COLLOQUIUM MATHEMATICUM

VOL. 142 2016 NO. 2

UPPER ESTIMATES ON SELF-PERIMETERS OF UNIT CIRCLES FOR GAUGES

BY

HORST MARTINI (Chemnitz) and ANATOLIY SHCHERBA (Cherkasy)

Abstract. Let M^2 denote a Minkowski plane, i.e., an affine plane whose metric is a gauge induced by a compact convex figure B which, as a unit circle of M^2, is not necessarily centered at the origin. Hence the self-perimeter of B has two values depending on the orientation of measuring it. We prove that this self-perimeter of B is bounded from above by the four-fold self-diameter of B. In addition, we derive a related non-trivial result on Minkowski planes whose unit circles are quadrangles.

1. Basic notions and main results. Let A^2 be an affine plane. In what follows, we identify the points of A^2 with their position vectors. Denote by $R^2 := (A^2, |\cdot|)$ the adjoint Euclidean plane with the Euclidean norm $|\cdot|$ which we use as an auxiliary metric. Let B be a compact convex figure on A^2 containing the origin O as an interior point. By ∂B and $\mathrm{int}(B)$ we denote the *boundary* and the *interior* of B, respectively. Each pair $(B;O)$ uniquely defines a *convex distance function* or *gauge* $g_B(x)$. Namely, if $x \in A^2$, $x \neq O$, and $\widehat{x} \in \partial B$ is on the ray $\overrightarrow{Ox}$, then

$$g_B(x) = |x|/|\widehat{x}| > 0. \tag{1}$$

The distance function $g_B(x)$ defines the *distance* between $x, y \in A^2$ by

$$\rho_B(x;y) = g_B(y-x). \tag{2}$$

Definition 1.1. An affine plane A^2 with metric ρ_B given by (2) and (1) is called a *Minkowski plane* M^2. The point O is called the *origin* of M^2. The figure B is called the *normalizing figure* or *unit circle* (or *gauge*) of M^2.

We note that the notion of "Minkowski plane" is frequently used also for the case of *normed planes*, where B has to be centered at O (see [18], [13], and [12]). However, it is to be noted for historical correctness that H. Minkowski, giving the axiomatic foundations of the relevant theory, also considered the general (non-symmetric) case.

2010 *Mathematics Subject Classification*: 28A75, 46B20, 52A10, 52A21, 52A38, 52A40.
Key words and phrases: convex distance functions, gauges, Minkowski plane, normalizing figures, self-diameter, self-perimeter.

DOI: 10.4064/cm142-2-3

In the following, we write ab, $\overrightarrow{ab}$, and (ab) for the *segment*, *ray*, and *line* determined by two distinct points $a, b \in A^2$ (with a as starting point in the second case), and we denote by $\angle abc$ the (oriented) angle with apex b. For *triangles* we write $\triangle abc$, for *quadrangles* $abcd$, and a *polygonal arc* is denoted by $\widehat{abc}$, with vertices a, b, c. The symbols $\sim$ and $\approx$ are used for *similarity* and *homothety*, respectively, and $\|$ stands for parallelity.

For a given segment ab in M^2, the distance $\rho_B(a; b)$ is called the *length* of this segment.

DEFINITION 1.2. For a given segment ab $(a \neq b)$ the position vector of the point $\widehat{b-a} \in \partial B$ defined by

$$\widehat{b-a} = (b-a)/\rho_B(a;b) \tag{3}$$

is called the *normalizing vector* of the segment.

Let K be a compact, convex figure in M^2. Denote by $L_B^+(K)$ the length of ∂K measured counter-clockwise, and by $L_B^-(K)$ the length of ∂K measured clockwise. Clearly, affine transformations of the plane preserve the collinearity of vectors (see [6, pp. 75–76]). Thus, from (1) and (2) it follows that the length of $\rho_B(a; b)$ and $L_B^{\pm}(K)$ are affine invariants of the plane M^2 (see also [13, p. 5]).

It is known that if M is a convex figure inside K, then (see [7, p. 110] and [18, p. 112]) then

$$L_B^{\mp}(M) \leq L_B^{\mp}(K). \tag{4}$$

In what follows, we call $L^-(B) = L_B^-(B)$ the *first self-perimeter of the unit circle* B, and $L^+(B) = L_B^+(B)$ denotes its *second self-perimeter*. Gołąb [2] proved that if B is symmetric with respect to the origin O (i.e., M^2 is a *normed plane*), then $L^-(B) = L^+(B) =: L(B)$, with the sharp estimates

$$6 \leq L(B) \leq 8. \tag{5}$$

If B is not centred at O, then still $L^{\mp}(B) \geq 6$. The equality $L^-(B) = 6$ or $L^+(B) = 6$ holds if and only if B is an affinely regular hexagon (see [3], [16], [17], and [11]). Simple examples show that there is no absolute constant that bounds the self-perimeters $L^{\mp}(B)$ for non-symmetric normalizing figures from above. Grünbaum [4] proved that it is possible to choose the origin O inside B in such a way that the self-perimeters satisfy

$$L^{\mp}(B) \leq 9. \tag{6}$$

The estimate (6) cannot be improved if B is a triangle $\triangle$, i.e., in fact $\min_{O \in \mathrm{int}(\triangle)} L^{\mp}(\triangle) = 9$. Further results in this direction were derived in [3], [16], [17], [9], and [10].

DEFINITION 1.3. The value

$$D(B) = \max_{x,y \in B} \rho(x;y) \tag{7}$$

is called the *self-diameter* of the normalizing figure B of M^2.

In the present paper we give upper estimates on the self-perimeters $L^{\mp}(B)$ in terms of the self-diameter $D = D(B)$ of the unit circle B of a Minkowski plane M^2. Our main results are summarized in the following theorems.

THEOREM 1.1. *If B is a unit circle of self-diameter $D = D(B)$, then*

$$L^{\mp}(B) \leq 4D(B). \tag{8}$$

We note that Theorem 1.1 is an almost immediate extension of the result of Gołąb [2], and it is sharp for centrally symmetric figures. On the other hand, our next theorem generalizes all three results: of Gołąb [2], of Grünbaum [4], and our Theorem 1.1.

THEOREM 1.2. *If P_4 is a normalizing quadrangle of diameter $D = D(P_4)$, then*

$$L^{\mp}(P_4) \leq 2(D(P_4))^2/(D(P_4) - 1). \tag{9}$$

This estimate is sharp.

It should be noticed that (9) implies (8), (6), and the right-hand inequality of (5) for all polygons with at most four vertices.

The proof of Theorem 1.2, via special constructions, can be reduced to the case when the quadrangle is a trapezium. These constructions are interesting in their own right, and we collect the related results in the following theorem.

THEOREM 1.3. *For a normalizing quadrangle P_4 there is a trapezium T such that*

(i) $O \in \mathrm{int}(T)$;

(ii) *the self-diameters of P_4 and T satisfy*

$$D(T) \leq D(P_4); \tag{10}$$

(iii) *the self-perimeters of P_4 and T satisfy*

$$L^{-}(T) \geq L^{-}(P_4). \tag{11}$$

2. Proofs and further results. To prove these theorems, we need some additional properties of self-diameters of normalizing figures. Without loss of generality, we consider the normalizing figure B as lying in the adjoint Euclidean plane R^2. We intend to prove that the diameter $D(B)$ uniquely defines the factor of symmetry $k = k(B)$ of the figure B with respect to the origin $O \in \mathrm{int}(B)$. The factor of symmetry (cf. Definition 2.2 below) was introduced by H. Minkowski and B. Neumann (see [14], [15], and [5, §6]).

Definition 2.1. A chord nm of the unit circle B is called *central* if it passes through the origin $O \in \operatorname{int}(B)$.

Set

$$g(nm) = \min\{|Om|/|On|; |On|/|Om|\} \leq 1,$$

where $n, m \in \partial B$ and $O \in nm$. Geometrically, $g(nm)$ is the ratio in which O divides the central chord nm of the figure B.

Definition 2.2. We define the *factor of symmetry* of the unit circle B by

$$k = \min_{nm} g(nm). \tag{12}$$

The *support function* $h_k(u)$, $|u| = 1$, of a compact convex figure $K \subset R^2$ is defined by

$$h_K(u) = \max\{\langle x, u\rangle : x \in K\},$$

where $\langle \cdot, \cdot \rangle$ means the *scalar product* of the Euclidean plane R^2 (see [1] and [7]).

B. Grünbaum [5, §6] remarks that the factor of symmetry $k(B)$ can, equivalently to (12), be defined as follows:

$$k = \min_{|u|=1}\{h_B(u)/h_B(-u); h_B(-u)/h_B(u)\}. \tag{13}$$

Proposition 2.1. *The diameter $D = D(B)$ and the factor of symmetry $k = k(B)$ of the unit circle B satisfy*

$$D(B) = 1 + 1/k. \tag{14}$$

Proof. Let nm be a central chord of B that provides the minimum in (12), and set $k = |Om|/|On|$. By (7) we have

$$D = \max_{x,y\in B} \rho(x; y) \geq \rho_B(n; m) = (|nO| + |Om|)/|Om| = 1 + 1/k.$$

To prove (14) it is sufficient to show that $D \leq 1 + 1/k$. Denote by pq the chord of B that provides the maximum in (7), i.e., $D = \rho_B(p; q) = |pq|/|On|$, where $n = \widehat{q - p}$ (see (3)). Set $\{m\} = (pO) \cap \partial B$. Since B is convex, there exists $\{l\} = On \cap qm$. The homothety $\triangle mOl \approx \triangle mpq$ implies

$$D = |pq|/|On| \leq |pq|/|Ol| = |pm|/|Om| = \rho_B(p; m) \leq D. \tag{15}$$

For the central chord pm it follows from (15) and (12) that

$$D = (|pO| + |Om|)/|Om| = 1 + 1/k. \blacksquare$$

Corollary 2.1. *If nm denotes a central chord of the unit circle B, then* $\max \rho_B(n; m) = D(B)$.

Corollary 2.2. *If pq is a chord of the unit circle B such that $\rho_B(p; q) = D(B)$, then the central chord pm has length $\rho_B(p; m) = D(B)$, and $qm \subset \partial B$.*

Indeed, (14) and (15) imply

$$1 + 1/k = D(B) = \rho_B(p; q) = |pq|/|On| = |pm|/|Om| = |pq|/|Ol|.$$

In this case $l = n = \widehat{q - p} \in qm$, and the convexity of B implies $qm \subset \partial B$.

PROPOSITION 2.2. *Let nm be a central chord of the unit circle B that provides the equality $\rho_B(n; m) = D(B)$. If $H(m)$ is a supporting line of B at $m \in \partial B$, then the line $H(n)$ that passes through $n \in \partial B$ in such a way that $H(n) \parallel H(m)$ is also a supporting line for B.*

Proof. By (14) we have $|Om|/|On| = k$, where $k = k(B)$ is the factor of symmetry. Assume that $H(n) \parallel H(m)$ is not a supporting line for B. Then there is a point $a \in \partial B$ such that $a \neq n$ and $aO \cap l(n) = b \neq a$. Write $\{c\} = H(m) \cap (aO)$ and $\{e\} = Oc \cap \partial B$. The homothety $\triangle Onb \approx \triangle Omc$ and the inequality $|Ob| < |Oa|$ imply $k = |Om|/|On| = |Oc|/|Ob| > |Oe|/|Oa|$. Since ae is a central chord, we get a contradiction to (12). ∎

COROLLARY 2.3. *Suppose that the polygon B with vertices $a_1, \ldots, a_l$ (in this order) is taken as a unit circle and a_ib_i are central chords of it* ($1 \leq i \leq l$)*. Then the factor of symmetry $k(B)$ is equal to*

$$k = \min\{|Ob_i|/|Oa_i| : 1 \leq i \leq l\}, \tag{16}$$

where the lengths of segments are given with respect to the auxiliary Euclidean metric.

Proof. Denote by nm a central chord of length $\rho_B(n; m) = D$, hence yielding $|Om|/|On| = k$. The existence of such a chord is guaranteed by Corollary 2.1. Consider first the case when m is one of the vertices of B, say $m = a_2$. Then the lines (a_1a_2) and (a_2a_3) are supporting ones for B at m. By Proposition 2.2, there are two different supporting lines $H_{1,2}(n)$ at $n \in \partial B$ such that $H_1(n) \parallel (a_1a_2)$ and $H_2(n) \parallel (a_2a_3)$. Therefore, n is also a vertex of B and (16) is fulfilled.

Now it is sufficient to consider the case when m and n do not coincide with a vertex of B. Suppose, for definiteness, that n is an interior point of a_1a_2. By Proposition 2.2, the supporting line $H(m)$ is parallel to a_1a_2. The line $H(m)$ contains one of the sides of B. Write $\{c_i\} = H(m) \cap (a_iO)$ and $\{b_i\} = \partial B \cap (a_iO)$ $(i = 1, 2)$. The homothety $\triangle Ona_i \approx \triangle Omc_i$ implies

$$k = |Om|/|On| = |Oc_i|/|Oa_i| \geq |Ob_i|/|Oa_i|.$$

Since a_ib_i are central chords of B, (12) implies $|Ob_i|/|Oa_i| = |Oc_i|/|Oa_i| = k$ and $c_i = b_i$. Moreover, the segment b_1b_2 is contained in ∂B. ∎

PROPOSITION 2.3. *Suppose that $O \in \operatorname{int}(B_1 \cap B_2)$, where B_1 and B_2 are compact, convex figures on R^2 with factors of symmetry $k(B_i) = k_i$ $(i = 1, 2)$. Then the factor of symmetry of the compact convex figure $B = B_1 \cap B_2$*

satisfies

$$k(B) \geq k_0 = \min\{k_1; k_2\}. \tag{17}$$

Proof. Denote by $h_i(u)$ ($|u| = 1$) the support functions for B_i ($i = 1, 2$). Then the support function for B is $h_B(u) = \min\{h_1(u); h_2(u)\}$ ($|u| = 1$). If

$$\begin{cases} h_B(u) = h_1(u), \\ h_B(-u) = h_1(-u), \end{cases} \quad \text{or} \quad \begin{cases} h_B(u) = h_2(u), \\ h_B(-u) = h_2(-u), \end{cases}$$

for some fixed unit vector u, then by (13) we have

$$k_0 \leq h_B(u)/h_B(-u) \leq 1/k_0.$$

Suppose, for definiteness, that $h_B(u) = h_1(u)$ and $h_B(-u) = h_2(-u)$. Then, again by (13), we have

$$\begin{aligned} k_0 &\leq h_1(u)/h_1(-u) \leq h_1(u)/h_2(-u) = h_B(u)/h_B(-u) \\ &\leq h_2(u)/h_2(-u) \leq 1/k_0, \end{aligned}$$

and (17) follows. ■

Corollary 2.4. *Suppose that $O \in M^2$ is an interior point of the segment nm. Denote by $H(n; m)$ the strip between two parallel lines $H(n) \parallel H(m)$ through n and m, respectively. If $k(B) = k$ and*

$$k_1 \leq |Om|/|On| \leq 1/k_1 \tag{18}$$

with respect to an auxiliary Euclidean metric, then the factor of symmetry of the convex figure $\widetilde{B} = B \cap H(n; m)$ satisfies

$$k(\widetilde{B}) \geq \min\{k; k_1\}. \tag{19}$$

Proposition 2.4. *If the unit circle of M^2 is the triangle $B = \triangle a_1a_2a_3$, then the factor of symmetry $k(B) = k$ satisfies $0 < k \leq 1/2$, and the oriented self-perimeters satisfy the following sharp estimates:*

$$5 + 4k + 1/k \leq L^{\mp}(B) \leq 3 + 2(1/k + k/(1-k)). \tag{20}$$

Proof. The factor of symmetry k and the self-perimeter of $B \subset M^2$ are invariant with respect to the choice of an auxiliary Cartesian metric in the adjoint plane R^2. Therefore, we may assume that $\triangle a_1a_2a_3$ is a right triangle. Denote by N the barycenter of $\triangle a_1a_2a_3$. Then we have $\triangle a_1a_2a_3 = \triangle a_1a_2N \cup \triangle a_2a_3N \cup \triangle a_3a_1N$. Write

$$\{b_1\} = a_2a_3 \cap (a_1O), \quad \{b_2\} = a_3a_1 \cap (a_2O), \quad \{b_3\} = a_1a_2 \cap (a_3O).$$

Let us prove that if $O \in \triangle a_3Na_2$, then $k = |Ob_1|/|Oa_1|$. By Corollary 2.3, it is sufficient to show that

$$|Ob_1|/|Oa_1| \leq |Ob_{2,3}|/|Oa_{2,3}|.$$

We present the proof for the first of them. Write $\{M\} = a_1b_1 \cap (a_3N)$ and $\{c\} = a_1a_3 \cap (a_2M)$. Since $\triangle a_1a_2a_3$ is a right triangle, we have $\triangle a_2Ma_1 \approx$

$\triangle cMb_1$ and $|Mb_1|/|Ma_1| = |cM|/|Ma_2|$. Take $g \in a_3b_1$ such that $cg \parallel a_1b_1$ and $\{e\} = cg \cap a_2b_2$. The homothety $\triangle a_2OM \approx \triangle a_2ec$ implies

$$|Ob_1|/|Oa_1| \le |Mb_1|/|Ma_1| = |cM|/|Ma_2| = |eO|/|a_2O| \le |b_2O|/|a_2O|.$$

Let $\{P\} = a_2a_3 \cap (a_1N)$, $Q \in NP$, and $OQ \parallel a_2a_3$. Then $\triangle a_1b_1P \approx \triangle a_1OQ$, and therefore

$$k = |Ob_1|/|Oa_1| = |PQ|/|a_1Q| \le |PN|/|a_1N| = 1/2.$$

Observe that, by duality, it is sufficient to prove (20) for $L^-(B)$ only. Mark the vertices of $\triangle a_1a_2a_3$ clockwise. Write $\{S\} = Na_3 \cap (OQ)$ and $\{T\} = Na_2 \cap (OQ)$. For every $V \in ST$, set $\{W\} = a_2a_3 \cap (a_1V)$. Evidently, $|VW|/|Va_1| = |Ob_1|/|Oa_1| = k$. Denote by $L_V^-(B)$ the first self-perimeter of $\triangle a_1a_2a_3$ in case when the origin $O \in M^2$ is located at V. *The function* $f(V) = L_V^-(B)$ *is strictly convex downwards for* $V \in ST$. This is a special case of a more general statement from [8]: the self-perimeter $L_V^\pm(B)$ is a strictly convex function of its center V, for any normalizing figure B of the plane M^2.

