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ON THE RELATION BETWEEN MAXIMAL RIGID OBJECTS AND τ-TILTING MODULES

BY

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Abstract. This note compares τ-tilting modules and maximal rigid objects in the context of 2-Calabi–Yau triangulated categories. Let $\mathcal{C}$ be a 2-Calabi–Yau triangulated category with suspension functor S. Let R be a maximal rigid object in $\mathcal{C}$ and let Γ be the endomorphism algebra of R. Let F be the functor $\mathrm{Hom}_{\mathcal{C}}(R, -) : \mathcal{C} \to \operatorname{mod} \Gamma$. We prove that any τ-tilting module over Γ lifts uniquely to a maximal rigid object in $\mathcal{C}$ via F, and in turn, that projection from $\mathcal{C}$ to $\operatorname{mod} \Gamma$ sends the maximal rigid objects which have no direct summands from $\operatorname{add} SR$ to τ-tilting Γ-modules, and in general, that the Γ-modules corresponding to the maximal rigid objects are the support τ-tilting modules.

1. Introduction. This note generalizes part of Adachi–Iyama–Reiten's theory of τ-tilting modules [1]. τ-tilting is a recent, important development in tilting theory where tilting modules are replaced by so-called τ-tilting modules. Adachi–Iyama–Reiten's main aim is to find modules which are closely related to the notion of mutation.

Mutation is one of the key steps in the definition of Fomin–Zelevinsky's cluster algebras [8]. It was shown in [16] that the combinatorics of cluster mutation are closely related to those of tilting theory in the representation theory of quivers and finite-dimensional algebras. This link was the main motivation for the study of cluster categories [5] and more general 2-Calabi–Yau categories [14]. These triangulated categories allow one to "categorify" cluster algebras. In doing so, cluster-tilting objects or maximal rigid objects play a central role; cf. for example the surveys [4, 10, 13, 17]. Cluster-tilting objects are always maximal rigid objects, while the converse is not true in general. There exist 2-Calabi–Yau triangulated categories in which maximal rigid objects are not cluster tilting [7, 3, 6].

One of the results in [1] is that cluster-tilting objects in a 2-Calabi–Yau triangulated category $\mathcal{C}$ correspond bijectively to support τ-tilting modules over a 2-Calabi–Yau tilted algebra associated with $\mathcal{C}$. Here 2-Calabi–Yau tilted algebras are algebras which appear as endomorphism rings of cluster-

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tilting objects in 2-Calabi–Yau triangulated categories. Motivated by this correspondence, in the current note we compare τ-tilting modules and maximal rigid objects. More precisely, let $\mathcal{C}$ be a 2-Calabi–Yau triangulated category with suspension functor S. Let R be a maximal rigid object in $\mathcal{C}$ and let Γ be the endomorphism algebra of R. Let F be the functor $\operatorname{Hom}_{\mathcal{C}}(R, -) : \mathcal{C} \to \operatorname{mod} \Gamma$. We prove that any τ-tilting module over Γ lifts uniquely via F to a maximal rigid object in $\mathcal{C}$ (Theorem 3.3), and in turn, that if the maximal rigid object M of $\mathcal{C}$ satisfies $\operatorname{add} M \cap \operatorname{add} SR = 0$, then the projection from $\mathcal{C}$ to $\operatorname{mod} \Gamma$ sends M to a τ-tilting Γ-module (Theorem 3.5). In general, if one allows direct summands of SR, then the Γ-modules which correspond to maximal rigid objects are support τ-tilting modules.

2. Preliminaries. In this section we review some useful notation and results.

2.1. τ-tilting modules. Let k be an algebraically closed field and Λ be a finite-dimensional basic algebra. Let $\operatorname{mod} \Lambda$ be the category of finite-dimensional left Λ-modules. As usual, we denote by τ the Auslander–Reiten translation. A Λ-module M is called *τ-tilting* if

- M is *τ-rigid*, i.e. $\operatorname{Hom}_{\Lambda}(M, \tau M) = 0$,
- the number of indecomposable direct summands of M (up to isomorphism) is the same as the number of simple Λ-modules, i.e. $|M| = |\Lambda|$.

This notation was introduced in [1].

The following result is useful.

