1 \times [0,1]/\sim$, where $(s,0) \sim (s,1)$. Consider ϕ as a flow on $T^2 \cong \mathbb{S}^1 \times [0,1]$ where we identify $\mathbb{S}^1 \times \{0\}$ and $\mathbb{S}^1 \times \{1\}$. After this change of coordinates, the resulting flow is a Cherry flow.*

Let now g be the first return map of the flow ϕ to the closed transversal. The existence of g is guaranteed by [5, Theorem 2.6.1] and by the absence of closed trajectories for Cherry flows. Observe that g is $\mathcal{C}^\infty$ everywhere except at one point which belongs to the stable manifold of the saddle point and which we will assume to be zero (we identify $\mathbb{S}^1$ with $[-1/2, 1/2]_{-1/2\sim 1/2}$). We denote by a and b respectively the left-side and the right-side limit of the orbit of the discontinuity point 0, and by U the interval (a,b).

We now consider the flow φ obtained by reversing the direction of ϕ. The repelling point of ϕ becomes an attractive point for φ, which is then a Cherry flow as in Cherry's example [1]. In this case, the first return map f of φ to the closed transversal is a circle endomorphism which is $\mathcal{C}^\infty$ everywhere except at a and b, where it is continuous, and it is constant on the interval $U = (a,b)$. Moreover, after a change of coordinates, on a half-open neighborhood of these two points, f can be written as $x^{\lambda_1/(-\lambda_2)}$ where $\lambda_1 > 0 > \lambda_2$ are the eigenvalues of the saddle point p_s of ϕ. The corresponding formula for g is clearly $x^{-\lambda_2/\lambda_1}$.

Rotation number. As f is a monotone circle map, it has a rotation number, measuring the rate at which an orbit winds around the circle. More precisely, if F is a lift of f to the real line, the rotation number of f is the limit

$$\rho(f) = \lim_{n\to\infty} \frac{F^n(x)}{n} \pmod 1.$$

This limit exists for every x and its value is independent of x.

We can then define the rotation number of any flow φ obtained by reversing the direction of a Cherry flow ϕ as follows:

Definition 1.7. The *rotation number* of φ is the rotation number of its first return map to any closed transversal.

It is easy to check

Fact 1.8. *The rotation number ρ of φ does not depend on the choice of the closed transversal.*

Consequently, the rotation number of any Cherry flow is defined:

Definition 1.9. Let ϕ be a Cherry flow as in Definition 1.3. The *rotation number* of ϕ is the rotation number of the flow φ obtained by reversing the direction of ϕ.

Since the flow under consideration does not have closed orbits, f has an irrational rotation number ρ which admits an expansion as an infinite continued fraction

$$\rho = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots}}},$$

where the a_i are positive integers.

If we cut off the portion of the continued fraction beyond the nth position, and write the resulting fraction in lowest terms as p_n/q_n, then the numbers q_n for $n \geq 1$ satisfy the recurrence relation

$$q_{n+1} = a_{n+1} q_n + q_{n-1}, \qquad q_0 = 1,\ q_1 = a_1. \tag{1.1}$$

The number q_n is the number of times we have to iterate the rotation by ρ in order that the orbit of any point makes its closest return so far to the point itself (see [2, Chapter I, Sect. I]).

Definition 1.10. Let ρ be an irrational number and let $(a_i)_{i\in\mathbb{N}}$ be the integers in the infinite continued fraction expansion of ρ. We say that ρ is *of* *bounded type* if there exists a positive real number M such that $a_i < M$ for any i.

First return time

Definition 1.11. Let z be a point of the closed transversal. The *first return time* $\tau(z)$ for ϕ to $\mathbb{S}^1$ is the number of iterations of g needed by z to come back to $\mathbb{S}^1$ for the first time.

Fact 1.12. *The first return time $\tau(z)$ for ϕ to $\mathbb{S}^1$ has a logarithmic singularity at* 0. *This means that for each $\epsilon > 0$, there exists a constant $C > 0$ such that, for all $z \in (-\epsilon, \epsilon)$, we have*

$$\frac{1}{C} \leq \frac{\tau(z)}{-\log|z|} \leq C.$$

In other words, $\tau(z)$ is of order of $-\log|z|$.