Since $f(V)$ is convex and symmetric with respect to $Q \in ST$, we have

$$\min_{V\in ST} L_V^-(B) = L_Q^-(B), \qquad \max_{V\in ST} L_V^-(B) = L_S^-(B) = L_T^-(B).$$

We calculate $L_S^-(B)$ in the adjoint plane R^2 with the Cartesian coordinate system such that the vertices of the relevant triangle get the coordinates

$$a_3(0;0),\ a_1(0;1+k),\ a_2(1+k;0).$$

Then the points S, T, and Q get the coordinates $S(k;k)$, $T(1-k;k)$, and $Q(1/2;k)$, respectively. It is easy to see that

$$\rho_S(a_3;a_1) = (1+k)/(1-k), \qquad \rho_S(a_1;a_2) = \rho_S(a_2;a_3) = (1+k)/k.$$

Therefore, $L^-(B) \le L_S^-(B) = 3 + 2(1/k + k/(1-k))$. For $L_Q^-(B)$ we have $\rho_Q(a_1;a_2) = (1+k)/k$ and $\rho_Q(a_2;a_3) = \rho_Q(a_3;a_1) = 2(1+k)$. Hence $L^-(B) \ge L_Q^-(B) = 5 + 1/k + 4k$. Evidently, the estimates in (20) are sharp, i.e., they can be achieved. ∎

Corollary 2.5. *If the normalizing quadrangle* P_4 *degenerates to a triangle, then the estimate* (9) *is still valid.*

Evidently, for $0 < k \le 1/2$ we have $2k/(1-k) \le 2k+1$. This inequality together with (20) and (14) implies $L^\mp(\triangle) \le 4 + 2(1/k + k) = 2D^2/(D-1)$.

The following example shows the sharpness of (9). The unit circle in this example is a quadrangle with given factor of symmetry.

Example 2.1. Endow a plane R^2 with a Cartesian coordinate system, origin $O(0;0)$, and a trapezium $a_1a_2a_3a_4$ with vertices

$$a_1(-k;-1),\ a_2(-k;k),\ a_3(t;k),\ a_4(1;-1), \quad k \in (0;1],\ t \in [k^2;1],$$

as a normalizing figure B.

To find the factor of symmetry $k(a_1a_2a_3a_4)$, mark the points $b_1(k^2;k) \in a_2a_3$ and $b_3(-k;-k^2/t) \in a_1a_2$. Since $|Oa_2|/|Oa_4| = k$, $|Ob_1|/|Oa_1| = k$, and $|Ob_3|/|Oa_3| = k/t$ ($\in [k;1/k]$), by (16) we have $k(a_1a_2a_3a_4) = k$. To find the self-perimeter $L^-(a_1a_2a_3a_4)$, evaluate the lengths of the sides of the trapezium using (1) and (3). Evidently, we have $(\widehat{a_1 - a_4})(-k;0)$ and $(\widehat{a_2 - a_1})(0;k)$, and hence $\rho(a_4;a_1) = \rho(a_1;a_2) = (1+k)/k$. Mark the points

$$c_1(t;0),\ c_2(1;0),\ c_3(0;-1), \widehat{a_3 - a_2} = c_4 \in a_3a_4,\ \widehat{a_4 - a_3} = c_5 \in a_4a_1.$$

Via the similarities $\triangle Oc_3c_5 \sim \triangle a_3c_1c_4 \sim \triangle a_4c_2c_4$, we find the points $c_4((k+t)/(k+1);0)$ and $c_5((1-t)/(k+1);-1)$. Then $\rho(a_2;a_3) = \rho(a_3;a_4) = 1+k$ and $L^-(a_1a_2a_3a_4) = 4 + 2(k+1/k)$. In accordance with (14) we have $L^-(a_1a_2a_3a_4) = 2D^2/(D-1)$.

Denote by $d(K_1;K_2)$ the *Hausdorff distance* between compact, convex sets K_1 and K_2 in R^2 (see, for instance, [5, §2]),

$$d(K_1;K_2) = \min\{\lambda \geq 0 : K_1 \subset K_2 + \lambda E,\ K_2 \subset K_1 + \lambda E\},$$

where E is the unit circle of R^2. A sequence of figures $B_1, B_2, \dots$ *converges to the figure* B if $d(B_\nu;B) \to 0$ as $\nu \to \infty$.

Proof of Theorem 1.1. For a compact, convex figure B with interior points, we apply a classical theorem on the approximation of B by polygons (see [1, §27]). There is a sequence $B_1, B_2, \dots$ of convex polygons which contain B and converge to it. By continuity for self-perimeters in M^2, we have

$$\lim_{v\to\infty} L^{\mp}(B_v) = L^{\mp}(B), \qquad \lim_{v\to\infty} D(B_v) = D(B).$$

Thus (8) is enough to prove our statement for a polygon B. Consider the centrally symmetric figure $\Delta B = \frac{1}{2}B + \frac{1}{2}(-B)$ (called the *central symmetral* of B), where $(-B) = (-1)B$. We can assume that B is a polygon with non-parallel sides. Then any side of ΔB is parallel either to a side of B or to a side of $-B$, and its length is half the length of the corresponding side of B or $-B$. Thus, for a normalizing figure C centered at O we have

$$L_C^{\mp}(\Delta B) = L_C^{\mp}(B). \tag{21}$$

According to Definition 2.2, for the symmetry coefficient k the inclusion $-B \subseteq \frac{1}{k}B$ holds. From this and from (14) we obtain

$$\Delta B = \frac{1}{2}(B - B) \subseteq \frac{1}{2}\left(1 + \frac{1}{k}\right)B = \frac{1}{2}DB,$$

i.e., DB contains $B+(-B)$ (the *difference body* of B). Therefore, the distance functions g_B and $g_{\Delta B}$ satisfy

$$g_B(x) = \frac{D}{2}g_{DB/2}(x) \leq \frac{D}{2}g_{\Delta B}(x)$$

(note that $g_{\Delta B}$ is an even function). Choosing in (21) the figure $C = \Delta B$, we obtain

$$L^{\mp}(B) = L^{\mp}_{B}(B) \le \frac{D}{2} L^{\mp}_{\Delta B}(\Delta B).$$

Applying (5) to the centrally symmetric figure ΔB, we come to (8), and Theorem 1.1 is proved. ∎

To prove Theorem 1.3 we need some auxiliary statements.

Proposition 2.5 (see [13] for details). *The equality in the triangle inequality $\rho_B(a;c) \le \rho_B(a;b) + \rho_B(b;c)$ for a Minkowski plane is only possible if the segment xy, where $x = \widehat{b-a}$ and $y = \widehat{c-b}$, lies on the boundary of the unit circle B.*

If the normalizing figure in M^2 is a polygon P_n, then we mark its vertices clockwise: $P_n = a_1 \dots a_n$. For completeness, we formulate here the analogues of Proposition 2 and Definitions 2 and 3 from [9] (see also [10, §3]).

Proposition 2.6. *Suppose the normalizing figure $P_4 = a_1a_2a_3a_4$ is not a trapezium. Then one can always choose an auxiliary metric and the order of the vertices in M^2 in such a way that the coordinates of the vertices become*

$$a_1(-(1+y)x/y; 1),\ a_2(1;1),\ a_3(1;0),\ a_4(0;-y),$$

where x and y are some positive numbers.

Definition 2.3. A normalizing quadrangle $a_1a_2a_3a_4 \subset M^2$ is called *canonically given* if it meets the requirements of Proposition 2.6.

Remark 2.1. In the notation of the canonically given quadrangle the first vertex is uniquely determined, i.e., if $a_1a_2a_3a_4$ is canonically given, then $a_2a_3a_4a_1$ is not.

Definition 2.4. If $a_1a_2a_3a_4$ is a canonically given quadrangle, then the point of intersection of the two lines through a_4 and a_3 which are parallel to a_3a_2 and a_2a_1, respectively, is called the *center* of the quadrangle.

Remark 2.2. In the auxiliary metric used for proving Proposition 2.6, the center g of the canonically given quadrangle $P_4 = a_1a_2a_3a_4$ coincides with the origin of the Cartesian coordinate system, i.e., $g = (0,0)$. We note that we will use also other auxiliary metrics on $\mathbb{R}^2$, with $g \neq (0,0)$; see, for example, the proof of Lemma 2.4.

Let $\{m\} = a_1a_3 \cap a_2a_4$. The diagonals a_1a_3 and a_2a_4 split the quadrangle $a_1a_2a_3a_4$ into four triangles, $\triangle a_1ma_4$, $\triangle a_2ma_1$, $\triangle a_3ma_2$, $\triangle\ a_4ma_3$.

Proposition 2.7. *Let $a_1a_2a_3a_4$ be a canonically given normalizing quadrangle. Let a_ib_i be its central chords $(0 \le i \le 4)$. With respect to our auxiliary metric, the factor of symmetry $k = k(a_1a_2a_3a_4)$ can be evaluated as follows:*

(a) *if the origin* O *is in* $\triangle a_1a_2a_4$*, then*

$$k = \min\{|Ob_i|/|Oa_i| : i \neq 3\}; \tag{22}$$

(b) *if* $O \in \triangle a_2a_3a_4$*, then*

$$k = |Ob_1|/|Oa_1|. \tag{23}$$

Proof. If $O \in \triangle a_1ma_4$, then $b_1 \in a_3a_4$, $b_{2,3} \in a_4a_1$, and $b_4 \in a_1a_2$. Find points $e_1 \in (a_1O)$ with $a_4e_1 \parallel a_1a_2$ and $e_2 \in (a_3O)$ with $b_2e_2 \parallel a_3a_2$. Since $a_1a_2a_3a_4$ is canonically given, we have $b_1 \in Oe_1$ and $e_2 \in Ob_3$. The homothety $\triangle Oa_2a_3 \approx \triangle Ob_2e_2$ implies

$$|Ob_2|/|Oa_2| = |Oe_2|/|Oa_3| \leq |Ob_3|/|Oa_3|. \tag{24}$$

If $O \in \triangle a_2ma_1$, then $b_1 \in a_2a_3$, $b_2 \in a_4a_1$, and $b_{3,4} \in a_1a_2$. Find e_3 in (Oa_3) with $a_4e_3 \parallel a_1a_2$. Since $a_1a_2a_3a_4$ is canonically given and $\triangle Ob_4b_3 \approx \triangle Oa_4e_3$, we have $|Ob_4|/|Oa_4| = |Ob_3|/|Oe_3| \leq |Ob_3|/|Oa_3|$. From this, together with (24) and (16), we obtain (22).

If $O \in \triangle a_3ma_2$, then $b_{1,4} \in a_2a_3$, $b_2 \in a_3a_4$, and $b_3 \in a_1a_2$. Find points e_i that satisfy $e_4 = (a_1a_2) \cap (a_4b_4)$; $e_1 \in (a_1b_1)$, $b_2e_1 \parallel a_1a_2$; $e_3 \in (a_4b_4)$, $a_3e_3 \parallel a_1a_2$; $e_2 \in (a_1b_1)$, $a_4e_2 \parallel a_3a_2$. The canonicity of $a_1a_2a_3a_4$ implies $b_4 \in Oe_4$, $b_1 \in Oe_1$, $e_3 \in Oa_4$, and $e_2 \in Oa_1$. The homotheties $\triangle Ob_1b_4 \approx \triangle Oe_2a_4$, $\triangle Oe_4b_3 \approx \triangle Oe_3a_3$, and $\triangle Oe_1b_2 \approx \triangle Oa_1a_2$ yield

$$\begin{aligned} |Ob_1|/|Oa_1| \leq |Ob_1|/|Oe_2| = |Ob_4|/|Oa_4| &\leq |Oe_4|/|Oa_4| \\ &\leq |Oe_4|/|Oe_3| = |Ob_3|/|Oa_3| \end{aligned}$$

and $|Ob_1|/|Oa_1| \leq |Oe_1|/|Oa_1| = |Ob_2|/|Oa_2|$. Combining this with (16), we get (23). ∎

If $O \in \triangle a_4ma_3$, then $b_{1,2} \in a_3a_4$, $b_3 \in a_4a_1$, and $b_4 \in a_2a_3$. Find points e_i that satisfy $e_1 \in (a_2b_2)$, $b_1e_1 \parallel a_1a_2$; $e_2 \in (a_4b_4)$, $b_2e_2 \parallel a_1a_2$; $e_3 \in (a_3b_3)$, $b_2e_3 \parallel a_2a_3$; $e_4 \in (a_4b_4)$, $b_1e_4 \parallel a_4a_1$. The canonicity of $a_1a_2a_3a_4$ implies $e_1 \in Ob_2$, $e_2 \in Oa_4$, $e_3 \in Ob_3$, and $e_4 \in Ob_4$. The homotheties $\triangle Ob_1e_1 \approx \triangle Oa_1a_2$, $\triangle Ob_2e_3 \approx \triangle Oa_2a_3$, and $\triangle Ob_1e_4 \approx \triangle Oa_1a_4$ yield

$$|Ob_1|/|Oa_1| = |Oe_1|/|Oa_2| \leq |Ob_2|/|Oa_2| = |Oe_3|/|Oa_3| \leq |Ob_3|/|Oa_3|;$$
$$|Ob_1|/|Oa_1| = |Oe_4|/|Oa_4| \leq |Ob_4|/|Oa_4|.$$

In combination with (16), we get (23). ∎

Our treatments essentially depend on the possible location of the origin O inside a canonically given quadrangle $a_1a_2a_3a_4$. *Denote by* g *the centre of the quadrangle* $a_1a_2a_3a_4$ and draw the lines (a_3g) and (a_4g). Set $\{u\} = a_4a_1 \cap (a_3g)$ and $\{w\} = a_1a_2 \cap (a_4g)$.

Definition 2.5. We use the following notation for normalizing vectors of the sides of a canonically given quadrangle $P_4 = a_1a_2a_3a_4$:

$$c_1 = \widehat{a_1 - a_4}, \quad c_2 = \widehat{a_2 - a_1}, \quad c_3 = \widehat{a_3 - a_2}, \quad c_4 = \widehat{a_4 - a_3}.$$

Observe that Definition 2.5 implies $c_1 \in a_1a_2$ and $c_4 \in a_4a_1$.

Set $\{v\} = a_1a_3 \cap a_4w$ and $\{n\} = a_2a_4 \cap a_3u$. Remember that we have already defined the points $\{g\} = a_3u \cap a_4w$ and $\{m\} = a_1a_3 \cap a_2a_4$. The chords a_3u, a_4w and the diagonals a_1a_3, a_2a_4 split the canonically given quadrangle $a_1a_2a_3a_4$ into nine parts: six triangles $\triangle a_1wv$, $\triangle a_3ma_2$, $\triangle uga_4$, $\triangle a_4gn$, $\triangle a_4na_3$, $\triangle nma_3$ and three quadrangles a_1vgu, wa_2mv, $vmng$. In view of Proposition 2.7 and Definition 2.5, the location of the origin O inside one of these parts uniquely defines the locations of c_i on the sides of $a_1a_2a_3a_4$ and implies either (22) or (23) for the factor of symmetry $k(a_1a_2a_3a_4)$.

Definition 2.6. We say that a normalizing quadrangle P_4 is *majorized by a trapezium* T if the trapezium meets all the requirements of Theorem 1.3, i.e., $O \in \operatorname{int}(T)$ and the inequalities (10) and (11) are satisfied.

Remark 2.3. In accordance with (14), it is possible to replace the inequality (10) in Definition 2.6 by the condition $k(P_4) \le k(T)$ on the respective factors of symmetry.

