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MAXIMAL FUNCTION AND CARLESON MEASURES IN THE THEORY OF BÉKOLLÉ–BONAMI WEIGHTS

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Abstract. Let ω be a Békollé–Bonami weight. We give a complete characterization of the positive measures μ such that

$$\int_{\mathcal{H}} |M_\omega f(z)|^q \, d\mu(z) \le C\Big(\int_{\mathcal{H}} |f(z)|^p \omega(z)\, dV(z)\Big)^{q/p}$$

and

$$\mu(\{z \in \mathcal{H} : Mf(z) > \lambda\}) \le \frac{C}{\lambda^q}\Big(\int_{\mathcal{H}} |f(z)|^p \omega(z)\, dV(z)\Big)^{q/p},$$

where M_ω is the weighted Hardy–Littlewood maximal function on the upper half-plane $\mathcal{H}$ and $1 \le p, q < \infty$.

1. Introduction. Let $\mathcal{H}$ be the upper half-plane, that is, the set $\{z = x + iy \in \mathbb{C} : x \in \mathbb{R} \text{ and } y > 0\}$. Given ω a non-negative locally integrable function on $\mathcal{H}$ (i.e. a *weight*), and $1 \le p < \infty$, we denote by $L^p_\omega(\mathcal{H})$ the set of functions f defined on $\mathcal{H}$ such that

$$\|f\|^p_{p,\omega} := \int_{\mathbb{D}} |f(z)|^p \omega(z)\, dV(z) < \infty$$

with dV being the Lebesgue measure on $\mathcal{H}$. We write $L^p(\mathcal{H})$ when $\omega(z) = 1$ for any $z \in \mathcal{H}$.

Given a weight ω and $1 < p < \infty$, we say ω is in the *Békollé–Bonami class* B_p if $[\omega]_{B_p} < \infty$, where

$$[\omega]_{B_p} := \sup_{I\subset\mathbb{R},\, I \text{ interval}} \Big(\frac{1}{|Q_I|}\int_{Q_I} \omega(z)\, dV(z)\Big)\Big(\frac{1}{|Q_I|}\int_{Q_I} \omega(z)^{1-p'}\, dV(z)\Big)^{p-1},$$

$Q_I := \{z = x + iy \in \mathbb{C} : x \in I \text{ and } 0 < y < |I|\}$, $|Q_I| = \int_{Q_I} dV(z)$, $pp' = p + p'$. This is the exact range of weights ω for which the orthogonal projection P from $L^2(\mathcal{H}, dV(z))$ to its closed subspace consisting of analytic functions is bounded on $L^p_\omega(\mathcal{H})$ (see [B, BB, PR]).

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Let $1 < p < \infty$, and $\omega \in B_p$. In this note we provide a full characterization of positive measures μ on $\mathcal{H}$ such that the Carleson-type embedding

$$\int_{\mathcal{H}} |M_\omega f(z)|^q \, d\mu(z) \le C\Big(\int_{\mathcal{H}} |f(z)|^p \omega(z)\, dV(z)\Big)^{q/p} \tag{1.1}$$

holds when $p \le q < \infty$ and when $p > q$, where M_ω is the weighted Hardy–Littlewood maximal function,

$$M_\omega f(z) := \sup_{I \text{ interval in } \mathbb{R},\, z \in Q_I} \frac{1}{|Q_I|_\omega} \int_{Q_I} |f(z)|\omega(z)\, dV(z),$$

$|Q_I|_\omega = \omega(Q_I) := \int_{Q_I} \omega(z)\, dV(z)$.

We also characterize those positive measures μ on $\mathcal{H}$ such that

$$\mu(\{z \in \mathcal{H} : Mf(z) > \lambda\}) \le \frac{C}{\lambda^q}\Big(\int_{\mathcal{H}} |f(z)|^p \omega(z)\, dV(z)\Big)^{q/p} \tag{1.2}$$

where M is the unweighted Hardy–Littlewood maximal function ($M = M_\omega$ with $\omega(z) = 1$ for all $z \in \mathcal{H}$).

Before stating our main results, let us see how the above questions are related to some others in complex analysis. We recall that the *Bergman space* $A^p_\omega(\mathcal{H})$ is the subspace of $L^p_\omega(\mathcal{H})$ consisting of holomorphic functions on $\mathcal{H}$. The *unweighted Bergman space* $A^p(\mathcal{H})$ is just the subspace of $L^p(\mathcal{H})$ consisting of holomorphic functions on $\mathcal{H}$. A positive measure on $\mathcal{H}$ is called a *q-Carleson measure* for $A^p_\omega(\mathcal{H})$ if there is a constant $C > 0$ such that for any $f \in A^p_\omega(\mathcal{H})$,

$$\int_{\mathcal{H}} |f(z)|^q \, d\mu(z) \le C\Big(\int_{\mathcal{H}} |f(z)|^p \omega(z)\, dV(z)\Big)^{q/p}. \tag{1.3}$$

Carleson measures are very useful in the study of many other questions in complex and harmonic analysis: Toeplitz operators, Cesàro-type integrals, embeddings between different analytic function spaces, etc. Carleson measures as defined by (1.3) in the case $p = q$ were first characterized by L. Carleson for Hardy spaces on the unit disc of the complex plane [CL1, CL2]. The case $p < q$ was handled by P. L. Duren [DP], while the result for $q < p$ is due to I. V. Videnskiĭ [VV]. The Bergman space version of the results of L. Carleson and P. L. Duren is due to Hastings [HW]. Estimations with loss ($q < p$) for these spaces are essentially due to D. Luecking [LD1]. We note that the full characterization of Carleson measures for Bergman spaces on the unit disc with Békollé–Bonami weights was obtained by O. Constantin [CO]. For some other developments, also in the setting of the unit ball of $\mathbb{C}^n$, we refer to [CW, GD2, HL, LD2, OP, SD].