Since f preserves the order and it does not have periodic points, by Poincaré's Theorem there exists a continuous order preserving degree one function $h : \mathbb{S}^1 \to \mathbb{S}^1$ such that $h \circ f = R_\rho \circ h$, where R_ρ is the rotation by ρ. Then the probability measure $\mu = h^*(\mathrm{Leb})$ supported on the minimal set is well defined and it is the only invariant ergodic measure for f.

By [10, Proposition 2] the measure μ can be extended to an invariant probability measure ν on the torus supported on the quasi-minimal set if $\int_{\mathbb{S}^1} \tau \, d\mu$ is convergent.

1.2. Discussion and statement of the results. In this paper we are interested in the physical measures for Cherry flows which are $\mathcal{C}^\infty$ flows on the torus $\mathbb{T}^2$, without closed orbits and with two singularities, a saddle point p_s and a repelling point p_r, both hyperbolic (see Figure 2).

This problem was first studied in [11] where the authors gave a description of the physical measures for some class of Cherry flows. They discovered that this problem is related to the variation of the divergence of the flow at the saddle point.

To be more precise, let $\lambda_1 > 0 > \lambda_2$ be the two eigenvalues at the saddle point. In the non-positive divergence case when $\lambda_1 \leq -\lambda_2$, [11] shows that the Dirac deltas at the singularities are the only ergodic invariant probability measures. Moreover [11] establishes that the Dirac delta at the saddle point is the physical measure for the flow.

On the other hand, in the positive divergence case, $\lambda_1 > -\lambda_2$, in addition to the Dirac deltas at the singularities, there exists another ergodic invariant probability measure ν, which is supported on the quasi-minimal set of the flow and it is different from the Dirac delta at the saddle point. Under the additional assumptions of strictly positive divergence, $\lambda_1 > -2\lambda_2$, and of the

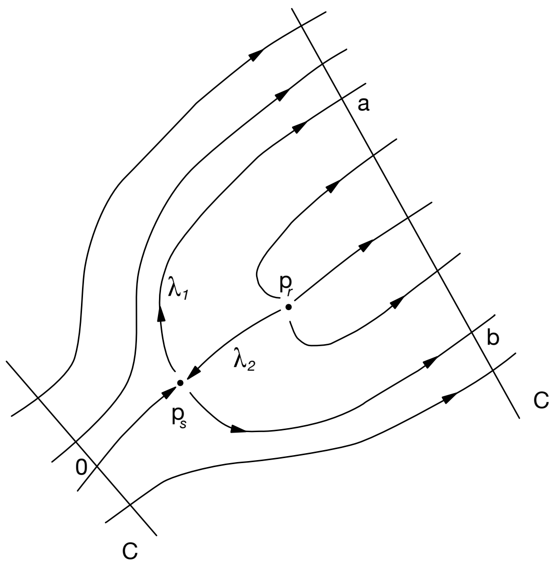

Fig. 2. Cherry flow

rotation number being of bounded type, the authors of [11] prove that ν is physical and they conjecture that this holds for any $\lambda_1 > -\lambda_2$.

In this paper we present some new results in this direction.

If the divergence at the saddle point is positive, under the hypothesis that the rotation number is of bounded type, we have:

THEOREM 1.13. *Let ϕ be a Cherry flow with eigenvalues $\lambda_1 > 0 > \lambda_2$ at the saddle point. If $\lambda_1 > -\lambda_2$ and ϕ has rotation number of bounded type, then the ergodic invariant probability measure ν supported on the quasi-minimal set is the physical measure for ϕ with attraction basin having full Lebesgue measure.*

If the divergence at the saddle point becomes strictly positive, *without any assumption* on the rotation number, we have:

THEOREM 1.14. *Let ϕ be a Cherry flow with eigenvalues $\lambda_1 > 0 > \lambda_2$ at the saddle point. If $\lambda_1 \geq -3\lambda_2$, then the ergodic invariant probability measure ν supported on the quasi-minimal set is the physical measure for ϕ with attraction basin having full Lebesgue measure.*

Theorem 1.13 together with the results of [11] provides a complete description of the physical measures for Cherry flows having rotation number of bounded type (bounded regime). In the unbounded regime the case $1 < \lambda_1/(-\lambda_2) < 3$ remains still open. We will comment on technical problems arising in this case in Remark 3.1.