Remark 2.4. Let l_0 be a line through the origin $O \in \operatorname{int}(B)$. Let B' be a figure axially symmetric with respect to l_0. Then $L^{\mp}(B) = L^{\pm}(B')$. In what follows, we refer to this fact as *duality*. Due to duality, it is sufficient to prove Theorem 1.3 for the first self-perimeter $L^{-}(P_4)$ of the quadrangle P_4.

Remark 2.5. In what follows, we mark the lengths and self-perimeters with respect to an old and new normalizing figure B with subscript "old" or "new", respectively. Namely, if P is an old normalizing polygon and P' is the new one, then we write $L^{-}(P) = L^{-}_{\text{old}}(P)$ in case $B = P$, and $L^{-}(P') = L^{-}_{\text{new}}(P')$ in case $B = P'$.

The following two corollaries are consequences of our main theorems.

Lemma 2.1. *If $O \in \triangle a_1wa_4 \cup \triangle a_4ga_3$, then the canonically given quadrangle $a_1a_2a_3a_4$ can be majorized by some trapezium T.*

Proof. Observe that $\triangle a_1wa_4 = \triangle a_1va_4 \cup \triangle a_1wv$.

1. If $O \in \triangle a_1va_4$, then the normalizing vectors c_i and the endpoints b_i of the central chords a_ib_i are located as follows: $c_3 \in a_4a_1$, c_2 is on the *polygonal arc* $\widehat{a_2a_3a_4}$, $b_1 \in a_3a_4$, $b_{2,3} \in a_4a_1$, $b_4 \in a_1a_2$ (see Definition 2.5 and (22)). Find points a_5 and b_5 that satisfy $a_5 \in (a_2b_1)$, $a_4a_5 \parallel a_1a_2$, and $\{b_5\} = a_1a_2 \cap (a_5O)$. Taking the trapezium $a_1a_2a_5a_4$ as a new normalizing figure of M^2, we see that $(\widehat{a_1 - a_4})_{\text{new}} = (\widehat{a_1 - a_4})_{\text{old}} = c_1$, $(\widehat{a_2 - a_1})_{\text{new}} = c_2' \in a_2b_1 \subset a_2a_5$ and $|Oc_2'| \le |Oc_2|$, where a_2b_1 subtends the *arc* $\widehat{a_2a_3b_1}$. Then

$$\rho_{\text{old}}(a_4; a_1) = \rho_{\text{new}}(a_4; a_1), \qquad \rho_{\text{old}}(a_1; a_2) \le \rho_{\text{new}}(a_1; a_2).$$

Let $c_4' = \widehat{a_4 - a_5}$ and $c_5' = \widehat{a_5 - a_2} = \widehat{b_1 - a_2}$. Since $c_{3,4}, c_{4,5}' \in a_4a_1$, by

Proposition 2.5 we have $\rho_{\mathrm{old}}(a_2;a_3)+\rho_{\mathrm{old}}(a_3;a_4)=\rho_{\mathrm{old}}(a_2;a_4)=\rho_{\mathrm{new}}(a_2;a_4) = \rho_{\mathrm{new}}(a_2;a_5)+\rho_{\mathrm{new}}(a_5;a_4)$.

The homothety $\triangle Oa_5a_4 \approx \triangle Ob_5b_4$ implies $|Ob_5|/|Oa_5| = |Ob_4|/|Oa_4|$. The segments a_ib_i $(i = 1,2,4)$ are central chords of $a_1a_2a_3a_4$ and $a_1a_2a_5a_4$. By (22), we have $k(a_1a_2a_5a_4) = k(a_1a_2a_3a_4) = k$. Therefore, the trapezium $T = a_1a_2a_3a_4$ majorizes $a_1a_2a_3a_4$.

2. If $O \in \triangle a_1wv$, then the points c_i and b_i are located as follows: $c_3 \in a_4a_1$, $b_1 \in a_2a_3$, $b_2 \in a_4a_1$, $b_{3,4} \in a_1a_2$, $c_2 \in a_2b_1 \subset a_2a_3$. By Proposition 2.5, $\rho(b_1;a_4) = \rho(b_1;a_3)+\rho(a_3;a_4)$ and $L^-(a_1a_2a_3a_4) = L^-(a_1a_2b_1a_4)$. The segments a_ib_i $(i = 1,2,4)$ are central chords of $a_1a_2a_3a_4$ and $a_1a_2b_1a_4$. Therefore, $k(a_1a_2b_1a_4) = k$.

The quadrangle $a_1a_2b_1a_4$ is evidently a canonical one. Denote by g_1 its center and set $\{v_1\} = a_4w \cap a_1b_1$. By construction, $O \in \triangle a_1v_1a_4 \subset a_1a_2b_1a_4$, which corresponds to the first case considered above.

3. If $O \in \triangle a_4ga_3$, then the points c_i, b_i are located as follows: $c_{2,3} \in a_3a_4$, $b_3 \in a_4a_1$, $b_1 \in a_3a_4$, and b_4 is on the polygonal arc $\widehat{a_1a_2a_3}$. Canonicity of $a_1a_2a_3a_4$ implies the existence of $a_5 \in a_1a_2$ such that $a_3a_5 \parallel a_4a_1$. The trapezium $a_1a_5a_3a_4$ can be taken as a new normalizing figure of M^2, and then $\widehat{a_5-a_1} = c_2$, $\widehat{a_1-a_4} = c_1 \in a_1a_5 \subset a_1a_2$, $\widehat{a_3-a_5} = c_3' \in c_2c_3 \subset a_3a_4$. By Proposition 2.5 we have

$$\begin{aligned}\rho_{\mathrm{old}}(a_1;a_2)+\rho_{\mathrm{old}}(a_2;a_3) &= \rho_{\mathrm{old}}(a_1;a_3)\\ &= \rho_{\mathrm{new}}(a_1;a_3) = \rho_{\mathrm{new}}(a_1;a_5)+\rho_{\mathrm{new}}(a_5;a_3)\end{aligned}$$

and $L^-(a_1a_2a_3a_4) = L^-(a_1a_5a_3a_4)$.

To estimate the factor of symmetry $k(a_1a_5a_3a_4)$, we use Corollary 2.4. We have $(a_1a_4) \parallel (a_5a_3)$. Choosing in (18)

$$k_1 = \min\{|Ob_3|/|Oa_3|; |Oa_3|/|Ob_3|\}, \qquad k_1 \geq k,$$

we infer from (19) that $k(a_1a_5a_3a_4) \geq k$. Therefore, the trapezium $T = a_1a_5a_3a_4$ majorizes $a_1a_2a_3a_4$. Lemma 2.1 is proved. ∎

LEMMA 2.2. *If $O \in wa_2a_3v$, then the canonically given normalizing quadrangle $a_1a_2a_3a_4$ can be majorized by some trapezium T.*

Proof. Observe that the trapezium wa_2a_3v equals $wa_2mv \cup \triangle a_2a_3m$.

1. If $O \in wa_2mv$, then the normalizing vectors c_i and the ends b_i of the central chords a_ib_i are located as follows: $c_2 \in a_2a_3$, $c_3 \in a_3a_4$, $b_1 \in a_2a_3$, $b_2 \in a_4a_1$, $b_{3,4} \in a_1a_2$. Remember that in this case formula (22) is satisfied. Find a point a_5 such that $a_4a_5 \parallel a_2a_1$ and $a_5a_1 \parallel a_3a_2$. *For the polygonal arc $\widehat{a_3a_5a_1}$, we consider* $\{b_6\} = (a_2O) \cap \widehat{a_3a_5a_1}$. Then either $b_6 \in a_5a_1$ or $b_6 \in a_3a_5$. If $b_6 \in a_5a_1$, then the end b_5 of the central chord a_5b_5 in the trapezium $a_1a_2a_3a_5$ is in a_1a_2. The homotheties $\triangle Oa_4a_5 \approx \triangle Ob_4b_5$ and $\triangle Oa_2b_1 \approx \triangle Ob_6a_1$ imply $|Ob_5|/|Oa_5| = |Ob_4|/|Oa_4|$ and $|Ob_6|/|Oa_2| =$

$|Oa_1|/|Ob_1|$. The segment a_3b_3 is a central chord in $a_1a_2a_3a_5$. Then formula (16) implies $k(a_1a_2a_3a_5) = k$. If $b_6 \in a_3a_5$, then the central chord a_5b_5 is such that $b_5 \in a_2a_3$. Find a point e_i on the line (a_2b_6) that satisfies $b_5e_3 \parallel a_3e_1 \parallel a_5e_2 \parallel a_1a_2$. The homotheties $\triangle Oa_3e_1 \approx \triangle Ob_3a_2$, $\triangle Oa_4e_2 \approx \triangle Ob_4a_2$, and $\triangle Oa_5a_1 \approx \triangle Ob_5b_1$ imply

$$|Oa_3|/|Ob_3| = |Oe_1|/|Oa_2| \leq |Ob_6|/|Oa_2| \leq |Oe_2|/|Oa_2| = |Oa_4|/|Ob_4|;$$
$$|Ob_1|/|Oa_1| = |Ob_5|/|Oa_5|.$$

By formula (16), we have $k(a_1a_2a_3a_5) \geq k$.

To estimate the self-perimeter of the trapezium $a_1a_2a_3a_5$, set $\widehat{a_1 - a_5} = c_1' \in a_1a_2$. The similarity $\triangle a_1a_4a_5 \sim \triangle Oc_1c_1'$ implies

$$\rho_{\text{old}}(a_4; a_1) = |a_4a_1|/|Oc_1| = |a_5a_1|/|Oc_1'| = \rho_{\text{new}}(a_5; a_1).$$

We have $(\widehat{a_3 - a_2})_{\text{new}} = c_3' \in Oc_3$, $(\widehat{a_2 - a_1})_{\text{new}} = c_2 \in a_2a_3$ and hence

$$\rho_{\text{old}}(a_2; a_3) \leq \rho_{\text{new}}(a_2; a_3), \qquad \rho_{\text{old}}(a_1; a_2) = \rho_{\text{new}}(a_1; a_2).$$

Set $\widehat{a_4 - a_3} = c_4 \in a_4a_1$, $(\widehat{a_5 - a_3})_{\text{new}} = c_5 \in a_5a_1$, and $\{e_4\} = Oc_4 \cap a_1a_3$. Find a point e_5 that satisfies $e_5 \in a_1a_5$ and $c_4e_5 \parallel a_4a_5$. The point a_1 is the centre of the homothety $\triangle e_4c_4e_5 \approx \triangle a_3a_4a_5$. Set $\{e_6\} = (c_4e_5) \cap (Oc_5)$ and consider the homothety $\triangle e_4c_4e_5 \approx \triangle Oc_4e_6$. Then $c_5 \in Oe_6$ and

$$\rho_{\text{old}}(a_3; a_4) = |a_3a_4|/|Oc_4| = |a_3a_5|/|Oe_6| \leq |a_3a_5|/|Oc_5| = \rho_{\text{new}}(a_3; a_5).$$

Therefore, $L^-(a_1a_2a_3a_5) \geq L^-(a_1a_2a_3a_4)$, and the trapezium $a_1a_2a_3a_5$ majorizes the given quadrangle $a_1a_2a_3a_4$.

2. If $O \in \triangle a_2a_3m$, then the points c_i and b_i are located as follows: $c_2 \in a_2a_3$, $c_3 \in a_3a_4$, $b_{1,4} \in a_2a_3$, $b_2 \in a_3a_4$, $b_3 \in a_1a_2$. By formula (23), the factor of symmetry is $k = |Ob_1|/|Oa_1|$. In complete analogy with item 1, we construct the trapezium $a_1a_2a_3a_5$ ($a_4a_5 \parallel a_2a_1$) and obtain the inequality $L^-(a_1a_2a_3a_5) \geq L^-(a_1a_2a_3a_4)$. Find $\{b_2'\} = a_3a_5 \cap (a_2O)$ such that

$$|Oa_3|/|Ob_3| \leq |Ob_2'|/|Oa_2| \leq |Ob_2|/|Oa_2|.$$

We have $\{b_5\} = (Oa_5) \cap (a_2a_3)$, $\triangle Oa_5a_1 \approx \triangle Ob_5b_1$, and $|Ob_5|/|Oa_5| = k$. Thus, the quadrangle $a_1a_2a_3a_4$ is majorized by the trapezium $T = a_1a_2a_3a_5$. Lemma 2.2 is proved. ■

To study the case $O \in \triangle nma_3$, we need the following statement.

PROPOSITION 2.8. *Let $\triangle abc$ be a triangle in the adjoint plane R^2. Let the points $d \in bc$, $e \in ca$, and $f \in ab$ be such that $de \parallel ba$, $df \parallel ca$, and $O \in df$. Set $\{h\} = (bO) \cap (de)$, $q \in dh \cap de$, and $\{p\} = bq \cap df$. Take $t = |eq|$ as a parameter. Then the function $y(t) = 1/|Op|$ is downwards convex over the interval $(t_1; t_2)$, where $t_2 = |ed|$ and $t_1 = 0$ if $de \subset dh$, while $t_1 = |eh|$ if $dh \subset de$.*

Proof. Set $\{l\} = ac \cap (bq)$. The homothety $\triangle bpf \approx \triangle bla$ implies $|pf| = |fb| \cdot |al|/|ab|$. Since $\triangle leq \approx \triangle lab$ and $|eq| = t$, we have $|ab|/t = |al|/|el| = |ae|/|el| + 1$. Therefore, $1/|el| = (|ab| - t)/(t \cdot |ae|)$. The similarity $\triangle bpf \sim \triangle qle$ implies $1/|pf| = t/(|el| \cdot |fb|) = (|ab| - t)/(|fb| \cdot |ea|)$. Set $\alpha = |ae| \cdot |fb|$, $\gamma = |Of|$, $\beta = |ab| > |af| = |ed| \geq t$. Then $|pf| = \alpha/(\beta - t)$. Observe that $|pf| \geq |Of|$, and hence $t \geq \beta - \alpha/\gamma$. If $O = f$, then $\gamma = 0$, and the function $y(t) = 1/|Op| = 1/|pf| = (\beta - t)/\alpha$ is linear with respect to the parameter t. If $O \neq f$, then use the equality $|Op| = |pf| - \gamma$ to deduce $y(t) = 1/|Op| = -1/\gamma + \alpha \cdot \gamma^{-2}/(t - (\beta - \alpha/\gamma))$. This means that for $t > \beta - \alpha/\gamma$ the graph of $y(t)$ is strictly downwards convex, namely the arc of a hyperbola. ∎

DEFINITION 2.7. Define r, z, s in such a way that $r \in a_4a_1$, $a_2r \parallel a_3a_4$, $\{z\} = a_1a_3 \cap a_2r$, and $\{s\} = a_2r \cap \widehat{ngw}$, where $\widehat{ngw}$ is a polygonal arc (the existence of r follows from the canonicity of $a_1a_2a_3a_4$).

In what follows, we use the figure $G = a_2a_3a_4r \cap \triangle gva_3$. Observe that

$$G = \begin{cases} \triangle gva_3 & \text{if } s \in vw, \\ gsza_3 & \text{if } s \in gv, \\ \triangle sza_3 & \text{if } s \in gn. \end{cases} \tag{25}$$

We will consider the cases when $O \in G$ or $O \notin G$.

Again, the next three corollaries follow from our main theorems.

LEMMA 2.3. *If $O \in G$, then the canonically given normalizing quadrangle $a_1a_2a_3a_4$ is majorized by some trapezium T.*

Proof. We restrict our considerations to the most general case of (25), when $G = gsza_3$. Since $r \in a_4a_1$, we have $\triangle nma_3 \subset G$ and $G = \triangle nma_3 \cup gszmn$. Observe that $\widehat{a_4 - a_3} = c_4 \in a_4r$, $\widehat{a_2 - a_1} = c_2 \in a_2a_3$. Set $\{a_7\} = (Oc_2) \cap (a_4a_3)$ and find points $a_{5,6}$ that satisfy $a_{5,6} \in (a_4a_3)$, $a_2a_5 \parallel a_1a_4$, and $a_2a_6 \parallel Oa_4$. Write

$$a_8 = \begin{cases} a_7 & \text{if } a_7 \in a_4a_5, \\ a_5 & \text{if } a_5 \in a_4a_7. \end{cases} \tag{26}$$

Let $M \in a_6a_8$, and take $t = |a_4M|$ as a parameter. Then $t \in [t_1; t_2]$, where $t_1 = |a_4a_6|$ and $t_2 = |a_4a_8|$. Set $t_0 = |a_4a_3|$. If $t = t_0$, then $M = a_3$. Take a canonically given quadrangle $a_1a_2Ma_4$ as the *new* normalizing figure of M^2. Consider the *self-perimeter* $L^-(a_1a_2Ma_4)$ *as a function* $f(t)$ of t, i.e., $f(t) = L^-(a_1a_2Ma_4)$ for $t \in [t_1; t_2]$. We have $\widehat{a_3 - a_2} = c_3 \in a_3a_4$, and write $(\widehat{a_5 - a_2})_{\text{new}} = c_5$ and $(\widehat{M - a_2})_{\text{new}} = c_M$. Since $\triangle a_1b_1a_4$ is non-degenerate and $Oc_5 \parallel a_1a_4$, by construction $c_5 \in b_1a_4 \subset a_3a_4$. Moreover, $c_M \in a_4c_5 \subset a_4a_3$. The similarity $\triangle a_2Ma_3 \sim \triangle Oc_Mc_3$ implies $\rho_{\text{new}}(a_2; M) = |a_2M|/|Oc_M| = |a_2a_3|/|Oc_3| = \rho_{\text{old}}(a_2; a_3)$.