Let $f \in A^p(\mathcal{H})$ and $z \in \mathcal{H}$. Let I be the unique interval such that Q_I is centered at z, and denote by $D(z, |I|/2)$ the disc with center z and radius $|I|/2$. By applying the mean value property and observing that $D(z,|I|/2) \subset Q_I$ and $|D(z,|I|/2)| \simeq |Q_I|$, we deduce that there is a constant $C > 0$ independent of f and z such that

$$|f(z)| \leq \frac{C}{|Q_I|} \int\limits_{Q_I} |f(w)|\,dV(w).$$

It follows that in the case $\omega(z) = 1$ for any $z \in \mathcal{H}$, any measure satisfying (1.1) is a q-Carleson measure for $A^p(\mathcal{H})$.

A full characterization of q-Carleson measures for $A^p(\mathcal{H})$ which uses Bergman balls can be found in [NS]. We are not aware of any result on q-Carleson measures for $A^p_\omega(\mathcal{H})$ for non-constant weights. Our results in this paper provide a full answer when on the left hand side of (1.3) the function is replaced by its weighted maximal function.

Our first main result is the following.

THEOREM 1.1. *Let $1 < p \leq q < \infty$, and let ω be a weight on $\mathcal{H}$. Assume that $\omega \in B_p$. Then the following assertions are equivalent:*

(i) *There exists a constant $C_1 > 0$ such that for any $f \in L^p_\omega(\mathcal{H})$,*

$$\Big(\int\limits_{\mathcal{H}} |M_\omega f(z)|^q\,d\mu(z)\Big)^{1/q} \leq C_1\|f\|_{p,\omega}. \tag{1.4}$$

(ii) *There is a constant C_2 such that for any interval $I \subset \mathbb{R}$,*

$$\mu(Q_I) \leq C_2(\omega(Q_I))^{q/p}. \tag{1.5}$$

Our second result provides estimations with loss.

THEOREM 1.2. *Let $1 < q < p < \infty$, and let ω be a weight on $\mathcal{H}$. Assume that $\omega \in B_p$. Then* (1.4) *holds if and only if the function*

$$K_\mu(z) := \sup_{I\subset\mathbb{R},\, I \text{ interval},\, z\in Q_I} \frac{\mu(Q_I)}{\omega(Q_I)} \tag{1.6}$$

belongs to $L^s_\omega(\mathcal{H})$ where $s = p/(p-q)$.

Note in particular that taking $d\mu(z) = \sigma(z)dV(z)$ with $\sigma \neq \omega$, we obtain a two-weight estimate with loss for the Hardy–Littlewood maximal function, which is in general a very hard question (see [GD1, VE]).

Our last result provides weak-type estimates.

THEOREM 1.3. *Let $1 \leq p \leq q < \infty$, and let ω be a weight on $\mathcal{H}$. Then the following assertions are equivalent:*

(a) *There is a constant $C_1 > 0$ such that for any $f \in L^p_\omega(\mathcal{H})$ and any $\lambda > 0$,*

$$\mu(\{z \in \mathcal{H} : Mf(z) > \lambda\}) \leq \frac{C_1}{\lambda^q}\Big(\int\limits_{\mathcal{H}} |f(z)|^p\omega(z)\,dV(z)\Big)^{q/p}. \tag{1.7}$$

(b) *There is a constant* $C_2 > 0$ *such that for any interval* $I \subset \mathbb{R}$,

$$|Q_I|^{-q/p}\left(\frac{1}{|Q_I|}\int_{Q_I} \omega^{1-p'}(z)\,dV(z)\right)^{q/p'}\mu(Q_I) \le C_2 \tag{1.8}$$

where $(|Q_I|^{-1}\int_{Q_I}\omega^{1-p'}(z)\,dV(z))^{1/p'}$ *is understood as* $(\inf_{Q_I}\omega)^{-1}$ *when* $p = 1$.

(c) *There exists a constant* $C_3 > 0$ *such that for any locally integrable function* f *and any interval* $I \subset \mathbb{R}$,

$$\left(\frac{1}{|Q_I|}\int_{Q_I} |f(z)|\,dV(z)\right)^{q}\mu(Q_I) \le C_3\left(\int_{Q_I} |f(z)|^p\omega(z)\,dV(z)\right)^{q/p}. \tag{1.9}$$

A special case of Theorem 1.3 appears when μ is a continuous measure with respect to the Lebesgue measure dV in the sense that $d\mu(z) = \sigma(z)\,dV(z)$; this provides a weak-type two-weight norm inequality for the maximal function.

To prove the sufficiency part in the three theorems above, we will observe that the matter can be reduced to the case of the dyadic maximal function. We then use an idea that comes from real harmonic analysis (see for example [CD, GR, SE]) and consists in discretizing integrals using appropriate level sets and, in our case, the nice properties of the upper halves of Carleson boxes when they are supported by dyadic intervals. The proof of Theorem 1.1 in particular consists just in rewriting the same type of proof from real harmonic analysis taking into account the fact that the second weight in our case is a measure, and the hypothesis on this measure. For the proof of the necessity in Theorem 1.2, let us observe that when it comes to estimations with loss for the case of the usual Carleson measures for analytic functions, one needs atomic decomposition of functions in the Bergman spaces to apply a method developed by D. Luecking [LD1]. We do not see how this can be extended here; we show instead that one can restrict to the dyadic case, and we then introduce a function g whose maximal function dominates the function K_μ. It turns out that to prove that the condition $K_\mu \in L^s_\omega(\mathcal{H})$ is necessary, it is enough to prove that if the embedding (1.4) holds, then g belongs to $L^s_\omega(\mathcal{H})$. To do so, we use boundedness of the maximal functions and a duality argument.

As Carleson-type embeddings considered in this note might be of some interest for researchers in analytic function spaces who are not necessarily accustomed to techniques of real harmonic analysis, we write down all the steps of the proofs, even those which to people in real harmonic analysis might seem useless.