1.3. Standing assumptions and notation. Let 0 be the discontinuity point of g. In order to simplify the notation we shall write

$$\underline{i} = f^i(0), \qquad \underline{i}_R = R^i_\rho(0).$$

We observe that, because of the properties of f, underlined non-positive integers of the type $\underline{-i}$ represent intervals.

Distance between points. We denote by $(a, b) = (b, a)$ the shortest open interval between a and b regardless of the order of these two points. The length of that interval in the natural metric on the circle will be denoted by $|a - b|$. Following [3], let us adopt these notational conventions for the distance between the preimages of the first return function f:

- $|\underline{-i}|$ stands for the length of the interval $\underline{-i}$.
- Consider a point x and an interval $\underline{-i}$ not containing it. Then the distance from x to the closest endpoint of $\underline{-i}$ will be denoted by $|(x, \underline{-i})|$, and the distance to the most distant endpoint by $|(x, \underline{-i}]|$.
- We define the distance between the endpoints of two intervals $\underline{-i}$ and $\underline{-j}$ analogously. For example, $|(\underline{-i}, \underline{-j})|$ denotes the distance between the closest endpoints of these two intervals while $|[\underline{-i}, \underline{-j})|$ stands for $|\underline{-i}| + |(\underline{-i}, \underline{-j})|$.

2. Proof of Theorem 1.13. We consider the sequence

$$\alpha_n = \frac{|(\underline{-q_n}, \underline{0})|}{|[\underline{-q_n}, \underline{0})|}$$

and we prove the following proposition:

Proposition 2.1. *Let $\lambda_1 > 0 > \lambda_2$ be the eigenvalues at the saddle point of ϕ and let f be the first return function of the reverse flow φ. If f has rotation number of bounded type and $\lambda_1/(-\lambda_2) \in (1, 2]$, then there exist constants $K > 0$ and $C < 1$ such that for n large enough,*

$$\frac{-\log \alpha_n}{q_{n+1}} \leq KC^n.$$

The constant C does not depend on the eigenvalues λ_1 and λ_2 at the saddle point.

Proof. We write $\ell = \lambda_1/(-\lambda_2)$. Before beginning the proof it is necessary to recall that:

(1) $q_0 = 1$, $q_1 = a_1$ and $q_{n+1} = a_{n+1}q_n + q_{n-1}$ by (1.1),

(2) by [3, Proposition 6], for n large enough, $\alpha_n \geq K_1 \alpha_{n-1}^{\frac{1-\ell^{-a_{n+1}}}{\ell-1}} \alpha_{n-2}^{\ell^{-a_n}}$ where K_1 is a positive constant.

In order to simplify the proof we will work assuming that, for n large enough,

$$\alpha_n \geq \alpha_{n-1}^{\frac{1-\ell^{-a_{n+1}}}{\ell-1}} \alpha_{n-2}^{\ell^{-a_n}}. \tag{2.1}$$

This hypothesis is not restrictive. Indeed, observe that for every n we have $q_n \geq \beta^n/K_2$ with $\beta = (1+\sqrt{5})/2$ and K_2 a positive constant. Then the estimation remains true also in the general case as one could add to each inequalities below the term $\log(K_1)/q_n$ which trivially satisfies the desired estimation.

To lighten the notation, we introduce a new sequence $(\theta_n)_{n\in\mathbb{N}}$ defined by $\theta_n := -\log \alpha_n$ for all n.

We fix $n_0 \in \mathbb{N}$ such that (2.1) is satisfied for all $n \geq n_0$.