LEMMA 2.1. *Let M be a Λ-module with a minimal projective presentation $P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} M \to 0$. Then for any $N \in \operatorname{mod} \Lambda$, $\operatorname{Hom}_{\Lambda}(N, \tau M) = 0$ if and only if the induced map $\operatorname{Hom}_{\Lambda}(d_1, N) : \operatorname{Hom}_{\Lambda}(P_0, N) \to \operatorname{Hom}_{\Lambda}(P_1, N)$ is surjective.*

Proof. This is standard. See for example [1, Prop. 2.4] ∎

2.2. Support τ-tilting modules. As before, let Λ be a finite-dimensional basic algebra, and let $\operatorname{proj} \Lambda$ be the category of projective Λ-modules. For a Λ-module M, let $\operatorname{add} M$ denote the full subcategory of $\operatorname{mod} \Lambda$ with objects all direct summands of direct sums of copies of M.

Then M is called *support τ-tilting* if there exists an idempotent e of Λ such that M is a τ-tilting $(\Lambda/\langle e \rangle)$-module. This is the original definition of support τ-tilting modules from [1], and it was proved in [1] that this terminology can be replaced by a support τ-tilting pair. A pair (M, P) with $M \in \operatorname{mod} \Lambda$ and $P \in \operatorname{proj} \Lambda$ is called *support τ-tilting* if

- (M, P) is a *τ-rigid* pair, i.e. M is τ-rigid and $\operatorname{Hom}_{\Lambda}(P, M) = 0$,
- $|M| + |P| = |\Lambda|$.

LEMMA 2.2 ([1, Prop. 2.3]). *Let (M,P) be a pair with $M \in \operatorname{mod} \Lambda$ and $P \in \operatorname{proj} \Lambda$. Let e be an idempotent of Λ such that $\operatorname{add} P = \operatorname{add} \Lambda e$. Then (M,P) is a support τ-tilting pair for Λ if and only if M is a τ-tilting $(\Lambda/\langle e\rangle)$-module. Moreover, M determines P and e uniquely.*

Recall that a Λ-module N is called *sincere* if $\operatorname{Hom}_\Lambda(P,N) \neq 0$ for all indecomposable projective Λ-modules P. It was proved in [1, Prop. 2.2] that the τ-tilting modules are precisely the sincere support τ-tilting modules.

2.3. Maximal rigid objects. Let $\mathcal{C}$ be a Krull–Remak–Schmidt triangulated k-linear category with split idempotents and suspension functor S. We suppose that all Hom-spaces of $\mathcal{C}$ are finite-dimensional and that $\mathcal{C}$ admits a Serre functor Σ, i.e. for any X, Y in $\mathcal{C}$, we have the following bifunctorial isomorphisms:

$$\operatorname{Hom}_{\mathcal{C}}(X,Y) \simeq D \operatorname{Hom}_{\mathcal{C}}(Y, \Sigma X),$$

where $D = \operatorname{Hom}_k(-,k)$ is the usual duality. We suppose that $\mathcal{C}$ is *Calabi–Yau of CY-dimension* 2, i.e. there is an isomorphism of triangle functors

$$S^2 \xrightarrow{\sim} \Sigma.$$

For $X, Y \in \mathcal{C}$ and $n \in \mathbb{Z}$, we set as usual

$$\operatorname{Ext}^n_{\mathcal{C}}(X,Y) = \operatorname{Hom}_{\mathcal{C}}(X, S^n Y).$$

Thus the Calabi–Yau property can be written as the following bifunctorial isomorphisms:

$$D \operatorname{Ext}^1(X,Y) \simeq \operatorname{Ext}^1(Y,X) \quad \text{for any } X, Y.$$

An object R of $\mathcal{C}$ is *maximal rigid* if

- R is *rigid*, i.e. $\operatorname{Ext}^1_{\mathcal{C}}(R,R) = 0$,
- $\operatorname{Ext}^1_{\mathcal{C}}(X \oplus R, X \oplus R) = 0$ implies that $X \in \operatorname{add} R$, the additive category of direct sums of direct summands of R.

A rigid object R is called *cluster tilting* if $\operatorname{Ext}^1_{\mathcal{C}}(X,R) = 0$ implies that $X \in \operatorname{add} R$.