The function $\rho_{\text{new}}(M; a_4) = |Ma_4|/|Oc_4| = t/|Oc_4|$ is linear in t, where $c_4 = (\widehat{a_4 - M})_{\text{new}} = \widehat{a_4 - a_3} \in a_4a_1$. Evidently, $(\widehat{a_1 - a_4})_{\text{new}} = c_1 \in a_1a_2$ and $\rho_{\text{new}}(a_4; a_1) = \rho_{\text{old}}(a_4; a_1)$. From (26) it follows that $(\widehat{a_2 - a_1})_{\text{new}} = c_2' \in a_2M$. By Proposition 2.8, if we take $b = a_2$, $p = c_2'$, $q = M$, and $e = a_4$, then we get the downwards convex function $y(t) = 1/|Oc_2'|$ and $\rho_{\text{new}}(a_1; a_2) = |a_1a_2|/|Oc_2'|$. Set

$$a_9 = \begin{cases} b_1 & \text{if } a_6 \in a_4b_1, \\ a_6 & \text{if } b_1 \in a_4a_6, \end{cases} \tag{27}$$

and $t_3 = |a_4a_9|$. Then $t_1 = |a_4a_6| \le |a_4a_9| = t_3 < |a_4a_3| \le |a_4a_8| = t_2$. Thus, the function $f(t) = L^-(a_1a_2Ma_4)$ *is downwards convex for* $t \in [t_3; t_2]$. Therefore,

$$\max_{[t_3;t_2]} f(t) = \max\{f(t_3); f(t_2)\}. \tag{28}$$

Consider the following four possible maxima of $f(t)$ on $[t_3; t_2]$ according to the conditions (26)–(28).

1. Suppose that $t = t_3$, $a_9 = b_1$, and $f_{\max} = f(t_3)$. If $O \in gszmn$, then all the chords a_ib_i ($i \ne 3$) remain central chords for the new canonical $a_1a_2b_1a_4$. If $O \in \triangle nma_3 \subset \triangle a_4a_2a_3$, then $k(a_1a_2b_1a_4) = |Ob_1|/|Oa_1|$ by (23). Thus, by (16) we have $k(a_1a_2b_1a_4) = k(a_1a_2a_3a_4)$. By construction, $c_M \in Ma_4$, $c_2' \in a_2M$, $O \in a_1b_1$ (a diagonal of $a_1a_2b_1a_4$), and hence $a_1a_2b_1a_4$ has all the properties of the normalizing quadrangle of Lemma 2.2.

2. Suppose that $f_{\max} = f(t_3)$ and $a_9 = a_6$. By construction, the new normalizing quadrangle $a_1a_2a_6a_4$ is canonically given, we have $b_1 \in a_6a_4$ and $c_6' = (\widehat{a_6 - a_2})_{\text{new}} = a_4$, and the central chords a_ib_i in this quadrangle are central for $a_1a_2a_3a_4$. Hence (22) and $O \in \triangle a_1a_2a_4$ imply $k(a_1a_2a_6a_4) = k(a_1a_2a_3a_4)$. Since $c_6' = a_4$, the quadrangle $a_1a_2a_6a_4$ has all the properties of the normalizing quadrangles of Lemma 2.1.

3. Suppose that $f_{\max} = f(t_2)$ and $a_8 = a_5$. By construction, $a_1a_2a_5a_4$ is a trapezium, the segments a_1b_1 and a_2b_2 are central chords for $a_1a_2a_3a_4$ as well, $(\widehat{a_2 - a_1})_{\text{new}} = c_2' \in a_2a_5$, and the central chord a_5b_5 is such that $b_5 \in a_4a_1$. If $O \in \triangle nma_3 \subset \triangle a_4a_2a_5$, then by (23) we have $k(a_1a_2a_5a_4) = |Ob_1|/|Oa_1| = k(a_1a_2a_3a_4)$. If $O \in gszmn \subset \triangle a_4a_1a_2$, then $\triangle Oa_5a_2 \approx \triangle Ob_5b_2$ implies $|Ob_5|/|Oa_5| = |Ob_2|/|Oa_2|$. By (16) and (22) we have $k(a_1a_2a_5a_4) = k(a_1a_2a_3a_4)$, and $T = a_1a_2a_5a_4$ is a majorizing trapezium.

4. Let $f_{\max} = f(t_2)$ and $a_8 = a_7$. Here we use the properties of the trapezium T from case 3, for which $k(a_1a_2a_5a_4) = k(a_1a_2a_3a_4)$. The chord a_1b_1 remains central for the quadrangle $a_1a_2a_7a_4$. If $O \in \triangle a_4a_2a_7$, then by (23) we have $k(a_1a_2a_7a_4) = k(a_1a_2a_3a_4)$. If $O \in gszmn$, then the chords a_1b_1, a_2b_2, and a_4b_4 are central for $a_1a_2a_7a_4 \supset a_1a_2a_3a_4$. By (22), we have

$k(a_1a_2a_7a_4) = k$. Since $a_7 = c_2' = (\widehat{a_2 - a_1})_{\text{new}}$, the new canonically given normalizing quadrangle $a_1a_2a_7a_4$ meets all the requirements of Lemma 2.1, and Lemma 2.3 is proved. ∎

To study the case $O \notin G$ in a canonically given quadrangle $a_1a_2a_3a_4$, we introduce the following definitions (see (25)).

Definition 2.8. A canonically given normalizing quadrangle $a_1a_2a_3a_4$ is called a *quadrangle of first special type* if

1) the origin satisfies

$$O \in \Omega \equiv \triangle ra_1a_2 \cap \triangle gva_3 \neq \emptyset, \tag{29}$$

2) the factor of symmetry satisfies

$$k(a_1a_2a_3a_4) = |Ob_2|/|Oa_2| = |Ob_4|/|Oa_4|. \tag{30}$$

Definition 2.9. A canonically given normalizing quadrangle $a_1a_2a_3a_4$ is called a *quadrangle of second special type* if (29) holds, but

$$k = k(a_1a_2a_3a_4) = |Ob_1|/|Oa_1| = |Ob_2|/|Oa_2|. \tag{31}$$

Lemma 2.4. *If a normalizing quadrangle $a_1a_2a_3a_4$ is of first special type, then it is majorized by some trapezium T.*

Proof. By (29), we have $O \in \triangle a_4a_1a_2$, and (22) yields $k \leq |Ob_1|/|Oa_1|$. Moreover, $\widehat{a_2 - a_1} = c_2 \in a_2a_3$, $b_1 \in a_3a_4$, $a_2r \parallel a_3a_4$, and $b_2 \in ra_1 \subset a_4a_1$, $\widehat{a_4 - a_3} = c_4 \in ra_1$. Choose a Cartesian coordinate system of R^2 in such a way that $b_4a_4 \subset Ox$, $b_2a_2 \subset Oy$ and $O(0;0)$, $a_4(1;0)$, $b_4(-k;0)$, $a_2(0;1)$, $b_2(0;-k)$. Here we use an auxiliary metric where the centre g of the canonically given quadrangle $a_1a_2a_3a_4$ does not in general coincide with the origin O of $\mathbb{R}^2$ (see Remark 2.2). Since $\{a_1\} = (a_1a_2) \cap (a_1a_4)$, we have $a_1(-k/(1-k); -k/(1-k))$. Find $a_{5,6} \in R^2$ such that $a_5a_4 \parallel a_2b_2$, $a_2a_5 \parallel a_1a_4$, $a_6 \in a_5a_4$, and $a_2a_6 \parallel a_4b_4$. It is easy to see that $a_5(1;1+k), a_6(1;1)$. The vertex a_3 is from $\triangle a_2a_5a_6$, because by (29) we have $c_4 \in a_4b_2$, $c_3 \in a_3a_4$, and $a_1a_2a_3a_4$ is canonically given.

Consider now a_3 as one of the points $M(a;b) \in \triangle a_2a_5a_6$. We also make the restriction $c_2 \in a_2M$. The coordinates of $c_2(x_2;y_2)$ satisfy

$$\begin{cases} y = x/k, \\ y - 1 = (b-1)/a \cdot x, \end{cases} \qquad \frac{1}{x_2} = \frac{1}{k} + \frac{1-b}{a},$$

and we have

$$\begin{aligned} \rho(a_1;a_2) = -x_1/x_2 &= k/(1-k) \cdot (1/k + (1-b)/a) \\ &= 1/(1-k) + k(1-b)/(a(1-k)). \end{aligned}$$

The coordinates of $\widehat{M - a_2} = c_M(x_3;y_3)$ satisfy $y = (b-1)/a \cdot x$, $y = b/(a-1) \cdot (x-1)$ and hence $1/x_3 = 1 + (a-1) \cdot (1-b)/(ab)$ and $\rho(a_2;M) = a/x_3 = a + (a-1) \cdot (1-b)/b$. The point $c_4(x_4;y_4)$ is on the lines $y =$

$b/(a-1)\cdot x$ and $y+k=kx$, and hence $-1/y_4 = 1/k + (1-a)/b$ and $\rho(M;a_4) = -b/y_4 = b/k + 1 - a$. The value $\rho(a_4;a_1)$ does not depend on the location of $M \in \triangle a_2a_5a_6$. Let us define a function

$$f(a;b) = \rho(a_1;a_2) + \rho(a_2;M) + \rho(M;a_4), \qquad M(a;b) \in \triangle a_2a_5a_6.$$

Thus, $f(a;b) = 2 + 1/(1-k) + k(1-b)/(a(1-k)) - a + (a-1)/b + b/k$, where $b \geq 1$ and $0 < a \leq 1$.

We calculate the derivatives:

$$f'_a = -\frac{k}{1-k}\cdot\frac{1-b}{a^2} - 1 + \frac{1}{b}, \qquad f'_b = -\frac{k}{1-k}\cdot\frac{1}{a} - \frac{a-1}{b^2} + \frac{1}{k}.$$

The stationary points of $f(a;b)$ are

$$(32) \qquad \begin{cases} \left[\begin{array}{l} b=1, \\ b=\frac{1-k}{k}a^2, \end{array}\right. \\ \frac{k}{1-k}\cdot\frac{1}{a} + \frac{a-1}{b^2} = \frac{1}{k}, \end{cases} \qquad \left[\begin{array}{l} b=1, \\ \frac{a}{b} + \frac{a-1}{b^2} = \frac{1}{k}, \end{array}\right. \qquad \left[\begin{array}{l} b=1, \\ a = \frac{b^2+k}{k(b+1)}. \end{array}\right.$$

We calculate the second derivatives:

$$f''_{aa} = \frac{2k}{1-k}\cdot\frac{1-b}{a^3}, \qquad f''_{bb} = 2\cdot\frac{a-1}{b^3}, \qquad f''_{ab} = \frac{k}{1-k}\cdot\frac{1}{a^2} - \frac{1}{b^2}.$$

We consider separately the case $b = (1-k)/k\cdot a^2$. In this case $f''_{ab} = (b-1)/b^2$ and

$$\begin{aligned} \triangle(a;b) = f''_{aa}\cdot f''_{bb} - [f''_{ab}]^2 &= 4\cdot\frac{k}{1-k}\cdot\frac{(b-1)(1-a)}{a^3\cdot b^3} - \frac{(b-1)^2}{b^4} \\ &= 4\cdot\frac{(b-1)(1-a)}{b^4\cdot a} - \frac{(b-1)^2}{b^4}. \end{aligned}$$

Taking into account (32), we obtain

$$\triangle(a;b) = \frac{b-1}{b^4}\left[\frac{4}{a} - (3+b)\right] = \frac{b-1}{b^4}\left[\frac{4k(b+1)}{b^2+k} - (3+b)\right].$$

Since $b > 1 \geq k$, we have $b^2+k > k(b+1)$ and $3+b > 4k(b+1)/(b^2+k)$. The inequality $\triangle(a;b) < 0$ implies that $f(a;b)$ achieves its maximum only at the boundary of $\triangle a_2a_5a_6$. Observe that if $b=1$, then $M(a;1) \in a_2a_6$.

We describe in detail the boundary of a polygon Σ that contains the vertex $M \in \Sigma \subset \triangle a_2a_5a_6$ of the canonically given quadrangle $a_1a_2Ma_4$ of first special type. By (29), we have $b_1 \in Ma_4$ and $c_2 \in a_2M$. Find a point e_0 such that $e_0 \in (a_1O)$, $O \in a_1e_0$, and $|Oe_0|/|Oa_1| = k$. Let e_3 be such that $e_3 \in a_2a_5$, $Oe_3 \parallel a_1a_2$. Set $\{e_1\} = (a_4e_0)\cap(a_2a_6)$, $\{e_2\} = (Oe_3)\cap(a_4e_0)$, $\{e_4\} = (a_4e_0)\cap(a_2a_5)$, and $\{e_5\} = (Oe_3)\cap a_2a_6$. We have $|Ob_1|/|Oa_1| \geq k$ and hence $M \in \triangle a_4e_4a_5$. If $e_1 \notin e_5a_6$, then $\Sigma = e_5e_3a_5a_6$. If $e_4 \in e_3a_5$, then $\Sigma = e_1e_4a_5a_6$. If $e_1 \in e_5a_6$ and $e_4 \notin e_3a_5$, then $\Sigma = e_2e_3a_5a_6e_1$. Observe that, by (22), $k(a_1a_2Ma_4) = k(a_1a_2a_3a_4) = k$ for the quadrangle of first special type, namely $k(a_1a_2Ma_4) = \min\{k;|Ob_1|/|Oa_1|\} = k$. We estimate the self-perimeter $L^{-}(a_1a_2Ma_4)$ when $M \in \partial\Sigma$ for the most general case when Σ is a pentagon, i.e., $\partial\Sigma = e_2e_3 \cup e_3a_5 \cup a_5a_6 \cup a_6e_1 \cup e_1e_2$.

1. Suppose that $M \in e_2e_3$. Then in the canonically given quadrangle $a_1a_2Ma_4$ we have $c_2 = M$. Such quadrangles were described in Lemma 2.1, and hence the conclusion of Lemma 2.4 holds.

2. Suppose that $M \in e_3a_5$. Then $a_2M \parallel a_1a_4$, and the majorizing trapezium is $T = a_1a_2Ma_4$.

3. Suppose that $M \in a_5a_6$. Then $a_2b_2 \parallel Ma_4$ and $r = b_2$. The case $O \in a_2r \subset a_2Ma_4r$ was considered in Lemmas 2.1–2.3, and hence the conclusion of Lemma 2.4 holds.

4. Suppose that $M \in a_2a_6$. Then $a_2M \parallel Oa_4$ and $\widehat{M - a_2} = c_M = a_4$. Thus, $O \in a_4w \subset \triangle a_4a_1w$, and we can apply Lemma 2.1.

5. Suppose that $M \in e_1e_2$. Then $e_0 = b_1$ and $|Ob_1|/|Oa_1| = k$. To study the properties of the quadrangle $a_1a_2Ma_4$ of first special type, it is convenient to use another adjoint plane R^2, namely such that $a_1(-1;0)$, $a_4(0;-1)$, $b_1(k;0)$, and $b_4(0;k)$. Set $\{a_7\} = (a_4b_1) \cap (Oc_2)$, $c_2 \in a_2M$, and $a_2 \in (a_1b_4)$. Let $a_2(x_2;y_2)$, $a_7(x_7;y_7)$, and $M(a;b)$. Then (see (30))

$$|Ob_1|/|Oa_1| = |Ob_2|/|Oa_2| = |Ob_4|/|Oa_4| = k.$$

Set $t = y_2/x_2$. Then a_2 belongs to the lines $y = tx$ and $y = kx+1$. The point $b_2(x_3;y_3)$ belongs to the lines $y = tx$ and $y = -x-1$. Solving the systems, we find $x_2 = 1/(t-k)$ and $x_3 = -1/(t+1)$. The ratios $|Ob_2|/|Oa_2| = -x_3/x_2 = (t-k)/(t+1) = k$ imply $t = 2k/(1-k)$ and $x_2 = (1-k)/(k+k^2)$. The point a_7 is on the lines $y = kx$ and $y = 1/k \cdot x - 1$, and therefore $x_7 = k/(1-k^2)$. By (29), we have $(\widehat{M - a_2}) = c_M \in Ma_4$, $c_2 \in a_2M$, and hence $x_2 \le a \le x_7$. In terms of k the latter means that $(1-k)/(k+k^2) \le a \le k/(1-k^2)$. The solution in a exists if $(1-k)^2 \le k^2$, i.e., $k \in [1/2;1]$. By the hypothesis, $O \in \Omega \subset \triangle ra_1a_2$, where $ra_2 \parallel Ma_4$. The case $O \in sz$ (see (25)) was considered in Lemma 2.3. Suppose that $O \notin sz$. Since the slope of a_2b_2 is equal to $t = 2k/(1-k)$ and the slope of a_4M is equal to $1/k$, we have $1/k > t$. In terms of k the latter inequality means that $2k^2 + k - 1 < 0$, i.e., $k \in (0;1/2)$. Thus $O \in sz$, and case 5 is settled.