For two positive quantities A and B, the notation $A \lesssim B$ (resp. $B \lesssim A$) will mean that there is a universal constant $C > 0$ such that $A \le CB$ (resp. $B \le CA$). When $A \lesssim B$ and $B \lesssim A$, we write $A \backsimeq B$.

2. Useful observations and results. For an interval $I \subset \mathbb{R}$, the *upper half* of the Carleson box Q_I associated to I is the subset T_I defined by

$$T_I := \{z = x + iy \in \mathbb{C} : x \in I \text{ and } |I|/2 < y < |I|\}.$$

Note that $|Q_I| \backsimeq |T_I|$.

We record the following weighted inequality.

LEMMA 2.1. *Let $1 < p < \infty$. Assume that ω belongs to the Békollé–Bonami class B_p. Then there is a constant $C > 0$ such that for any interval $I \subset \mathbb{R}$,*

$$\omega(Q_I) \le C[\omega]_{B_p}\omega(T_I).$$

Proof. Using Hölder's inequality and the definition of Békollé–Bonami weight, we obtain

$$\begin{aligned}\frac{|T_I|^p}{|Q_I|^p} &\le \frac{1}{|Q_I|^p}\Big(\int_{T_I} \omega(z)\,dV(z)\Big)\Big(\int_{T_I} \omega^{-p'/p}(z)\,dV(z)\Big)^{p/p'}\\ &\le \frac{1}{|Q_I|^p}\Big(\int_{T_I} \omega(z)\,dV(z)\Big)\Big(\int_{Q_I} \omega^{-p'/p}(z)\,dV(z)\Big)^{p/p'} \le [\omega]_{B_p}\frac{\omega(T_I)}{\omega(Q_I)}.\end{aligned}$$

Thus $\omega(Q_I) \le [\omega]_{B_p}(|Q_I|/|T_I|)^p\omega(T_I) \backsimeq [\omega]_{B_p}\omega(T_I)$. ∎

We will also need the following lemma.

LEMMA 2.2. *Let $1 \le p, q < \infty$ and suppose that ω is a weight and μ a positive measure on $\mathcal{H}$. Then the following assertions are equivalent:*

(i) *There exists a constant $C_1 > 0$ such that for any interval $I \subset \mathbb{R}$,*

$$|Q_I|^{-q/p}\Big(\frac{1}{|Q_I|}\int_{Q_I} \omega^{1-p'}(z)\,dV(z)\Big)^{q/p'}\mu(Q_I) \le C_1 \tag{2.1}$$

where $(|Q_I|^{-1}\int_{Q_I} \omega^{1-p'}(z)\,dV(z))^{1/p'}$ is understood as $(\inf_{Q_I}\omega)^{-1}$ when $p = 1$.

(ii) *There exists a constant $C_2 > 0$ such that for any locally integrable function f and any interval $I \subset \mathbb{R}$,*

$$\Big(\frac{1}{|Q_I|}\int_{Q_I} |f(z)|\,dV(z)\Big)^q\mu(Q_I) \le C_2\Big(\int_{Q_I} |f(z)|^p\omega(z)\,dV(z)\Big)^{q/p}. \tag{2.2}$$

Proof. That (ii)⇒(i) follows by testing (ii) with $f(z) = \chi_{Q_I}(z)\omega^{1-p'}(z)$ if $p > 1$. For $p = 1$, take $f(z) = \chi_S(z)$ where S is a subset of Q_I. One obtains

$$\frac{\mu(Q_I)}{|Q_I|^q} \le C_2\Big(\frac{\omega(S)}{|S|}\Big)^q.$$

As this happens for any subset S of Q_I, it follows that for any $z \in Q_I$,

$$\frac{\mu(Q_I)}{|Q_I|^q} \le C_2(\omega(z))^q,$$

which implies (2.1) for $p = 1$.

Let us check that (i)⇒(ii). Applying Hölder's inequality (in case $p > 1$) to L, the left hand side of (2.2), we obtain

$$\begin{aligned} L &\le |Q_I|^{-q}\Big(\int_{Q_I} \omega^{-p'/p}(z)\,dV(z)\Big)^{q/p'}\mu(Q_I)\Big(\int_{Q_I} |f(z)|^p\omega(z)\,dV(z)\Big)^{q/p} \\ &\le C\Big(\int_{Q_I} |f(z)|^p\omega(z)\,dV(z)\Big)^{q/p}. \end{aligned}$$

For $p = 1$, we easily get

$$L \le \frac{(\inf_{Q_I}\omega)^{-q}}{|Q_I|^q}\Big(\int_{Q_I} |f(z)|\omega(z)\,dV(z)\Big)^q\mu(Q_I) \le C\Big(\int_{Q_I} |f(z)|\omega(z)\,dV(z)\Big)^q. \ \blacksquare$$

Next, we consider the following system of dyadic grids:

$$\mathcal{D}^\beta := \{2^j([0,1) + m + (-1)^j\beta) : m, j \in \mathbb{Z}\} \quad \text{for } \beta \in \{0, 1/3\}.$$

For more on this system and its applications, we refer to [APR, HP, LA, LOPTT, PR]. When $\beta = 0$, we use the notation $\mathcal{D} = \mathcal{D}^0$ that we call the *standard dyadic grid* of $\mathbb{R}$. When I is a dyadic interval, we denote by I^- and I^+ its left half and its right half respectively. We make the following observation which is surely known.

LEMMA 2.3. *Any interval I of $\mathbb{R}$ can be covered by at most two adjacent dyadic intervals I_1 and I_2 in the same dyadic grid such that*

$$|I| < |I_1| = |I_2| \le 2|I|.$$

Proof. Without loss of generality, we can suppose that $I = [a, b)$. For $x \in \mathbb{R}$, we denote by $[x]$ the unique integer such that $[x] \le x < [x] + 1$. If $I \in \mathcal{D}$, then there is nothing to say. If $|I| = 1$, then the dyadic interval $[k, k + 1)$, where $k = [a]$, covers I.