Let R be a maximal rigid object in $\mathcal{C}$. An object M of $\mathcal{C}$ is *finitely presented* by R if there is a triangle in $\mathcal{C}$:

$$R_1 \to R_0 \xrightarrow{f} M \to SR_1$$

with R_0, R_1 in $\operatorname{add} R$. The morphism f is necessarily a right $\operatorname{add} R$-approximation of M, and conversely, the cone of any $\operatorname{add} R$-approximation of an object M finitely presented by R belongs to $\operatorname{add} SR$ (see [20]). Let $\operatorname{pr}(R)$ denote the (full) subcategory of $\mathcal{C}$ of objects finitely presented by R.

The endomorphism algebras of maximal rigid objects in 2-Calabi–Yau triangulated categories have been investigated in [19, 20, 15]. Let Γ be the endomorphism algebra of R. The following result expresses a close relationship between 2-Calabi–Yau triangulated categories and the module categories of endomorphism algebras of maximal rigid objects in 2-Calabi–Yau triangulated categories (see [12, 20]).

LEMMA 2.3. *The functor* $F = \mathrm{Hom}_{\mathcal{C}}(R, -) : \mathcal{C} \to \operatorname{mod} \Gamma$ *induces an equivalence*

$$\mathrm{pr}(R)/\operatorname{add} SR \xrightarrow{\sim} \operatorname{mod} \Gamma,$$

where the category on the left has the same objects as $\mathrm{pr}(R)$*, with morphisms given by morphisms in* $\mathcal{C}$ *modulo maps factoring through objects in* $\operatorname{add} SR$*.*

Thus the functor F induces an equivalence from $\operatorname{add} R$ to the category of projective modules in $\operatorname{mod} \Gamma$, and each Γ-module has the form FM for M in $\mathrm{pr}(R)$. Since our focus is on the quotient category $\mathrm{pr}(R)/\operatorname{add} SR$, when we mention a Γ-module FM we shall always assume that $\operatorname{add} M \cap \operatorname{add} SR = 0$.

3. Main results. Let $\mathcal{C}$ be a 2-Calabi–Yau category with a maximal rigid object R, let Γ be the endomorphism algebra of R in $\mathcal{C}$ and let F be the functor $\mathrm{Hom}_{\mathcal{C}}(R, -) : \mathcal{C} \to \operatorname{mod} \Gamma$. We write $\mathcal{C}(X, Y)$ for the set of morphisms from X to Y in the category $\mathcal{C}$, and $\Gamma(X, Y)$ for the one in the category $\operatorname{mod} \Gamma$.

3.1. Rigid and τ-rigid. This subsection compares τ-rigid Γ-modules and rigid objects in $\mathcal{C}$. First we show that a τ-rigid Γ-module lifts to a rigid object in $\mathcal{C}$.

LEMMA 3.1. *If two indecomposable* Γ*-module* FM, FN *satisfy*

$$\Gamma(FM, \tau FN) = 0 = \Gamma(FN, \tau FM),$$

then

$$\mathrm{Ext}^1_{\mathcal{C}}(M, N) = 0 = \mathrm{Ext}^1_{\mathcal{C}}(N, M).$$

Proof. Let

$$R_1^M \xrightarrow{p_1^M} R_0^M \xrightarrow{p_0^M} M \xrightarrow{\eta} SR_1^M \tag{1}$$

be the approximation triangle for M. Applying the functor $\mathrm{Hom}_{\mathcal{C}}(-, SN)$ to (1), we have the following exact sequence:

$$\begin{aligned} \mathcal{C}(SR_0^M, SN) &\xrightarrow{\mathcal{C}(Sp_1^M, SN)} \mathcal{C}(SR_1^M, SN) \\ &\to \mathcal{C}(M, SN) \xrightarrow{\mathcal{C}(p_0^M, SN)} \mathcal{C}(R_0^M, SN). \end{aligned} \tag{2}$$

On the other hand, applying $F = \mathrm{Hom}_{\mathcal{C}}(R, -)$ to (1) gives the minimal projective presentation

$$FR_1^M \xrightarrow{Fp_1^M} FR_0^M \to FM \to 0.$$

Note that

$$\Gamma(FN, \tau FM) = 0,$$

and Lemma 2.1 implies that $\Gamma(Fp_1^M, FN)$ is surjective. Since R is rigid,

$$\Gamma(Fp_1^M, FN) = \mathcal{C}(p_1^M, N).$$

Hence $\mathcal{C}(p_1^M, N)$ is surjective. Combining with (2) we get

$$\operatorname{Ker}\mathcal{C}(p_0^M, SN) \simeq \operatorname{Coker}\mathcal{C}(Sp_1^M, SN) \simeq \operatorname{Coker}\mathcal{C}(p_1^M, N) = 0.$$