Hence Lemma 2.4 is proved. ∎

LEMMA 2.5. *If a normalizing quadrangle $a_1a_2a_3a_4$ is of second special type, then it is majorized by some trapezium T.*

Proof. By (29), we have $c_2 \in a_2a_3$, $c_3 \in a_3a_4$, $c_4 \in rb_2 \subset a_4a_1$, and $a_2r \parallel a_3a_4$. By (31), $|Ob_1|/|Oa_1| = |Ob_2|/|Oa_2| = k \le |Ob_4|/|Oa_4|$, and hence $\triangle Ob_1b_2 \approx \triangle Oa_1a_2$. Find points a_5, b_5, a_6, e_1 that satisfy $a_4 \in a_1a_5$, $\{b_5\} = a_1a_2 \cap (a_5O)$, $|Ob_5|/|Oa_5| = k$; $a_6 \in (b_2b_1)$, $a_2a_6 \parallel a_1a_4$; and $\{e_1\} =$

$a_2b_1 \cap Oc_2$ (*the chords a_ib_i are central ones*). Set $\{a_7\} = (a_2a_6) \cap (a_5b_1)$, $\{e_2\} = (Oe_1) \cap (a_5b_1)$, and $\{e_3\} = (Oe_1) \cap a_2a_6$. By construction, the trapezium $b_1e_1e_3a_6$ contains the point a_3 of the initial quadrangle $a_1a_2a_3a_4$.

Define a polygon Σ depending on the location of a_7 with respect to the segment a_2e_3:

$$\Sigma = \begin{cases} b_1e_2e_3a_6 & \text{if } a_7 \in a_2e_3, \\ b_1e_1e_3a_6 & \text{if } a_2 \in a_7e_3, \\ b_1a_7a_6 & \text{if } a_7 \in e_3a_6. \end{cases} \tag{33}$$

Take a point $M \in \Sigma$ and find a point e_4 such that $e_4 \in (Mb_1)$ and $Oe_4 \parallel a_2M$. Set $\{a_8\} = (Mb_1) \cap a_1a_5$ and $\{b_8\} = (a_8O) \cap a_1a_2$. We have $O \in a_1b_1$. The non-degeneracy of $\triangle a_1b_1a_8$ implies $c_6 = \widehat{a_6 - a_2} \in b_1a_4$. Consider the quadrangle $a_1a_2Ma_8$ of second special type in the capacity of a normalizing quadrangle of M^2. Observe that if $M = a_3 \in \Sigma$, then it coincides with the initial one, i.e., $a_1a_2a_3a_4$. Canonicity of $a_1a_2Ma_8$ and the inclusions $a_8 \in a_5b_2 \subset a_5a_1$ and $b_8 \in b_5a_2 \subset a_1a_2$ yield $k = |Ob_5|/|Oa_5| \le |Ob_8|/|Oa_8| \le |Oa_2|/|Ob_2| = 1/k$. The latter inequality and the equalities (22) and (31) imply $k(a_1a_2Ma_8) = k(a_1a_2a_3a_4) = k$.

To estimate the self-perimeter $L^-(a_1a_2Ma_8)$, we calculate the lengths of the sides by using (1)–(3). For the normalizing vectors we have $c_2' = (\widehat{a_2 - a_1})_{\text{new}} \in a_2M$, $c_8 = \widehat{a_8 - M} \in a_8a_1 \subset a_5a_1$, $c_1 = \widehat{a_1 - a_4} = \widehat{a_1 - a_8}$, and $\widehat{M - a_2} = c_M \in \widehat{b_1a_8a_1}$, where $\widehat{b_1a_8a_1}$ is again a polygonal arc. If c_M is in b_1a_8, then $c_M = e_4$ and $\rho(a_2; M) = |a_2M|/|Oe_4|$. If $c_M \in a_8a_1$, then $c_M \in Oe_4$ and $\rho(a_2; M) \ge |a_2M|/|Oe_4|$. Define a function of $M \in \Sigma$ by

$$f(M) = \rho(a_1; a_2) + \rho(M; a_8) + \rho(a_8; a_1) + |a_2M|/|Oe_4|,$$

where the distance function is meant with respect to $a_1a_2Ma_8$. We have $a_3 \in \Sigma$, and by (29) we get $\widehat{a_3 - a_2} = c_3 \in b_1a_4$. Hence

$$\max_{\Sigma} f(M) \ge L^-(a_1a_2a_3a_4). \tag{34}$$

Evidently,

$$f(M) \le L^-(a_1a_2Ma_8), \quad M \in \Sigma. \tag{35}$$

We want to prove that $f(M)$ attains its maximum at the boundary of the polygon Σ, i.e., when $M \in \partial\Sigma$. We choose a Cartesian system of coordinates in the adjoint plane R^2 in such a way that $O(0;0)$, $a_2(0;1)$, $a_1(-1;0)$, $b_1(k;0)$, $b_2(0;-k)$, and we set $M(a;b)$ (see Remark 2.2). Since $a_1a_2a_6b_2$ is a parallelogram, $\triangle b_1a_2a_6$ is in the first quadrant and $0 \le a, b \le 1$. The case $b = 0$ means that $M = b_1$ and hence $O \in a_1M$. Also this case was considered in Lemma 2.2. If $a = 0$, then $M = a_2$ and $a_1a_2Ma_8 = a_1a_2b_1a_8$. For the canonically given quadrangle $a_1a_2b_1a_8$ we have $O \in a_1b_1$. This case was considered in Lemma 2.2. Thus, we suppose that $a, b \in (0;1]$. Taking into account that $M \in b_1e_1e_3a_6$, we find the abscissa of $\{c_2'\} = (Oe_1) \cap a_2M$ by solving the system $y = x$, $y-1 = (b-1)/a \cdot x$, i.e., $(1+(1-b)/a) \cdot x = 1$. Hence $\rho(a_1; a_2) = |a_1a_2|/|Oc_2'| = 1 + (1-b)/a$. The point $\{e_4\} = (Oe_4) \cap (Mb_1)$ is defined by $y = x \cdot (b-1)/a$ and $y = b \cdot (x-k)/(a-k)$. Thus, for $e_4 = (x_e; y_e)$ we have $1/x_e = 1/k + (a-k)(1-b)/(kba)$ and $|a_2M|/|Oe_4| = a/x_e = a/k + (a-k)(1-b)/(kb)$.

Set $\{b_M\} = a_8a_1 \cap (MO)$. The similarity $\triangle Ma_8b_M \sim \triangle Oc_8b_M$ implies

$$\rho(M; a_8) = |Ma_8|/|Oc_8| = |Mb_M|/|Ob_M| = 1 + |OM|/|Ob_M|.$$

The point $b_M(x_b; y_b)$ is on the lines $y = b \cdot x/a$ and $y + k = -kx$. Hence $-1/x_b = (k + b/a)/k$ and $\rho(M; a_8) = 1 + a + b/k$. The points $\{c_1\} = (Oc_1) \cap a_1a_2$ and $\{a_8\} = (Mb_1) \cap (a_1b_2)$ can be found as solutions of the systems $y = x + 1$, $y = -kx$ and $y + k = -kx$, $y = b(x-k)/(a-k)$, respectively. If one writes $c_1(x_c; y_c)$ and $a_8(x_8; y_8)$, then $-1/x_c = 1 + k$ and $x_8 = k \cdot (b - (a-k))/(b + k(a-k))$. Finally,

$$\rho(a_8; a_1) = |a_8a_1|/|Oc_1| = -(1 + x_8)/x_c = b(1+k)^2/(b + k(a-k)).$$

We express the function $f(M)$ by means of the coordinates of $M(a; b)$:

$$f(a; b) = 2 + (1-b)/a + (a+b)/k + (a-k) \cdot (1-b)/(kb) \\ + a + b(1+k)^2/(b + k(a-k)).$$

Evidently, $f_a' = 1 - (1-b) \cdot a^{-2} + 1/(kb) - (1+k)^2 \cdot b \cdot k \cdot (b + k(a-k))^{-2}$. Then

$$f_{aa}'' = 2 \cdot (1-b) \cdot a^{-3} + 2(1+k)^2 \cdot b \cdot k^2 \cdot (b + k(a-k))^{-3}. \tag{36}$$

Find a point c_1' that satisfies $c_1' \in a_1a_2$ and $b_1c_1' \parallel a_2a_6 \parallel b_2a_1$. In a parallelogram $b_1c_1'a_2a_6$, the equation of the side (b_1c_1') is $y = -k(x-k)$. By the hypothesis, $M(a; b) \in \Sigma \subset \triangle b_1a_2a_6 \subset b_1c_1'a_2a_6$, and hence $b > -k(a-k)$. Combining $0 < a, b \le 1$ and the equality (36), we get $f_{aa}'' > 0$. Thus, the function $f = f(M)$, where $M \in \Sigma$, can achieve its maximal value only at $\partial\Sigma$. To estimate $f_{\max}$ from above, consider, in accordance with (33), the following five cases:

1. If $M \in a_6b_1$, $M \neq b_1$, then $a_1a_2Ma_8 = a_1a_2Mb_2$ is a trapezium.

2. If $M \in e_3a_6$, then $a_8a_1 \parallel a_2M$ and $a_1a_2Ma_8$ is a trapezium.

3. If $M \in e_1e_3$, then $M = c_2' = \widehat{a_2 - a_1}$, and the canonically given quadrangle $a_1a_2Ma_8$ meets the requirements of Lemma 2.1. By the inequalities (34) and (35) we have $L^-(a_1a_2a_3a_4) \le f(M) \le L^-(a_1a_2Ma_8)$. Thus, for the quadrangle $a_1a_2a_3a_4$ there exists a majorizing trapezium T.

4. If $M \in b_1e_1$ and $M \neq b_1$, then the quadrangle $a_1a_2Ma_8$ degenerates to $\triangle a_1a_2a_8$. By Corollary 2.5, we have $L^-(\triangle) \le 2D^2/(D-1)$. A suitable choice of the adjoint plane R^2 transforms the isosceles trapezium $T = a_1a_2b_1b_2$ into the trapezium from our Example 2.1, showing the sharpness of (9) (for $t = k^2$). Thus $L^-(\triangle) \le 2D^2/(D-1) = L^-(T)$, and $a_1a_2b_1b_2$ is the majorizing trapezium.

5. If $M \in b_1e_2$, $M \ne b_1$, then $a_8 = a_5$ and $a_1a_2Ma_8 = a_1a_2Ma_5$. Here $|Ob_1|/|Oa_1| = |Ob_2|/|Oa_2| = |Ob_5|/|Oa_5| = k$, $c_2' = \widehat{a_2 - a_1} \in a_2M$, and $c_8 = \widehat{a_5 - M} = c_5 \in a_5a_4$. Since $a_4 \in b_2a_5$, there is a point $r' \in a_5r$ such that $a_2r' \parallel Ma_5$ and $O \in \triangle a_1a_2r'$. If $\widehat{M - a_2} = c_M \in Ma_5$, then the canonically given quadrangle $a_1a_2Ma_5$ is of first special type as described in Lemma 2.4. If $c_M \in a_5a_1$, then the normalizing quadrangle meets the requirements of Lemma 2.1, and Lemma 2.5 is proved. ∎

Proof of Theorem 1.3. If the normalizing quadrangle $P_4 = a_1a_2a_3a_4$ is a trapezium, then the statement of the theorem is obvious. By Proposition 2.6, we may restrict our considerations to canonically given quadrangles $a_1a_2a_3a_4 \subset M^2$. According to Definition 2.4, denote by g the center of $a_1a_2a_3a_4$. Set $\{u\} = a_4a_1 \cap (a_3g)$, $\{w\} = a_1a_2 \cap (a_4g)$, and $\{v\} = a_1a_3 \cap a_4w$. We have $a_2r \parallel a_3a_4$, where $r \in a_4a_1$. The theorem is already proved in Lemmas 2.1–2.3 for three particular locations of the origin O inside $a_1a_2a_3a_4$. Namely, if $O \in \triangle a_1wa_4 \cup \triangle ga_3a_4 \cup wa_2a_3v \cup ra_2a_3a_4$, then for the normalizing quadrangle $a_1a_2a_3a_4$ there is a majorizing trapezium T (see Definition 2.6). Keep the notation for the polygon $\Omega \equiv \triangle ra_1a_2 \cap \triangle gva_3$ in correspondence with (29). If $\Omega = \emptyset$, then the proof is complete. If $O \in \Omega$, then the proof is completed by Lemmas 2.4 and 2.5 for normalizing quadrangles $a_1a_2a_3a_4$ of first and second special type (see Definitions 2.8 and 2.9).

Introducing some auxiliary metric for M^2, i.e., the metric of the adjoint plane R^2, we now prove the theorem in the case of $O \in \Omega$ for an arbitrary canonically given normalizing quadrangle $a_1a_2a_3a_4$. Since $\Omega \subset \triangle a_1a_2a_4$, we consider two cases in accordance to (22): either $k(a_1a_2a_3a_4) = k = |Ob_2|/|Oa_2|$, or $\min\{|Ob_1|/|Oa_1|; |Ob_4|/|Oa_4|\} = k < |Ob_2|/|Oa_2|$.

1. Suppose that $k = |Ob_2|/|Oa_2| \le |Ob_1|/|Oa_1|$ and $O \in \Omega$. Find a point e_1 that satisfies $e_1 \in Ob_1$ and $b_2e_1 \parallel a_1a_2$, i.e., $\triangle Oa_1a_2 \approx \triangle Oe_1b_2$. Set $\{e_2\} = Ob_1 \cap a_3u$ and

$$e_3 = \begin{cases} e_1 & \text{if } e_1 \in b_1e_2, \\ e_2 & \text{if } e_2 \in b_1e_1, \end{cases} \qquad \{e_4\} = a_4a_1 \cap (a_3e_3).$$

If $e_3 = e_2$, then $e_4 = u$. To apply Proposition 2.8, we introduce the following notation:

$$\widehat{a_3 - a_2} = c_3 \in a_3a_4, \quad \{d\} = (Oc_3) \cap (a_1a_4), \quad b := a_3, \quad h := b_3, \quad e := a_1,$$

where $h \in ed$. Find points c and a that satisfy $c \in (bd)$, $a_1c \parallel a_2b$; $a \in (a_1c)$, and $ab \parallel ed$. Write $\{f\} = ab \cap (dO)$, $t_1 = |a_1b_3| = |eh| > 0$, and $t_2 = |a_1d|$. Let $q \in e_4d \subset hd$. If one writes $t_3 = |ee_4|$ and $t = |eq|$, then $t_1 \le t_3 \le t \le t_2$. Set $\{p\} = Od \cap a_3q$. For the new canonically given quadrangle $a_1a_2a_3q \subset M^2$

we have $p = (\widehat{a_3 - a_2})_{\rm new} \in a_3q$ and $\rho_{\rm new}(a_2; a_3) = |a_2a_3|/|Op|$. By Proposition 2.8, the function $y(t) = |a_2a_3|/|Op|$ is downwards convex for $t \in [t_3; t_2]$. Set $c_1 = \widehat{a_1 - q} = \widehat{a_1 - a_4} \in a_1a_2$, $c_2 = \widehat{a_2 - a_1} \in a_2a_3$, $c_q = \widehat{q - a_3} \in qa_1 \subset da_1$, and $c_4 = \widehat{a_4 - a_1} \in a_4a_1 \subset da_1$. Since $\triangle a_3a_4q \sim \triangle Oc_4c_q$, we have $\rho_{\rm new}(a_3; q) = |a_3q|/|Oc_q| = |a_3a_4|/|Oc_4| = \rho_{\rm old}(a_3; a_4) = \text{const}$, $t \in [t_3; t_2]$. The function $\rho_{\rm new}(q; a_1) = |qa_1|/|Oc_1| = t/|Oc_1|$ is linear in t, and $\rho_{\rm new}(a_1; a_2) = \rho_{\rm old}(a_1; a_2)$. Thus, the self-perimeter function $f(t) \equiv L^-(a_1a_2a_3q)$ is downwards convex in $t \in [t_3; t_2]$. Among the quadrangles $\{a_1a_2a_3q\}$ we consider those for which $k(a_1a_2a_3q) \geq k(a_1a_2a_3a_4)$. Take the points $a_5 \in (a_1a_4)$ and $\{b_5\} = (a_1a_2) \cap (a_5O)$. If $a_5 \in a_4e_4$, then the canonicity of $a_1a_2a_3a_4$ implies $|Ob_5|/|Oa_5| \geq |Ob_4|/|Oa_4| \geq k$. If a_5 satisfies the conditions $a_4 \in e_4a_5$ and $|a_1a_5| \to \infty$, then $|Ob_5|/|Oa_5| \to 0$. By continuity, there is a point a_5 such that $a_4 \in a_1a_5$ and $|Ob_5|/|Oa_5| = k$. Set

$$a_6 = \begin{cases} d & \text{if } d \in a_4a_5, \\ a_5 & \text{if } a_5 \in a_4d, \end{cases}$$

and $t_4 = |a_1a_6|$, where $t_3 \leq t_4 \leq t_2$. The convexity of $f(t)$, $t \in [t_3; t_4]$, implies

$$\max_{[t_3;t_4]} f(t) = \max\{f(t_3); f(t_4)\}. \tag{37}$$

Consider the following four possible maxima of $f(t)$ in (37).