We shall prove the proposition by induction on $n \geq n_0$. We take

$$C = C(\ell) = \sup_i \left(\frac{1-\ell^{-a_i}}{(\ell-1)a_i} \right)^{1/n_0} \quad \text{and} \quad K \geq \max\{\theta_{n_0-2}, \theta_{n_0-1}\}.$$

We observe that, for all $1 < \ell \leq 2$, we have $C < 1$; if we consider C as a function of ℓ, then, in the interval $(1,2]$, $C(\ell)$ is continuous, decreasing, and moreover $\lim_{\ell\to 1} C(\ell) = 1$ and $C(2) < 1$.

We observe that, for any natural number $i \geq 1$,

$$\ell^{-a_i} \leq \frac{1-\ell^{-a_i}}{(\ell-1)a_i} \leq C^{n_0} \leq C. \tag{2.2}$$

We now begin the proof by induction.

- Let n_0 be as above. By (2.1) and (2.2), we have

$$\begin{aligned} \theta_{n_0} &\leq \frac{1-\ell^{-a_{n_0+1}}}{\ell-1}\theta_{n_0-1} + \ell^{-a_{n_0}}\theta_{n_0-2} \\ &\leq C^{n_0} K a_{n_0+1} + C^{n_0} K \leq K C^{n_0}(a_{n_0+1}+1). \end{aligned}$$

By point (1),

$$\theta_{n_0} \leq K C^{n_0} q_{n_0+1}. \tag{2.3}$$

We now prove the assertion for n_0+1. By (2.1)–(2.3),

$$\begin{aligned} \theta_{n_0+1} &\leq \frac{1-\ell^{-a_{n_0+2}}}{\ell-1}\theta_{n_0} + \ell^{-a_{n_0+1}}\theta_{n_0-1} \\ &\leq K C^{n_0} \frac{1-\ell^{-a_{n_0+2}}}{(\ell-1)a_{n_0+2}} a_{n_0+2} q_{n_0+1} + K C^{n_0}; \end{aligned}$$

and by point (1) and (2.2),

$$\theta_{n_0+1} \leq K C^{n_0+1} \left(a_{n_0+2} q_{n_0+1} + \frac{C^{n_0}}{C} \right) \leq K C^{n_0+1} q_{n_0+2}.$$

• We now assume that the assertion is true for $n-2$ and for $n-1$, and we prove it for n. By (2.1) and the inductive hypothesis we have

$$\begin{aligned}\theta_n &\le \frac{1-\ell^{-a_{n+1}}}{\ell-1}\theta_{n-1}+\ell^{-a_n}\theta_{n-2}\\ &\le K\biggl(\frac{(1-\ell^{-a_{n+1}})}{(\ell-1)a_{n+1}}C^{n-1}a_{n+1}q_n+C^{n-2}\ell^{-a_n}q_{n-1}\biggr).\end{aligned}$$

Finally, by (2.2) and by point (1),

$$\theta_n \le KC^n\biggl(a_{n+1}q_n+\frac{\ell^{-a_n}}{C^2}q_{n-1}\biggr)\le KC^n q_{n+1}.$$

So, the assertion of the lemma is true for all $n\in\mathbb{N}$ large enough. ∎

As a direct consequence of Proposition 2.1 we have the following.

Corollary 2.2. *Let $\lambda_1>0>\lambda_2$ be the eigenvalues at the saddle point of ϕ and let f be the first return function of the reverse flow φ. If f has rotation number of bounded type and $\lambda_1/(-\lambda_2)\in(1,2]$, then there exist constants $K>0$ and $C<1$ such that for n large enough, $-\log|(\underline{q_n},\underline{0})|/q_{n+1}\le KC^n$. The constant C does not depend on the eigenvalues λ_1 and λ_2 at the saddle point.*

We recall the following theorem proved in [3]:

Theorem 2.3. *Let $\lambda_1>0>\lambda_2$ be the eigenvalues at the saddle point of ϕ and let f be the first return function of the reverse flow φ. If $\lambda_1>-\lambda_2$, then $\bigcup_{i=0}^{\infty}f^{-i}(U)$ has full Lebesgue measure on $\mathbb{S}^1$.*

The proof of Theorem 1.13 uses the main ideas of [11, proof of Theorem 3].