Let us suppose in general that I is not dyadic. Let j be the unique integer such that

$$2^{-j} \le b - a = |I| < 2^{-j+1}, \tag{2.3}$$

and define

$$E_{a,b} := \{l \in \mathbb{Z} : a < l2^{-j} \le b\}.$$

Then $E_{a,b}$ is not empty. To see this, observe that the interval $J = [a2^j, b2^j)$ has length $1 \le |J| < 2$, and consequently J contains at least one integer. Let

$$k_0 := \max\{k : k \in E_{a,b}\}.$$

Then we necessarily have $(k_0 - 2)2^{-j} \le a$, since otherwise $|I| = b - a > k_0 2^{-j} - a > 2^{-j+1}$ and this contradicts (2.3).

As from the definition of k_0 we have $b \le (k_0 + 1)2^{-j}$, it follows that if $(k_0 - 1)2^{-j} \le a$, then the union $[(k_0 - 1)2^{-j}, k_0 2^{-j}) \cup [k_0 2^{-j}, (k_0 + 1)2^{-j})$ covers I, and taking I_1 and I_2 such that $I_1^+ = [(k_0 - 1)2^{-j}, k_0 2^{-j})$ and $I_2^- = [k_0 2^{-j}, (k_0 + 1)2^{-j})$ we get the lemma. If $(k_0 - 1)2^{-j} > a$, then $I \subset I_1 \cup I_2$ where $I_1 = [(l_0 - 1)2^{-j+1}, l_0 2^{-j+1})$, $I_2 = [l_0 2^{-j+1}, (l_0 + 1)2^{-j+1})$ with $k_0 = 2l_0$ if k_0 is even, and $k_0 = 2l_0 + 1$ otherwise. ∎

3. Proof of the main results. Let us start with some observations. Recall that the upper half of Q_I is $T_I = \{x + iy \in \mathcal{H} : x \in I \text{ and } |I|/2 < y < |I|\}$. It is clear that the family $\{T_I\}_{I\in\mathcal{D}}$ where $\mathcal{D}$ is a dyadic grid in $\mathbb{R}$ provides a tiling of $\mathcal{H}$.

Next we recall from [PR] that given an interval $I \subset \mathbb{R}$, there is a dyadic interval $K \in \mathcal{D}^\beta$ for some $\beta \in \{0, 1/3\}$ such that $I \subseteq K$ and $|K| \le 6|I|$. It follows in particular that $|Q_K| \le 36|Q_I|$. Also, proceeding as in the proof of Lemma 2.1 one obtains $\omega(Q_K) \lesssim [\omega]_{B_p}\omega(Q_I)$. Therefore

$$\frac{1}{\omega(Q_I)} \int\limits_{Q_I} |f(z)|\omega(z)\,dV(z) \le C\frac{1}{\omega(Q_K)} \int\limits_{Q_K} |f(z)|\omega(z)\,dV(z)$$

and consequently, for any locally integrable function f,

$$M_\omega f(z) \le C \sum_{\beta\in\{0,1/3\}} M_{d,\omega}^\beta f(z), \qquad z \in \mathcal{H}, \tag{3.1}$$

where $M_{d,\omega}^\beta$ is defined as M_ω but with the supremum taken only over dyadic intervals of the dyadic grid $\mathcal{D}^\beta$. When $\omega \equiv 1$, we use the notation M_d^β, and if moreover $\beta = 0$, we just write M_d.

3.1. Proof of Theorem 1.1. First suppose that (1.4) holds and observe that for any interval $I \subset \mathbb{R}$, $1 \le M_\omega \chi_{Q_I}(z)$ for any $z \in Q_I$. It follows that

$$(\mu(Q_I))^{1/q} \le \Big(\int\limits_{\mathcal{H}} (M_\omega \chi_{Q_I}(z))^q\,d\mu(z)\Big)^{1/q} \le C_1\|\chi_{Q_I}\|_{p,\omega} = (\omega(Q_I))^{1/p},$$

which implies that for any interval $I \subset \mathbb{R}$,

$$\mu(Q_I) \le C_1\,(\omega(Q_I))^{q/p}.$$

That is, (1.5) holds.

To prove that (ii)⇒(i), by the observations made at the beginning of this section it is enough to prove the following.

LEMMA 3.1. *Let $1 < p \le q < \infty$. Assume that ω is a weight in the class B_p such that* (1.5) *holds. Then there is a positive constant C such that for* *any* $f \in L^p_\omega(\mathcal{H})$ *and any* $\beta \in \{0, 1/3\}$,

$$\Big(\int_{\mathcal{H}} |M^\beta_{d,\omega} f(z)|^q \, d\mu(z)\Big)^{1/q} \leq C\|f\|_{p,\omega}. \tag{3.2}$$

Proof. Let $a \geq 2$. To each integer k we associate the set

$$\Omega_k := \{z \in \mathcal{H} : a^k < M^\beta_{d,\omega} f(z) \leq a^{k+1}\}.$$

We observe that $\Omega_k \subset \bigcup_{j=1}^\infty Q_{I_{k,j}}$, where $Q_{I_{k,j}}$ $(I_{k,j} \in \mathcal{D}^\beta)$ is a dyadic square maximal (with respect to inclusion) such that

$$\frac{1}{\omega(Q_{I_{k,j}})} \int_{Q_{I_{k,j}}} |f(z)|\omega(z) \, dV(z) > a^k.$$