(a) Let $f_{\max} = f(t_3)$ and $e_3 = e_1$. Then in $a_1a_2a_3e_4$ the central chord a_1e_1 satisfies $|Oe_1|/|Oa_1| = |Ob_2|/|Oa_2| = k$, $c_2 \in a_2a_3$, and $c_3 \in a_3a_4$, and the quadrangle is of second special type. Lemma 2.5 completes the proof.

(b) Let $f_{\max} = f(t_3)$ and $e_3 = e_2$. Then $a_1a_2a_3e_4$ contains a trapezium ($e_4 = u$, $a_1a_2 \parallel a_3u$).

(c) Let $f_{\max} = f(t_4)$ and $a_6 = d$. Then $a_1a_2a_3q = a_1a_2a_3d$, $d = \widehat{a_3 - a_2}$, $\{w_1\} = a_1a_2 \cap (dO)$, $dw_1 \parallel a_4w$, and $O \in \triangle a_1w_1d$. This case was considered in Lemma 2.1.

(d) Let $f_{\max} = f(t_4)$ and $a_6 = a_5$. Then $a_1a_2a_3q = a_1a_2a_3a_5$ and $|Ob_5|/|Oa_5| = |Ob_2|/|Oa_2| = k$, $c_2 \in a_2a_3$, $c_3 \in a_3a_5$, and $r \in b_2a_4 \subset b_2a_5$. This means that $a_1a_2a_3a_5$ is a quadrangle of first special type. The result of Lemma 2.4 completes the proof.

2. Suppose that $|Ob_2|/|Oa_2| > k = k(a_1a_2a_3a_4)$. Take auxiliary points as follows: $e_1 \in Ob_1$, $|Oe_1|/|Oa_1| = k$; $e_2 \in a_4a_1$, $e_1e_2 \parallel a_1a_2$; $e_7 \in (a_4a_3), Oe_7 \parallel a_1a_2$; $a_8 \in a_1a_2$, $a_8O \parallel a_3a_4$; $\{r'\} = a_4a_1 \cap (a_8O)$; $\{a_5\} = (a_1a_2) \cap (e_2O)$; $a_6 \in (a_1a_2)$, $a_6F \parallel Oa_4$. Further, we use the point $\{F\} = (a_1b_1) \cap (a_2a_3)$. Since $O \in \Omega$, we have $b_1 \in a_1F$, $a_3 \in a_2F$, and $b_2 \in e_2r' \subset a_1r$. Set $\{a_7\} = (a_1a_2) \cap (Fe_7)$, $a_9 \in a_1a_2$ and $Fa_9 \parallel a_4a_1$; $\{e_i\} = (a_4a_3) \cap (Fa_i)$, where $i = 5, 6, 7, 9$. Write $t_1 = |a_1a_9|$ and $t_2 = \min\{|a_1a_i| : 5 \leq i \leq 7\}$. Denote by a_{10} the point such that $a_{10} \in (a_1a_2)$ and $|a_1a_{10}| = t_2$. Canonicity of $a_1a_2a_3a_4$ yields

$$a_1a_9 \subset a_1a_2 \subset a_1a_{10} \subset \bigcap_{5\le i\le 7} a_1a_i. \tag{38}$$

Consider an arbitrary point $M \in a_9a_{10}$ and introduce a parameter $t = |a_1M|$, where $t \in [t_1; t_2]$. Set $\{N\} = MF \cap (a_4a_3)$. If $|a_1a_2| = t_0$, then for $t = t_0 \in [t_1; t_2]$ we have $MN = a_2a_3$. The canonically given quadrangle a_1MNa_4 plays the role of a new normalizing figure of M^2.

Let us show that *the self-perimeter function*

$$f(t) \equiv L^-(a_1MNa_4), \quad t_1 \le t \le t_2, \tag{39}$$

is downwards convex in t. Evidently, $(\widehat{a_1 - a_4})_{\text{new}} = c_1 \in a_1a_9 \subset a_1a_2$ and

$$\rho_{\text{new}}(a_4; a_1) = \rho_{\text{old}}(a_4; a_1). \tag{40}$$

By (38), we have $c_M = (\widehat{M - a_1})_{\text{new}} \in MN$ and $c_2 = \widehat{a_2 - a_1} \in a_2a_3$. The factors of homothety for the triangles $\triangle a_1MF \approx \triangle Oc_MF$ and $\triangle a_1a_2F \approx \triangle Oc_2F$ are the same, so (1) implies

$$\begin{aligned}\rho_{\text{new}}(a_1; M) &= |a_1M|/|Oc_M| = |a_1F|/|OF| \\ &= |a_1a_2|/|Oc_2| = \rho_{\text{old}}(a_1; a_2), \quad M \in a_9a_{10}.\end{aligned} \tag{41}$$

Set $c_N = (\widehat{N - M})_{\text{new}} \in Na_4$ and $c_3 = \widehat{a_3 - a_2} \in a_3a_4$. Find a point τ that satisfies $\tau \in (Oc_3)$ and $c_N\tau \parallel a_1a_2$. The similarity $\triangle FNa_3 \sim \triangle Oc_Nc_3$ implies

$$\rho_{\text{new}}(M; N) = |MF|/|Oc_N| - |NF|/|Oc_N| = |MF|/|Oc_N| - |a_3F|/|Oc_3|.$$

Set $\gamma_1 = |a_3F|/|Oc_3|$. Then

$$\rho_{\text{new}}(M; N) = |MF|/|Oc_N| - \gamma_1. \tag{42}$$

The similarity $\triangle FMa_2 \sim \triangle Oc_N\tau$ implies $|MF|/|Oc_N| = |Fa_2|/|O\tau|$. This ratio does not depend on the choice of the metric of R^2, and hence we may assume $\angle a_1a_2a_3 = \pi/2$. Let $\angle c_3Oc_N = \phi$ and $\angle c_Nc_3O = \alpha$. In $\triangle Oc_Nc_3$ we find $|Oc_3| = |O\tau| \cdot (1 + \cot\alpha \cdot \tan\phi)$. From this and the equality $\angle a_2FM = \angle c_3Oc_N = \phi$ we conclude

$$\begin{aligned}|Fa_2|/|O\tau| &= (|Fa_2| + \cot\alpha \cdot |Ma_2|)/|Oc_3| \\ &= |Fa_2|/|Oc_3| + \cot\alpha \cdot (|a_1a_2| - t)/|Oc_3| = \gamma_2 - \gamma_3 \cdot t,\end{aligned}$$

where $\gamma_2 = |Fa_2|/|Oc_3| + \cot\alpha \cdot |a_1a_2|/|Oc_3|$ and $\gamma_3 = \cot\alpha/|Oc_3|$ are constants. By (42), the function

$$\rho_{\text{new}}(M; N) = (\gamma_2 - \gamma_1) - \gamma_3 \cdot t, \quad t \in [t_1; t_2], \tag{43}$$

is linear in t. By construction, $b_1 \in a_4N$ and $c_4 = \widehat{a_4 - a_3} = \widehat{a_4 - N}$. Then

$$\begin{aligned}\rho_{\text{new}}(N; a_4) &= |Na_4|/|Oc_4| = |a_4b_1|/|Oc_4| + |b_1N|/|Oc_4| \\ &\equiv \gamma_4 + |b_1N|/|Oc_4|.\end{aligned} \tag{44}$$

Find the points P and P_1 that satisfy $P \in FN$, $b_1P \parallel a_1a_2$; $P_1 \in b_1F$, $PP_1 \parallel Nb_1$. The homothety $\triangle Fa_1M \approx \triangle Fb_1P$ implies that $|b_1P| = |a_1M| \cdot |b_1F|/|a_1F| = \gamma_5 t$, where $\gamma_5 = |b_1F|/|a_1F|$ is a constant. We write $\angle b_1c_3O = \omega$ and $\angle Pb_1P_1 = \beta$. In $\triangle b_1PP_1$ we have $\angle b_1PP_1 = \pi/2 - \omega$ and $\angle PP_1b_1 = \pi/2 + \omega - \beta$. The sine theorem implies $|b_1P|/\cos(\omega - \beta) = |b_1P_1|/\cos\omega = |PP_1|/\sin\beta$. From this and the homothety $\triangle FP_1P \approx \triangle Fb_1N$ we obtain

$$\begin{aligned} |b_1N| &= |PP_1| \cdot \frac{|b_1F|}{|P_1F|} = \frac{|b_1P| \cdot \sin\beta}{\cos(\omega-\beta)} \cdot \frac{|b_1F|}{|b_1F| - |b_1P_1|} \\ &= \frac{|b_1F| \cdot \sin\beta}{\cos\omega} \cdot \frac{|b_1P| \cdot \cos\omega/\cos(\omega-\beta)}{|b_1F| - \cos\omega \cdot |b_1P|/\cos(\omega-\beta)}. \end{aligned}$$

From (44) we get

$$\rho_{\text{new}}(N; a_4) = \gamma_4 - \frac{|b_1F| \cdot \sin\beta}{|Oc_4| \cdot \cos\omega} + \frac{|b_1F|^2 \cdot \sin\beta \cdot \cos(\omega-\beta)/\cos^2\omega}{|b_1F| \cdot \cos(\omega-\beta)/\cos\omega - |b_1P|} \cdot \frac{1}{|Oc_4|}.$$

Introducing positive constants

$$\begin{aligned} \gamma_6 &= |b_1F| \cdot \sin\beta/(\cos\omega \cdot |Oc_4|), \\ \gamma_7 &= |b_1F|^2 \cdot \sin\beta \cdot \cos(\omega-\beta)/(\cos^2\omega \cdot |Oc_4|), \\ \gamma_8 &= |b_1F| \cdot \cos(\omega-\beta)/\cos\omega, \end{aligned}$$

we have

$$\rho_{\text{new}}(N; a_4) = \gamma_4 - \gamma_6 + \gamma_7/(\gamma_8 - \gamma_5 \cdot t). \tag{45}$$

Since $|b_1F| > |b_1P_1|$, we have $\gamma_8 - \gamma_5 \cdot t > 0$ for $t \in [t_1; t_2]$. The right-hand side of (45) is a downwards convex function of t. By (40), (41), (43), and (45), the function (39), that is, $f(t) = L^-(a_1MNa_4)$ $(t_1 \le t \le t_2)$, is downwards convex in t. Therefore, $\max f(t) = \max\{f(t_1); f(t_2)\}$. Consider the following four possible maxima of $f(t)$ on $[t_1; t_2]$:

(a) Suppose that $f_{\max} = f(t_1)$ and $a_1MNa_4 = a_1a_9e_9a_4$ is a trapezium $(a_4a_1 \parallel e_9a_9)$. Since $b_1 \in a_4e_9$, it follows that $(e_9O) \cap (a_4a_1) = \{b_9\}$ is in a_4a_1. We have $|Ob_9|/|Oe_9| \in [k; 1/k]$, and from (19) we get $k(a_1a_9e_9a_4) \ge k$. The trapezium $T = a_1a_9e_9a_4$ majorizes $a_1a_2a_3a_4$.

(b) Suppose that $f_{\max} = f(t_2)$ and $a_{10} = a_7$. Then $a_1MNa_4 = a_1a_7e_7a_4$. In the canonically given quadrangle $a_1a_7e_7a_4$ the points $c_7 = \widehat{a_7 - a_1} = e_7$, $\widehat{e_7 - a_7} \in e_7a_4$, and the origin O meet the requirements of Lemma 2.1.

(c) Suppose that $f_{\max} = f(t_2)$ and $a_{10} = a_6$. Then $a_1MNa_4 = a_1a_6e_6a_4$. In the canonically given quadrangle we have $\widehat{e_6 - a_6} = a_4$, $\{w_1\} = a_1a_6 \cap (a_4O)$, $w_1a_4 \parallel a_6e_6$, and $O \in \triangle a_4a_1w_1$. This case was considered in Lemma 2.1.

(d) Suppose that $f_{\max} = f(t_2)$ and $a_{10} = a_5$. Then $a_1MNa_4 = a_1a_5e_5a_4$. By construction, $|Oe_2|/|Oa_5| = k$, $\widehat{a_5 - a_1} = c_5 \in a_5e_5$, $\widehat{e_5 - a_5} \in e_5a_4$.

Take r_1 such that $r_1 \in a_4a_1$, $a_5r_1 \parallel e_5a_4 \parallel a_3a_4$. Since $a_2 \in a_1a_5$, we have $a_1r_1 \supset a_1r$ and $O \in \triangle r_1a_1a_5$. Moreover, if g_1 is a center of the canonically given $a_1a_5e_5a_4$, $\{w_1\} = a_1a_5 \cap (a_4g_1)$, and $\{v_1\} = a_4w_1 \cap a_1e_5$, then the inclusion $a_4a_3 \subset a_4e_5$ implies $O \in \triangle g_1v_1e_5$. In analogy with (29), consider $\Omega_1 = (\triangle r_1a_1a_5) \cap (\triangle g_1v_1e_5)$ with $O \in \Omega_1$. Therefore, case (d) is reduced to case 1 of the proof.

Thus, Theorem 1.3 is proved. ∎

Remark 2.6. In what follows, we mark the vertices of the trapezium $T = a_1a_2a_3a_4$ clockwise in such a way that $a_4a_1 \parallel a_2a_3$ and $|a_4a_1| \geq |a_2a_3|$ with respect to the metric of the adjoint plane R^2. In this case always $c_1 \in a_1a_2$, $c_3 \in a_3a_4$, and $c_4 \in a_4a_1$.

Lemma 2.6. *Let $a_1a_2a_3a_4$ be a normalizing parallelogram, $\{m\} = a_1a_3 \cap a_2a_4$, and $O \in \triangle a_1a_2m$. Then the corresponding factor of symmetry satisfies*

$$k = k(a_1a_2a_3a_4) = |Ob_3|/|Oa_3| = |Ob_4|/|Oa_4|,$$

and for the self-perimeter we get

$$L^-(a_1a_2a_3a_4) \leq 4 + 2(1/k + k) = 2D^2/(D-1). \tag{46}$$

Proof. The central chords a_3b_3 and a_4b_4 form homothetic triangles $\triangle Ob_3b_4 \approx \triangle Oa_3a_4$. Moreover $|Ob_3|/|Oa_1| = |Ob_4|/|Oa_4|$. We look for points $e_{3,4}$ that satisfy $e_3 \in a_2b_2$, $b_3e_3 \parallel a_2a_3$ and $e_4 \in a_1b_1$, $b_4e_4 \parallel a_1a_4$, respectively. Since the chords a_ib_i are central ones, we have $e_3 \in Ob_2$, $e_4 \in Ob_1$ and $\triangle Ob_4e_4 \approx \triangle Oa_4a_1$, $\triangle Ob_3e_3 \approx \triangle Oa_3a_2$. Therefore $|Ob_4|/|Oa_4| = |Oe_4|/|Oa_1| \leq |Ob_1|/|Oa_1|$ and $|Ob_3|/|Oa_3| = |Oe_3|/|Oa_2| \leq |Ob_2|/|Oa_2|$, and hence $k = |Ob_{3,4}|/|Oa_{3,4}|$.