Proof of Theorem 1.13. By [11, Theorem 2] we know that the flow ϕ has an invariant probability measure ν supported on the quasi-minimal set which corresponds to the extension of the f-invariant measure μ (defined by $\mu=h^\star(\mathrm{Leb})$). It remains to prove that ν is a physical measure for ϕ and that its basin of attraction has full Lebesgue measure.

By Theorem 2.3, it is sufficient to prove that the points of the wandering set of φ are in the basin of attraction of ν. Since all points of the wandering set pass through the flat interval of f, we just have to prove that any point of U is in the basin of attraction of ν.

Let $z\in U$, $\underline{n}_g=g^{n-1}(z)$ and $t_n=\tau(\underline{n}_g)$. For all $t>0$ there exists $N\in\mathbb{N}$ such that $t=t_1+\cdots+t_N+\tilde{t}$ where $0<\tilde{t}\le t_{N+1}$. Moreover, let $n\in\mathbb{N}$ be such that $q_n\le N<q_{n+1}$. Since τ is uniformly bounded below, we have

$$t\ge CN \tag{2.4}$$

with C a positive constant.

Let m_t be the probability measure introduced in Definition 1.1. Since the only invariant probability measures are δ_s, δ_r and ν (for more details see [11, Theorem 2]) and since p_r is repelling, the limits for m_t will be of the form

$$\gamma\delta_s + (1-\gamma)\nu \tag{2.5}$$

for some $\gamma \in [0,1]$. To prove that z is in the basin of attraction of ν, which means that $\lim_{t\to\infty} m_t = \nu$, we have to prove that $\gamma = 0$.

We fix $0 < n_0 < n$ and we will prove that the trajectory of z under ϕ spends most of the time outside of

$$A_{n_0} = \{\phi_s(w) : w \in (\underline{q_{n_0}}, \underline{q_{n_0+1}}),\ 0 \le s \le \tau(w)\}.$$

We will show that by choosing n_0 correctly, the time $t_{A_{n_0}}$ which the trajectory $\phi_s(z)$, $0 \le s \le t$, spends in A_{n_0} can be made arbitrarily small in comparison to t, for all t large enough.

To do so we divide A_{n_0} and we start by estimating the time t_{B_l} spent by the trajectory $\phi_s(z)$, $0 \le s \le t$, in

$$B_l = \{\phi_s(w) : w \in (\underline{q_l}, \underline{q_{l+2}}),\ 0 \le s \le \tau(w)\}.$$

We observe that, since f is the first return function of the flow obtained by reversing the direction of ϕ, if h is the semiconjugation between f and the rotation R_ρ, then

$$h(\underline{n}_g) = h(g^{n-1}(z)) = h(f^{-n+1}(z)) = h(f^{-n}(0)) = \underline{-n}_R.$$

Then for all $l \in \mathbb{N}$ we have $\underline{q_l}_g \in (\underline{q_{l-1}}, \underline{q_{l+1}})$ and $\underline{q_l} \in (\underline{q_{l-1}}_g, \underline{q_{l+1}}_g)$. So the number of points $\underline{i}_g$, $1 \le i \le N$, in $(\underline{q_l}, \underline{q_{l+2}})$ is equal to the number of points $\underline{-i}_R$, $1 \le i \le N$, in $(\underline{q_l}_R, \underline{q_{l+2}}_R)$.

Now we estimate the number N_l of points $\underline{-i}_R$, $1 \le i \le N$, which are in $(\underline{q_l}_R, 0)$. Since $|(\underline{q_l}_R, 0)|$ is of the order of $1/q_{l+1}$ and since the rotation is a bijection preserving the distance, we can divide the circle into exactly q_{l+1} disjoint images of $(\underline{q_l}_R, 0)$, and any image has N_l points $\underline{-i}_R$, $1 \le i \le N$.

In conclusion,

$$q_{l+1} N_l \le N$$

and the number of points $\underline{-i}_R$, $1 \le i \le N$, which are in $(\underline{q_l}_R, \underline{q_{l+2}}_R)$ is less than or equal to N/q_{l+1}.