It follows from Lemma 2.1 that

$$\begin{aligned}
L &:= \int_{\mathcal{H}} (M^\beta_{d,\omega} f(z))^q \, d\mu(z) = \sum_k \int_{\Omega_k} (M^\beta_{d,\omega} f(z))^q \, d\mu(z) \\
&\leq a^q \sum_k a^{kq} \mu(\Omega_k) \leq a^q \sum_{k,j} a^{kq} \mu(Q_{I_{k,j}}) \\
&\leq a^q \sum_{k,j} \Big(\frac{1}{\omega(Q_{I_{k,j}})} \int_{Q_{I_{k,j}}} |f(z)|\omega(z) \, dV(z)\Big)^q \mu(Q_{I_{k,j}}) \\
&\lesssim a^q \sum_{k,j} \Big(\frac{1}{\omega(Q_{I_{k,j}})} \int_{Q_{I_{k,j}}} |f(z)|\omega(z) \, dV(z)\Big)^q (\omega(Q_{I_{k,j}}))^{q/p} \\
&\lesssim a^q \Big(\sum_{k,j} \Big(\frac{1}{\omega(Q_{I_{k,j}})} \int_{Q_{I_{k,j}}} |f(z)|\omega(z) \, dV(z)\Big)^p \omega(Q_{I_{k,j}})\Big)^{q/p} \\
&\lesssim a^q \Big(\sum_{k,j} \Big(\frac{1}{\omega(Q_{I_{k,j}})} \int_{Q_{I_{k,j}}} |f(z)|\omega(z) \, dV(z)\Big)^p [\omega]_{B_p} \omega(T_{I_{k,j}})\Big)^{q/p} \\
&\lesssim \Big(\sum_{k,j} \int_{T_{I_{k,j}}} \Big(\frac{1}{\omega(Q_{I_{k,j}})} \int_{Q_{I_{k,j}}} |f(z)|\omega(z) \, dV(z)\Big)^p \omega(w) \, dV(w)\Big)^{q/p} \\
&\lesssim \Big(\sum_{k,j} \int_{T_{I_{k,j}}} (M_{d,\omega} f(w))^p \omega(w) \, dV(w)\Big)^{q/p} \lesssim \Big(\int_{\mathcal{H}} |f(z)|^p \omega(z) \, dV(z)\Big)^{q/p}.
\end{aligned}$$

The proof of Lemma 3.1, and hence of Theorem 1.1, is complete. ■

Taking $d\mu(z) = \sigma(z) dV(z)$ where σ is a weight, we obtain the following result.

COROLLARY 3.2. *Let* $1 < p \leq q < \infty$, *and let* ω, σ *be two weights on* $\mathcal{H}$. *Assume that* $\omega \in B_p$. *Then the following assertions are equivalent:*

(i) *There exists a constant $C_1 > 0$ such that for any $f \in L^p_\omega(\mathcal{H})$,*

$$\Big(\int_{\mathcal{H}} |M_\omega f(z)|^q \sigma(z)\, dV(z)\Big)^{1/q} \le C_1 \|f\|_{p,\omega}. \tag{3.3}$$

(ii) *There is a finite constant $C_2 > 0$ such that for any interval $I \subset \mathbb{R}$,*

$$|Q_I|^{1/q-1/p}\bigg(\frac{1}{|Q_I|}\int_{Q_I} \omega^{1-p'}(z)\, dV(z)\bigg)^{1/p'}\bigg(\frac{1}{|Q_I|}\int_{Q_I} \sigma(z)\, dV(z)\bigg)^{1/q} \le C_2. \tag{3.4}$$

3.2. Proof of Theorem 1.2. Let us start with the following lemma.

LEMMA 3.3. *Let $1 \le q < p < \infty$, and let ω be a weight in the class B_p. Assume that μ is a positive measure on $\mathcal{H}$ such that the function K_μ defined by (1.6) belongs to $L^s_\omega(\mathcal{H})$, $s = p/(p-q)$. Then (3.2) holds for any $f \in L^p_\omega(\mathcal{H})$.*

Proof. We proceed as in the proof of Lemma 3.1, using the same notation. For $\beta \in \{0, 1/3\}$, we define

$$K^\beta_{d,\mu}(z) := \sup_{I \in \mathcal{D}^\beta,\, z \in Q_I} \frac{\mu(Q_I)}{\omega(Q_I)}.$$

Then $K^\beta_{d,\mu}(z) \le K_\mu(z)$ for any $z \in \mathcal{H}$. Using Hölder's inequality and Lemma 2.1, we obtain

$$\begin{aligned}
L &:= \int_{\mathcal{H}} (M^\beta_{d,\omega} f(z))^q \, d\mu(z) = \sum_k \int_{\Omega_k} (M^\beta_{d,\omega} f(z))^q \, d\mu(z) \\
&\le a^q \sum_k a^{kq} \mu(\Omega_k) \le a^q \sum_{k,j} a^{kq} \mu(Q_{I_{k,j}}) \\
&\le a^q \sum_{k,j} \bigg(\frac{1}{\omega(Q_{I_{k,j}})} \int_{Q_{I_{k,j}}} |f(z)|\omega(z)\, dV(z)\bigg)^q \mu(Q_{I_{k,j}}) \\
&= a^q \sum_{k,j} \bigg(\frac{1}{\omega(Q_{I_{k,j}})} \int_{Q_{I_{k,j}}} |f(z)|\omega(z)\, dV(z)\bigg)^q \frac{\mu(Q_{I_{k,j}})}{\omega(Q_{I_{k,j}})} \omega(Q_{I_{k,j}}) \le A^{q/p} B^{1/s}
\end{aligned}$$

where

$$\begin{aligned}
A &= \sum_{k,j} \bigg(\frac{1}{\omega(Q_{I_{k,j}})} \int_{Q_{I_{k,j}}} |f(z)|\omega(z)\, dV(z)\bigg)^p \omega(Q_{I_{k,j}}), \\
B &= \sum_{k,j} \bigg(\frac{\mu(Q_{I_{k,j}})}{\omega(Q_{I_{k,j}})}\bigg)^s \omega(Q_{I_{k,j}}).
\end{aligned}$$