Denote by $L^-_V(a_1a_2a_3a_4)$ the self-perimeter of the parallelogram $a_1a_2a_3a_4$ in case the origin $O \in M^2$ is at some point V. Find points e_1, e_2 that satisfy $e_1 \in a_1a_3$, $e_2 \in a_2a_4$, $e_1e_2 \parallel a_1a_2$, and $O \in e_1e_2$. As mentioned in the proof of Proposition 2.4, the function $f(V) = L^-_V(a_1a_2a_3a_4)$, where $V \in e_1e_2$, is strictly downwards convex. By symmetry, $\max L^-_V(a_1a_2a_3a_4) = f(e_1) = f(e_2) = L^-_e(a_1a_2a_3a_4)$, where $e = e_2$. In case $O = e$ we have $\rho(a_4;a_1) = \rho(a_1;a_2)$ and $\rho(a_2;a_3) = \rho(a_3;a_4)$. Using the homotheties $\triangle a_2Oc_2 \approx \triangle a_2a_4a_3$ and $\triangle a_4Oc_4 \approx \triangle a_4a_2a_1$, where $c_2 = \widehat{a_2 - a_1} \in a_2a_3$, we calculate

$$\rho(a_1;a_2) = |a_4a_3|/|Oc_2| = |a_4a_2|/|Oa_2| = 1 + |Oa_4|/|Oa_2| = 1 + 1/k,$$
$$\rho(a_3;a_4) = |a_2a_1|/|Oc_4| = |a_4a_2|/|Oa_4| = 1 + |Oa_2|/|Oa_4| = 1 + k.$$

The latter equalities and (14) imply (46). ∎

Lemma 2.7. *Let the vertices of the normalizing trapezium $a_1a_2a_3a_4$ be marked as in Remark 2.6, $O \in \triangle a_1a_2a_4$, and $\widehat{a_2 - a_1} = c_2 \in a_3a_4$. If $M \in a_2a_3$, then the self-perimeters of the trapeziums $a_1a_2a_3a_4$ and $a_1a_2Ma_4$*

satisfy

$$L^{-}(a_1a_2a_3a_4) \le L^{-}(a_1a_2Ma_4). \tag{47}$$

Proof. By the hypothesis, $\widehat{a_2 - a_1} = c_2 \in a_3a_4$ and $\widehat{a_3 - a_1} = e_1 \in c_2c_3 \subset a_3a_4$. Proposition 2.5 implies that $\rho_{\text{old}}(a_1; a_2) + \rho_{\text{old}}(a_2; a_3) = \rho_{\text{old}}(a_1; a_3)$. Set $\widehat{a_4 - M} = c_4' \in a_4a_1$. Then $\triangle Oc_4'c_4 \sim \triangle a_4Ma_3$. If the trapezium $a_1a_2Ma_4$ is taken as a new normalizing figure of M^2, then $\rho_{\text{new}}(a_4; a_1) = \rho_{\text{old}}(a_4; a_1)$ and

$$\rho_{\text{new}}(M; a_4) = |Ma_4|/|Oc_4'| = |a_3a_4|/|Oc_4| = \rho_{\text{old}}(a_3; a_4). \tag{48}$$

The endpoint b_1 of the central chord a_1b_1 in the trapezium $a_1a_2a_3a_4$ belongs to a_3a_4, i.e., $b_1 \in a_3a_4$. We look for a point e_2 on the chord Mb_1 and, at the same time, on the side of $\triangle Ma_3b_1$ such that $e_1e_2 \parallel a_3M \parallel a_2a_3$.

The homotheties $\triangle b_1e_1e_2 \approx \triangle b_1a_3M$, $\triangle b_1Oe_2 \approx \triangle b_1a_1M$, and $\triangle Oe_1e_2 \approx \triangle a_1a_3M$ imply $|a_1a_3|/|Oe_1| = |a_1b_1|/|Ob_1| = |a_1M|/|Oe_2|$. For a new normalizing trapezium $a_1a_2Ma_4$, we have $(\widehat{a_2 - a_1})_{\text{new}} = c_2' \in Ma_4$, $(\widehat{M - a_2})_{\text{new}} = c_M \in Ma_4$, and $(\widehat{M - a_1})_{\text{new}} = e_3 \in Ma_4$, $\{e_3\} = Oe_2 \cap Ma_4$. By Proposition 2.5,

$$\begin{aligned}\rho_{\text{new}}(a_1; a_2) + \rho_{\text{new}}(a_2; M) &= \rho_{\text{new}}(a_1; M)\\ &= |a_1M|/|Oe_3| \ge |a_1M|/|Oe_2| = |a_1a_3|/|Oe_1| = \rho_{\text{old}}(a_1; a_3).\end{aligned}$$

From this and (48) we get (47). ∎

DEFINITION 2.10. A normalizing trapezium $T = a_1a_2a_3a_4$ is called *distinctive* if its vertices are marked in accordance with Remark 2.6, $\widehat{a_2 - a_1} = c_2 \in a_3a_4$, and the central chords a_1b_1 and a_2b_2 are such that $|Ob_1|/|Oa_1| = |Ob_2|/|Oa_2|$.

LEMMA 2.8. *The self-perimeter of a distinctive trapezium $T = a_1a_2a_3a_4$ satisfies*

$$L^{-}(T) \le 4 + 2(1/k + k), \tag{49}$$

where $k = k(T)$ is the factor of symmetry of T.

Proof. The cases of degeneration of T into a triangle or a parallelogram were considered in Corollary 2.5 and Lemma 2.6. In what follows, we assume that $|a_4a_1| > |a_2a_3| > 0$. By Definition 2.10, the central chords a_ib_i satisfy

$$|Ob_1|/|Oa_1| = |Ob_2|/|Oa_2| = |Ob_3|/|Oa_3|,\ b_1 \in a_3a_4,\ b_{2,3} \in a_4a_1,\ b_4 \in a_1a_2.$$

We also have $\widehat{a_3 - a_1} = e_1 \in c_2b_1 \subset c_2c_3 \subset a_3a_4$. We first consider the following particular cases.

1. Suppose that $k = |Ob_i|/|Oa_i|$, $0 \le i \le 4$ (see (16)). Find a point e_2 that satisfies $e_2 \in a_4a_1$ and $a_3e_2 \parallel a_2a_1$. We intend to calculate the self-perimeter $L^{-}(a_1a_2a_3a_4)$.

The homothety $\triangle b_4Oc_1 \approx \triangle b_4a_4a_1$ implies

(50) $|a_1a_4| = |Oc_1|\cdot|b_4a_4|/|Ob_4| = |Oc_1|\cdot(1+|Oa_4|/|Ob_4|) = |Oc_1|\cdot(1+1/k)$.

Therefore, $\rho(a_4; a_1) = 1 + 1/k$. Since $\triangle a_3c_3O \approx \triangle a_3a_4b_3$, we have

$$\rho(a_3; a_4) = |a_3a_4|/|Oc_4| = |a_3b_3|/|Ob_3| = 1 + |Oa_3|/|Ob_3| = 1 + 1/k.$$

By Proposition 2.5, $\rho(a_1; a_3) = \rho(a_1; a_2) + \rho(a_2; a_3)$. The homothety $\triangle b_1Oe_1 \approx \triangle b_1a_1a_3$ implies $\rho(a_1; a_3) = |a_1a_3|/|Oe_1| = |a_1b_1|/|Ob_1| = 1 + |Oa_1|/|Ob_1| = 1 + 1/k$. Finally,

$$L^-(a_1a_2a_3a_4) = 3(1 + 1/k). \tag{51}$$

Let us prove (49) for case 1. Since c_2 is in a_3a_4, we have $|Oc_1| \geq |a_2a_3| = |a_1e_2| = |a_1a_4| - |e_2a_4|$. Since $\triangle a_1c_1O \approx \triangle a_1b_4b_1$, we get $|b_4b_1| = |Oc_1| \cdot |a_1b_1|/|Oa_1| = |Oc_1|(1 + k)$. The figure $a_1b_4b_1b_2$ is a parallelogram, $|a_1b_2| = |b_4b_1|$, and hence $|b_2a_4| = |a_1a_4| - |a_1b_2| = |Oc_1| \cdot (1/k - k)$. Using subsequently the homotheties $\triangle a_4b_1b_2 \approx \triangle a_4a_3e_2$, $\triangle a_1a_3e_2 \approx \triangle b_3b_1b_2$, and $\triangle Oa_1a_3 \approx \triangle Ob_1b_3$, we obtain $|e_2a_4| = |b_2a_4| \cdot |a_3e_2|/|b_1b_2| = |b_2a_4| \cdot |a_1a_3|/|b_1b_3| = |b_2a_4| \cdot |Oa_3|/|Ob_3| = |b_2a_4|/k$. Then we have $|e_2a_4| = |Oc_1| \cdot (1 - k^2)/k^2$, and using (50) we obtain $|Oc_1| \geq |a_1e_2| = |Oc_1| \cdot (1 + 1/k) - |Oc_1| \cdot (1 - k^2)/k^2 \geq 0$. From this we obtain $1 \geq (2k^2 + k - 1)/k^2 \geq 0$ or $1/2 \leq k \leq (\sqrt{5} - 1)/2$. If $k \geq 1/2$, then $1/k \leq 2k + 1$, and together with (51) this gives (49).

2. Suppose that $k = |Ob_4|/|Oa_4| \leq |Ob_1|/|Oa_1|$. Write $\{e_3\} = a_1b_1 \cap a_2a_4$, and find a point e_4 that satisfies $e_4 \in Ob_1$ and $e_4b_4 \parallel a_1a_4$.

2.1. If $e_3 \in e_4b_1$, then

$$|Ob_4|/|Oa_4| = |Oe_4|/|Oa_1| \leq |Oe_3|/|Oa_1| \leq |Ob_1|/|Oa_1| = |Ob_2|/|Oa_2|.$$

In view of (16), the latter means that $k(\triangle a_1a_2a_4) = k = k(T)$. By Lemma 2.7 and Corollary 2.5, inequality (47) implies (49).

2.2. If $e_4 \in e_3b_1$, then take the point $\{a_5\} = a_2a_3 \cap (a_4e_4)$. By Lemma 2.7, for the trapezium $a_1a_2a_5a_4$ we have $L^-(a_1a_2a_3a_4) \leq L^-(a_1a_2a_5a_4)$. Since $\triangle Oe_4b_4 \approx \triangle Oa_1a_4$, we have $k = |Oe_4|/|Oa_1| = |Ob_4|/|Oa_4| \leq |Ob_2|/|Oa_2| = |Ob_3|/|Oa_3|$ and $k(a_1a_2a_5a_4) = k$. Set $\{a_6\} = (a_1a_2) \cap (a_4a_5)$. Find a point e_5 that satisfies $e_5 \in a_1a_4$ and $e_4e_5 \parallel a_1a_2$. Write $\{a_7\} = (e_5O) \cap (a_1a_2)$. With respect to the new normalizing trapezium $a_1a_2a_5a_4$ we have $(\widehat{a_2 - a_1})_{\text{new}} = c_2' \in a_5a_4$, $\widehat{a_5 - a_2} = c_5 \in a_5a_4$, $\widehat{a_1 - a_4} = c_1 \in a_1a_2$, and $(\widehat{a_4 - a_5})_{\text{new}} = c_4' \in a_4a_1$. If $a_6 \in a_2a_7$, then the homothety $\triangle Oe_4e_5 \approx \triangle Oa_1a_7$ implies $k(\triangle a_1a_6a_4) = k$. By construction, $a_1a_2a_5a_4 \subset \triangle a_1a_6a_4$, $\widehat{a_6 - a_1} = c_2'$, and $\widehat{a_4 - a_6} = c_4'$. Therefore, (4) implies $L^-(a_1a_2a_5a_4) \leq L^-(\triangle a_1a_6a_4)$. The latter inequality and Corollary 2.5 imply (49). If $a_7 \in a_2a_6$, then find a point a_8 that satisfies $a_8 \in (a_4a_5)$ and $a_7a_8 \parallel a_1a_4$. Since $\triangle Oa_1a_7 \approx \triangle Oe_4e_5$, evidently $k(a_1a_7a_8a_4) = k$. In view of (4) and the relations $(\widehat{a_7 - a_1})_{\text{new}}$

$= c_2' \in a_5a_4$, $(\widehat{a_8 - a_7})_{\rm new} = c_5 \in a_5a_4$, $a_1a_7a_8a_4 \supset a_1a_2a_5a_4$, the self-perimeter of the trapezium $a_1a_7a_8a_4$ satisfies $L^-(a_1a_7a_8a_4) \geq L^-(a_1a_2a_5a_4) \geq L^-(a_1a_2a_3a_4)$. Since $|Ob_i|/|Oa_i| = k$, $i = 1, 4, 7, 8$, by construction case 2.2 is reduced to case 1.

3. Suppose that $k = |Ob_1|/|Oa_1| \leq |Ob_4|/|Oa_4|$. Set $\{e_6\} = Ob_4 \cap a_1a_3$, and find a point e_7 that satisfies $e_7 \in Ob_4$ and $b_1e_7 \parallel a_4a_1$, where $\triangle Oa_4a_1 \approx \triangle Oe_7b_1$. Observe that $c_{2,3} \in a_3a_4$. The normalizing vector for the point $M \in a_2a_3$ is $\widehat{M - a_1} = c_M \in a_3a_4$, and by Proposition 2.5 we have $\rho(a_1; a_3) = \rho(a_1; M) + \rho(M; a_3)$. With respect to the new normalizing trapezium $a_1Ma_3a_4 \subset M^2$ we have $(\widehat{a_1 - a_4})_{\rm new} = c_1'$ which is $Oc_1 \cap a_1M$, $|Oc_1'| \leq |Oc_1|$, and $\rho_{\rm new}(a_4; a_1) \geq \rho_{\rm old}(a_4; a_1)$. Evidently, $\rho_{\rm new}(a_3; a_4) = \rho_{\rm old}(a_3; a_4)$. Thus

$$L^-(a_1a_2a_3a_4) \leq L^-(a_1Ma_3a_4), \qquad M \in a_2a_3. \tag{52}$$

3.1. If $e_6 \in b_4e_7$, then the central chords a_1b_1, a_3b_3, a_4e_6 of $\triangle a_1a_3a_4$ satisfy $k = |Ob_1|/|Oa_1| = |Ob_3|/|Oa_3| = |Oe_7|/|Oa_4| \leq |Oe_6|/|Oa_4|$. By (16), we have $k(\triangle a_1a_3a_4) = k$, and by (52) with $M = a_3$ we have $L^-(a_1a_2a_3a_4) \leq L^-(\triangle a_1a_3a_4)$. With Corollary 2.5, we get (49).

3.2. If $e_7 \in b_4e_6$, then let $\{a_5\} = a_2a_3 \cap (a_1e_7)$ and $\{b_5\} = (a_5O) \cap a_4a_1$. The self-perimeter of the new normalizing trapezium $a_1a_5a_3a_4 \subset M^2$ satisfies (52) with $M = a_5$. The central chords a_1b_1, a_5b_5, a_3b_3, and a_4e_7 satisfy $k = |Ob_1|/|Oa_1| = |Ob_5|/|Oa_5| = |Ob_3|/|Oa_3| = |Oe_7|/|Oa_4|$. Thus, case 3.2 is reduced to case 1, and Lemma 2.8 is proved. ∎

Proof of Theorem 1.2. Let $k(P_4)$ and $k(T)$ be the factors of symmetry for a given normalizing quadrangle P_4 and its majorizing trapezium T, respectively. The latter exists by Theorem 1.3. In view of (14), condition (10) is equivalent to $k(P_4) \leq k(T)$. If (49) holds for an arbitrary trapezium, then the estimate (9) for the first self-perimeter holds due to the inequalities

$$\begin{aligned} L^-(P_4) &\leq L^-(T) \leq 4 + 2(1/k(T) + k(T)) \\ &\leq 4 + 2(1/k(P_4) + k(P_4)) = 2D^2/(D-1). \end{aligned} \tag{53}$$

The inequality (9) for the second self-perimeter $L^+(P_4)$ follows by duality.

Denote the vertices of the trapezium T in accordance with Remark 2.6, i.e., $T = a_1a_2a_3a_4$, $a_4a_1 \parallel a_2a_3$ and $|a_4a_1| \geq |a_2a_3|$ in the adjoint plane R^2. Find a point $u \in a_4a_1$ such that $ua_3 \parallel a_1a_2$. Write $\{m\} = a_1a_3 \cap a_2a_4$ and $\{n\} = ua_3 \cap a_2a_4$. The chord ua_3 and the diagonals a_1a_3 and a_2a_4 split T into six parts: $a_1a_2a_3a_4 = \triangle a_2a_3m \cup \triangle a_1a_2m \cup a_1mnu \cup \triangle una_4 \cup \triangle nma_3 \cup \triangle a_4na_3$.

Our reasonings depend on the possible location of the origin $O \in M^2$ with respect to the above mentioned parts of T.

1. Suppose that $O \in \triangle a_2a_3m \subset \triangle a_2a_3a_4$. Similarly to (23) (Proposition 2.7), we have $k = |Ob_i|/|Oa_i|$, $i = 1, 4$, where a_ib_i are central chords in T. Take a point a_5 in such a way that $a_1a_5a_3a_4$ is a parallelogram. Select $M \in b_4a_5$. Introduce a parameter $t = |b_1M|$ and set $t_1 = |b_1b_4|$ and $t_2 = |b_1a_5|$. Observe that $t_1 \le t \le t_2$. Consider the new normalizing trapezium $a_1Ma_3a_4 \subset M^2$, and define the self-perimeter function

$$f(t) = L^-(a_1Ma_3a_4), \quad t \in [t_1; t_2].$$

Write $(\widehat{a_1 - a_4})_{\text{new}} = c_1' \in a_1M$, $(\widehat{M - a_1})_{\text{new}} = c_M \in b_4b_1 \subset a_2a_3$, and $\widehat{a_3 - M} = \widehat{a_3 - a_2} = c_3 \in a_3a_4$. Evidently, $\rho_{\text{new}}(a_3; a_4) = \rho_{\text{old}}(a_3; a_4)$. The similarity $\triangle a_1Ma_2 \sim \triangle Oc_Mc_2$ implies $\rho_{\text{new}}(a_1; M) = |a_1M|/|Oc_M| = |a_1a_2|/|Oc_2| = \rho_{\text{old}}(a_1; a_2)$. The function $\rho_{\text{new}}(M; a_3) = |Ma_3|/|Oc_3| = (t + |b_1a_3|)/|Oc_3|$ is linear in t. The homothety $\triangle a_1Mb_1 \approx \triangle a_1c_1'O$ yields $\rho_{\text{new}}(a_4; a_1) = |a_1a_4|/|Oc_1'| = |a_1a_4|\cdot|a_1b_1|/(|Oa_1|\cdot t)$. Thus, the function $f(t)$ is downwards convex on $[t_1; t_2]$, and hence $\max f(t) = \max\{f(t_1); f(t_2)\}$.