By Fact 1.12, equation (2.4) and Corollary 2.2 we have

$$\begin{aligned}\frac{t_{A_{n_0}}}{t} = \frac{1}{t}\sum_{l=n_0}^{n-1} t_{B_l} &\le \frac{C_3 N}{t}\sum_{l=n_0}^{n-1} \frac{-\log|(\underline{q_{l+2}}, \underline{0})|}{q_{l+1}} \\ &\le \frac{C_3}{C}\sum_{l=n_0}^{n-1} \frac{-\log|(\underline{q_{l+2}}, \underline{0})|}{q_{l+1}} \le \frac{C_3}{CC_4}\sum_{l=n_0}^{n-1} (C_5)^{l+2}.\end{aligned}$$

Observe that we are assuming a supplementary hypothesis on the eigenvalues $\lambda_1 > 0 > \lambda_2$ of the saddle point: $\lambda_1 \le -2\lambda_2$. The case $\lambda_1 > -2\lambda_2$ is proved in [11, Theorem 3]. Finally, since $\sum_{l=n_0}^{\infty}(C_5)^l$ is convergent, taking n_0 large enough, we can make $t_{A_{n_0}}/t$ as small as we want.

We observe that we have the same result if in place of A_{n_0} we consider A_{n_0-c} with $c > 0$ and $A_{n_0} \Subset A_{n_0-c}$.

It remains to prove that if $\lim_{n_0\to\infty} t_{A_{n_0}}/t = 0$ then $\gamma = 0$.

We suppose for contradiction that $\gamma > 0$, and we recall that there exists a strictly non-decreasing sequence $(t_n)_{n\in\mathbb{N}}$ of positive reals with $t_n \to \infty$ as $n \to \infty$ such that $\lim_{t_n\to\infty} m_{t_n}(z) = \gamma\delta_s + (1-\gamma)\nu$ (see (2.5)).

Let us fix $\epsilon > 0$. There exists $T > 0$ such that for all $n \in \mathbb{N}$ for which $t_n > T$ and for any continuous $\alpha : \mathbb{T}^2 \to \mathbb{R}$,

$$\Big|\int_{\mathbb{T}^2} \alpha\, dm_{t_n}(z) - \int_{\mathbb{T}^2} \alpha\, d(\gamma\delta_s + (1-\gamma)\nu)\Big| < \epsilon. \tag{2.6}$$

Now let $c > 0$ be such that $A_{n_0} \Subset A_{n_0-c}$. Let α be a bump function with compact support such that $\alpha(x) = 1$ for all $x \in A_{n_0}$, and $\alpha(x) = 0$ for all $x \in (A_{n_0-c})^c$. We observe that

$$\frac{t_{A_{n_0}}}{t_n} \le \int_{\mathbb{T}^2} \alpha\, dm_{t_n}(z) \le \frac{t_{A_{n_0-c}}}{t_n}$$

and we recall that by hypothesis $\lim_{n_0\to\infty} t_{A_{n_0}}/t_n = \lim_{n_0\to\infty} t_{A_{n_0-c}}/t_n = 0$.

Then, for n large enough, we deduce from (2.6) that

$$\gamma - \epsilon < \gamma + (1-\gamma)\nu(A_{n_0}) - \epsilon < \epsilon, \tag{2.7}$$

which contradicts the hypothesis that $\gamma > 0$.

So $\gamma = 0$ and $\lim_{t\to\infty} m_t(z) = \nu$ in the weak-$\star$ topology. By Definition 1.2, z is in the basin of attraction of ϕ. ∎

3. Proof of Theorem 1.14. The idea of the proof is similar to the proof of Theorem 1.13. The main technical tool is that, under the condition $\lambda_1 \ge -3\lambda_2$, without any assumption on the rotation number, the sequence $|(\underline{0}, \underline{q_n})|/|(\underline{0}, \underline{q_{n-2}})|$ is bounded away from zero [9, Theorem 1.2, second claim].

Proof of Theorem 1.14. By [9, Theorem 1.2, second claim] we can assume that there exist $n_0 \in \mathbb{N}$ and a constant $\alpha \in (0,1)$ such that $|(\underline{0}, \underline{q_n})|/|(\underline{0}, \underline{q_{n-2}})| > \alpha^2$ for $n \ge n_0 > 0$. Then, by induction,

$$|(\underline{0}, \underline{q_n})| > C\alpha^n \tag{3.1}$$

for some $C > 0$.