From the proof of Lemma 3.1, we already know how to estimate A. Let us estimate B:

$$\begin{aligned} B &\lesssim \sum_{k,j}\left(\frac{\mu(Q_{I_{k,j}})}{\omega(Q_{I_{k,j}})}\right)^s \omega(T_{I_{k,j}}) \lesssim \sum_{k,j}\int\limits_{T_{I_{k,j}}}\left(\frac{\mu(Q_{I_{k,j}})}{\omega(Q_{I_{k,j}})}\right)^s \omega(z)\,dV(z) \\ &\lesssim \sum_{k,j}\int\limits_{T_{I_{k,j}}}(K^\beta_{d,\mu}(z))^s\omega(z)\,dV(z) \lesssim \int\limits_{\mathcal{H}}(K^\beta_{d,\mu}(z))^s\omega(z)\,dV(z) \\ &= C\|K^\beta_{d,\mu}\|^s_{s,\omega} < \infty. \blacksquare \end{aligned}$$

We can now prove Theorem 1.2.

Proof of Theorem 1.2. Sufficiency follows from Lemma 3.3 and the observations made at the beginning of this section. Let us prove necessity. For this we make the following observations: first, (1.4) implies that there exists a constant $C > 0$ such that for any $f \in L^p_\omega(\mathcal{H})$,

$$\int\limits_{\mathcal{H}}(M^\beta_{d,\omega}f(z))^q\,d\mu(z) \le C\|f\|^q_{p,\omega}. \tag{3.5}$$

We recall that

$$K^\beta_{d,\mu}(z) := \sup_{I\in\mathcal{D}^\beta,\, z\in Q_I}\frac{\mu(Q_I)}{\omega(Q_I)}.$$

It is easy to see that there is a constant $C > 0$ such that for any $z \in \mathcal{H}$,

$$K_\mu(z) \le C\sum_{\beta\in\{0,1/3\}}K^\beta_{d,\mu}(z).$$

Thus to prove that $K_\mu \in L^s_\omega(\mathcal{H})$ if (1.4) holds, it is enough to prove that (3.5) implies $K^\beta_{d,\mu} \in L^s_\omega(\mathcal{H})$.

Fix $\beta \in \{0, 1/3\}$. For $z \in \mathcal{H}$, we write $Q_z = Q_{I_z}$ ($I_z \in \mathcal{D}^\beta$) for the smallest Carleson box containing z, and consider the weighted box kernel

$$K^\beta_{d,\omega}(z_0, z) := \frac{1}{\omega(Q_{z_0})}\chi_{Q_{z_0}}(z).$$

For f a locally integrable function, we define

$$K^\beta_{d,\omega}f(z_0) = \int\limits_{\mathcal{H}}K^\beta_{d,\omega}(z_0,z)f(z)\omega(z)\,dV(z) = \frac{1}{\omega(Q_{z_0})}\int\limits_{Q_{z_0}}f(z)\omega(z)\,dV(z).$$

Finally, we define a function g_β on $\mathcal{H}$ by

$$g_\beta(z) := \int\limits_{\mathcal{H}}K^\beta_{d,\omega}(\xi,z)\,d\mu(\xi) = \int\limits_{\mathcal{H}}\frac{\chi_{Q_\xi}(z)}{\omega(Q_\xi)}\,d\mu(\xi).$$

For any (dyadic) Carleson box Q_I, $I \in \mathcal{D}^\beta$, writing Q for Q_I we obtain

$$\begin{aligned}\frac{1}{\omega(Q)}\int_Q g_\beta(z)\omega(z)\,dV(z) &= \frac{1}{\omega(Q)}\int_Q\Big(\int_{\mathcal{H}} K^\beta_{d,\omega}(w,z)\,d\mu(w)\Big)\omega(z)\,dV(z)\\ &= \int_{\mathcal{H}}\int_{\mathcal{H}}\frac{1}{\omega(Q)}\,\frac{\chi_{Q_w}(z)\chi_Q(z)}{\omega(Q_w)}\omega(z)\,dV(z)\,d\mu(w)\\ &\geq \int_Q\frac{1}{\omega(Q)}\int_{\mathcal{H}}\frac{\chi_{Q_w\cap Q}(z)}{\omega(Q_w)}\omega(z)\,dV(z)\,d\mu(w)\\ &\geq \frac{1}{\omega(Q)}\int_Q d\mu(w) = \frac{\mu(Q)}{\omega(Q)}.\end{aligned}$$

Thus for any $z \in \mathcal{H}$,

$$M^\beta_{d,\omega}g_\beta(z) \gtrsim \sup_{I\in\mathcal{D}^\beta,\, z\in Q_I}\frac{\mu(Q_I)}{\omega(Q_I)} := K^\beta_{d,\mu}(z).$$

Hence if the function g_β belongs to $L^s_\omega(\mathcal{H})$, then

$$\|K^\beta_{d,\mu}\|_{s,\omega} \lesssim \|M^\beta_{d,\omega}g_\beta\|_{s,\omega} \lesssim \|g_\beta\|_{s,\omega} < \infty.$$

To finish the proof, we only need to check that $g_\beta \in L^s_\omega(\mathcal{H})$ whenever (3.5) holds.