(a) If $f_{\max} = f(t_2)$, then $a_1Ma_3a_4 = a_1a_5a_3a_4$ is a parallelogram. We have $O \in \triangle a_3ma_2 \subset \triangle a_3m'a_5$, where $\{m'\} = a_1a_3 \cap a_5a_4$. Since $k = |Ob_4|/|Oa_4|$, by Lemma 2.6 we have $k(a_1a_5a_3a_4) = k$ and (46) holds. In combination with (53) we get (9).

(b) If $f_{\max} = f(t_1)$, then $a_1Ma_3a_4 = a_1b_4a_3a_4$. The line through a_4 parallel to a_1b_4 is a supporting one for the trapezium $a_1b_4a_3a_4$. We have $|Ob_4|/|Oa_4| = k = k(a_1a_2a_3a_4)$ by hypothesis, and $k(a_1b_4a_3a_4) = k$ by Corollary 2.4. By construction, $|b_4a_3| \le |a_1a_4|$ and $b_1 \in b_4a_3$, and hence $a_1b_4a_3a_4$ is affinely equivalent to the trapezium from Example 2.1 that shows the sharpness of inequality (9).

2. Suppose that $O \in a_1a_2nu = (\triangle a_1a_2m) \cup (a_1mnu)$. We have $b_4 \in a_1a_2$. Construct a parallelogram $e_1a_5a_3a_4$ such that $e_1 \in a_4a_1$, $b_4 \in e_1a_5$, and $a_2 \in a_5a_3$. Mark the points $\widehat{a_4 - a_3} = c_4 \in a_4e_1 \subset a_4a_1$, $(\widehat{a_1 - a_4})_{\text{old}} = c_1 \in a_1a_2$, $(\widehat{a_1 - a_4})_{\text{new}} = c_1' \in e_1a_5$, $\widehat{a_2 - a_1} = c_2 \in a_2a_3$, $(\widehat{a_5 - a_1})_{\text{new}} = c_5 \in a_5a_3$, and $\widehat{a_3 - a_2} = \widehat{a_3 - a_5} = c_3 \in a_3a_4$. The homotheties $\triangle b_4Oc_1' \approx \triangle b_4a_4e_1$ and $\triangle b_4Oc_1 \approx \triangle b_4a_4a_1$ imply $\rho_{\text{new}}(a_4; e_1) = |a_4e_1|/|Oc_1'| = |a_4b_4|/|Ob_4| = |a_4a_1|/|Oc_1| = \rho_{\text{old}}(a_4; a_1)$. The similarities $\triangle Oc_5c_2 \sim \triangle b_4a_5a_2 \sim \triangle b_4e_1a_1$ yield $\rho_{\text{new}}(e_1; a_5) = \rho_{\text{old}}(a_1; a_2)$.

Evidently, $\rho(a_5; a_3) \ge \rho(a_2; a_3)$ and $\rho_{\text{new}}(a_3; a_4) = \rho_{\text{old}}(a_3; a_4)$. Hence we have $L^-(a_1a_2a_5a_4) \le L^-(e_1a_5a_3a_4)$. Set $\{m'\} = e_1a_3 \cap a_5a_4$. By construction, $O \in \triangle a_4e_1a_5$. If $O \in \triangle e_1a_5m'$, then by Lemma 2.6 we have $k(e_1a_5a_3a_4) = |Ob_4|/|Oa_4| \ge k(a_1a_2a_3a_4)$. If $O \in \triangle a_4e_1m'$, then $k(e_1a_5a_3a_4) = |Ob_3|/|Oa_3| \ge k$. Combining this with (46) and (53), we get (9).

3. Suppose that $O \in \triangle una_4$, $c_{2,3} \in a_3a_4$, $b_1 \in a_3a_4$, $b_{2,3} \in a_4u$, and $b_4 \in a_1a_2$. Find a point e_1 that satisfies $e_1 \in a_4a_1$ and $e_1b_1 \parallel a_1a_2$. Set $\{a_5\} = (a_1a_2) \cap (a_4a_3)$, $\{b_5\} = a_4a_1 \cap (a_5O)$, and $\{a_6\} = (a_1a_2) \cap (e_1O)$. The homothety $\triangle Ob_1e_1 \approx \triangle Oa_1a_6$ implies $|Ob_1|/|Oa_1| = |Oe_1|/|Oa_6|$. Observe that $\widehat{a_6 - a_1} = \widehat{a_5 - a_1} = c_2 \in a_3a_4$.

(a) If $|Ob_1|/|Oa_1| \leq |Ob_2|/|Oa_2|$, then $\{b_2'\} = Ob_2 \cap b_1e_1$, $e_1 \in b_2u$, $a_2 \in a_1a_6$. If $a_5 \in a_2a_6$, then $\triangle a_1a_5a_4$ is a new normalizing figure of M^2. Evidently, $|Ob_5|/|Oa_5| \geq |Oe_1|/|Oa_6|$. By (16) we have $k(\triangle a_1a_5a_4) = k$. The inclusion $a_1a_2a_3a_4 \subset \triangle a_1a_5a_4$, $c_2 \in a_3a_4$, and inequality (4) imply $L^-(a_1a_2a_3a_4) \leq L^-(\triangle a_1a_5a_4)$. Combining this with Corollary 2.5 we get (9). If $a_6 \in a_2a_5$, then the trapezium $T = a_1a_6a_7a_4$, where $a_7 \in (a_4a_3)$ and $a_7a_6 \parallel a_4a_1$, is a new normalizing figure of M^2. Set $\{b_7\} = a_4a_1 \cap (a_7O)$ and $b_6 = e_1$. Since $|Ob_6|/|Oa_6| = |Ob_1|/|Oa_1| = |Ob_7|/|Oa_7|$, we obtain $k(a_1a_6a_7a_4) = k$, and the trapezium T is distinctive. The estimate (49) of Lemma 2.8 implies (9).

(b) If $|Ob_2|/|Oa_2| \leq |Ob_1|/|Oa_1|$, then $a_6 \in a_1a_2$. Find points e_2, e_3 that satisfy $e_2 \in Ob_1$, $e_2b_2 \parallel a_2a_1$, and $e_3 \in Ob_1 \cap a_2a_4$. If $e_2 \in Oe_3$, then $\triangle a_1a_2a_4$ is a new normalizing figure of M^2. Formula (16) and $|Ob_2|/|Oa_2| = |Oe_2|/|Oa_1| \leq |Oe_3|/|Oa_1|$ imply $k(\triangle a_1a_2a_4) = k$. By Lemma 2.7 with $M = a_2$ in (47), and Corollary 2.5, we get (9). If $e_3 \in Oe_2$, then the trapezium $T = a_1a_2a_7a_4$, where $\{a_7\} = a_2a_3 \cap (a_4e_2)$, is a new normalizing figure of M^2. Since $(\widehat{a_2 - a_1})_{\text{new}} = c_2' \in a_7a_4$, $|Oe_2|/|Oa_1| = |Ob_2|/|Oa_2|$, $|a_2a_7| \leq |a_1a_4|$, and $a_2a_7 \parallel a_1a_4$, it follows that $T = a_1a_2a_7a_4$ is a distinctive trapezium and $k(T) = k$. By Lemma 2.7 we have $L^-(a_1a_2a_3a_4) \leq L^-(T)$. Together with (49) we get (9).

4. Suppose that $O \in \triangle a_4na_3$, $b_{1,2} \in a_3a_4$, $b_3 \in a_4a_1$, $b_4 \in a_2a_3$, and $c_{2,3} \in a_3a_4$. For this kind of trapezium, in analogy with the proof of Proposition 2.7, case (b), we can prove (23), i.e., $k(a_1a_2a_3a_4) = |Ob_1|/|Oa_1|$. Take the trapezium $a_1b_4a_3a_4$ in the capacity of a new normalizing one of M^2. The chords a_4b_4, a_3b_3, and a_1b_1 are simultaneously central ones for the trapeziums $a_1a_2a_3a_4$ and $a_1b_4a_3a_4$. From (16) we get $k(a_1b_4a_3a_4) = k = |Ob_1|/|Oa_1|$. For normalizing points we have $c_{2,3} \in a_3a_4$ and $\widehat{b_4 - a_1} = c_b \in c_2c_3$. Then, by Proposition 2.5,

$$\begin{aligned}\rho_{\text{new}}(a_1; b_4) + \rho_{\text{new}}(b_4; a_3) &= \rho_{\text{new}}(a_1; a_3) = \rho_{\text{old}}(a_1; a_3)\\ &= \rho_{\text{old}}(a_1; a_2) + \rho_{\text{old}}(a_2; a_3).\end{aligned}$$

Evidently, $\rho_{\text{new}}(a_3; a_4) = \rho_{\text{old}}(a_3; a_4)$. We have $(\widehat{a_1 - a_4})_{\text{old}} = c_1 \in a_1a_2$ and $(\widehat{a_1 - a_4})_{\text{new}} = c_1' \in a_1b_4$. Therefore $|Oc_1'| \leq |Oc_1|$ and $\rho_{\text{new}}(a_4; a_1) \geq \rho_{\text{old}}(a_4; a_1)$. Then $L^-(a_1a_2a_3a_4) \leq L^-(a_1b_4a_3a_4)$, where the origin $O \in \triangle a_1b_4a_4$ is in the normalizing trapezium $a_1b_4a_3a_4 \subset M^2$. Thus, case 4 is reduced to cases 2 and 3, where the origin $O \in \triangle a_1a_2a_4$ is in the normalizing trapezium $a_1a_2a_3a_4$.

5. Suppose that $O \in \triangle nma_3$, $b_{1,2} \in a_3a_4$, $b_3 \in a_4a_1$, $b_4 \in a_2a_3$, and $\widehat{a_2 - a_1} = c_2 \in a_2a_3$. In analogy with case 4, we have $k = |Ob_1|/|Oa_1|$. Set

$\{e_1\} = (a_1b_1) \cap (a_2a_3)$, and find points e_2, e_3 that satisfy $e_2 \in a_2a_3$, $e_2a_1 \parallel a_3O$; $e_3 \in a_4b_4$, $e_3b_1 \parallel a_2a_3$; and $\{e_4\} = a_2a_3 \cap (a_1e_3)$. For the parallelogram $a_1a_5a_3a_4$, the vertex a_5 is in (a_2a_3). Define

$$e_5 = \begin{cases} e_2 & \text{if } e_4 \in e_1e_2, \\ e_4 & \text{if } e_2 \in e_1e_4. \end{cases}$$

Write $t_1 = |e_1e_5|$ and $t_2 = |e_1a_5|$. Let $M \in a_5e_5$ and take $t = |e_1M| \in [t_1; t_2]$ as a parameter. In analogy with case 1, the function $f(t) = L^{-}(a_1Ma_3a_4)$, $t \in [t_1; t_2]$, is downwards convex.

(a) If $f_{\max} = f(t_2)$, then $a_1Ma_3a_4 = a_1a_5a_3a_4$ is a parallelogram. The origin O is in $\triangle nma_3 \subset \triangle a_4m'a_3$, where $\{m'\} = a_1a_3 \cap a_4a_5$. By Lemma 2.6, we have $k(a_1a_5a_3a_4) = |Ob_1|/|Oa_1| = k$. Using (46), we get (9).

(b) If $f_{\max} = f(t_1)$, then $a_1Ma_3a_4 = a_1e_5a_3a_4$ is a trapezium. Denote by $a_4b'_4$ and e_5e_6 the central chords in $a_1e_5a_3a_4$ that correspond to a_4 and e_5, respectively. By definition of e_5, we have $a_4e_3 \subseteq a_4b'_4$. Since $\triangle Oe_3b_1 \approx \triangle Oa_4a_1$, it follows that $k = |Ob_1|/|Oa_1| = |Oe_3|/|Oa_4| \leq |Ob'_4|/|Oa_4|$. The chord e_5e_6 is also central in the trapezium $a_1a_2a_3a_4$. Hence $k \leq |Oe_6|/|Oe_5| \leq 1/k$. By (16), we have $k(a_1e_5a_3a_4) = k$. If $e_5 = e_4$, then $\{e_3\} = a_4b_4 \cap a_1e_4$, and the origin $O \in \triangle a_1e_5a_4$ is located inside the new normalizing trapezium $a_1e_5a_3a_4$. Such a location of the origin in the normalizing trapezium has been considered in cases 2 and 3 (this is the case when $O \in \triangle a_1a_2a_4$ in the trapezium $a_1a_2a_3a_4$). If $e_5 = e_2$, we have $\widehat{e_5 - a_1} = \widehat{e_2 - a_1} = a_3$. Then $O \in \triangle a_4b_3a_3$, where the chord a_3b_3 is central. The latter means that O is inside the normalizing trapezium of cases 3 and 4 (in these cases $O \in \triangle a_4ua_3$ in the trapezium $a_1a_2a_3a_4$).

Summarizing, Theorem 1.2 is proved. ∎

REFERENCES

[1] T. Bonnesen and W. Fenchel, *Theorie der konvexen Körper*, Springer, Berlin, 1934.

[2] S. Gołąb, *Some metric problems in the geometry of Minkowski planes*, Prace Akademii Górniczej w Krakowie 6 (1932), 1–79 (in Polish, with French summary).

[3] B. Grünbaum, *Self-circumference of convex sets*, Colloq. Math. 13 (1964), 55–57.

[4] B. Grünbaum, *The perimeter of Minkowski unit discs*, Colloq. Math. 15 (1966), 135–139.

[5] B. Grünbaum, *Studies in Combinatorial Geometry and the Theory of Convex Bodies*, Nauka, Moscow, 1971 (in Russian).

[6] F. Klein, *Elementary Mathematics from an Advanced Standpoint: Geometry*, Dover, New York, 2004.

[7] K. Leichtweiss, *Konvexe Mengen*, Deutsch. Verlag Wiss., Berlin, 1980.

[8] V. V. Makeev, *On upper estimates for the perimeter of non-symmetric unit circles of Minkowski plane*, Zap. Nauchn. Sem. LOMI 299 (2003), 262–266 (in Russian).

[9] H. Martini and A. I. Shcherba, *On the self-perimeter of quandrangles for gauges*, Beitr. Algebra Geom. 52 (2011), 191–203.

[10] H. Martini and A. I. Shcherba, *On the self-perimeter of pentagonal gauges*, Aequationes Math. 84 (2012), 157–183.
[11] H. Martini and A. I. Shcherba, *On the stability of the unit circle with minimal self-perimeter in normed planes*, Colloq. Math. 131 (2013), 69–87.
[12] H. Martini and K. J. Swanepoel, *The geometry of Minkowski spaces—a survey, Part II*, Expo. Math. 22 (2004), 93–144.
[13] H. Martini, K. J. Swanepoel and G. Weiss, *The geometry of Minkowski spaces—a survey, Part I*, Expo. Math. 19 (2001), 97–142.
[14] H. Minkowski, *Theorie der konvexen Körper, insbesondere Begründung ihres Oberflächenbegriffs*, in: Gesammelte Abhandlungen, Bd. 2, Berlin, 1911, 131–229.
[15] B. H. Neumann, *On some affine invariants of closed convex regions* J. London Math. Soc. 14 (1939), 262–272.
[16] A. I. Shcherba, *On estimates for the self-perimeter of the unit circle of a Minkowski plane*, Tr. Rubtsovsk. Ind. Inst. 12 (2003), 96–107 (in Russian).
[17] A. I. Shcherba, *The unit disk of smallest self-perimeter in a Minkowski plane*, Mat. Zametki 81 (2006), 125–135 (in Russian); English transl.: Math. Notes 81 (2007), 108–116.
[18] A. C. Thompson, *Minkowski Geometry*, Cambridge Univ. Press, Cambridge, 1996.

Horst Martini
Faculty of Mathematics
Technical University of Chemnitz
09107 Chemnitz, Germany
E-mail: martini@mathematik.tu-chemnitz.de

Anatoliy Shcherba
Department of Industrial Computer Technologies
Cherkasy State Technological University
Shevchenko Blvd., 460
Cherkasy, 18006, Ukraine
E-mail: shcherbaanatoly@gmail.com

Received 22 May 2014;
revised 27 April 2015 (6271)