Let α be any morphism in $\mathcal{C}(M, SN)$ with $\alpha \cdot p_0^M = 0$. Then α factors through SR_1^M. In turn, if α factors through objects in $\operatorname{add} SR$, then $\alpha \cdot p_0^M = 0$. It follows that if we denote by $\mathcal{C}_{SR}(M, SN)$ the class of maps $\alpha : M \to SN$ which factor through an object from $\operatorname{add} SR$, then

$$\mathcal{C}_{SR}(M, SN) = \operatorname{Ker}\mathcal{C}(p_0^M, SN) = 0. \tag{3}$$

Dually, letting $\mathcal{C}_{SR}(N, SM)$ be the class of maps $\phi : N \to SM$ which factor through an object from $\operatorname{add} SR$, one has

$$\mathcal{C}_{SR}(N, SM) = 0.$$

Applying the functor $\operatorname{Hom}_{\mathcal{C}}(N, -)$ to the triangle (1) we get the following exact sequence:

$$\mathcal{C}(N, SR_0^M) \xrightarrow{\mathcal{C}(N, Sp_0^M)} \mathcal{C}(N, SM) \xrightarrow{\mathcal{C}(N, S\eta)} \mathcal{C}(N, S^2R_1^M).$$

Since

$$\operatorname{Im}\mathcal{C}(N, Sp_0^M) \subset \mathcal{C}_{SR}(N, SM) = 0,$$

we know that

$$\mathcal{C}(N, S\eta) : \mathcal{C}(N, SM) \hookrightarrow \mathcal{C}(N, S^2R_1^M)$$

is injective. Thus by the definition of Serre functor and the 2-Calabi–Yau property there is an injective morphism

$$D\mathcal{C}(M, SN) \hookrightarrow D\mathcal{C}(SR_1^M, SN).$$

Then the dual $\mathcal{C}(SR_1^M, SN) \to \mathcal{C}(M, SN)$ is surjective, which yields

$$\mathcal{C}(M, SN) = \mathcal{C}_{SR}(M, SN).$$

Hence (3) yields

$$\operatorname{Ext}^1_{\mathcal{C}}(M, N) = 0.$$

Thanks to the 2-Calabi–Yau property, we get $\operatorname{Ext}^1_{\mathcal{C}}(N, M) = 0$, too. ■

We now prove that projection from $\operatorname{pr}(R)$ to $\operatorname{mod}\Gamma$ sends rigid objects to τ-rigid Γ-modules.

LEMMA 3.2. *Let FM, FN be two indecomposable Γ-modules. If*

$$\operatorname{Ext}^1_{\mathcal{C}}(M, N) = 0 = \operatorname{Ext}^1_{\mathcal{C}}(N, M)$$

then

$$\Gamma(FM, \tau FN) = 0 = \Gamma(FN, \tau FM).$$

Proof. First we show that $\Gamma(FN, \tau FM) = 0$. Let

$$R_1^M \xrightarrow{p_1^M} R_0^M \xrightarrow{p_0^M} M \to SR_1^M \tag{4}$$

be the "minimal add T-approximation triangle" of M. Applying F to (4), we get an exact sequence in mod Γ:

$$FR_1^M \xrightarrow{Fp_1^M} FR_0^M \xrightarrow{Fp_0^M} FM \to 0,$$

which is a minimal projective presentation for FM. Combining this with Lemma 2.1, we only need to show that

$$\Gamma(Fp_1^M, FN) : \Gamma(FR_0^M, FN) \to \Gamma(FR_1^M, FN)$$

is surjective. Note that R is rigid; it follows that

$$\Gamma(Fp_1^M, FN) = \mathcal{C}(p_1^M, N)$$

by the definition of quotient category. Thus it suffices for us to prove that $\mathcal{C}(p_1^M, N)$ is surjective. For this, applying the functor $\mathrm{Hom}_{\mathcal{C}}(-, N)$ to (4), one gets the following exact sequence:

$$\mathcal{C}(M, N) \to \mathcal{C}(R_0^M, N) \xrightarrow{\mathcal{C}(p_1^M, N)} \mathcal{C}(R_1^M, N) \to \mathcal{C}(S^- M, N).$$

But

$$\mathcal{C}(S^- M, N) \simeq \mathcal{C}(M, SN) = \mathrm{Ext}^1_{\mathcal{C}}(M, N) = 0.$$

Thus $\mathcal{C}(p_1^M, N)$ is a surjective map, which implies that $\Gamma(FN, \tau FM) = 0$. We can dually prove that $\Gamma(FM, \tau FN) = 0$. ∎

3.2. Lifting to a maximal rigid object. This subsection is devoted to proving that a support τ-tilting Γ-module can be lifted via F to a maximal rigid object in $\mathcal{C}$.