Let us start from the following inequality between $K^\beta_{d,\omega}f$ and $M^\beta_{d,\omega}f$. Fix z_0 in $\mathcal{H}$. For any $\xi \in Q_{z_0}$, we have

$$|K^\beta_{d,\omega}f(z_0)| := \left|\frac{1}{\omega(Q_{z_0})}\int_{Q_{z_0}} f(z)\omega(z)\,dV(z)\right| \leq M^\beta_{d,\omega}f(\xi).$$

Thus

$$|K^\beta_{d,\omega}f(z)|^{1/q} \leq M^\beta_{d,\omega}((M^\beta_{d,\omega}f)^{1/q})(z) \quad \text{for any } z\in\mathcal{H}. \tag{3.6}$$

Now, for any $f \in L^{p/q}_\omega(\mathcal{H})$, using (3.5), (3.6) and the boundedness of the maximal function, we obtain

$$\begin{aligned}\left|\int_{\mathcal{H}} g_\beta(z)f(z)\omega(z)\,dV(z)\right| &= \left|\int_{\mathcal{H}}\Big(\int_{\mathcal{H}} K^\beta_{d,\omega}(\xi,z)\,d\mu(\xi)\Big)f(z)\omega(z)\,dV(z)\right|\\ &= \left|\int_{\mathcal{H}}\Big(\int_{\mathcal{H}} K^\beta_{d,\omega}(\xi,z)f(z)\omega(z)\,dV(z)\Big)\,d\mu(\xi)\right|\end{aligned}$$

$$= \Big|\int_{\mathcal{H}} K_{d,\mu}^{\beta} f(\xi)\, d\mu(\xi)\Big| \le \int_{\mathcal{H}} |K_{d,\mu}^{\beta} f(\xi)|\, d\mu(\xi)$$
$$= \int_{\mathcal{H}} (|K_{d,\mu}^{\beta} f(\xi)|^{1/q})^q\, d\mu(\xi) \le C \int_{\mathcal{H}} (M_{d,\omega}^{\beta}((M_{d,\omega}^{\beta} f)^{1/q})(\xi))^q\, d\mu(\xi)$$
$$\le C\Big(\int_{\mathcal{H}} (M_{d,\omega}^{\beta} f(z))^{p/q}\omega(z)\, dV(z)\Big)^{q/p} \le C\Big(\int_{\mathcal{H}} |f(z)|^{p/q}\omega(z)\, dV(z)\Big)^{q/p}.$$

Thus there is a constant $C > 0$ such that

$$\|g_\beta\|_{s,\omega} := \sup_{f\in L_\omega^{p/q}(\mathcal{H}),\, \|f\|_{p/q,\omega}\le 1} \Big|\int_{\mathcal{H}} g_\beta(z) f(z)\omega(z)\, dV(z)\Big| \le C.$$

The proof of Theorem 1.2 is complete. ∎

3.3. Proof of Theorem 1.3. We start from the following lemma which tells us that it will suffice to restrict to level sets involving the dyadic maximal function.

LEMMA 3.4. *Let f be a locally integrable function. Then for any $\lambda > 0$,*

$$\{z \in \mathcal{H} : Mf(z) > \lambda\} \subset \{z \in \mathbb{D} : M_d f(z) > \lambda/68\}. \tag{3.7}$$

Proof. Set

$$A := \{z \in \mathcal{H} : Mf(z) > \lambda\}, \qquad B := \{z \in \mathcal{H} : M_d f(z) > \lambda/68\}.$$

Recall that there is a family $\{Q_{I_j}\}_{j\in\mathbb{N}_0}$ of maximal (with respect to inclusion) dyadic Carleson boxes (i.e. $I_j \in \mathcal{D}$) such that

$$\frac{4\lambda}{68} \ge \frac{1}{|Q_{I_j}|}\int_{Q_{I_j}} |f|\, dV > \frac{\lambda}{68}$$

so that $B = \bigcup_{j\in\mathbb{N}_0} Q_{I_j}$.

Let $z \in A$ and suppose that $z \notin B$. We know that there is an interval I (not necessarily dyadic) such that $z \in Q_I$ and

$$\frac{1}{|Q_I|}\int_{Q_I} |f|\, dV > \lambda. \tag{3.8}$$

Recall from Lemma 2.3 that I can be covered by at most two adjacent dyadic intervals J_1 and J_2 (in this order) such that $|I| < |J_1| = |J_2| \le 2|I|$ so that $Q_I \subset Q_{J_1} \cup Q_{J_2}$. Of course, z belongs to one and only one of the associated boxes Q_{J_1} and Q_{J_2}. Suppose $z \in Q_{J_1}$. Then necessarily Q_{J_1} is not contained in B, since if so then z would belong to B, contrary to our hypothesis. Thus $Q_{J_1} \cap B = \emptyset$ or $Q_{J_1} \supset Q_{I_j}$ for some j, and in both cases, because of the maximality of the I_js, we deduce that

$$\frac{1}{|Q_{J_1}|}\int_{Q_{J_1}} |f|\, dV \le \frac{\lambda}{68}.$$

For the other interval J_2, we have the following possibilities:

$$\begin{aligned} &J_2 = I_j \quad \text{for some } j,\\ &J_2 \subset I_j \quad \text{for some } j,\\ &J_2 \supset I_j \quad \text{for some } j,\\ &J_2 \cap B = \emptyset. \end{aligned}$$

If $J_2 \supset I_j$ for some j or $J_2 \cap B = \emptyset$, then because of the maximality of the I_js,

$$\frac{1}{|Q_{J_2}|} \int_{Q_{J_2}} |f|\, dV \le \frac{\lambda}{68}.$$

If $J_2 = I_j$ for some j, then, of course,

$$\frac{1}{|Q_{J_2}|} \int_{Q_{J_2}} |f|\, dV \le \frac{4\lambda}{68}.$$

It remains to consider the case where $J_2 \subset I_j$ for some j. If $J_2 \subset I_j$, then we can have

$$J_2 = I_j^-, \quad J_2 \subset I_j^- \quad \text{or} \quad J_2 \subseteq I_j^+,$$

where I_j^- and I_j^+ denote the left and right halves of I_j respectively. If $J_2 \subset I_j^-$ or $J_2 \subseteq I_j^+$, then $J_1 \cap I_j \ne \emptyset$, and this necessarily implies that $J_1 \subset I_j$. Thus $z \in Q_{J_1} \subset Q_{I_j} \subset B$, contrary to hypothesis. Hence the only possible case is $J_2 = I_j^-$, which leads to the estimate

$$\frac{1}{|Q_{J_2}|} \int_{Q_{J_2}} |f|\, dV \le \frac{4}{|Q_{I_j}|} \int_{Q_{I_j}} |f|\, dV \le \frac{16\lambda}{68}.$$