By [11, Theorem 2], there exists an invariant probability measure ν supported on the quasi-minimal set. We prove that the basin of attraction of ν has full Lebesgue measure, so ν is a physical measure for ϕ.

As in the proof of Theorem 1.13, we prove that any point of U is in the basin of attraction of ν.

Let $z \in U$, $\underline{n}_g = g^{n-1}(z)$ and $t_n = \tau(\underline{n}_g)$. For all $t > 0$ there exists $N \in \mathbb{N}$ such that $t = t_1 + \cdots + t_N + \tilde{t}$ where $0 < \tilde{t} \le t_{N+1}$ and there exists $n \in \mathbb{N}$ such that $q_n \le N < q_{n+1}$. Since τ is uniformly bounded below, we have

$$t \ge C_1 N \tag{3.2}$$

with $C_1 > 0$ a positive constant.

Let m_t be the probability measure as introduced in Definition 1.1. The possible limits for m_t must have the form $\gamma\delta_s + (1-\gamma)\nu$ for some $\gamma \in [0,1]$. We have to prove that γ is zero (for the details see the proof of Theorem 1.13).

We fix $0 < n_0 < n$ and we prove that the orbit of z under ϕ spends most of the time outside of

$$A_{n_0} = \{\phi_s(w) : w \in (\underline{q_{n_0}}, \underline{q_{n_0+1}}),\ 0 \le s \le \tau(w)\}.$$

The time $t_{A_{n_0}}$ spent in A_{n_0} will be calculated as the sum of the times t_{B_l} spent in small pieces of A_{n_0} of the form

$$B_l = \{\phi_s(w) : w \in (\underline{q_l}, \underline{q_{l+2}}),\ 0 \le s \le \tau(w)\}.$$

For these reasons, we have to estimate the number of points $\underline{i}_g$, $1 \le i \le N$, in $(\underline{q_l}, \underline{q_{l+2}})$, which, just as in Theorem 1.13, coincides with the number of points $\underline{-i}_R$, $1 \le i \le N$, in $(\underline{q_l}_R, \underline{q_{l+2}}_R)$, which is less than or equal to N/q_{l+1}.

Finally by (3.1), (3.2), Fact 1.12 and the fact that $q_l \ge \beta^l/C_6$ for $\beta = (1+\sqrt{5})/2$ we have

$$\begin{aligned}\frac{t_{A_{n_0}}}{t} &= \frac{1}{t}\sum_{l=n_0}^{n-1} t_{B_l} \le \frac{C_5 N}{t}\sum_{l=n_0}^{n-1} \frac{-\log|(\underline{q_{l+2}}, \underline{0})|}{q_{l+1}} \le \frac{C_5}{C_1}\sum_{l=n_0}^{n-1} \frac{-\log|(\underline{q_{l+2}}, \underline{0})|}{q_{l+1}} \\ &\le \frac{C_5 C\alpha}{C_1}\sum_{l=n_0}^{n-1} \frac{l}{q_{l+1}} \le \frac{C_5 C\alpha C_6}{C_1}\sum_{l=n_0}^{n-1} \frac{l}{\beta^{l+1}}.\end{aligned}$$

In conclusion, taking n large enough, we can make t_{A_n}/t as small as we want; hence, as in the proof of Theorem 1.13, $\gamma = 0$ and z is in the basin of attraction of ν. ∎

Remark 3.1. The proof of Theorem 1.13 hinges on the recursive estimate (2.1) for α_n (used to demonstrate Proposition 2.1). In the case of rotation number of bounded type such an estimate was found in [3] and is sufficient to conduct the proof. The case of unbounded type is more problematic, since the estimate provided by [9] is not sufficient any more. To circumvent this problem we assume additionally that the ratio of the eigenvalues at the saddle point is greater than or equal to 3, which ensures that the sequence α_n is bounded away from zero (Theorem 1.14). We note that an improved estimate may lead to a complete description of the physical measures for Cherry flows without any assumption on the rotation number.

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