THEOREM 3.3. *Let $\mathcal{C}$ be a 2-Calabi–Yau triangulated category with a maximal rigid object R, let Γ be the endomorphism algebra of R in $\mathcal{C}$ and let F be the functor* $\mathrm{Hom}_{\mathcal{C}}(T, -) : \mathcal{C} \to \mathrm{mod}\,\Gamma$. *Let (FL', FR_L) be a support τ-tilting pair for Γ. Then $L = L' \oplus SR_L$ is a maximal rigid object in $\mathcal{C}$. In particular, if FL is a τ-tilting module over Γ, then FL lifts uniquely to a maximal rigid object in $\mathcal{C}$.*

Proof. By definition, FL' is a τ-rigid module over Γ. By Lemma 3.1, L' is a rigid object in $\mathcal{C}$. Since $R_L \in \mathrm{add}\,R$, we have

$$\Gamma(FR_L, FL') = \mathcal{C}(R_L, L').$$

Note that (FL', FR_L) is a τ-rigid pair,

$$\mathrm{Ext}^1_{\mathcal{C}}(SR_L, L') = \mathcal{C}(SR_L, SL') \simeq \mathcal{C}(R_L, L') = \Gamma(FR_L, FL') = 0.$$

By the 2-Calabi–Yau property,

$$\mathrm{Ext}^1_{\mathcal{C}}(L', SR_L) \simeq D\,\mathrm{Ext}^1_{\mathcal{C}}(SR_L, L') = 0.$$

It follows that $L = L' \oplus SR_L$ is a rigid object in $\mathcal{C}$.

Suppose there is an indecomposable object X such that $\mathrm{Ext}^1_{\mathcal{C}}(X\oplus L, X\oplus L) = 0$. There are two cases to consider. First we claim that if $X \in \mathrm{add}\, SR$ then $X \in \mathrm{add}\, SR_L$. In fact, let $X = SR_X$ for some $R_X \in \mathrm{add}\, R$ and $R_X \notin \mathrm{add}\, R_L$. Since (FL', FR_L) is a support τ-tilting pair, FL' is a τ-tilting, thus sincere, $(\Gamma/\langle e\rangle)$-module, where e is an idempotent of Γ such that $\mathrm{add}\, FR_L = \mathrm{add}\, \Gamma e$. It follows that

$$\Gamma(FR_X, FL') \neq 0. \tag{5}$$

On the other hand, we clearly have

$$0 = \mathrm{Ext}^1_{\mathcal{C}}(X, L') = \mathcal{C}(SR_X, SL') \simeq \mathcal{C}(R_X, L') = \Gamma(FR_X, FL'), \tag{6}$$

which contradicts (5). Hence $X \in \mathrm{add}\, L$.

The second case is that $X \notin \mathrm{add}\, SR$. Note that in the 2-Calabi–Yau triangulated category $\mathcal{C}$, every rigid object belongs to $\mathrm{pr}(R)$ [21, Cor. 2.5]. Then by Lemma 3.2, $FX \oplus FL'$ is a τ-rigid Γ-module. Moreover,

$$\Gamma(FR_L, FX \oplus FL') = \mathcal{C}(R_L, X \oplus L') \simeq \mathrm{Ext}^1_{\mathcal{C}}(SR_L, X \oplus L') = 0.$$

That is, $(FX \oplus FL', FR_L)$ is a τ-rigid pair. Since (FL', FR_L) is a τ-tilting pair, we have

$$|FX| + |FL'| + |FR_L| \le |\Gamma| = |FL'| + |FR_L|.$$

It follows that $X \in \mathrm{add}\, L'$, hence $X \in \mathrm{add}\, L$. Putting all this together we deduce that L is maximal rigid in $\mathcal{C}$.