Thus from all the above analysis, we obtain

$$\begin{aligned} \frac{1}{|Q_I|} \int_{Q_I} |f|\, dV &= \frac{1}{|Q_I|}\Big(\int_{Q_I \cap Q_{J_1}} |f|\, dV + \int_{Q_I \cap Q_{J_2}} |f|\, dV \Big)\\ &\le \frac{|Q_{J_1}|}{|Q_I|}\bigg(\frac{1}{|Q_{J_1}|} \int_{Q_{J_1}} |f|\, dV + \frac{1}{|Q_{J_2}|} \int_{Q_{J_2}} |f|\, dV \bigg)\\ &\le 4\bigg(\frac{\lambda}{68} + \frac{16\lambda}{68}\bigg) = \lambda, \end{aligned}$$

which clearly contradicts (3.8). ∎

Proof of Theorem 1.3. Note that by Lemma 2.2, we have (b)⇔(c). Let us prove that (a)⇔(c).

Let f be a locally integrable function and I an interval. Fix λ such that

$$0 < \lambda < \frac{1}{|Q_I|} \int_{Q_I} |f|\, dV.$$

Then $Q_I \subset \{z \in \mathcal{H} : M(\chi_{Q_I} f) > \lambda)\}$. It follows from (1.7) that

$$\mu(Q_I) \leq \frac{C}{\lambda^q}\Big(\int\limits_{Q_I} |f(z)|^p \omega(z)\, dV(z)\Big)^{q/p}.$$

As this happens for all $\lambda > 0$, we see in particular that

$$\mu(Q_I)\bigg(\frac{1}{|Q_I|}\int\limits_{Q_I} |f|\, dV(z)\bigg)^q \leq C\Big(\int\limits_{Q_I} |f(z)|^p \omega(z)\, dV(z)\Big)^{q/p}.$$

Next suppose that (1.9) holds. We observe by Lemma 3.4 that to obtain (1.7), we only have to prove

$$\mu\bigg(\bigg\{z \in \mathcal{D} : M_d f(z) > \frac{\lambda}{68}\bigg\}\bigg) \leq \frac{C}{\lambda^q}\|f\|_{p,\omega}^q. \tag{3.9}$$

We recall that

$$\bigg\{z \in \mathcal{H} : M_d f(z) > \frac{\lambda}{68}\bigg\} = \bigcup_{j\in\mathbb{N}_0} Q_{I_j}$$

where the I_js are maximal dyadic intervals with respect to inclusion and such that

$$\frac{1}{|Q_{I_j}|}\int\limits_{Q_{I_j}} |f|\, dV > \frac{\lambda}{68}.$$

Our hypothesis implies in particular that

$$\mu(Q_{I_j}) \leq C\bigg(\frac{|Q_{I_j}|}{\int_{Q_{I_j}} |f|\, dV}\bigg)^q \Big(\int\limits_{Q_{I_j}} |f|^p \omega\, dV\Big)^{q/p}.$$

Thus

$$\begin{aligned}
&\mu\bigg(\bigg\{z \in \mathcal{H} : M_d f(z) > \frac{\lambda}{68}\bigg\}\bigg)\\
&\quad = \sum_j \mu(Q_{I_j}) \leq \sum_j \bigg(\frac{|Q_{I_j}|}{\int_{Q_{I_j}} |f|\, dV}\bigg)^q \Big(\int\limits_{Q_{I_j}} |f|^p \omega\, dV\Big)^{q/p}\\
&\quad \leq \Big(\frac{68}{\lambda}\Big)^q \sum_j \bigg(\int\limits_{Q_{I_j}} |f|^p \omega\, dV\bigg)^{q/p} \leq \Big(\frac{68}{\lambda}\Big)^q \Big(\sum_j \int\limits_{Q_{I_j}} |f|^p \omega\, dV\Big)^{q/p}\\
&\quad \leq \Big(\frac{68}{\lambda}\Big)^q \Big(\int\limits_{\mathcal{H}} |f(z)|^p \omega(z)\, dV(z)\Big)^{q/p} = \Big(\frac{68}{\lambda}\Big)^q \|f\|_{p,\omega}^q.
\end{aligned}$$

The proof of Theorem 1.3 is complete. ∎

Taking $d\mu(z) = \sigma(z)dV(z)$, we obtain the following result.

COROLLARY 3.5. *Let $1 \leq p \leq q < \infty$, and let ω, σ be two weights on $\mathcal{H}$. Then the following assertions are equivalent:*

(a) *There is a constant $C_1 > 0$ such that for any $f \in L^p_\omega(\mathcal{H})$ and any $\lambda > 0$,*

$$\sigma\left(\{z \in \mathcal{H} : Mf(z) > \lambda\}\right) \le \frac{C_1}{\lambda^q}\Big(\int_{\mathcal{H}} |f(z)|^p \omega(z)\, dV(z)\Big)^{q/p}.$$

(b) *There is a constant $C_2 > 0$ such that for any interval $I \subset \mathbb{R}$,*

$$|Q_I|^{1/q-1/p}\bigg(\frac{1}{|Q_I|}\int_{Q_I} \omega^{1-p'}(z)\, dV(z)\bigg)^{1/p'}\bigg(\frac{1}{|Q_I|}\int_{Q_I} \sigma(z)\, dV(z)\bigg)^{1/q} \le C_2$$

where $(|Q_I|^{-1}\int_{Q_I} \omega^{1-p'}(z)\, dV(z))^{1/p'}$ is understood as $(\inf_{Q_I} \omega)^{-1}$ when $p = 1$.

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