Now let FL be a τ-tilting Γ-module. We only need to prove that the projection functor $\mathrm{pr}(R) \to \mathrm{pr}(R)/\mathrm{add}\, SR$ does provide a unique lift. Note that the preimage of FL has the form $L \oplus X$ for some $X \in \mathrm{add}\, SR$. By Lemma 3.1, $L \oplus X$ is a rigid object in $\mathcal{C}$. But this cannot be true if X is not zero. To see this, let $X = SR_X$ for some $R_X \in \mathrm{add}\, R$; then

$$0 = \mathrm{Ext}^1_{\mathcal{C}}(X, L) = \mathcal{C}(SR_X, SL) \simeq \mathcal{C}(R_X, L) = \Gamma(FR_X, FL),$$

which contradicts FL being sincere. ∎

Since tilting modules over Γ are all τ-tilting [2, Cor. IV.2.14], we recover the following result directly.

Corollary 3.4. *Let R be a maximal rigid object in the 2-Calabi–Yau triangulated category $\mathcal{C}$, let Γ be the endomorphism algebra of R, and let FL be a tilting module over Γ. Then FL lifts uniquely to a maximal rigid object in $\mathcal{C}$.*

This result, which generalizes a corresponding one in [18] in the case of cluster-tilted algebras and of [9, 11] in the case of 2-Calabi–Yau tilted algebras, was proved in [15].

3.3. Projecting to a support τ-tilting module. This subsection proves that projection from $\mathrm{pr}(R)$ to $\operatorname{mod} \Gamma$ sends maximal rigid objects to support τ-tilting Γ-modules.

THEOREM 3.5. *Let $\mathcal{C}$ be a 2-Calabi–Yau triangulated category with a maximal rigid object R, let Γ be the endomorphism algebra of R in $\mathcal{C}$ and let F be the functor $\mathrm{Hom}_{\mathcal{C}}(R,-) : \mathcal{C} \to \operatorname{mod} \Gamma$. Let $L = L' \oplus SR_L$ be an object in $\mathcal{C}$ such that SR_L is the maximal direct summand of L which belongs to $\operatorname{add} SR$. If L is maximal rigid, then the pair (FL', FR_L) is support τ-tilting for Γ. In particular, for any maximal rigid object M, if $\operatorname{add} M \cap \operatorname{add} SR = 0$, then FM is a τ-tilting Γ-module.*

Proof. Note that L' is rigid in $\mathcal{C}$, and so Lemma 3.2 implies that FL' is τ-rigid in $\operatorname{mod} \Gamma$. Since $R_L \in \operatorname{add} R$ and $L' \oplus SR_L$ is maximal rigid,

$$\Gamma(FR_L, FL') = \mathcal{C}(R_L, L') \simeq \mathcal{C}(SR_L, SL') = \mathrm{Ext}^1_{\mathcal{C}}(SR_L, L') = 0.$$

Hence (FL', FR_L) is a τ-rigid pair, or equivalently, FL' is a τ-rigid $(\Gamma/\langle e\rangle)$-module, where e is an idempotent of Γ such that $\operatorname{add} FR_L = \operatorname{add} \Gamma e$. As any τ-rigid module is a direct summand of some τ-tilting module [1, Theorem 0.2], there exists some FX such that $FL' \oplus FX$ is a τ-tilting $(\Gamma/\langle e\rangle)$-module. That is, $(FL' \oplus FX, FR_L)$ is a support τ-tilting pair. By Theorem 3.3, we have

$$\mathrm{Ext}^1_{\mathcal{C}}(L' \oplus X \oplus SR_L, L' \oplus X \oplus SR_L) = \mathrm{Ext}^1_{\mathcal{C}}(L \oplus X, L \oplus X) = 0.$$

Since L is maximal, we have $X \in \operatorname{add} L$, i.e. $FX \in \operatorname{add} FL'$. Thus (FL', FR_L) is support τ-tilting for Γ.

Clearly, if M is a maximal rigid object with $\operatorname{add} M \cap \operatorname{add} SR = 0$, then FM is a τ-tilting Γ-module. ∎

Combining Theorems 3.3 and 3.5, we recover the following result about the number of indecomposable direct summands of maximal rigid objects, which was first proved in [21].

COROLLARY 3.6. *All maximal rigid objects in a 2-Calabi–Yau triangulated category have the same number of indecomposable direct summands (up to isomorphism).*

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